Trees of Primitive Pythagorean Triples

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Abstract

All and only primitive Pythagorean triples are generated by three trees of Firstov, among which are the UAD tree of Berggren et al. and the Fibonacci boxes FB tree of Price and Firstov.

Alternative proofs are offered here for the conditions on primitive Pythagorean triple preserving matrices and that there are only three trees with a fixed set of matrices and single root.

Some coordinate and area results are obtained for the UAD tree. Further trees with varying children are possible, such as filtering the Calkin-Wilf tree of rationals.

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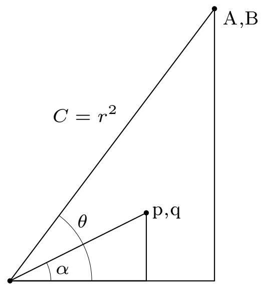
1 Pythagorean Triples

A Pythagorean triple is integers A,B,C satisfying $A^2 + B^2 = C^2$. These triples can be parameterized by two integers p,q as from Euclid [8,9],

$$\begin{aligned} A &= p^2 - q^2 && A \text{ leg odd} \\ B &= 2pq && B \text{ leg even} \\ C &= p^2 + q^2 && \text{hypotenuse} \\ p &= \sqrt{\tfrac{C+A}{2}} \qquad q = \sqrt{\tfrac{C-A}{2}} && \text{converse} \end{aligned} \tag{1}$$

If A,B and p,q are treated as points in the plane with polar coordinates C, θ and r, α respectively then they are related as



$$\begin{aligned} C &= r^2 && \text{square distance} \\ \theta &= 2\alpha && \text{double angle} \end{aligned} \tag{2}$$

This is complex squaring giving 2α and r^2,

$$(r\, e^{\alpha i})^2 = r^2\, e^{2\alpha i}$$
$$(p + iq)^2 = (p^2 - q^2) + 2pq\,i \;=\; A + Bi$$

Or the angle is since the ratio B/A written in terms of q/p is the tan double-angle formula,

$$\tan\theta = \frac{B}{A} = \frac{2pq}{p^2 - q^2} = \frac{2(q/p)}{1 - (q/p)^2} = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha \tag{3}$$

Kalman [14] calls an acute angle θ “Pythagorean” if it occurs in a Pythagorean triangle A, B, C and notes that this corresponds to $\sin\theta$ and $\cos\theta$ both rational.

1.1 Geometry

The usual geometric interpretation of p, q is that a Pythagorean triple divided by hypotenuse C corresponds to a rational point on the unit circle. Then a line drawn from the point back to $x = -1$, $y = 0$ has a rational slope and can be written q/p.

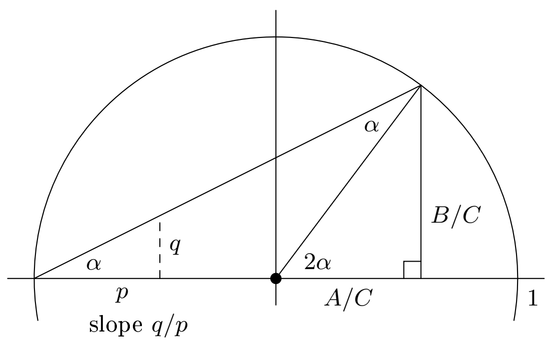
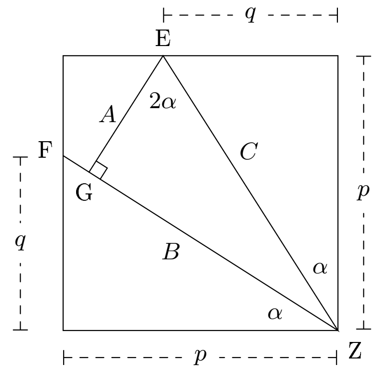


Figure 1:
rational points
on the unit circle

$$\left(\tfrac{A}{C}\right)^2 + \left(\tfrac{B}{C}\right)^2 = 1$$
$$\tfrac{q}{p} = \tfrac{B/C}{A/C+1}$$

Angle α for p,q doubling to 2α for A,B is per Euclid book III proposition 20.

Other constructions which double an angle q/p can make a Pythagorean triangle A,B,C too, possibly with some scale factor. Bonsangue [5] gives the following form in a square of side p. Line $\overline{EG}$ is dropped down from E perpendicular to line $\overline{FZ}$.



1.2 Primitive Pythagorean Triples

A primitive triple has $\gcd(A, B, C) = 1$. All triples are a multiple of some primitive triple. For A,B,C to be a primitive triple with positive A,B, the parameters p,q must be

$$\begin{aligned} p &> q && \text{so } p \geq 2 \\ q &\geq 1 && \\ p + q &\equiv 1 \bmod 2 && \text{opposite parity} \\ \gcd(p, q) &= 1 && \text{coprime} \end{aligned} \tag{4}$$

A 3×3 matrix can be applied by left multiplication to transform a triple to a new triple.

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}$$

A 2×2 matrix can do the same on p,q.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Palmer, Ahuja and Tikoo [20] give the following formula for the 3×3 matrix corresponding to a 2×2.

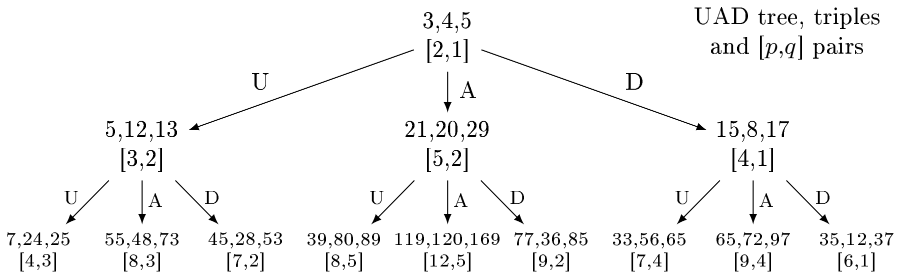
$$\begin{pmatrix} \frac{(a^2-c^2)-(b^2-d^2)}{2} & ab - cd & \frac{(a^2-c^2)+(b^2-d^2)}{2} \\ ac - bd & ad + bc & ac + bd \\ \frac{(a^2+c^2)-(b^2+d^2)}{2} & ab + cd & \frac{(a^2+c^2)+(b^2+d^2)}{2} \end{pmatrix} \tag{5}$$

They show the correspondence is one-to-one, so finding triple preserving matrices can be done in either 3×3 or 2×2. Generally the 2×2 is more convenient since it reduces primitive triples to pairs p,q coprime and not both odd.

$$\begin{tikzcd}
\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} p^2-q^2 \\ 2pq \\ p^2+q^2 \end{pmatrix} \arrow[rr, "3\times3"'] \arrow[d, leftrightarrow] & & \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} p'^2-q'^2 \\ 2p'q' \\ p'^2+q'^2 \end{pmatrix} \arrow[d, leftrightarrow] \\
\begin{pmatrix} p \\ q \end{pmatrix} \arrow[rr, "2\times2", "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] & & \begin{pmatrix} p' \\ q' \end{pmatrix}
\end{tikzcd}$$

2 UAD Tree

The UAD tree by Berggren[3, 1934] and independently Barning[2, 1963], Hall [12, 1970], Kanga[15, 1990] and Alperin[1, 2005], uses three matrices U,A,D to make a tree of all primitive Pythagorean triples.



$$U = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \qquad = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$\begin{aligned}
\text{row-wise A leg} &= 3,\ \ 5, 21, 15,\ \ 7, 55, 45, 39, 119, 77, 33, 65, 35, \ldots && \text{A321768} \\
\text{B leg} &= 4, 12, 20,\ \ 8, 24, 48, 28, 80, 120, 36, 56, 72, 12, \ldots && \text{A321769} \\
\text{C leg} &= 5, 13, 29, 17, 25, 73, 53, 89, 169, 85, 65, 97, 37, \ldots && \text{A321770}
\end{aligned}$$

$$\begin{aligned}
\text{row-wise } p &= 2, 3, 5, 4, 4, 8, 7, 8, 12, 9, 7, 9, 6, \ldots && \text{A321782} \\
q &= 1, 2, 2, 1, 3, 3, 2, 5,\ \ 5, 2, 4, 4, 1, \ldots && \text{A321783}
\end{aligned}$$

The pairs shown [2,1] etc are the p,q parameters for each triple. The matrices multiply onto a column vector of the triplet or pair. For example 15,8,17 and 4,1 in the second row descends by the U branch to

$$U \begin{pmatrix} 15 \\ 8 \\ 17 \end{pmatrix} = \begin{pmatrix} 33 \\ 56 \\ 65 \end{pmatrix} \qquad U \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

The 2×2 UAD matrices send a given p,q to the disjoint regions in figure 2.

$$
\begin{array}{lllr}
U & p' = 2p - q \; < \; 2p = 2q' & \text{so } p' < 2q' & (7) \\
A & p' = 2p + q \; > \; 2p = 2q' & & \\
 & p' = 2p + q \; < \; 3p = 3q' & \text{so } 2q' < p' < 3q' & (8) \\
D & p' = p + 2q \; > \; 3q = 3q' & \text{so } p' > 3q' &
\end{array}
$$

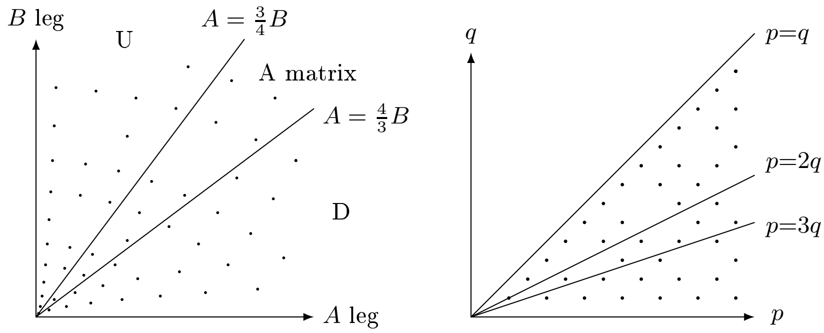


Figure 2: UAD tree regions of A,B legs and p,q points

Point 2,1 is on the $p = 2q$ line and is the tree root. Thereafter $p = 2q$ does not occur since it would have common factor q. The line $p = 3q$ is never touched at all since 3,1 is not opposite parity and anything bigger would have common factor q.

A,B leg points are double the angle of a p,q as per (2), so a ratio for p,q becomes a ratio for the A,B legs too and hence the 3×3 UAD matrices fall in corresponding disjoint regions. If $p = kq$ then, as from the double-angle (3),

$$
\begin{aligned}
B/A &= \frac{2(q/p)}{1 - (q/p)^2} = \frac{2/k}{1 - (1/k)^2} && \text{from } q/p = 1/k \\
A &= \frac{k^2 - 1}{2k} B
\end{aligned}
$$

$k = 2$ gives	$A = \frac{3}{4}B$	U matrix region
$k = 3$ gives	$A = \frac{4}{3}B$	A matrix region

Figure 3 shows how the 2×2 matrices transform a vertical line of p,q points. The dashed line $k, 1$ through $k, k{-}1$ becomes the solid lines by the respective U,A,D matrices.

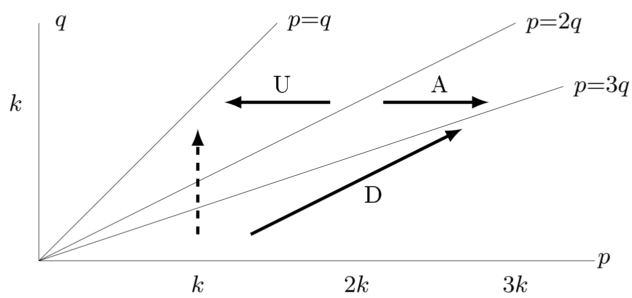


Figure 3: UAD tree, line transformations

D is a shear, as can be seen from its matrix $\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$. The line for D is longer but has the same number of coprime not-both-odd points as the original dashed line.

U is a rotate +90° then shear across to $p = 2q$. This shear is the same as D.

$$U = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \qquad \text{rotate then shear} \tag{9}$$
$$= D\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

A is the same as U but with a reflection to go to the right. This reflection $\left(\begin{smallmatrix}1 & 0\\0 & -1\end{smallmatrix}\right)$ gives the negative determinant $\det(A) = -1$.

$$A = U\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \text{mirror then same as U} \tag{10}$$
$$= D\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \text{mirror, rotate, shear}$$
$$= D\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

Repeated matrix U is the left-most side of the tree. These p, q and resulting triples A,B,C are

$$U^k\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}k{+}2\\k{+}1\end{pmatrix} \tag{11}$$

$$U^k\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}2k+3\\2(k{+}1)(k{+}2)\\2(k{+}1)(k{+}2)+1\end{pmatrix} = \begin{cases}3,\ 5,\ 7,\ 9, 11, \dots\\4, 12, 24, 40, 60, \dots & \text{A046092}\\5, 13, 25, 41, 61, \dots & \text{A099776}\end{cases}$$

This family of triples was known to Pythagoras [8] and is all those with leg difference $C - B = 1$. In the p, q parameterization, such a difference requires

$$p^2 + q^2 - 2pq = (p-q)^2 = 1 \qquad \text{so } p - q = 1$$

so a single index k suffices for p and q. Since $\gcd(k{+}2, k{+}1) = 1$, all these triples are primitive. In figure 2, they are the line of p,q points immediately below the $p{=}q$ diagonal. In A,B, the double-angle sends them to beside the B axis as a parabola $B = \frac{1}{2}(A^2 - 1)$ for A odd ≥ 3.

Repeated matrix A is the middle of each tree row. These p, q and resulting triples are

$$A^k\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}Pell_{k+2}\\Pell_{k+1}\end{pmatrix} = \begin{cases}2, 5, 12, 29, 70, \dots\\1, 2,\ 5, 12, 29, \dots\end{cases}$$

$$A^k\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}2\,Pell_{k+2}\,Pell_{k+1} - (-1)^k\\2\,Pell_{k+2}\,Pell_{k+1}\\Pell_{2k+3}\end{pmatrix} = \begin{cases}3, 21, 119, 697, \dots & \text{A046727}\\4, 20, 120, 696, \dots & \text{A046729}\\5, 29, 169, 985, \dots & \text{A001653}\end{cases}$$

where $Pell_k$ is the Pell numbers. Matrix A is precisely their recurrence,

$$Pell_k = 2\,Pell_{k-1} + Pell_{k-2} \qquad \text{starting } Pell_0 = 0,\ Pell_1 = 1 \tag{12}$$
$$= 0, 1, 2, 5, 12, 29, 70, 169, 408, \dots \qquad \text{A000129}$$

Leg C is the odd Pells by a usual identity $Pell_{k+2}^2 + Pell_{k+1}^2 = Pell_{2k+3}$.

Leg A is what are sometimes called Pell oblongs. A little recurrence manipulation shows (13).

$$\begin{aligned} PellOblong_k &= (Pell_{k+1} + Pell_k)\,(Pell_{k+1} - Pell_k) \\ &= 2\,Pell_{k+1} Pell_k + (-1)^k \qquad (13) \\ &= 1, 3, 21, 119, 697, 4059, 23661, \ldots \qquad \text{A084159} \end{aligned}$$

$(-1)^k$ means leg difference $|A - B| = 1$. Triples with such a leg difference were set as a challenge by Fermat (as related for instance by Mack and Czernezkyj [17]). Fermat's solution is the A matrix descent. Forget and Larkin [11] work though this leg difference requirement in terms of solutions to the Pell equation since in p,q the difference requires $p^2 - q^2 - 2pq = \pm 1$ which is a Pell equation $d^2 - 2q^2 = \pm 1$ for $d = p-q$.

In figure 2, these A,B points are immediately each side of the 45° leading diagonal $A{=}B$ which is the middle of the A matrix region. There are relatively few such since $Pell_k$ grows rapidly. The half angle in p,q is 22.5° in the middle of the 2×2 A matrix region by angle, which is slope

$$\frac{Pell_{k+1}}{Pell_{k+2}} \to \sqrt{2} - 1 \ = 0.414213\ldots \qquad \text{A188582}$$

Repeated matrix D is the right-most side of the tree. These p, q and resulting triples are

$$D^k \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2k{+}2 \\ 1 \end{pmatrix} \qquad (14)$$

$$D^k \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} (2k{+}2)^2 - 1 \\ 4(k{+}1) \\ (2k{+}2)^2 + 1 \end{pmatrix} = \begin{cases} 3, 15, 35, 63, \ \ 99, \ldots & \text{A000466} \\ 4, \ \ 8, 12, 16, \ \ 20, \ldots & \\ 5, 17, 37, 65, 101, \ldots & \text{A053755} \end{cases}$$

This family of triples was known to Plato [8] and is all those with leg difference $C - A = 2$. In the p, q parameterization, such a difference requires

$$p^2 + q^2 - (p^2 - q^2) \ = \ 2q^2 \ = 2 \qquad \text{so } \ q = 1$$

And then p, q not both odd means p must be even. Even p has gcd($2k{+}2$, 1) = 1 so is primitive. In figure 2, this is the line of p,q points immediately above the p axis. In A,B, the double-angle sends them to above the A axis as a shallow parabola $A = \frac{1}{4}B^2 - 1$.

2.1 UAD Tree Row Totals

Theorem 1. *In the UAD tree, the total leg difference $d_k = \sum B - A$ in tree row k is $d_k = (-1)^k$.*

Proof. Depth 0 is the single point $A{=}3$, $B{=}4$ and its leg difference is $B - A = 1$ $= (-1)^0$.

Let $(A_1, B_1, C_1), \ldots, (A_n, B_n, C_n)$ be the triples at depth $k{-}1$. The children of those points, at depth k, are then

$$U\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}, A\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}, D\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}, \;\ldots,\; U\begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix}, A\begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix}, D\begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix}$$

Their total is

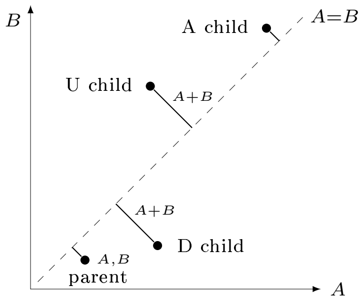
$$\begin{pmatrix}\alpha_k\\\beta_k\\\gamma_k\end{pmatrix} = (U{+}A{+}D)\left(\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix} + \cdots + \begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix}\right) = \begin{pmatrix}1&2&6\\2&1&6\\2&2&9\end{pmatrix}\begin{pmatrix}\alpha_{k-1}\\\beta_{k-1}\\\gamma_{k-1}\end{pmatrix}$$

So the difference

$$\begin{aligned} d_k &= \beta_k - \alpha_k \\ &= (2\alpha_{k-1} + \beta_{k-1} + 6\gamma_{k-1}) - (\alpha_{k-1} + 2\beta_{k-1} + 6\gamma_{k-1}) \\ &= \alpha_{k-1} - \beta_{k-1} \\ &= -d_{k-1} \end{aligned}$$

□

The following diagram shows the geometric interpretation of this in a plot of legs A and B. The leg difference $A-B$ is the distance from the leading diagonal $A{=}B$.



UAD children
leg difference
symmetry

For the U and D children, the leg difference is $A+B$ of the original parent point A,B, each side of the leading diagonal.

$$\begin{aligned} A'_U &= A - 2B + 2C && \text{U matrix child legs} \\ B'_U &= 2A - B + 2C \\ B'_U - A'_U &= A + B && \text{U child leg difference} \\ A'_D &= -A + 2B + 2C && \text{D matrix child legs} \\ B'_D &= -2A + B + 2C \\ B'_D - A'_D &= -(A + B) && \text{D child leg difference} \end{aligned}$$

The leg differences of U and D are opposite sign and so cancel out in the total,

$$(B'_U - A'_U) + (B'_D - A'_D) \;=\; 0$$

Matrix A mirrors a given A,B leg pair across the leading diagonal. It changes the sign of the leg difference but not the magnitude. On further descent, this mirroring applies to both the U and D children so their leg differences continue to cancel in further tree levels.

$$A'_A = A + 2B + 2C \qquad \text{A matrix child legs}$$

$$B'_A = 2A + B + 2C$$
$$B'_A - A'_A = -(B - A)$$

The initial triple 3,4,5 with leg difference $4-3=1$ alternates 1 and -1 as the A matrix is repeatedly applied to it at each tree level. For all other children, the leg differences cancel out.

Theorem 2. *In the UAD tree, total p and total q in row k are successive terms of the Pell sequence* (12)*,*

$$\begin{aligned} ptotal_k &= \sum_{\textit{at depth } k} p \quad = Pell_{2k+2} \qquad \textit{even Pell} \\ &= 2, 12, 70, 408, 2378, 13860, 80782, \ldots \qquad \text{A001542} \\ qtotal_k &= \sum_{\textit{at depth } k} q \quad = Pell_{2k+1} \qquad \textit{odd Pell} \\ &= 1, 5, 29, 169, 985, 5741, 33461, \ldots \qquad \text{A001653} \end{aligned}$$

Proof. Depth 0 is the single point $p{=}2, q{=}1$ so that $ptotal_0 = 2 = Pell_2$ and $qtotal = 1 = Pell_1$.

Let $p_1, q_1, \ldots, p_n, q_n$ be the points at depth k. The children of these points, at depth $k{+}1$ are

$$U\begin{pmatrix}p_1\\q_1\end{pmatrix}, A\begin{pmatrix}p_1\\q_1\end{pmatrix}, D\begin{pmatrix}p_1\\q_1\end{pmatrix}, \;\ldots,\; U\begin{pmatrix}p_n\\q_n\end{pmatrix}, A\begin{pmatrix}p_n\\q_n\end{pmatrix}, D\begin{pmatrix}p_n\\q_n\end{pmatrix}$$

Their sum is

$$\begin{pmatrix}ptotal_{k+1}\\qtotal_{k+1}\end{pmatrix} = (U + A + D)\left(\begin{pmatrix}p_1\\q_1\end{pmatrix} + \cdots + \begin{pmatrix}p_n\\q_n\end{pmatrix}\right) = \begin{pmatrix}5 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}ptotal_k\\qtotal_k\end{pmatrix}$$

$qtotal_{k+1} = 2ptotal_k + qtotal_k$ is the Pell recurrence (12).

$ptotal_{k+1} = 5ptotal_k + 2qtotal_k$ is the Pell recurrence $Pell_{2k+4}$ in terms of $Pell_{2k+2}$ and $Pell_{2k+1}$,

$$\begin{aligned} Pell_{2k+4} &= 2Pell_{2k+3} + Pell_{2k+2} \\ &= 2\,(2Pell_{2k+2} + Pell_{2k+1}) + Pell_{2k+2} \\ &= 5Pell_{2k+2} + 2Pell_{2k+1} \end{aligned}$$ □

2.2 UAD Tree Iteration

The points of the UAD tree can be iterated row-wise by a method similar to what Newman[19] gave for the Calkin-Wilf tree. At a given p,q, a division finds how many trailing D, and from that go up, across, and back down.

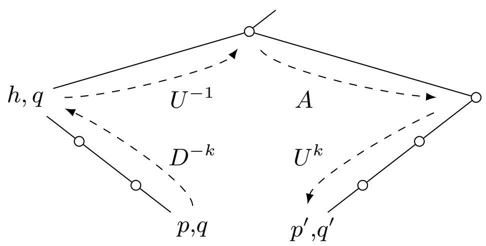


Figure 4:
p,q with $k \geq 0$
steps of D down from
U child h,q

Theorem 3. *The next point after p,q going row-wise across the UAD tree, and from the end of a row to the start of the next, is*

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{1}{2}p + 2 \\ \frac{1}{2}p + 1 \end{pmatrix} & \textit{if } q = 1 \\ \begin{pmatrix} p + 2q - \frac{1}{2}(m{+}3)\,r \\ p \qquad - \frac{1}{2}(m{+}1)\,r \end{pmatrix} & \textit{if } m \textit{ odd} \\ \begin{pmatrix} \frac{1}{2}p \qquad + \frac{1}{2}(m{+}1)\,r \\ \frac{1}{2}p - q + \frac{1}{2}(m{-}1)\,r \end{pmatrix} & \textit{if } m \textit{ even} \end{cases} \tag{15}$$

where m and r are quotient and remainder from division of p by q,

$$m = \lfloor p/q \rfloor, \ \ \textit{remainder } r = p - m.q \ \ \textit{range } 0 \le r < q \tag{16}$$

Proof. If h,q in figure 4 is the root 2,1, rather than a child, then p,q is per (14)

$$\begin{pmatrix} p \\ q \end{pmatrix} = D^k \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2k+2 \\ 1 \end{pmatrix}$$

Its next p', q' is the first of the next row, which is the first case in (15)

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = U^{k+1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} k+3 \\ k+2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}p + 2 \\ \frac{1}{2}p + 1 \end{pmatrix}$$

Otherwise, $k \ge 0$ many descents down from h,q is

$$\begin{pmatrix} p \\ q \end{pmatrix} = D^k \begin{pmatrix} h \\ q \end{pmatrix} = \begin{pmatrix} h + k.2q \\ q \end{pmatrix}$$

Dividing $p = h + k.2q$ by q at (16) determines k.

If h,q is a U child then $q < h < 2q$ per (7) so the division gives m odd and $k = \frac{1}{2}(m{-}1)$, $h = r{+}q$. At h,q, go up U then down A and down k steps of U. This is the second case in (15)

$$U^k.A.U^{-1} \begin{pmatrix} h \\ q \end{pmatrix} = \begin{pmatrix} (m{+}2)q - \frac{1}{2}(m{+}1)r \\ mq - \frac{1}{2}(m{-}1)r \end{pmatrix} = \begin{pmatrix} p + 2q - \frac{1}{2}(m{+}3)\,r \\ p \qquad - \frac{1}{2}(m{+}1)\,r \end{pmatrix}$$

If h,q is an A child then $2q < h < 3q$ per (8) so the division gives m even and $k = \frac{1}{2}m - 2$, $h = r + 2q$. At h,q, go up A then down D and down k steps of U,

$$U^k.D.A^{-1} \begin{pmatrix} h \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{2}mq + \frac{1}{2}(m{+}2)\,r \\ \frac{1}{2}(m{-}2)\,q + \frac{1}{2}mr \end{pmatrix} = \begin{pmatrix} \frac{1}{2}p \qquad + \frac{1}{2}(m{+}1)\,r \\ \frac{1}{2}p - q + \frac{1}{2}(m{-}1)\,r \end{pmatrix} \qquad \square$$

At (15), the division when $q{=}1$ gives $r{=}0$. This is the only case $r{=}0$, since $\gcd(p,q) = 1$ for a primitive triple. So $r{=}0$ can be used to distinguish the end-of-row case if preferred. It can be noticed the resulting $m{=}p$ would be the m even case, however that does not give the necessary $\frac{1}{2}p{+}2$, $\frac{1}{2}p{+}1$ and hence a separate $q{=}1$ case.

2.3 UAD Tree Low to High

A complete ternary tree has 3^k points at depth k. Positions across the row can be written in ternary with k digits. The tree descent U,A,D at each vertex follows those digits from high to low, ie. most significant to least significant, and multiply on the left for each digit.

If digits are instead taken low to high then figure 5 shows how the tree branches remain in the separate regions of figure 2. This can be thought of recursively as the tree at a given point being a copy of the whole tree with the matrices which reach that point applied.

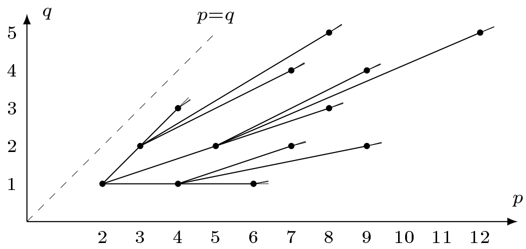


Figure 5:
UAD tree branches,
digits low to high

Theorem 4. *The p parameter across each row of the UAD tree is the same for ternary digits high to low or low to high. This is since for any product of the $2{\times}2$ U,A,D matrices,*

$$X = x_1\, x_2 \cdots x_{k-1}\, x_k \qquad \textit{each } x_i = U \text{ or } A \text{ or } D$$

the reversed product $Rev(X) = x_k\, x_{k-1} \cdots x_2\, x_1$ is given by

$$Rev\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 2c+d & 2a+b-4c-2d \\ c & a-2c \end{pmatrix} \tag{17}$$

Proof. The empty product is the identity matrix and for it $Rev\big(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\big) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ which is per (17). Proceed then by induction, supposing the theorem is true for all products up to k matrices. Further $x_{k+1} = U$ on the right of $X = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is

$$\begin{aligned} Rev(X\,.U) &= \begin{pmatrix} 3c+2d & 3a+2b-6c-4d \\ 2c+d & 2a+b-4c-2d \end{pmatrix} \\ &= U\,.Rev(X) \end{aligned}$$

So the reversal formula holds for another U. For the A and D matrices similarly $Rev(X.A) = A.Rev(X)$ and $Rev(X.D) = D.Rev(X)$, and so any product of $k{+}1$ matrices.

The p parameter is from $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ or $Rev\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ applied to $\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$. They give the same $p = 2a{+}b$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+b \\ 2c+d \end{pmatrix} \qquad Rev\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+b \\ a \end{pmatrix} \tag{18}$$

□

The q parameter at (18) is respectively $2c{+}d$ and a which are not identical, and indeed q does differ variously between low to high and high to low for depth ≥ 2.

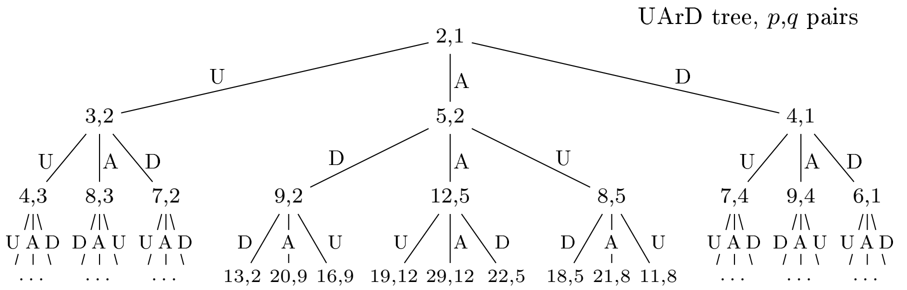
In *Rev* at (17), the coefficients for a, b, c, d are found by considering what they must do for a selection of particular X matrices. Suppose the top left element of *Rev* is $\alpha a + \beta b + \gamma c + \delta d$. Then for example $X = U.A = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and its reversal $A.U = \left(\begin{smallmatrix} 5 & \cdots \\ \cdots & \cdots \end{smallmatrix}\right)$ requires

$$\alpha.3 + \beta.2 + \gamma.2 + \delta.1 \;=\; 5$$

The 6 combinations of U,A,D pairs give 6 equations in 4 unknowns. Some linear algebra shows they have a unique solution for each entry in *Rev*. The task in the theorem is then to show *Rev* works for any X, not just those used to find the coefficients.

3 UArD Tree

A variation on the UAD tree can be made by applying a left–right mirror image under each A matrix. Call this UArD. The three children at each node are the same but the order is reversed when under an odd number of A descents.



The entire sub-tree under each A is mirrored. Under an even number of A matrices, the mirrorings cancel out to be plain again. For example the middle 12,5 shown is under A,A and so its children are U-A-D again.

The effect of these mirrorings is to apply matrices by ternary reflected Gray code digits high to low. For example the row shown at depth 2 goes

U-A-D, D-A-U, U-A-D

Theorem 5. *Across a row of the UArD tree, the p,q points take steps alternately horizontal (q unchanged) and diagonal (same increment p and q).*

The steps can be illustrated in the row at depth 2. ($hdist_2$ and $ddist_2$ are for theorem 6 below.)

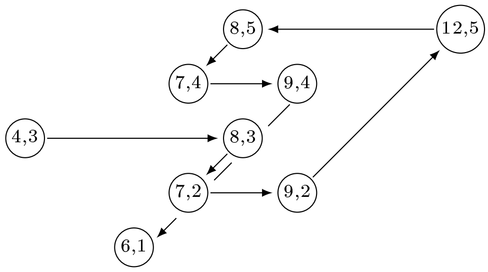


Figure 6:
UArD steps of p,q
in row at depth k=2

$hdist_2 = 4{+}2{+}4{+}2 = 12$

$ddist_2 = 1{+}3{+}1{+}3 = 8$

Proof. A step between U and A children is always horizontal since

$$A\begin{pmatrix}p\\q\end{pmatrix} - U\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}2q\\0\end{pmatrix} \tag{19}$$

A step between A and D children is always diagonal since

$$A\begin{pmatrix}p\\q\end{pmatrix} - D\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}p-q\\p-q\end{pmatrix} \tag{20}$$

A UArD row steps between children in sequence either U-A-D or D-A-U and with each always followed by its reversal.

```
                 p,q
          /       |       \
        U         A         D
      / | \     / | \     / | \
      U A D     D A U     U A D
```

When going across a D-D gap, the common ancestor is U-A since D is at the right edge of the plain sub-tree beneath U and the left edge of the reflected sub-tree beneath A. This U-A ancestor is a horizontal step (19). The D descents multiply on the left and preserve the horizontal step and its distance since horizontal p to $p+x$ has

$$D\begin{pmatrix}p+x\\q\end{pmatrix} - D\begin{pmatrix}p\\q\end{pmatrix} = D\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} \tag{21}$$

When going across a U-U gap, the common ancestor is A-D since U is the right edge of the reflected sub-tree beneath A and the left edge of the plain sub-tree beneath D. This A-D ancestor is a diagonal step (20). The U descents multiply on the left and preserve the diagonal and its distance since

$$U\begin{pmatrix}p+x\\q+x\end{pmatrix} - U\begin{pmatrix}p\\q\end{pmatrix} = U\begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}2&-1\\1&0\end{pmatrix}\begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}x\\x\end{pmatrix} \tag{22}$$

So sequence U-A-D-D-A-U-U-A-D etc across a row takes steps alternately horizontal and diagonal. □

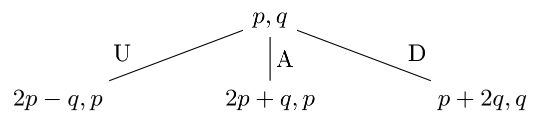
Theorem 6. *In row k of the UArD tree, the total distance of horizontal p,q steps is an even index Pell number,*

$$hdist_k = Pell_{2k} \qquad \textit{even Pells}$$

and the total distance (in one coordinate) of diagonal p,q steps is the sum of Pell numbers with alternating signs

$$\begin{aligned} ddist_k &= Pell_{2k} - Pell_{2k-1} + \cdots + Pell_2 - Pell_1 \\ &= 6\,ddist_{k-1} - ddist_{k-2} + 2 \quad \textit{starting } ddist_{0,1} = 0, 1 \\ &= 0, 1, 8, 49, 288, 1681, 9800, \ldots \qquad \text{A001108} \end{aligned}$$

Proof. Take a p,q point at depth $k-1$ and let h and d be the horizontal and diagonal distances between its three children. The diagonal distance d is the change in each coordinate.



$h = (2p+q) - (2p-q) = 2q \qquad d = (p+2q) - (2p+q) = p - q$

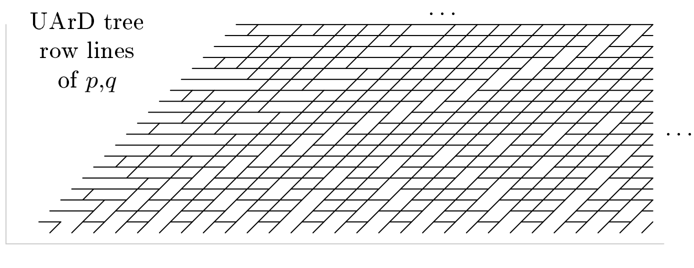
$hdist_k$ is U-A step $h = 2q$, where q is each from depth $k-1$, plus the D-D steps between triples at depth k. Per (21), D-D steps are U-A steps propagated down from preceding depth levels. So $hdist$ is $2q$ of all points in all depth levels $< k$. Final $Pell_{2k}$ is the usual sum of odd Pells,

$$hdist_k = \sum_{\substack{\text{depths} \\ 0 \text{ to } k-1}} 2q \;= 2\sum_{i=0}^{k-1} qtotal_i \;= 2\sum_{i=0}^{k-1} Pell_{2i+1} \;= Pell_{2k}$$

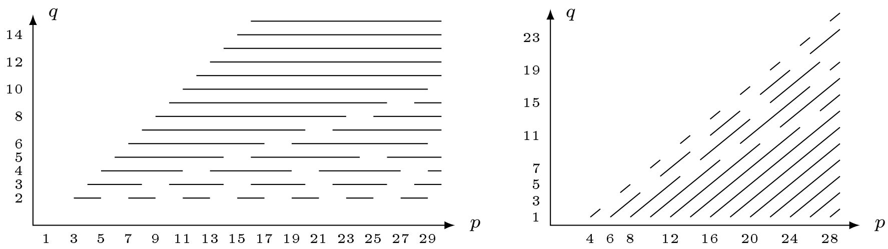
$ddist_k$ is the A-D steps $d = p - q$, where p,q is each from $k-1$, plus the U-U steps between triples at depth k. Per (22), U-U steps are A-D propagated down from preceding depth levels. So $ddist$ is $p-q$ of all points in all depth levels $< k$.

$$ddist_k = \sum_{\substack{\text{depths} \\ 0 \text{ to } k-1}} p - q \;= \sum_{i=0}^{k-1} ptotal_i - qtotal_i \;= \sum_{i=0}^{k-1} Pell_{2i+2} - Pell_{2i+1} \qquad \square$$

The UArD row lines variously overlap. The k=2 sample in figure 6 has one such overlap. The following diagram shows all row lines plotted together. They continue above and right.



The pattern of gaps can be seen by separating the horizontals and diagonals. The diagonals are always on odd differences $p-q$ since p,q are opposite parity.



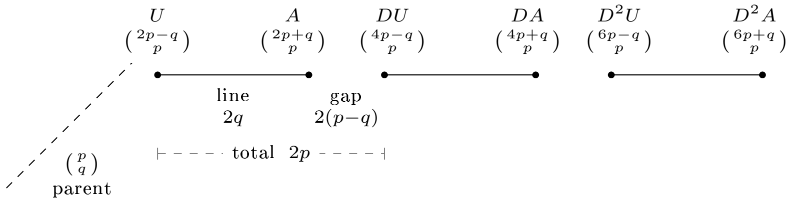
horizontals
lengths $2q{-}2 = 2, 4, 6, 8, \ldots$
with gaps 2

diagonals
lengths $p{-}q{-}2 = 1, 3, 5, 7, \ldots$
with gaps 2

UArD tree horizontal and diagonal steps of p,q

Theorem 7. *When all UArD tree row lines are plotted, the horizontals are length $2q' - 2$ with gap 2 in between and the diagonals are length $p' - q' - 2$ with gap 2 in between.*

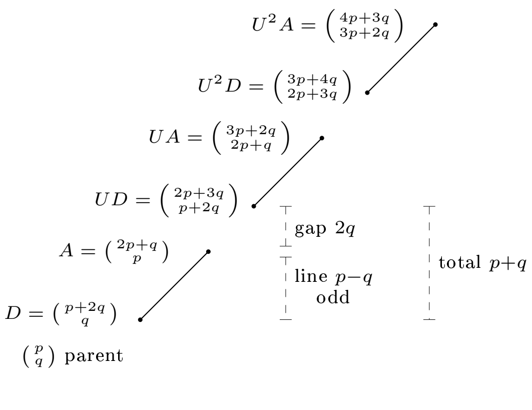
Proof. As from theorem 5 above, horizontal lines arise from the U child to A child of a p,q parent. The D matrix on them in subsequent tree levels preserves the length. The D matrix preserves the child q' but shears the line across so that it repeats in the following way,



When the parent point is $q = p{-}1$, the gap is $2\,(p{-}q) = 2$ and the lines are length $2(p-1)$ which with $q' = p$ is $2q' - 2$.

Other parent points with the same p give line endings $2p \pm q$ so they are centred at $2p$, $4p$, etc. The longest line is $q = p - 1$ and it overlaps all others.

Also from theorem 5, diagonal lines arise from the A child to D child of a given p,q parent. The U matrix on them in subsequent tree levels preserves the direction and length. U also preserves the $d' = p' - q'$ diagonal position but shifts it up in the following way.



When the parent is q=1, the gap $2q$=2 and the lines are length $p-q=p$−1. Measured from the child $d'=p'-q'$, this is $p'-q'-2=p+2q-q-2=p-1$.

Lines from other parent p,q which fall on the same diagonal are centred on the line midpoint W. The longest is when $p-q$ is the maximum which is q=1 and this longest line overlaps all others.

$$\begin{aligned} W &= \begin{pmatrix} (3p+3q)/2 \\ (p+q)/2 \end{pmatrix} \\ D\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} p+2q \\ q \end{pmatrix} = W - \begin{pmatrix} (p-q)/2 \\ (p-q)/2 \end{pmatrix} \\ A\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} 2p+q \\ p \end{pmatrix} = W + \begin{pmatrix} (p-q)/2 \\ (p-q)/2 \end{pmatrix} \end{aligned} \qquad \square$$

3.1 UArD Tree Low to High

The mirroring in UArD can be taken low to high too. The Gray code is applied first, then digits are taken low to high. The mirroring under each A compensates for the reflection in the A matrix (10).

n	ternary	Gray	matrices	
0	000	000	U.U.U $\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)=\left(\begin{smallmatrix}5\\4\end{smallmatrix}\right)$	
1	001	001	U.U.A	
2	002	002	U.U.D	
3	010	012	U.A.D	
4	011	011	U.A.A	reflect
5	012	010	U.A.U	
6	020	020	A.A.U	
7	021	021	A.A.A	
8	022	022	A.A.D	
...				
26	333	333	D.D.D $\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)=\left(\begin{smallmatrix}8\\1\end{smallmatrix}\right)$	

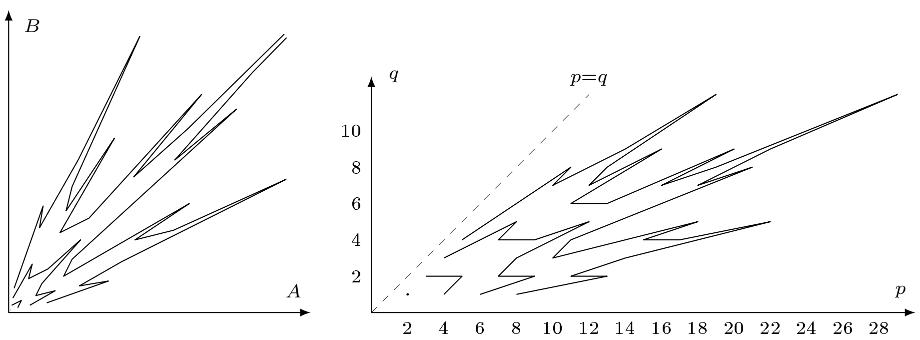
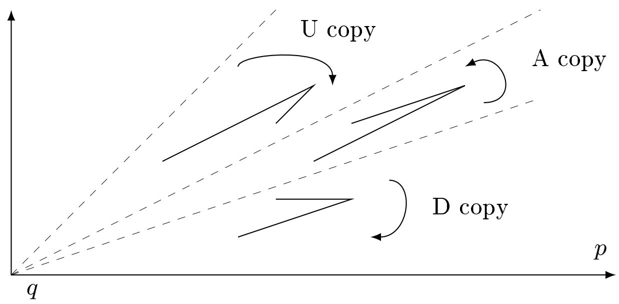


Figure 7: UArD tree rows, digits low to high

Theorem 8. *The A,B legs and the p,q points in a row of the low-to-high UArD tree go clockwise when plotted as points.*

Proof. In row 0 there is a single point p=2, q=1.

For a subsequent row, the new matrices U,A,D multiply on the left and so copy the previous row p,q points into their respective matrix regions as from figure 2, which is also in the manner of the low-to-high branches figure 5.



The D copy is a shear and so maintains the clockwise order. The U copy is a rotate and shear (9) and so also maintains clockwise order.

The A copy includes a reflection so reverses to anti-clockwise. But for UArD, the points under A are mirrored so clockwise order is restored.

The A,B legs, as points, are at double the angle of the corresponding p,q per (3). So A,B points go clockwise too. □

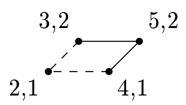
3.2 UArD Tree Row Area

The row lines of the UArD low to high (figure 7) do not intersect preceding rows and so give the shape of an expanding region of p,q or A,B coverage by the tree. This area is the same for all UAD tree forms, only the order of points within a row differs. The area of this expanding region, in p,q, can be calculated as follows.

Theorem 9. *For the UAD tree in p,q coordinates, the area r_k between row k and $k+1$, and the total area R_k up to row k, are*

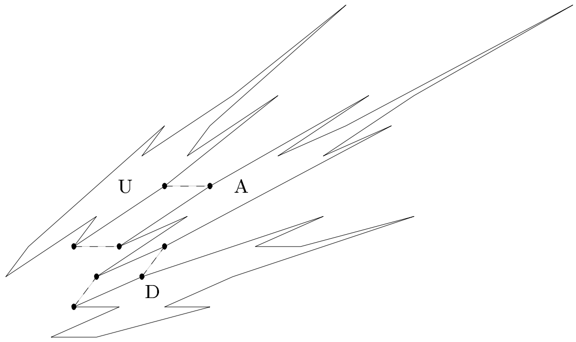
$$
\begin{aligned}
r_k &= 5.3^k - 3 && \textit{area between rows } k \textit{ and } k{+}1 \\
&= 2, 12, 42, 132, 402, 1212, 3642, 10932, \dots && 2{\times}\mathrm{A134931} \\
R_k &= \sum_{i=0}^{k-1} r_i \;= \tfrac{5}{2}(3^k{-}1) - 3k && \textit{total area to row } k \\
&= 0, 2, 14, 56, 188, 590, 1802, 5444, 16376, \dots
\end{aligned}
$$

Proof. The area between row 0 and row 1 is the initial parallelogram 2,1–4,1–5,2–3,2 of area 2 which is $r_0 = 5.3^0 - 3 = 2$.

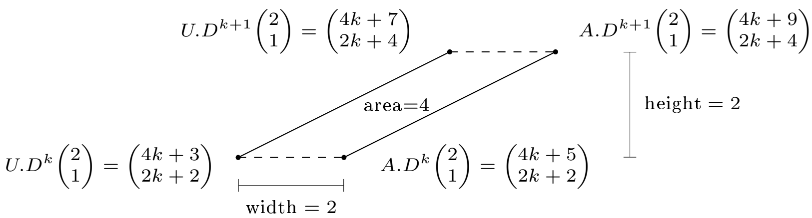


initial parallelogram
area = 2

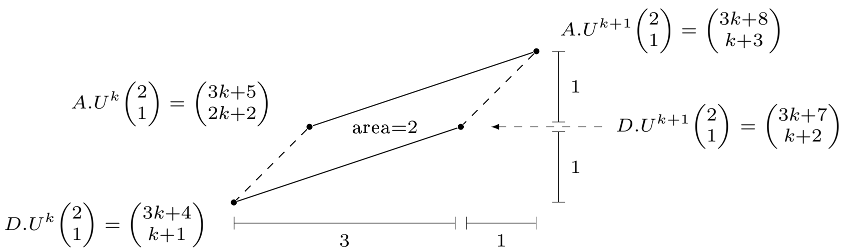
For a subsequent row, the area between rows k and $k+1$ is three copies of the preceding $k-1$ to k area transformed by multiplication on the left by U,A,D. Those transformations don't change the area. Between the copies are two gaps show below by dashed lines.



The upper gap U to A is a parallelogram. The right edge of the U block is $U.D^k$. The left edge of the A block is $A.D^k$ since A is a reflection. The result is area 4.



The lower gap A to D is a parallelogram. The upper edge of the D block is $D.U^k$. The lower edge of the A block is $A.U^k$ since A is a reflection. The result is area $3.1 - 1.1 = 2$.



So the gaps are $4+2=6$ and the area between rows is thus per the theorem

$$r_k = 3r_{k-1} + 6 \qquad \square$$

A recurrence for the total area R using (23) is

$$\begin{aligned} R_k &= r_0 + \sum_{i=1}^{k-1} (3r_{i-1} + 6) \\ &= 2 + 6(k-1) + 3R_{k-1} \ = 3R_{k-1} + 6k - 4 \end{aligned} \tag{23}$$

The parallelograms making up the rows are a tiling of the eighth of the plane $p > q \geq 1$ using parallelograms of areas 2 and 4, but the repeated shears soon make them very elongated.

3.3 UArD as Filtered Stern-Brocot

The Stern-Brocot tree enumerates all rationals $p/q \geq 1$. It can be filtered to pairs $p > q \geq 1$ not-both-odd to give primitive Pythagorean triples. Katayama [16] shows this is the UArD tree low-to-high.

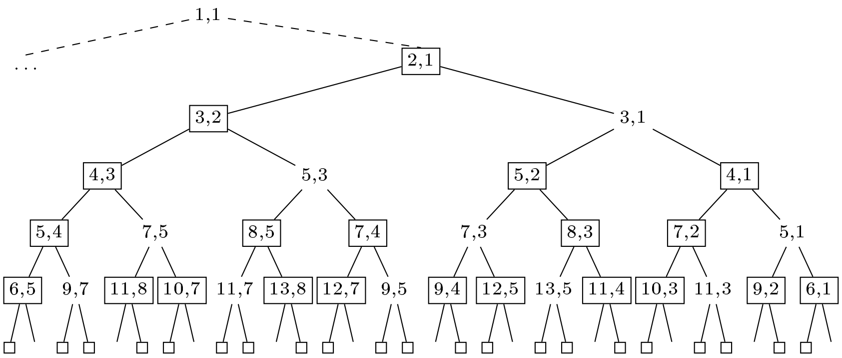


Figure 8: Stern-Brocot tree filtered to $p > q$ not-both-odd

Every third node is odd/odd when read by rows wrapping around at each row end. Those nodes can be removed to leave the nodes shown boxed in figure 8. The two children of a removed node are adopted by their grandparent to make a ternary tree.

The Stern-Brocot tree applies matrices by bits of the row position taken low-to-high. This means left and right sub-trees are defined recursively by a matrix L or R multiplied onto the points of the entire tree.

tree

L.tree	R.tree
$\mathrm{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\mathrm{R} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$1/(1 + 1/\text{tree})$	$1 + \text{tree}$

The equivalence to UArD by Katayama can be outlined as follows. The sub-tree at 3,2 is $R.L$.tree and has same structure as the sub-tree at 2,1. Map from 2,1 to 3,2 using a matrix U. This is seen to be the U matrix of the UAD tree.

$$
\begin{aligned}
U \,.\, R.\text{tree} &= R.L.\text{tree} \\
U \,.\, R &= R.L \\
U &= R.L.R^{-1} = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)
\end{aligned}
$$

Similarly mapping 2,1 to 4,1 is the D matrix of the UAD tree.

$$
\begin{aligned}
D \,.\, R.\text{tree} &= R.R.R.\text{tree} \\
D &= R.R = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)
\end{aligned}
$$

The not-both-odd points of the sub-tree under 5,2 have a left-to-right mirroring. Swapping $p \leftrightarrow q$ performs such a mirroring in the Stern-Brocot tree.

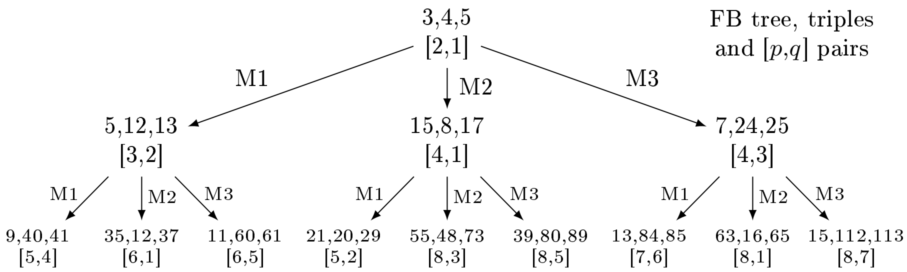
Apply that first with matrix $S = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and then descend. Mapping 2,1 to 5,2 is then by a matrix A as follows and which is seen to be the A matrix of the UAD tree.

$$\begin{aligned}
&A\,.\,R.\text{tree} = R.R.L.S.\text{tree}\\
&A.R = R.R.L.S\\
&A = R.R.L.S.R^{-1} = \left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)
\end{aligned}$$

If the filtered tree is read left to right then that reading includes the mirroring under each A. For example under 5,2 points 9,4–12,5–8,3 are sub-trees D–A–U. That mirroring is per UArD. The clockwise order of pairs in the Stern-Brocot rows corresponds to the clockwise order in the UArD rows of theorem 8.

4 FB Tree

Firstov [10] and Price [22] independently give another tree using a different set of three matrices M1,M2,M3 (in Price's naming).

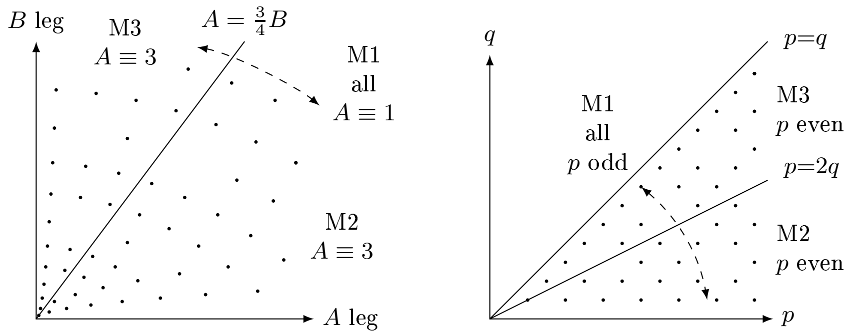


$$\begin{aligned}
&\mathit{M1} = \begin{pmatrix}2&1&-1\\-2&2&2\\-2&1&3\end{pmatrix} \qquad \mathit{M2} = \begin{pmatrix}2&1&1\\2&-2&2\\2&-1&3\end{pmatrix} \qquad \mathit{M3} = \begin{pmatrix}2&-1&1\\2&2&2\\2&1&3\end{pmatrix}\\
&\quad\;\; = \begin{pmatrix}1&1\\0&2\end{pmatrix} \qquad\qquad\qquad = \begin{pmatrix}2&0\\1&-1\end{pmatrix} \qquad\qquad\qquad = \begin{pmatrix}2&0\\1&1\end{pmatrix}
\end{aligned} \tag{24}$$

Matrix M1 is all p odd. Matrices M2 and M3 are p even and in disjoint regions M3 above and M2 below the $p\,{=}\,2q$ line.

$$\begin{array}{llll}
\text{M2} & p' = 2p \;>\; 2p - 2q = 2q' & & p' > 2q' \\
\text{M3} & p' = 2p \;<\; 2p + 2q = 2q' & & p' < 2q'
\end{array} \tag{25}$$

When p is even, leg $A \equiv 3 \bmod 4$. When p is odd, leg $A \equiv 1 \bmod 4$.



FB tree, regions of p,q

Figure 9 shows how the matrices transform a vertical line of points $k,1$ through $k,k{-}1$ out to bigger p,q.

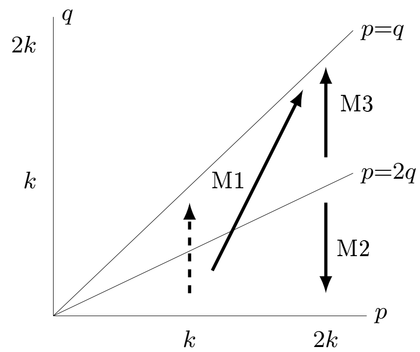


Figure 9:
FB tree
line transformations

M3 is the simplest, just a shift up to point p,p.

$$M3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

M2 is the same as M3 but negating q first so it's mirrored to go downwards instead.

$$M2 = M3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{negate to mirror } q\text{, then M3}$$

M1 is a horizontal shear, followed by a vertical stretch.

$$M1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{shear then stretch}$$

Repeated M1 is the left-most side of the tree. These p,q and resulting triples are

$$M1^k \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^k + 1 \\ 2^k \end{pmatrix} \tag{26}$$

$$M1^k \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2^{k+1} + 1 \\ 2^{k+1}(2^k + 1) \\ 2^{k+1}(2^k + 1) + 1 \end{pmatrix} = \begin{cases} 3,\ 5,\ 9,\ 17,\ 33, \ldots & \text{A000051} \\ 4, 12, 40, 144, 544, \ldots & \text{A028403} \\ 5, 13, 41, 145, 545, \ldots & \text{A085601} \end{cases}$$

(26) has $p = q{+}1$ like matrix U at (11), but only $p = 2^k{+}1$, not all integers, so a subset of U repeatedly.

Repeated M2 is the middle of each tree row. These p, q and resulting triples are

$$M2^k \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2^{k+1}\\ \mathit{Jacobsthal}_{k+1}\end{pmatrix}$$

$$M2^k \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}4^{k+1} - \mathit{Jacobsthal}_{k+1}^2\\ 2^{k+1}\,\mathit{Jacobsthal}_{k+1}\\ 4^{k+1} + \mathit{Jacobsthal}_{k+1}^2\end{pmatrix} = \begin{cases} 3, 15, 55, 231,\ \ 903, \dots & \text{A015249}\\ 4,\ \ 8, 48, 160,\ \ 704, \dots & \text{A054881}\\ 5, 17, 73, 281, 1145, \dots & \tfrac{1}{2}\text{A108924}\end{cases}$$

where the Jacobsthal numbers are

$$\begin{aligned}\mathit{Jacobsthal}_k &= \tfrac{1}{3}\left(2^k - (-1)^k\right)\\ &= 0, 1, 1, 3, 5, 11, 21, 43, 85, \dots \qquad \text{A001045}\end{aligned}$$

The A leg values are sometimes called Jacobsthal oblongs since they're not squares but rather consecutive product

$$4^{k+1} - \mathit{Jacobsthal}_{k+1}^2 \ = \ \mathit{Jacobsthal}_{k+2}\,\mathit{Jacobsthal}_{k+3}$$

Repeated M3 is the right-most side of the tree. These p, q and resulting triples are

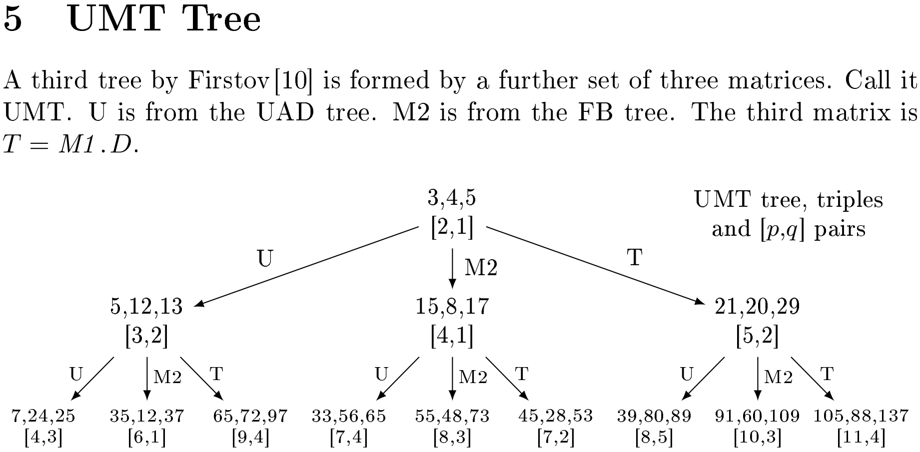
$$M3^k \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2^{k+1}\\ 2^{k+1} - 1\end{pmatrix} \tag{27}$$

$$M3^k \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}2^{k+2} - 1\\ 2^{k+2}(2^{k+1} - 1)\\ 2^{k+2}(2^{k+1} - 1) + 1\end{pmatrix} = \begin{cases} 3,\ \ 7,\ \ 15,\ \ 31,\ \ \ 63, \dots & \text{A000225}\\ 4, 24, 112, 480, 1984, \dots & \text{A059153}\\ 5, 25, 113, 481, 1985, \dots & \text{A092440}\end{cases}$$

(27) has $p = q{+}1$ like M1 above and like matrix U at (11), but only $p = 2^{k+1}$, not all integers, so M3 repeatedly has leg difference $C - B = 1$ and is another subset of U repeatedly.

5 UMT Tree

A third tree by Firstov [10] is formed by a further set of three matrices. Call it UMT. U is from the UAD tree. M2 is from the FB tree. The third matrix is $T = M1\,.D$.



$$
\begin{aligned}
U &= \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix} &\qquad M2 &= \begin{pmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} &\qquad T &= \begin{pmatrix} -2 & 3 & 3 \\ -6 & 2 & 6 \\ -6 & 3 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} & &= \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} & &= \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \\
& & & & &= M1 \,.\, D
\end{aligned} \tag{28}
$$

The matrix sum is the same as in UAD and so like theorem 2 the total p and q at depth k of the UMT are the Pell numbers.

$$
U + M2 + T = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = U + A + D
$$

Matrix U is all points $p < 2q$ (7). Matrix M2 is points $p > 2q$ with p even (25). Matrix T is points $p > 2q$ with p odd since

$$
T \qquad p' = p + 3q \ > \ 4q = 2q' \qquad\qquad \text{so } p' > 2q' \tag{29}
$$

Together M2 and T are all points $p > 2q$.

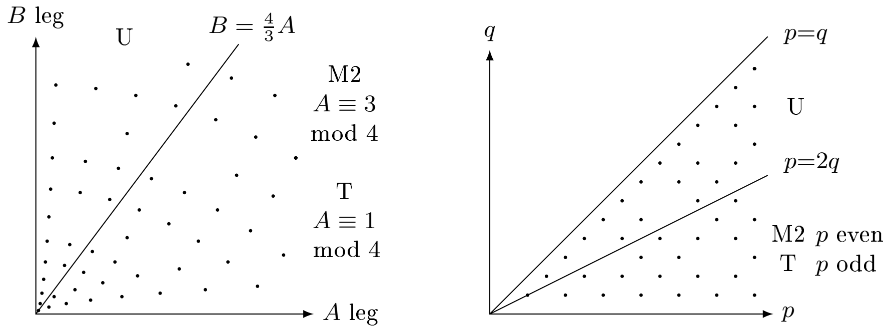


Figure 10: UMT tree, regions of p,q

Theorem 10 (Firstov). *The UMT tree visits all and only primitive Pythagorean triples without duplication.*

Proof. It's convenient to work in p,q pairs. The tree visits only primitive Pythagorean triples because for a given p,q each of the three children p',q' satisfy the PQ conditions (4). For U,

$$
\begin{aligned}
\begin{pmatrix} p' \\ q' \end{pmatrix} &= U \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2p - q \\ p \end{pmatrix} \\
p' &= 2p - q = p + (p - q) \ > \ p = q' \qquad\qquad \text{since } p > q \\
q' &= p \ \geq \ 1 \\
\gcd(p', q') &= \gcd(2p - q, p) = \gcd(q, p) = 1 \\
p' + q' &= 3p - q \equiv p + q \equiv 1 \bmod 2
\end{aligned}
$$

For M2,

$$
\begin{pmatrix} p' \\ q' \end{pmatrix} = M2 \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2p \\ p - q \end{pmatrix}
$$

$$\begin{aligned}
p' &= 2p \;>\; p-q = q' \\
q' &= p-q \;\ge\; 1 && \text{since } p>q \\
\gcd(p',q') &= \gcd(2p, p-q) \\
&= \gcd(p, p-q) && \text{since } p-q \text{ odd} \\
&= \gcd(p,q) = 1 \\
p'+q' &= 3p-q \equiv p+q \equiv 1 \bmod 2
\end{aligned}$$

For T,

$$\begin{aligned}
\begin{pmatrix} p' \\ q' \end{pmatrix} &= T \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+3q \\ 2q \end{pmatrix} \\
p' &= p+3q \;>\; 2q = q' \\
q' &= 2q \;\ge\; 1 && \text{since } q \ge 1 \\
\gcd(p',q') &= \gcd(p+3q, 2q) \\
&= \gcd(p+3q, q) && \text{since } p+3q \text{ odd} \\
&= \gcd(p,q) = 1 \\
p'+q' &= p+5q \equiv p+q \equiv 1 \bmod 2
\end{aligned}$$

No pair is duplicated in the tree because the children of U are above the $p=2q$ line (7) whereas M2 and T are below (25) (29). Then M2 gives p even whereas T gives p odd. Therefore two paths ending with a different matrix cannot reach the same point.

Conversely, every pair p',q' occurs in the tree because it can be reversed according to its region and parity.

$$\begin{aligned}
&\text{if } p' < 2q' \text{ then} && \begin{pmatrix} p \\ q \end{pmatrix} = U^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix} = \begin{pmatrix} q' \\ 2q'-p' \end{pmatrix} \\
&\text{if } p' < 2q',\ p' \text{ even, } q' \text{ odd} && \begin{pmatrix} p \\ q \end{pmatrix} = M2^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix} = \begin{pmatrix} p'/2 \\ p'/2-q' \end{pmatrix} \\
&\text{if } p' < 2q',\ p' \text{ odd, } q' \text{ even} && \begin{pmatrix} p \\ q \end{pmatrix} = T^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} 1 & -3/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p'-3q'/2 \\ q'/2 \end{pmatrix}
\end{aligned}$$

The parent p,q satisfies the p,q conditions (4).

For reversing U have $p' < 2q'$ which is $p' \le 2q'-1$.

$$\begin{aligned}
q &= 2q'-p' = q'-(p'-q') \;<\; q' = p && \text{so } p>q \\
q &= 2q'-p' \;\ge\; 2q'-(2q'-1) = 1 \\
\gcd(p,q) &= \gcd(q', 2q'-p') = \gcd(q',p') = 1 \\
p+q &= 3q'-p' \equiv p'+q' \equiv 1 \bmod 2 \\
p &= q' < p'
\end{aligned}$$

For reversing M2 have $p' > 2q'$ which is $p' \ge 2q'+2$ since p' is even.

$$\begin{aligned}
q &= p'/2-q' \;<\; p'/2 = p && \text{so } p>q \\
q &= p'/2-q' \;\ge\; (2q'+2)/2-q' = 1 \\
\gcd(p,q) &= \gcd(p'/2, p'/2-q') = \gcd(p'/2, q') \\
&= \gcd(p',q') = 1 && \text{since } p' \text{ even, } q' \text{ odd}
\end{aligned}$$

$$\begin{aligned}
p+q &= p'-q' \equiv p'+q' \equiv 1 \bmod 2 \\
p &= p'/2 < p'
\end{aligned}$$

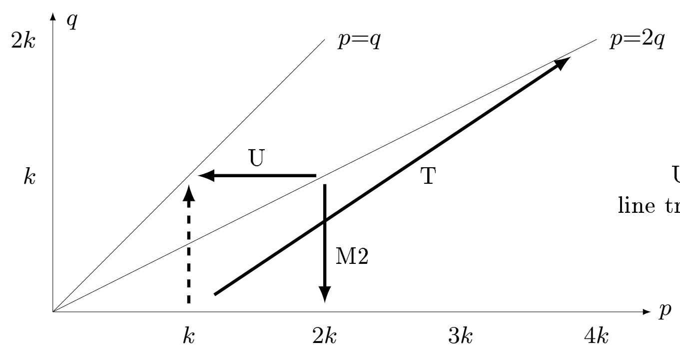
For reversing T have $p' > 2q'$ and q' is even.

$$\begin{aligned}
p &= p'-3q'/2 \;>\; 2q'-3q'/2 = q'/2 = q \\
q &= q'/2 \;\ge\; 1 && \text{since } q' \text{ even} \\
\gcd(p,q) &= \gcd(p'-3q'/2, q'/2) = \gcd(p', q'/2) \\
&= \gcd(p', q') = 1 && \text{since q' even} \\
p+q &= p'+3q'/2-q'/2 = p'+q' \equiv 1 \bmod 2 \\
p &= p'-3q'/2 < p'
\end{aligned}$$

□

Each ascent has $p < p'$ so repeatedly taking the parent this way is a sequence of strictly decreasing p which must eventually reach the root 2,1. The matrix to reverse goes according to the region and parity of p',q' which are where the respective matrices descend.

U,M2,T transform a vertical line of points $k,1$ through $k,k{-}1$ out to bigger p,q as follows



UMT tree
line transformations

U is described with the UAD tree figure 3. M2 is described with FB tree figure 9.

T shears p across by 3 to give $4k$ and then doubles q to put it on even points.

$$T = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix} \qquad \text{shear then stretch } q$$

For T, the line endpoint $k,k{-}1$ becomes $4k{-}3, 2k{-}2$. This longer line still has the same number of coprime not-both-odd points as the original.

Repeated T is the right-most side of the tree. These p,q and resulting triples A,B,C are

$$T^k\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3.2^k-1\\ 2^k\end{pmatrix} = \begin{cases}2,5,11,23,47,\ldots & \text{A055010}\\ 1,2,\;\;4,\;\;8,16,\ldots & \text{A000079}\end{cases}$$

$$T^k\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}(2^{k+2}-1)\,(2^{k+1}-1)\\ (3.2^k-1).2^{k+1}\\ 10.4^k-6.2^k+1\end{pmatrix} = \begin{cases}3,21,105,465,\ldots & \text{A134057}\\ 4,20,\;\;88,368,\ldots & \text{A093357}\\ 5,29,137,593,\ldots\end{cases}$$

These points are $p = 3q - 1$ so immediately above a line $p{=}3q$ in figure 10, but only those $p = 3.2^k - 1$. In A,B, the double-angle sends them to immediately above a line $B = \frac{3}{4}A$. Leg difference $C - A = 2q^2 = 2^{2k+1}$ is odd powers of 2, since in general

$$T^k \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p + 3(2^k - 1)q \\ 2^k q \end{pmatrix}$$
$$C - A = (p'^2 + q'^2) - (p'^2 - q'^2) \ = 2q'^2 \ = 2^{2k+1} q^2$$

6 Triple Preserving Matrices

The set of all matrices which preserve primitive Pythagorean triples are characterized in $2{\times}2$ form by Firstov[10]. The conditions are stated in slightly different form here below in theorem 11.

A primitive triple preserving matrix is to send a p,q point to a new p',q' satisfying the PQ conditions (4), and without duplication meaning that two different p,q do not map to the same child p',q'.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Palmer, Ahuja and Tikoo[20] give a set of conditions on a,b,c,d which are sufficient, but not necessary. As they note, their R-3 determinant condition $\Delta = ad - bc = \pm 1$ is sufficient, but not necessary. $\Delta{=}\pm 1$ ensures that $\gcd(p, q) = 1$ implies $\gcd(p', q') = 1$ over all integers, but for Pythagorean triples only p,q of opposite parity need be considered.

For p,q opposite parity, a determinant $\Delta = \pm 2^r$ (41) maintains $\gcd(p', q') = 1$. The argument in this part of the proof largely follows an answer by Thomas Jager[13] for the all integers case (on $n{\times}n$ matrices). A little care is needed that the p,q constructed to induce a common factor obeys $p > q \geq 1$ not-both-odd.

Matrices M1, M2, M3 (24) and T (28) have $\Delta = \pm 2$. Higher powers of 2 occur from products of those matrices (and believe also from matrices not a product of others).

$$\det(M^r) = (\det M)^r = \pm 2^r$$

Theorem 11 (variation of Firstov)**.** *Conditions* (30) *through* (41) *are necessary and sufficient for a matrix* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *to preserve* p,q *pairs without duplication, and hence for its corresponding* $3{\times}3$ *matrix to preserve primitive Pythagorean triples without duplication.*

$$a, b, c, d \ \textit{integers} \tag{30}$$
$$a \geq 1 \tag{31}$$
$$c \geq 0 \tag{32}$$
$$a > c \tag{33}$$
$$a + b \geq 1 \qquad \textit{so } b \geq -a + 1 \tag{34}$$
$$c + d \geq 0 \qquad \textit{so } d \geq -c \tag{35}$$
$$a + b \geq c + d \tag{36}$$
$$\gcd(a, c) = 1 \tag{37}$$

$$\begin{aligned}
\gcd(b,d) &= 1 && && (38)\\
a+c &\equiv 1 \bmod 2 && \textit{opposite parity} && (39)\\
b+d &\equiv 1 \bmod 2 && \textit{opposite parity} && (40)\\
ad-bc &= \pm 2^r,\ \ r\ge 0 && \textit{determinant} && (41)
\end{aligned}$$

The identity matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ preserves p,q pairs without duplication and satisfies the conditions.

The GCD is taken as $\gcd(n,0)=|n|$ in the usual way. So $\gcd(a,c)=1$ and $a\ge 1$ together mean

$$\text{if } c=0 \text{ then can only have } a=1 \tag{42}$$

Similarly $\gcd(b,d)=1$ means

$$\text{if } d=0 \text{ then can only have } b=\pm 1$$

Proof of Theorem 11. First the necessity, that if a matrix sends all good p,q to good p',q' and never duplicates p',q' then its a,b,c,d are as described.

Consider p=2, q=1 and p=3, q=2,

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2a+b\\2c+d\end{pmatrix}=\begin{pmatrix}p'_1\\q'_1\end{pmatrix}$$
$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}3a+2b\\3c+2d\end{pmatrix}=\begin{pmatrix}p'_2\\q'_2\end{pmatrix}$$

These are solved for a,b,c,d in terms of p'_1,p'_2,q'_1,q'_2

$$\begin{aligned}
a &= 2p'_1-p'_2=\text{integer} & b &= -3p'_1+2p'_2=\text{integer}\\
c &= 2q'_1-q'_2=\text{integer} & d &= -3q'_1+2q'_2=\text{integer}
\end{aligned}$$

p'_1,p'_2,q'_1,q'_2 are all integers so a,b,c,d are all integers (30).

Consider $p=2k$, $q=1$,

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2k\\1\end{pmatrix}=\begin{pmatrix}2ka+b\\2kc+d\end{pmatrix}=\begin{pmatrix}p'\\q'\end{pmatrix}$$

Must have $a\ge 0$ otherwise big enough k gives $p'<2$. Similarly $c\ge 0$ (32) otherwise $q'<1$. Must have $a\ge c$ otherwise big enough k gives $p'\le q'$. But cannot have both $a=0$ and $c=0$ otherwise constant $p'=b$, $q'=d$ is duplicated, so $a\ge 1$ (31). b and d must be opposite parity (40) so that p' and q' are opposite parity. If b,d have a common factor $g=\gcd(b,d)>1$ then $k=g$ gives that common factor in p',q', so must have $\gcd(b,d)=1$ (38).

Consider $p=2k{+}1$, $q=2k$,

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2k+1\\2k\end{pmatrix}=\begin{pmatrix}2k(a+b)+a\\2k(c+d)+c\end{pmatrix}=\begin{pmatrix}p'\\q'\end{pmatrix}$$

Must have $a{+}b\ge 0$ otherwise big enough k gives $p'<2$. Similarly $c{+}d\ge 0$ (35) otherwise $q'<1$. Must have $a{+}b\ge c{+}d$ (36) otherwise big enough k gives $p'\le q'$. But cannot have both $a{+}b=0$ and $c{+}d=0$ otherwise constant $p'=c$, $q'=d$ is duplicated, so $a{+}b\ge 1$ (34). a and c must be opposite parity (39) so that p'

and q' are opposite parity. Since a,c are opposite parity, cannot have $a = c$ so $a \geq c$ above becomes $a > c$ (33). If a,c have a common factor $g = \gcd(a, c) > 1$ then $k = g$ gives that common factor in p',q', so must have $\gcd(a, c) = 1$ (37).

The determinant $\Delta = ad - bc \neq 0$ because if it was 0 then $ad = bc$ and with $a \geq 1$, $\gcd(a, c) = 1$ and $\gcd(b, d) = 1$ could only have $d = c$ and $b = a$. In that case

$$\begin{pmatrix} a & a \\ c & c \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3a \\ 3c \end{pmatrix}$$

has common factor 3 in p',q' if $c \neq 0$ or has $q' = 0$ if $c = 0$. Therefore $\Delta \neq 0$.

Let δ be the odd part of the determinant so

$$\Delta = ad - bc = \delta\, 2^r \qquad \delta \text{ odd}$$

Since $\gcd(a, c) = 1$, there exist integers x, y with

$$-xc + ya = 1 \qquad \text{since } a,c \text{ coprime} \tag{43}$$

Consider p,q pair

$$\begin{aligned} p &= xd - yb + \delta k && \text{some integer } k \\ q &= 1 \end{aligned}$$

Choose k the same parity as $xd - yb$ so as to make p even (since δ is odd). Choose k big enough positive or negative to give $p \geq 2$ (possible since $\delta \neq 0$). The resulting p,q gives

$$\begin{aligned} p' &= a(xd - yb + \delta k) + b \\ &= xad - yab + a\delta k + b \\ &= xad - (1 + xc)b + a\delta k + b && \text{since } ya = 1 + xc \text{ (43)} \\ &= x(ad - bc) + a\delta k \\ &= x\delta 2^r + a\delta k && \text{multiple of } \delta \end{aligned}$$

$$\begin{aligned} q' &= c(xd - yb + \delta k) + d \\ &= xcd - ybc + c\delta k + d \\ &= (ya - 1)d - ybc + c\delta k + d && \text{since } xc = ya - 1 \text{ (43)} \\ &= y(ad - bc) + c\delta k \\ &= y\delta 2^r + c\delta k && \text{multiple of } \delta \end{aligned}$$

δ is a common factor in p',q' and so must have $\delta = \pm 1$ and therefore $\Delta = ad - bc = \pm 2^r$ (41).

As a remark, this p,q pair arises from M and its adjoint (inverse times determinant),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\Delta \\ y\Delta \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta k \\ 0 \end{pmatrix} \right) = \begin{pmatrix} x\Delta + a\delta k \\ y\Delta + c\delta k \end{pmatrix}$$

This is common factor δ in p',q' on the right, provided the vector part on the left

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta k \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

is an acceptable p,q pair or can be made so. One way to make it so is q=1 by x,y from $\gcd(a,c) = 1$ and then $p \geq 2$ and even using k.

$\gcd(a,c) = 1$ gives a whole class of solutions to $-cx + ay = 1 = q$,

$$\begin{aligned} x &= x_0 + fa && \text{integer } f \\ y &= y_0 + fc \end{aligned}$$

Taking a different f adds $f(ad-bc)$ to p. This is a multiple of the determinant and so maintains factor δ in the resulting p'. Could choose f to make $p \geq 2$, but if Δ is even then f cannot force p to even. It's necessary to have k and the odd part δ for that.

Now for sufficiency, ie. that if the conditions of the theorem hold then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends all good p,q to good p',q' (4) without duplication.

$$\begin{aligned} p' &= ap + bq \\ &= ap + (1-a)q && a+b \geq 1 \text{ (34) so } b \geq 1-a \\ &= a(p-q) + q \\ &\geq 2 && \text{since } a \geq 1 \text{ (31)} \end{aligned}$$

If c=0 then c+d $\geq$ 0 (35) means $d \geq 0$. Determinant $ad - bc = \pm 2^r \neq 0$ means not c=0, d=0, so $d \geq 1$ giving

$$\begin{aligned} q' &= cp + dq \\ &= dq \ \geq \ 1 \end{aligned}$$

If $c > 0$ then

$$\begin{aligned} q' &= cp + dq \\ &= cp + (-c)q && c+d \geq 0 \text{ (35) so } d \geq -c \\ &= c(p-q) \ \geq \ 1 \end{aligned}$$

For the relative magnitude of p' and q',

$$\begin{aligned} p' &= ap + bq \\ &= a(p-q) + (a+b)q \\ &> c(p-q) + (c+d)q && a > c \text{ (33) and } a+b \geq c+d \text{ (36)} \\ &= cp + dq = q' \end{aligned}$$

For the parity of p' and q',

$$p' + q' = (a+c)p + (b+d)q$$

$$\begin{aligned} &\equiv p+q \bmod 2 && \text{by } a{+}c \equiv 1 \text{ (39)}, b{+}d \equiv 1 \text{ (40)} \\ &\equiv 1 && \end{aligned} \tag{44}$$

Any p',q' is reached from just one p,q since $\Delta \neq 0$ means M is invertible so p,q is uniquely determined by p',q'.

Since $\gcd(p,q) = 1$, there exist integers x,y satisfying

$$xp + yq = 1 \qquad \text{as } p,q \text{ coprime}$$

and the inverses for p,q in terms of p',q' give

$$\begin{aligned} x(dp' - bq')/\Delta + y(-cp' + aq')/\Delta = 1 \\ (xd - yc)p' + (-xb + ya)q' = \pm 2^r \qquad \text{by } \Delta = \pm 2^r \text{ (41)} \end{aligned}$$

which is integer multiples of p',q' adding up to $\pm 2^r$. So $\gcd(p',q')$ must be a divisor of 2^r. But p',q' are opposite parity (44) so one of them is odd which means $\gcd(p',q')$ is odd and the only odd divisor of 2^r is $\gcd(p',q') = 1$. □

Since a,c are opposite parity (39), and b,d are opposite parity (40), there are 4 possible combinations of odd/even among the matrix terms. All 4 occur in the trees.

The parity of the terms control whether the child p',q' either keeps, swaps, or has fixed parity.

Combination	$\begin{pmatrix} E & O \\ O & E \end{pmatrix}$	$\begin{pmatrix} O & E \\ E & O \end{pmatrix}$	$\begin{pmatrix} E & E \\ O & O \end{pmatrix}$	$\begin{pmatrix} O & O \\ E & E \end{pmatrix}$
Matrices	U, A	D	M2, M3	M1,T
PQ	swap parity $p' \equiv q$ $q' \equiv p$	keep parity $p' \equiv p$ $q' \equiv q$	always p' even q' odd	always p' odd q' even
Determinant	odd $\Delta = \pm 1$	odd $\Delta = \pm 1$	even $\Delta = \pm 2^r$ $r \geq 1$	even $\Delta = \pm 2^r$ $r \geq 1$

7 No Other Trees

Firstov [10] shows that the only trees which can be made from a fixed set of matrices are the UAD, FB and UMT. The proof offered here takes points according to increasing p whereas Firstov goes by sum p+q. The conditions of the theorem allow for any number of matrices but it happens that the trees all have 3 matrices each.

Lemma 1. *A triple preserving matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$*, other than the identity matrix, advances* p*, ie.* $p' = ap + bq > p$.

Proof. If $a = 1$ then $a > c \geq 0$ (33)(32) means $c = 0$, and $a + b \geq 1$ (34) means $b \geq 0$. If $b = 0$ then $\gcd(b,d) = 1$ (38) and $c + d \geq 0$ (35) together mean $d = 1$ which is the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. So if $a = 1$ then a non-identity matrix has $b \geq 1$ and

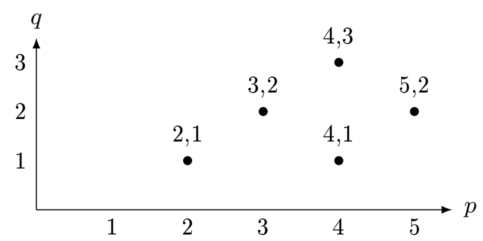
$$p' = ap + bq \geq p + q > p$$

Otherwise if $a \geq 2$ then

$$\begin{aligned} p' &= ap + bq \\ &\geq ap + (1-a)(p-1) && \text{by } b \geq 1-a \text{ (34) and } p-1 \geq q > 0 \\ &= p + a - 1 \\ &> p && a \geq 2 \end{aligned}$$ □

Theorem 12 (Firstov)**.** *The UAD, FB and UMT trees are the only trees which generate all and only primitive Pythagorean triples in least terms without duplicates using a fixed set of matrices from a single root.*

Proof. By lemma 1, the matrices always advance p. So a given p',q' at 3,2 onwards must have as its parent some p,q with smaller p. The following cases consider the ways points up to 5,2 can be reached, culminating in the diagram of figure 11 on page 34.



Case. *2,1 to 3,2 — U and M1*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{45}$$

$$\begin{aligned} a + (a{+}b) &= 3 \\ c + (c{+}d) &= 2 \end{aligned} \tag{46}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
1	2	1	0	2	2	matrix M1
2	1	-1	0			fails $\gcd(a,c) = 1$ (37)
2	1	-1	1	1	0	matrix U

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is to send $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)$ per (45). This is written in terms of a, $a{+}b$ and c, $c{+}d$ (46) since those terms are ≥ 1 and ≥ 0. The table headings are reminders of the conditions on those quantities (31),(34), (32),(35).

The solutions are listed in the table by increasing a and then increasing c as long as $a > c$ (33). The b column is derived from a and $a{+}b$. The d column is derived from c and $c{+}d$.

The note beside each combination is either the matrix name or how the values fail the matrix conditions. For example the middle row of the table above fails $\gcd(a,c) = 1$ (37). This GCD failure is of the "if c=0 then only a=1" kind (42). This a and c also fail opposite parity (39).

Case. *2,1 to 4,1 — M2 and D*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{aligned} a + (a{+}b) &= 4 \\ c + (c{+}d) &= 1 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
1	3	2	0	1	1	matrix D
2	2	0	0			fails $\gcd(a,c) = 1$ (37)
2	2	0	1	0	-1	matrix M2
3	1	-2	0			fails $\gcd(a,c) = 1$ (37)
3	1	-2	1	0		fails $a + c \equiv 1 \bmod 2$ (39)

Case. *2,1 to 4,3 — M3 and X1*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \begin{aligned} a + (a{+}b) &= 4 \\ c + (c{+}d) &= 3 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
1	3	2	0	3	3	fails $ad - bc = \pm 2^r$ (41)
2	2	0	0			fails $\gcd(a,c) = 1$ (37)
2	2	0	1	2	1	matrix M3
3	1	-2	0	3		fails $\gcd(a,c) = 1$ (37)
3	1	-2	1	2		fails $a{+}b \geq c{+}d$ (36)
3	1	-2	2	1	-1	matrix $X1 = U.U$

$$X1 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = U.U \tag{47}$$

As a remark, in general any matrix with c=0 such as $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right)$ in the first line of the table must not have $g = \gcd(a{+}b, d) > 1$ odd (here $g{=}3$), otherwise $p = g{+}1$, $q = 1$ leads to a common factor g in p', q'.

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} g+1 \\ 1 \end{pmatrix} &= \begin{pmatrix} ag + (a{+}b) \\ d \end{pmatrix} \\ &= \begin{pmatrix} ag + s'g \\ d'g \end{pmatrix} \qquad\qquad a{+}b = s'g \text{ and } d = d'g \end{aligned}$$

Case. *2,1 to 5,2 — matrices A, T, X2*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} a + (a{+}b) &= 5 \\ c + (c{+}d) &= 2 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
1	4	3	0	2	2	matrix T
2	3	1	0			fails $\gcd(a,c) = 1$ (37)
2	3	1	1	1	0	matrix A
3	2	-1	0			fails $\gcd(a,c) = 1$ (37)
3	2	-1	1			fails $a + c \equiv 1 \bmod 2$ (39)
3	2	-1	2	0	-2	matrix $X2 = M1.M2$
4	1	-3	0			fails $\gcd(a,c) = 1$ (37)
4	1	-3	1	1	0	fails $\gcd(b,d) = 1$ (38)
4	1	-3	2			fails $a + c \equiv 1 \bmod 2$ (39)

$$X2 = \begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix} = M1 \, . \, M2 \tag{48}$$

Case. *3,2 to 4,1 — no matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{aligned} a + 2(a{+}b) &= 4 \\ c + 2(c{+}d) &= 1 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
2	1	-1	1	0	-1	fails $b + d \equiv 1 \bmod 2$ (40)

Case. *3,2 to 4,3 — matrix U only*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \begin{aligned} a + 2(a{+}b) &= 4 \\ c + 2(c{+}d) &= 3 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
2	1	-1	1	1	0	matrix U

Case. *3,2 to 5,2 — no matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} a + 2(a{+}b) &= 5 \\ c + 2(c{+}d) &= 2 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
1	2	-1	0	1	1	fails $b + d \equiv 1 \bmod 2$ (40)
3	1	-2	0			fails $\gcd(a,c) = 1$ (37)
3	1	-2	2	0	-2	fails $b + d \equiv 1 \bmod 2$ (40)

Case. *4,1 to 5,2 — M1 only*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} 3a + (a{+}b) &= 5 \\ 3c + (c{+}d) &= 2 \end{aligned}$$

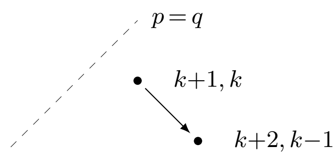
$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
1	2	1	0	2	2	matrix M1

Case. *4,3 to 5,2 — no matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} a + 3(a{+}b) &= 5 \\ c + 3(c{+}d) &= 2 \end{aligned}$$

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d	
2	1	-1	2	0		fails $a > c$ (33)

As a remark, in general no matrix can give a unit step diagonally down from just under the leading diagonal like 4,3 to 5,2 here and case 3,2 to 4,1 above.



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k+1 \\ k \end{pmatrix} = \begin{pmatrix} k+2 \\ k-1 \end{pmatrix} \qquad \begin{aligned} a + (a{+}b)\,k &= k+2 \\ c + (c{+}d)\,k &= k-1 \end{aligned} \qquad \text{with } k \geq 3$$

which has a single solution,

$a \geq 1$	$a{+}b \geq 1$	b	$c \geq 0$	$c{+}d \geq 0$	d
2	1	-1	$k{-}1$	0	$-(k{-}1)$

If $k \geq 3$ then this solution fails $a > c$ (33), as per 4,3 to 5,2. If $k = 2$ (or for that matter any k even) then this solution fails $b{+}d \equiv 1 \bmod 2$ (40) as per 3,2 to 4,1.

Figure 11 summarises the above cases. These are the only ways to go to each point shown.

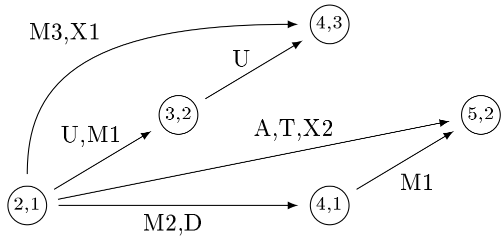


Figure 11:
possible p,q
point descents

X1 $= U.U$ (47) goes 2,1 to 4,3 by U twice. *X2* $=$ *M1* . *M2* (48) goes 2,1 to 5,2 by M2 then M1. It is this order because the matrix multiplication is on the left so *X2* $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) =$ *M1* . *M2* $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$ is step by M2 first then M1.

3,2 is reached from 2,1 by either U or M1. Suppose firstly it is M1. M1 repeated goes

$$2,1 \;\rightarrow\; 3,2 \;\rightarrow\; 5,4 \;\rightarrow \cdots \qquad \text{as at (26)}$$

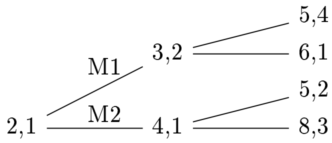
So 4,1 and 4,3 are not visited. Consider 4,1 first. The candidates for its parent are

2,1 to 4,1	M2 or D
3,2 to 4,1	none

It cannot be D since M1 and D overlap at 7,2.

$$M1 \begin{pmatrix} 6 \\ 1 \end{pmatrix} = D \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \qquad \text{M1 and D overlap}$$

M1 and M2 together visit



Point 4,3 is still not visited. The candidates for its parent are

2,1 to 4,3	M3 or X1
3,2 to 4,3	U

Its parent cannot be U since U overlaps with M1 at 3,2 (both U and M1 send 2,1 to 3,2).

$$U \begin{pmatrix} 2 \\ 1 \end{pmatrix} = M1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \text{U and M1 overlap} \tag{49}$$

Neither can it be X1 because X1 overlaps M1 at 5,4,

$$M1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = X1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \qquad \text{M1 and X1 overlap}$$

So when M1 is included the only possible combination is M1,M2,M3 which is the FB tree.

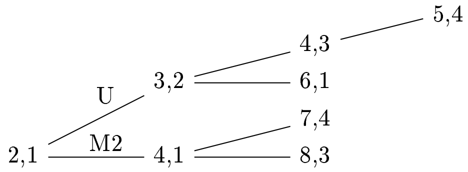
Return to instead take U for 2,1 to 3,2. Repeated U goes

$$2,1 \to 3,2 \to 4,3 \to 5,4 \to \cdots \qquad \text{as at (11)}$$

Point 4,1 is the smallest p not visited. The candidates for its parent are

2,1 to 4,1	M2 or D
3,2 to 4,1	none

Take first U and M2. Together they visit



The smallest p not visited yet is 5,2. The candidates for its parent are

2,1 to 5,2	A, T, X2
3,2 to 5,2	none
4,1 to 5,2	M1
4,3 to 5,2	none

(50)

M1 is excluded because it overlaps with U at 3,2 per (49).

A is excluded because it overlaps with M2 at 8,3.

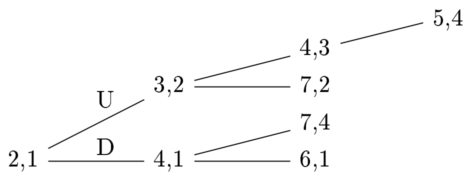
$$A \begin{pmatrix} 3 \\ 2 \end{pmatrix} = M2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \qquad \text{A and M2 overlap}$$

X2 is excluded because it overlaps with U at 11,6,

$$U\begin{pmatrix}6\\1\end{pmatrix} = X2\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}11\\6\end{pmatrix} \qquad \text{X2 and U overlap} \tag{51}$$

So when U and M2 are in the tree the only combination is U,M2,T which is the UMT tree.

Return to U and D. Together they visit



Again the smallest p not visited is 5,2 and its parent candidates are per (50).

M1 and X2 are excluded because they overlap U by (49),(51). T is excluded because it overlaps D at 11,6,

$$D\begin{pmatrix}3\\2\end{pmatrix} = T\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}7\\2\end{pmatrix} \qquad \text{D and T overlap}$$

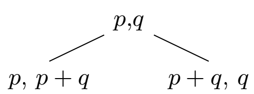
So when U and D are included the only combination is UAD by Berggren etc. □

It might be noted for the proof that only some of the conditions on triple preserving a,b,c,d from theorem 11 are needed.

It would be enough here to have the ranges of a,b,c,d giving the 29 matrices which are the rows of the case tables, then for each of the 18 failing matrices exhibit a particular p,q which makes a bad p',q'. Those p,q would follow the general conditions but be just particular integer values.

8 Calkin-Wilf Tree Filtered

The tree of rationals by Calkin and Wilf[7] arranges the Stern diatomic sequence into tree rows which descend as



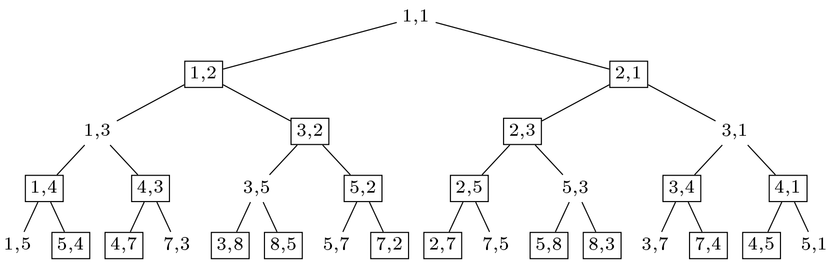
The diatomic sequence goes in a repeating pattern of odd and even

O O E O O E O O E O O E ...

So when taking adjacent pairs every third is odd/odd, and is to be filtered out for the purpose of Pythagorean triples.

O/O, O/E, E/O, O/O, O/E, E/O, O/O, O/E, E/O ...

Removing the odd/odd points leaves two tree roots 1,2 and 2,1.



The orphaned children can be adopted by their grandparent to make a 3-point descent. The rule for the children then varies according to whether the parent p is odd or even. It happens that the D matrix (6) is one of the legs.

if p odd, q even: p,q

$p, 2p+q$	$2p+q, p+q$	$p+q, q$
$\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$	$\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$
= D matrix transposed		

if p even, q odd: p,q

$p,\ p+q$	$p+q,\ p+2q$	$p+2q,\ q$
$\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$
		= D matrix

Points $p>q$ and $p<q$ are intermingled in the tree. Each left leg has $p<q$ and each right leg has $p>q$.

The $p<q$ points give leg $A=p^2-q^2<0$. For example 1,2 becomes -3,4,5. So two copies of the primitive triples are obtained, one with A positive and the other A negative. If desired, the negatives could instead be taken to mean swap A and B giving triples such as 4,3,5 which is A even, B odd. That would be all positive primitive triples with A,B both ways around.

9 Parameter Variations

9.1 Parameter Difference

Triples can also be parameterized by d,q where d is a difference $d=p-q$.

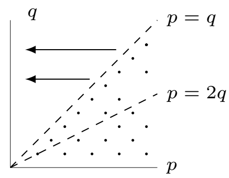
$$d \ge 1, \text{ odd integer}$$
$$q \ge 1, \text{ any integer}$$
$$\gcd(d,q)=1$$

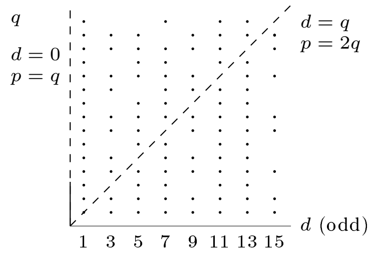
$$A = d^2 + 2dq \qquad A \text{ leg odd}$$
$$B = \quad 2dq + 2q^2 \qquad B \text{ leg even}$$
$$C = d^2 + 2dq + 2q^2 \qquad \text{hypotenuse}$$
$$d = \sqrt{C-B} \qquad q = \sqrt{\tfrac{C-A}{2}}$$

The effect is to shear p,q coordinates left to use the whole first quadrant. This for example changes the UArD tree steps (theorem 5) from diagonals to verticals.

$$\begin{pmatrix} d \\ q \end{pmatrix} = H \begin{pmatrix} p \\ q \end{pmatrix} \qquad \text{where shear } H = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

shear for $d = p - q$





Tong[23] takes d,q as a symmetric parameterization of all triples. It is symmetric in that d and q can be swapped to give a conjugate triple.

$$\begin{pmatrix} q \\ d \end{pmatrix} = J_{dq} \begin{pmatrix} d \\ q \end{pmatrix} \quad \text{conjugate,} \quad \text{where swap } J_{dq} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Primitive triples with q odd have a primitive conjugate. Primitive triples with q even do not have a primitive conjugate since it would be d even.

Braza, Tong and Zhan[6] consider 3×3 matrix transformations on Pythagorean triples and show that the only invertible transformation is an L which corresponds to the d,q conjugate operation.

$$L = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix}$$

L can be had from the 2×2 → 3×3 formula (5), which is for p,q, by expressing the conjugate operation in p,q,

$$J_{pq} = H^{-1}.J_{dq}.H = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\text{then } L = \text{2to3}(J_{pq})$$

J_{pq} transforms p,q to $p, p-q$. Geometrically, this is a reversal of the q values within a column. The swap of d,q is a reversal in an anti-diagonal which corresponds to a column in p,q.

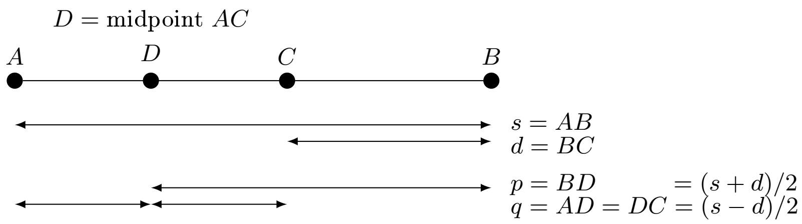
9.2 Parameters Sum and Difference

Triples can also be parameterized by sum $s = p+q$ and difference $d = p-q$.

$$s > d \geq 1, \text{ integers, both odd}$$
$$\gcd(s, d) = 1$$

$$\begin{aligned} A &= sd && A \text{ leg odd} \\ B &= (s^2 - d^2)/2 && B \text{ leg even} \\ C &= (s^2 + d^2)/2 && \text{hypotenuse} \\ s &= \sqrt{C+B} \qquad d = \sqrt{C-B} \end{aligned} \tag{52}$$

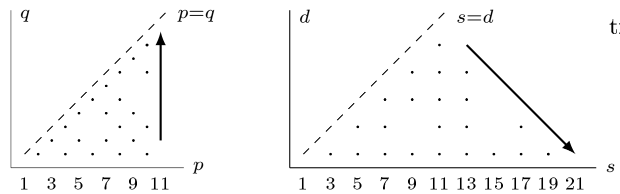
s,d is the original parameterization in Euclid [9] and also noted for instance by Mitchell [18]. In Euclid, s is a length AB and a point C on that line defines d. From there p is the midpoint (mean) of A and C at D. q is the distance to that midpoint.



Algebraically, the difference of two squares in the A leg of p,q (1) and in the B leg of s,d (52) each suggest taking sum and difference, which gives respectively the other parameterization.

$$\begin{array}{rcccl} A &=& p^2 - q^2 &=& (p-q)(p+q) \\ B &=& \frac{1}{2}(s^2 - d^2) &=& \frac{1}{2}(s-d)(s+d) \end{array}$$

The geometric interpretation of s,d coordinates is to transform a column in p,q to a downward diagonal



transformation
for
$s = p{+}q$
$d = p{-}q$

Points in the eighth of the plane $x > y \geq 1$ with $\gcd(x,y) = 1$ are either opposite parity or both odd. They are never both even since that would be common factor 2. p,q are the opposite parity points. s,d are the both-odd points. The sum and difference is a one-to-one mapping between the two classes.

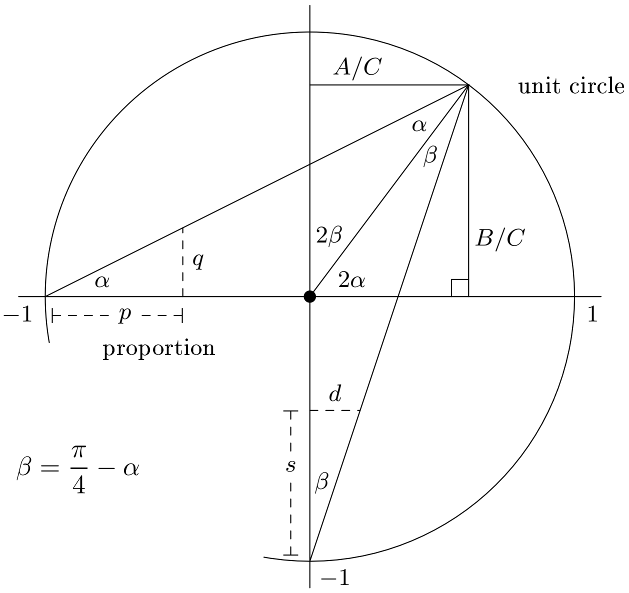
Write the mapping as a matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{so} \quad \begin{pmatrix} s \\ d \end{pmatrix} = F \begin{pmatrix} p \\ q \end{pmatrix}$$

The U,A,D matrices transformed to act on s,d are D,A,U.

$$U_{sd} = F.U.F^{-1} = D$$
$$A_{sd} = F.A.F^{-1} = A$$
$$D_{sd} = F.D.F^{-1} = U$$

In the geometry of figure 1, a pair s,d is a slope from the Y axis at $y = -1$, rather than the X axis at $x = -1$.



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