# Iterations of the Alternate Paperfolding Curve

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**Abstract**

Various properties of finite iterations of the alternate paperfolding curve, including coordinates, boundary, area, Golay-Rudin-Shapiro sequence, twin alternate, area tree, and some fractionals.

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### Notation

Bits of an integer written in binary are numbered starting from 0 for the least significant bit (the lowest bit). Odd and even bit positions follow from this numbering.

Some formulas have terms going in a repeating pattern of say 4 values according as an index $k \equiv 0 \,\text{to}\, 3 \bmod 4$. They are written for example

$$[5,\ 8,\ -5,\ 9] \qquad \text{values according as } k \bmod 4$$

meaning 5 when $k \equiv 0 \bmod 4$, or 8 when $k \equiv 1 \bmod 4$, etc. Likewise periodic patterns of other lengths.

Periodic patterns like this can be expressed by powers of $-1$ or complex $i$ (or other roots of unity), but except in simple cases that tends to be less clear than the values.

# 1 Alternate Paperfolding

The alternate paperfolding curve of Davis and Knuth [8, section 4] is defined as repeated unfolding of a copy of itself beginning from a unit line segment. The unfoldings are alternately to the left and right sides.

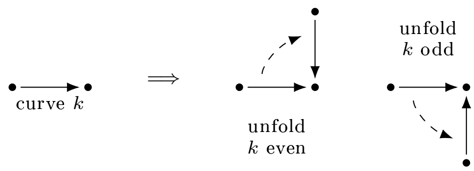
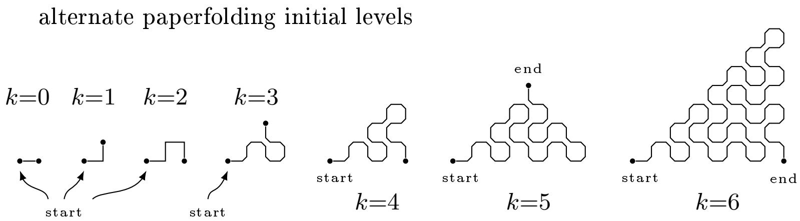
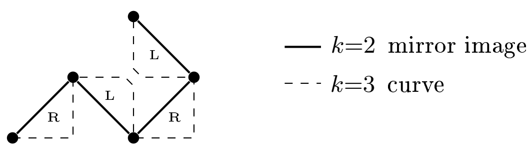



Figure 1:
unfolding

When the unfolding is 90° the curve touches itself at level $k$=3 onwards. In the following diagram the corners are chamfered off to better see the path taken.




An equivalent definition is to form the next level by mirror image and expand even segments on the right and odd segments on the left. The whole curve is rotated suitably to keep the first segment East.




This can be seen explicitly for the expansion of $k$=2, and for subsequent levels it holds by the unfolding. The mirror image each time effectively alternates the unfolding.

It's convenient to draw even segments directed forward and odd segments reverse. This corresponds to the unfolding, and drawn this way the expansion is always on the right after mirror image.

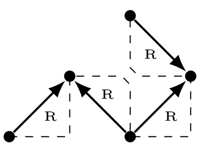



Figure 2:
segments mirror image
then expand on the right

Applying two expansions is two mirror images which cancel out, giving the following plain segment replacement.

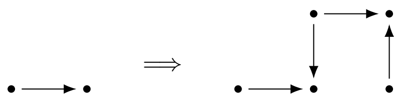
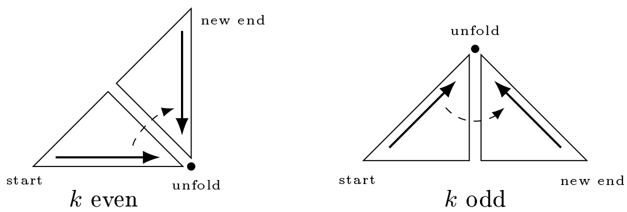


Figure 3:
two expansions

The successive unfolding means the shape is triangular and traverses all segments in the eighth of the plane $0 \le y \le x$ except for odd segments on the $x$ axis.




## 2 Turn

Davis and Knuth [8] give the alternate paperfolding curve turn sequence in the form $+1$ left and $-1$ right, for $n \ge 1$,

$$\begin{aligned} \mathit{turn}(2n) &= -\mathit{turn}(n) && \text{even negated} && (1) \\ \mathit{turn}(2n{+}1) &= (-1)^n && \text{odd alternately L, R} && (2) \\ &= +--+++--+---++-+\dots && && \text{A209615} \end{aligned}$$

This can be calculated from $n$ in binary, again for $n \ge 1$,

$$\mathit{turn}(n) = \begin{cases} +1 \ \text{(left)} & \text{if } \begin{array}{c} \mathit{BitAboveLowestOne}(n) \\ +\ \mathit{CountLowZeros}(n) \end{array} \text{ is even} \\ -1 \ \text{(right)} & \text{if odd} \end{cases} \qquad (3)$$
$$= (-1)^{\mathit{BitAboveLowestOne}(n) + \mathit{CountLowZeros}(n)}$$

$$\begin{aligned} \mathit{BitAboveLowestOne}(n) &= 0,0,1,0,0,1,1,0,0,0,1,\dots \quad n \ge 1 && \text{A038189} \\ \mathit{CountLowZeros}(n) &= 0,1,0,2,0,1,0,3,0,1,0,2,0,\dots \quad n \ge 1 && \text{A007814} \end{aligned}$$

The effect of *CountLowZeros* is an $0 \leftrightarrow 1$ flip of *BitAboveLowestOne* when that bit is at an odd position (least significant bit as position 0).

For computer calculation in a single machine word, *BitAboveLowestOne* can be located by some bit-twiddling. The flip at odd positions can be done by XOR of binary 1010…10 before applying the location mask (similar to for example Arndt [2]).

$$\mathit{turn}(n) = \begin{cases} +1 & \text{if BITAND}\left( \begin{array}{l} \mathit{MaskAboveLowestOne}(n), \\ \text{BITXOR}(n,\ 1010\dots10) \end{array} \right) = 0 \\ -1 & \text{if } \ne 0 \end{cases} \qquad (4)$$

$$\begin{aligned} \mathit{MaskAboveLowestOne}(n) &= \text{BITXOR}(n,\ n{-}1) + 1 \qquad n \ge 1 && (5) \\ &= 2,4,2,8,2,4,2,16,2,4,2,8,2,4,2,32,\dots && \text{A171977} \end{aligned}$$

*MaskAboveLowestOne* is a 1-bit located immediately above the lowest 1-bit of $n$. In (5), the $n{-}1$ changes low zeros $\ldots 1000$ to $\ldots 0111$ and XORing gives 0001111 which is a mask up to and including the lowest 1-bit. Then +1 gives 0010000 which is the bit above.

This bit-twiddling uses carry propagation in the CPU adder to locate the lowest 1-bit. It's common for the adder on a single machine word to be faster than a *CountLowZeros* and test-$j$th-bit.

The next turn, ie. the turn at point $n{+}1$, after segment $n$, is given similarly but above the lowest 0-bit.

$$\begin{aligned} \mathit{turn}(n{+}1) &= \begin{cases} +1 \text{ (left)} & \text{if } \begin{matrix} \mathit{BitAboveLowestZero}(n) \\ +\ \mathit{CountLowOnes}(n) \end{matrix} \text{ is even} \\ -1 & \text{if odd} \end{cases} \\ &= (-1)^{\mathit{BitAboveLowestZero}(n) + \mathit{CountLowOnes}(n)} \end{aligned} \tag{6}$$

$\mathit{turn}(n)$ and $\mathit{turn}(n{+}1)$ are related simply by $n{+}1$ changing low "0111" to "1000",

| | | | | |
|---|---|---|---|---|
| $n$ | $\cdots$ | $t$ | 0 | $1\cdots1$ |
| $n{+}1$ | $\cdots$ | $t$ | 1 | $0\cdots0$ |

Figure 4:
bits, turn per $t$
and its position

$\mathit{turn}(n)$ is multiplicative, as noted by Davis and Knuth. This follows from the recurrence (1) or the bits (4). In the bits, multiplication adds the counts of low zeros, then odd parts $1 \text{ or } 3 \equiv \pm 1 \bmod 4$ multiply.

$$\mathit{turn}(m.n) = \mathit{turn}(m)\,.\,\mathit{turn}(n) \qquad \text{multiplicative}$$

Michael Somos in OEIS A209615 gives a generating function for *turn*,

$$\mathit{gturn}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2^k}}{1 + x^{2^{k+1}}}$$

This follows from recurrence (1). Term $k$ is those $n$ with $\mathit{CountLowZeros}(n) = k$. The first term $k{=}0$ is signs at odd terms per (2) which is generating function $x/(1+x^2)$. Further $\mathit{turn}(2n)$ is by substituting $x^2$ to have $2n$, and negate by $(-1)^k$.

Predicates for left and right turns are

$$\begin{aligned} \mathit{TurnLpred}(n) &= n{\ge}1 \text{ and } \mathit{turn}(n) = 1 \\ &= 1,0,0,1,1,1,0,0,1,0,0,0,1,1,0,1,1,0,\ldots && \text{A106665} \\ \mathit{TurnRpred}(n) &= n{\ge}1 \text{ and } \mathit{turn}(n) = -1 \\ &= 0,1,1,0,0,0,1,1,0,1,1,1,0,0,1,0,0,1,\ldots && \text{A292077} \end{aligned}$$

Generating functions can be formed by $k$ many low 0s (like *gturn*). A left turn is then bits 01 when $k$ even, or bits 11 when $k$ odd. These are $x^{2^k}$ or $x^{3.2^k}$ respectively. Likewise but opposite for a right turn.

$$\mathit{gTurnLpred}(x) = \sum_{k=0}^{\infty} \frac{x^{[1,3]\,.\,2^k}}{1 - x^{4.2^k}} \tag{7}$$

$$gTurnRpred(x) = \sum_{k=0}^{\infty} \frac{x^{[3,1]\,.\,2^k}}{1 - x^{4.2^k}}$$

Difference of the predicates is *turn*,

$$turn(n) = TurnLpred(n) - TurnRpred(n)$$

In the generating functions, the 1,3 cases become $(-1)^k$ in *gturn* for which way around, and $1{-}x^{2.2^k}$ cancels between numerator and denominator.

$$gturn(n) = \sum_{k=0}^{\infty} \frac{x^{[1,3]_k\,.\,2^k} - x^{[3,1]_k\,.\,2^k}}{1 - x^{4.2^k}} = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2^k}(1 - x^{2.2^k})}{1 - x^{4.2^k}}$$

Turn runs follow from recurrence (1). The odd turns alternate L,R and each even turn between is the same as one or the other, forming runs of lengths 1, 2 or 3. (See section 12.2 on how this pattern falls at curve locations in the plane.)

$$\begin{array}{ccccccccccc|l}
 & \boxed{-t_1} & & \boxed{-t_2} & & \boxed{-t_3} & & \boxed{-t_4} & & \boxed{-t_4} & & turn(2n) \quad \text{evens} \\
\text{L} & & \text{R} & & \text{L} & & \text{R} & & \text{L} & & \text{R} & turn(2n{+}1) \ \text{odds} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
\text{run} & & \text{run} & & \text{run} & & \text{run} & & \text{run} & & & \\
m{=}0 & & 1 & & 2 & & 3 & & 4 & & &
\end{array}$$

Figure 5: turn runs

Counting the first run as $m$=0, the run lengths are then, using *turn*,

$$TurnRun(m) = \begin{cases} 1 & \text{if } m{=}0 \ \text{(lefts)} \\ 2 - \frac{1}{2}\big(turn(m) + turn(m{+}1)\big) & \text{if } m \text{ even } {\ge}2 \ \text{(lefts)} \\ 2 + \frac{1}{2}\big(turn(m) + turn(m{+}1)\big) & \text{if } m \text{ odd } \ \text{(rights)} \end{cases} \tag{8}$$

$$= 1, 2, 3, 2, 1, 3, 2, 1, 2, 2, 3, 1, 2, 3, 2, 2, 1, 2, 3, 2, 1, 3, \ldots$$

For a curve of finite $k \ge 2$, the run lengths end with a final 1 which is an unfold of the initial run length 1.

The pairs of turns $n, n{+}1$ in (8) can be written together as a sum *sturn*. This occurs in the midpoint curve ahead in section 10 too.

$$\begin{aligned} sturn(n) &= turn(n) + turn(n{+}1) & (9) \\ &= (-1)^{\lfloor n/2 \rfloor} - turn(\lceil n/2 \rceil) & (10) \\ &= 0, -2, 0, 2, 2, 0, -2, 0, 0, -2, -2, 0, 2, 0, \ldots \quad n{\ge}1 \\ gsturn(x) &= -1 + \left(1 + \tfrac{1}{x}\right) gturn(x) \end{aligned}$$

Form (10) is by a pair of integers $n$ and $n{+}1$ having one odd and one even. *turn* of the odd one alternates $\pm 1$ and the even one is $-turn$ per (1). Floor and ceil of $n/2$ combine them.

*TurnRun* using *sturn* is then

$$TurnRun(m) = \begin{cases} 1 & \text{if } m{=}0 \\ 2 - \frac{1}{2}(-1)^m\, sturn(m) & \text{if } m \ge 1 \end{cases}$$

$$gTurnRun(x) = -\tfrac{1}{2} + \frac{2}{1{-}x} + \tfrac{1}{2}\left(-1{+}\frac{1}{x}\right) gturn(-x) \tag{11}$$

$gturn(-x)$ in (11) is $(-1)^m\, turn(m)$. In $gturn$ sum (11), only the $k$=0 term is changed by $-x$. That term could be taken separately, or an adjustment applied, if desired

$$\begin{aligned} gturn(-x) &= gturn(x) - \frac{2x}{1+x^2} \\ (-1)^n\, turn(n) &= turn(n) + [0,-2,0,2] \end{aligned} \tag{12}$$

A predicate for $n$ which is the start of a turn run is

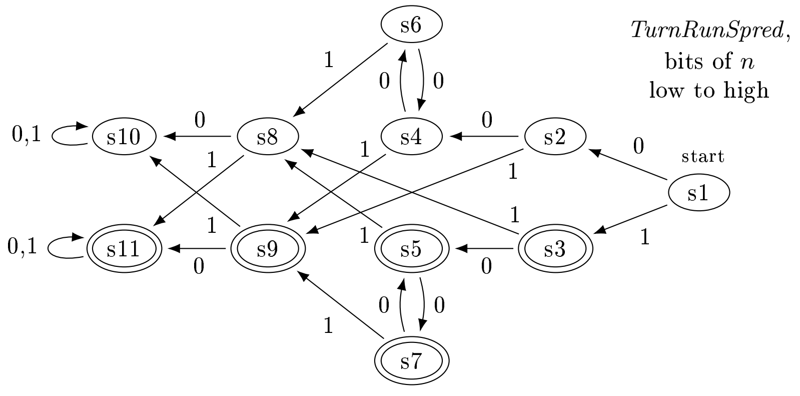
$$\begin{aligned} TurnRunSpred(n) &= \begin{cases} 1 & \text{if } n{=}1 \text{ or } turn(n{-}1) \neq turn(n) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } n{=}1 \text{ or } turn(\lfloor n/2 \rfloor) = -(-1)^{\lceil n/2 \rceil} \\ 0 & \text{otherwise} \end{cases} \\ &= 1,1,0,1,0,0,1,0,1,1,0,0,1,0,1,1,\ldots \qquad n{\geq}1 \end{aligned} \tag{13}$$

Form (13) is by considering cases $n$ odd or even. Each pair $n = 2j, 2j{+}1$ has the run start at either $2j$ or $2j{+}1$. The odd is $(-1)^j$ and hence compare $(-1)^{\lceil n/2 \rceil}$.

A state machine for $TurnRunSpred$ can be made using state machines for the bit patterns of $TurnLpred$ and $TurnRpred$. A run start is left preceded by right or vice versa.

$$TurnRunSpred(n) = \begin{cases} n = 1 \\ \text{or} \quad TurnLpred(n) \text{ and } TurnRpred(n{-}1) \\ \text{or} \quad TurnRpred(n) \text{ and } TurnLpred(n{-}1) \end{cases}$$

Usual state machine manipulations to increment bit strings makes a state machine for those $n$ where $TurnLpred(n{-}1) = 1$. Similarly $TurnRpred(n{-}1)$. Low bits of $n$ determine the turns, and low to high suits $TurnRunSpred$ too. The double-circled accepting states are where $TurnRunSpred(n) = 1$.




A generating function for $TurnRunSpred$ can be formed by the odd and even cases in figure 5. Each odd $n$ is the start of a run, unless its preceding even $n$ is the same turn in which case that even is the run start.

$n \equiv 1 \bmod 4$ is a left turn. So if $0 \bmod 4$ is a $TurnLpred$ then that's a run start. $0 \bmod 4$ is two low 0 bits so terms $k \geq 2$ in $gTurnLpred$ at (7).

If instead 0 mod 4 is a right turn then 1 mod 4 is the run start. So terms $k \geq 2$ in *gTurnRpred* shifted up by a factor $x$.

$n \equiv 2, 3 \bmod 4$ cases alternate so their run starts are $n \equiv 2, 7 \bmod 8$.

$$gTurnRunSpred(x) = x + \frac{x^2 + x^7}{1 - x^8} + \sum_{k=2}^{\infty} \frac{x^{[1,3].2^k} + x^{[3,1].2^k+1}}{1 - x^{4.2^k}}$$

The sequence of those $n$ which are the start of a run ($TurnRunSpred(n) = 1$) follows from the odd/even cases too. Counting the first run as $m{=}0$, each odd turn is at $n = 2m{+}1$. If preceded by the same turn then its run starts 1 earlier. This can be written as an expression (15).

$$\begin{aligned} TurnRunStart(m) &= 1 + \sum_{j=0}^{m-1} TurnRun(j) && (14) \\ &= \begin{cases} 2m & \text{if } m{\geq}1 \text{ and } turn(m) = -(-1)^m \\ 2m+1 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } m{=}0 \\ 2m + \tfrac{1}{2} + \tfrac{1}{2}(-1)^m\, turn(m) & \text{if } m \geq 1 \end{cases} && (15) \\ &= 1, 2, 4, 7, 9, 10, 13, 15, 16, 18, 20, 23, 24, \ldots \end{aligned}$$

In sum (14), *TurnRun* formula (8) gives *turn* in pairs with alternating signs. They cancel out leaving $\frac{1}{2}turn(1) = \frac{1}{2}$ at the start and $(-1)^m \frac{1}{2} turn(m)$ at the end, hence (15).

A generating function follows from (15) and *gturn*. Like at (12), it's possible to adjust so $(-x)$ becomes just $x$, but doing so is a bigger expression.

$$gTurnRunStart(x) = \tfrac{1}{2} - \tfrac{3}{2}\frac{1}{1{-}x} + \frac{2}{(1{-}x)^2} + \tfrac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k\,(-x)^{2^k}}{1 + x^{2^{k+1}}}$$

Shallit [21] considers certain sums of powers of powers of 2 (which are among types Kempner [14] showed are transcendental),

$$C(u,k) = \sum_{j=0}^{k} \frac{(-1)^j}{u^{2^j}} \qquad (16)$$

and gives the continued fraction representation by the following successive "unfolding". The continued fraction integer part is $a_0 = 0$ always since the sums are $C(u,k) < 1$.

$$\begin{aligned} C(u,0) &= [0, u] \qquad C(u,1) = [0, u{+}1, u{-}1] && (17) \\ C(u,k) &= [a_0,\, a_1, \ldots, a_n] \\ C(u,k{+}1) &= [a_0,\, a_1, \ldots, a_{n-1},\, a_n{-}(-1)^k, a_n{+}(-1)^k,\, a_{n-1}, \ldots, a_1] && (18) \end{aligned}$$

The $k{+}1$ continued fraction (18) is the $k$ continued fraction taken forward and reverse and offsets $\mp(-1)^k$ on the elements each side of the middle. Case $u{=}2$ is

$$C(2,0) = [0,\ 2] \qquad = \tfrac{1}{2}$$

$$
\begin{aligned}
C(2,1) &= [0,\ 3,1] &&= \tfrac{1}{2} - \tfrac{1}{4} \\
C(2,2) &= [0,\ 3,2,0,3] &&= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{16} \\
C(2,3) &= [0,\ 3,2,0,2,4,0,2,3] &&= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{16} - \tfrac{1}{256} \\
C(2,\infty) &= [0,\ 3,2,0,2,4,0,2,4,2,2,0,4,\ldots]
\end{aligned}
\tag{19}
$$

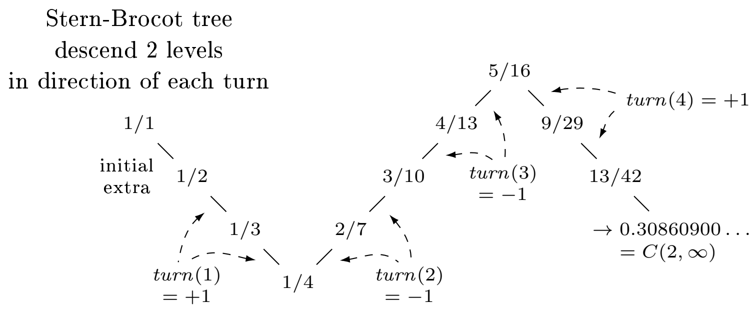
$C(2,k)$ corresponds to turn runs in alternate paperfolding curve $k$. Each term is $2r$ for run length $r = \mathit{TurnRun}$, except first and last terms are $2r{+}1$.

This follows since the expansion at (18) is the same as curve unfolding. Existing terms are appended in reverse, per the curve unfold. The last term $2r{+}1$ is each side of the middle and adding $\pm(-1)^k$ gives 2 for the new *turn* at $n{=}2^k$ either on the lefts run or rights run alternating according as $k$ odd or even, per the curve unfold.

For $u{=}2$, the general $C$ formula starting (17) has some 0 terms (one in each block of 4 terms). For the turn run lengths they can be collapsed by summing each side of the empty run. In a continued fraction, the same applies, ie. a continued fraction with a 0 term is equal to sum of the terms each side. If $C$ is started from collapsed $C(2,2) = [0,\ 3,5]$ instead of (19) then there are no 0s.

$$
C(2,\infty) = \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{4 + \cfrac{1}{\cdots}}}}}
\qquad
\begin{aligned}
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{16} - \frac{1}{256} + \cdots + \frac{(-1)^k}{2^{2^k}} \cdots \\
&= 0.30860900\ldots \qquad \text{A275975} \\
&= 0.010011110\ldots \text{ binary} \qquad \text{A030300} \\
&= 3,4,6,4,2,6,\ldots \text{ cont frac}
\end{aligned}
$$

Continued fraction terms $a_1, a_2, \ldots$ are descents down the Stern-Brocot tree of rationals by $a_1$ many levels left, $a_2$ many right, etc, alternating left and right. Continued fraction terms which are in fact run lengths are therefore successive descents left or right according to the original sequence, in this case the alternate paperfolding curve turn sequence with each value taken twice, and extra initial left.




The tree here is drawn across the page. A left descent is downwards and a right descent is upwards. The initial 1 in the continued fraction goes to 1/2 and that is the starting point for the turns.

At each fraction, descent is to the left (the smaller child) if $\mathit{turn} = +1$. Descent is to the right (the larger child) if $\mathit{turn} = -1$. For example 1/2 has children 1/3 and 2/3. Since $\mathit{turn}(1) = +1$ go to 1/3. Take two such descents for each curve turn value.

Any binary sequence can be used as directions down the Stern-Brocot tree like this. At a given node the values in all deeper nodes are within a wedge-shaped area. The children divide that into non-overlapping smaller wedges, so any descent sequence converges towards some constant.

**Theorem 1.** *The $n$ which is the $m$'th left or right turn is given by mutual recurrences, with first turn as $m$=0,*

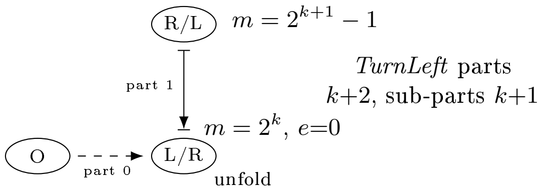
$$\text{for } m = 2^k + e \text{ with } 0 \le e < 2^k \tag{20}$$

$$\mathit{TurnLeft}(m) = \begin{cases} 1 & \text{if } m{=}0 \\ 2^{k+2} - [0,2]_k & \text{if } e = 2^k{-}1 \\ 2^{k+2} - \mathit{TurnRight}\big(2^k - [2,1]_k - e\big) & \text{otherwise} \end{cases} \tag{21}$$
$$= 1, 4, 5, 6, 9, 13, 14, 16, 17, 20, 21, 22, \ldots$$

$$\mathit{TurnRight}(m) = \begin{cases} 2,3,7 & \text{if } m = 0 \text{ to } 2 \\ 2^{k+2} - [1,0]_k & \text{if } e = 2^k{-}1 \text{ and } m{\ge}3 \\ 2^{k+2} - \mathit{TurnLeft}\big(2^k - [1,2]_k - e\big) & \text{otherwise} \end{cases} \tag{22}$$
$$= 2, 3, 7, 8, 10, 11, 12, 15, 18, 19, 23, 26, \ldots$$

*Proof.* Among the turns $n = 1 \text{ to } 2^k$ inclusive, for $k \ge 1$ there are $2^{k-1}$ lefts and $2^{k-1}$ rights. This follows from the unfolding since the unfolding swaps lefts and rights and the turn between, which is the final new turn $2^k$, is alternately left and right.

The turns $n = 1$ to $2^{k+2}$ are in sub-curves level $k$+1,




The $m$ which is the L after the unfold point, so $n > 2^{k+1}$, is the number of L preceding, which is $m$=$2^k$. For $m{\ge}1$, taking $k, e$ per (20) gives $e$ ranging from 0 after the unfold L/R up to but not including the opposite R/L at the unfold after part 1.

The unfolding swaps turns L↔R, so the L sought is an R of part 1 and measuring back from the end. The last $m = 2^{k+1} - 1$ is $e = 2^k{-}1$. If $k$+1 is even then this is the R/L end at $n = 2^{k+2}$. If $k$+1 is odd then this $m$ is the L preceding that end, which is $n = 2^{k+2} - 2$.

For other $e$, measure back in part 1 to seek an R of index $2^k{-}1 - e$, or when $k$+1 even the L at the end of part 1 reduces that to $2^k{-}2$.

Similarly *TurnRight*. When $k$+1 odd the end of part 1 is an R, or when $k$+1 even the last R is $n = 2^{k+2} - 1$ being an unfold of the initial L at $n$=1. Likewise opposite reduction $2^k{-}1 - e$ or $2^k{-}2 - e$. □

Both *TurnLeft* and *TurnRight* are close to $2m$, roughly since there are $2^k$ of each turn among $n = 1 \text{ to } 2^{k+1}$ inclusive. Or algebraically in (21),(22) an

$m = 2^k$ subtracted past the unfold adds $2^{k+1}$ to the resulting $n$ (without the reversal). Offsets from $2m$ can be expressed

$$\begin{aligned} \mathit{TurnLeftOff}(m) &= 2m - \mathit{TurnLeft}(m) \\ &= -1, -2, -1, 0, -1, -3, -2, -2, -1, -2, -1, 0, 0, 1, -1, \ldots \\ \mathit{TurnRightOff}(m) &= \mathit{TurnRight}(m) - 2m \\ &= 2, 1, 3, 2, 2, 1, 0, 1, 2, 1, 3, 4, 3, 2, 3, \ldots \end{aligned} \tag{23}$$

Substituting into (21),(22) gives mutual recurrences,

where $m = 2^k + e$, with $0 \le e < 2^k$

$$\mathit{TurnLeftOff}(m) = \begin{cases} -1, -2 & \text{if } m = 0, 1 \\ [-2, 0]_k & \text{if } e = 2^k{-}1 \\ \mathit{TurnRightOff}\left(2^k{-}1 - e - [1{,}0]_k\right) \ - [4{,}2]_k & \text{otherwise} \end{cases}$$

$$\mathit{TurnRightOff}(m) = \begin{cases} 2, 1, 3 & \text{if } m = 0, 1, 2 \\ [1, 2]_k & \text{if } e = 2^k{-}1 \\ \mathit{TurnLeftOff}\left(2^k{-}1 - e - [0{,}1]_k\right) \ + [2{,}4]_k & \text{otherwise} \end{cases}$$

The offsets at (23) are taken in opposite directions away from $2m$ in order to have the mutual recurrences descending to each other as positives (then $\pm$2,4).

Both offsets can be arbitrarily large positive or negative. (The first right negative is $\mathit{TurnRightOff}(26) = -1$.) Algebraically this is by choosing $m$ so that its high bits recurse with successive $k$ giving the larger or smaller of each $-[4, 2]$ and $+[2, 4]$ term.

$m = 2^k{+}e$ takes a high bit, then the reversal is a bit flip, so the descent into the opposite Left/Right finds the next 0-bit below. The recurrences can be expressed staying in left or right by taking a high run of 1s from $m$.

$$\begin{array}{cccccccc} k & & & & l & & \overbrace{\qquad}^{e} \\ \hline \multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\ldots} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{\ldots} \\ \hline \text{high} & & & & & & \text{low} \end{array} \qquad \text{high run of 1s of } m$$

for $m = 2^k - 2^l + e$ with $0 \le e < 2^{l-1}$, and $d = [0, 1]_k - [1, 0]_l$

$$\mathit{TurnLeftOff}(m) = \begin{cases} -1 & \text{if } m{=}0 \\ \ \ 0 & \text{if } e{+}d < 0 \\ -2 & \text{if } e{=}0,\ k \text{ odd},\ l{=}0 \\ -3 + d & \text{if } e{=}2^{l-1}{-}1,\ k \text{ odd} \\ \mathit{TurnLeftOff}(e{+}d) \ + 2d & \text{otherwise} \end{cases} \tag{24}$$

$$\mathit{TurnRightOff}(m) = \begin{cases} 2 & \text{if } m{=}0 \\ 0 & \text{if } e{-}d < 0 \\ 1 - d & \text{if } l{=}0 \\ 3 & \text{if } l{=}1,\ k \text{ even} \\ 2 - d & \text{if } l{\ge}2,\ k \text{ even},\ e{=}0 \\ 4 + 2d & \text{if } l{\ge}2,\ k \text{ even},\ e{=}2^{l-1}{-}1 \\ \mathit{TurnRightOff}(e{-}d) \ - 2d & \text{otherwise} \end{cases} \tag{25}$$

Offset $d = +1, 0, -1$ arises from the offsets in the mutual recurrences going to the other and back again. It can also be written as a function of the curve direction $dir$ (ahead in section 3).

$$\begin{aligned} d &= 0,0,-1,0,0,1,0,0,0,0,0,-1,-1,0,-1,\dots \qquad m{\ge}1 \\ &= \begin{cases} \tfrac{1}{2}\big(dir(2^k-2^l)-1\big) & \text{if } l=0 \\ \tfrac{1}{2}\ \ dir(2^k-2^l) & \text{if } l\ge 1 \end{cases} \\ &= \tfrac{1}{2}\,dir\big(4.(2^k-2^l)\big) \end{aligned}$$

In left (24), the $e{+}d < 0$ case is when $e{=}0$, $k$ even, $l$ even so $d{=}{-1}$. The corresponding $e{-}d < 0$ in the right (25) is $k$ odd, $l$ odd so $d{=}{+1}$. The cases $e{=}2^{l-1}{-}1$ are $m$ with a single 0-bit like 11101111.

Increments between $n$ with turns successively L or R are

$$\begin{aligned} dTurnLeft(m) &= TurnLeft(m{+}1) - TurnLeft(m) \\ &= 3,1,1,3,4,1,2,1,3,1,1,2,1,4,1,3,3,\dots \\ dTurnRight(m) &= TurnRight(m{+}1) - TurnRight(m) \\ &= 1,4,1,2,1,1,3,3,1,4,3,1,1,3,1,2,1,\dots \end{aligned}$$

The expansions in figure 5 show steps are always 1, 2, 3, 4. The $m$'th such increment can be expressed by mutual recurrences.

for $m = 2^k + e$ with $0 \le e < 2^k$

$$dTurnLeft(m) = \begin{cases} 3 & \text{if } m{=}0 \\ [2,1]_k & \text{if } e = 2^k{-}2 \\ [1,3]_k & \text{if } e = 2^k{-}1 \\ dTurnLeft\big(2^k{-}2 - e - [1,0]_k\big) & \text{otherwise} \end{cases}$$

$$dTurnRight(m) = \begin{cases} 1,4 & \text{if } m = 0,1 \\ [3,1]_k & \text{if } e = 2^k{-}2 \\ [4,2]_k & \text{if } e = 2^k{-}1 \text{ and } m{\ge}2 \\ dTurnRight\big(2^k{-}2 - e - [0,1]_k\big) & \text{otherwise} \end{cases}$$

In the unfolding, the direction reverses so the two turns which are the delta step swap positions. This makes it necessary to descend to 1 smaller $2^k{-}2-e$ back from the end, to stay across the same step.

In these recurrences, nothing is accumulated, just descend down $m$ by unfolds until reaching one of the final 1, 2, 3, 4 cases.

## 2.1 Dean's $\alpha$

A form of the turn sequence occurs in Dean [9] (found from a reference in the OEIS) who constructs a 4-symbol infinite sequence with consecutive terms of alternating parity and which is "square-free" in the sense that no repeat E,E occurs for any block E of any length.

$$\alpha_n = 1,2,3,4,\ 1,4,3,2,\ 1,2,3,2,\ 1,4,3,4,\dots \qquad n\ge 1 \qquad \text{A003324}$$

Dean's construction is by repeated doubling of the sequence with quarters in the new part permuted,

$$\alpha_{1\ldots 4.2^k} = ABCD \qquad \text{starting } \alpha_{1\ldots 4} = 1234$$
$$\alpha_{1\ldots 8.2^k} = ABCD\ ADCB \qquad \text{block lengths } 2^k$$

This is equivalent to a morphism expanding each sequence element to two new elements (Dean's collapse $\alpha^\star = \alpha$).

$$\alpha = \quad 1 \to 1,2 \quad 2 \to 3,4 \quad 3 \to 1,4 \quad 4 \to 3,2 \qquad \text{starting from } 1$$

Such an equivalence holds for block doubling in general by thinking of how blocks of level $k$ (length $2^k$ each) came from doubled blocks of $k-1$, down until singles $A_0 = 1, B_0 = 2$ etc.

| $A_k$ | $B_k$ | $C_k$ | $D_k$ |
|---|---|---|---|

| $A_{k-1}$ | $B_{k-1}$ | $C_{k-1}$ | $D_{k-1}$ | $A_{k-1}$ | $D_{k-1}$ | $C_{k-1}$ | $B_{k-1}$ |
|---|---|---|---|---|---|---|---|

Each morphism expansion is a bit from high to low of an index $m = n-1 \geq 0$ in a state machine. A 0-bit goes to the first new term in the morphism, and a 1-bit goes to the second.

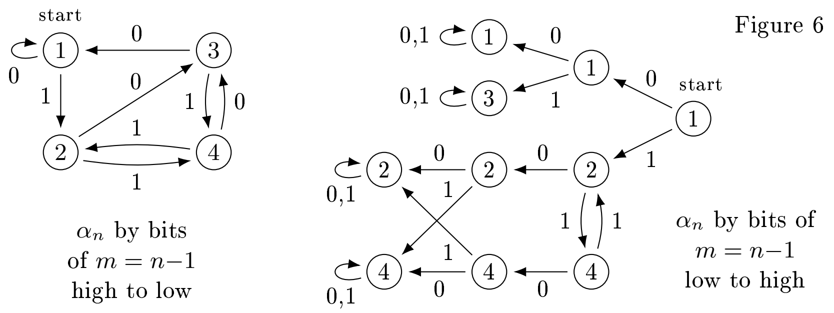



Figure 6

Dean notes 1 and 3 are $m \equiv 0, 2 \bmod 4$ just by the doubling construction. For bits high to low, and with high 0-bits reckoned above, 1 or 3 are reached by two bits 00 or 10 respectively from any state, so their cases are distinguished by the bit above the low 0. A 1-bit then goes from 1 or 3 to 2 or 4 respectively and further 1-bits alternate between 2 and 4. So and 2 or 4 are for $m$ odd and are determined by bit above lowest 0 and bit position odd or even of that bit.

Some usual manipulations reversing to bits of $m$ low to high gives the state machine at the right in figure 6. It makes clearer that 2,4 alternate with each low 1-bit and flip again if the bit above lowest 0 is another 1. In any case, this combination indexed by $m \geq 0$ is the alternate paperfolding next turn (6), or indexed by $n \geq 1$ is $\mathit{turn}(n)$.

$$\begin{aligned} \alpha_n &= \begin{cases} 1,3 & \text{if } n \equiv 1,3 \bmod 4 \\ 3 + \mathit{turn}(n) & \text{if } n \text{ even} \end{cases} \\ &= \tfrac{5}{2} + (-1)^n \left(\tfrac{1}{2} + \mathit{turn}(n)\right) \end{aligned} \tag{26}$$

$\alpha_n$ values $2, 4$ differ by 2 so $\mathit{turn} = \pm 1$ also differing by 2 is convenient. The unified (26) has a flip $(-1)^n$ since odd cases $1, 3$ take their "bit-above" the opposite way around to what the $2, 4$ form would be on $n$ odd.

$$\alpha_n = \begin{cases} 2 - turn(n) & \text{if } n \text{ odd} \\ 3 + turn(n) & \text{if } n \text{ even} \end{cases}$$

See ahead at page 23 for $\alpha$ related to starts of runs in the Golay-Rudin-Shapiro sequence.

Dean's word is among words Kao et al[13] construct from generalized paperfolding sequences by adding an alternating "parity" at odd terms. They show that for $k$ odd, taking each $k$'th term is a square-free word. This extends the same result by Carpi on the regular paperfolding sequence with alternating parity. Odd $k$ includes $k$=1 which is the whole of each sequence, and odd $k$ means the subsequences are all alternating parity.

**Theorem 2.** *The only palindromes of consecutive values in $\alpha$ have odd lengths $\le 13$ and each such length occurs infinitely.*

*Proof.* This follows mechanically from state machine manipulations building up those $n$ with matching values at suitable offsets. An explicit argument can be made too.

There are no palindromes of even length since alternating parity would make the first and last terms different.

A palindrome of odd length cannot have an even $n$ in the middle (except trivially length 1), since alternating $\alpha_{\mathrm{odd}} = 1 \text{ or } 3$ would be different each side.

A palindrome of odd length with an odd $n$ in the middle is an even length palindrome of even *turns*,

Figure 7

| | | | | middle | | | | |
|---|---|---|---|---|---|---|---|---|
| odd A | *turn*(2*n*) +3 | odd B | *turn*(2*n*+2) +3 | odd A | *turn*(2*n*+4) +3 | odd B | *turn*(2*n*+6) +3 | odd A |

$turn(2n) = -turn(n)$ as from (1) means these turns are palindromes in the full *turn* sequence. Allouche[1] shows the only even length palindromes in *turn* are lengths 2,4,6 (and trivially 0), so the only palindromes in $\alpha$ are odd lengths up to $2.6 + 1 = 13$.

Some explicit calculation finds initial examples up to 13. *turn* and $\alpha$ are determined by low bits of $n$ and higher bits are arbitrary, giving infinite replications. □

In figure 7, a length 3 palindrome always has the same odd "B" each side, so all 3s are the middles of 5s. But some 5s are maximal in that they have $turn(2n) \neq turn(2n{+}6)$ so are not just the middles of 7s.

7s have the same odd "A" each side so are only the middles of 9s. Some 9s are maximal (not middles of 11s). Finally 11s have odd "B" each side so are only the middles of 13s.

# 3 Direction

The direction of segment $n$, numbered as $n$=0 for the first segment, is net sum of turns preceding it

$$dir(n) = \sum_{j=1}^{n} turn(j) \qquad \text{empty sum when } n{=}0, \text{ so } dir(0){=}0 \tag{27}$$

$$= 0, 1, 0, -1, 0, 1, 2, 1, 0, 1, 0, -1, -2, -1, 0, -1, 0, 1, \ldots$$

$$gdir(x) = \frac{1}{1-x}\, gturn(x) \;= \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2^k}}{1+x^{2^{k+1}}} \tag{28}$$

**Theorem 3** (Mendès France and Tenenbaum[18]). *Write $n$ = binary $a_k \ldots a_1 a_0$, where $a_0$ is the least significant bit and at least one high 0-bit so $a_k = 0$. $dir(n)$ is sum $\pm1$ at each bit transition with sign according as even or odd bit position,*

$$dir(n) = \sum_{j=0}^{k-1} \begin{cases} +1 & \text{if } a_j \neq a_{j+1} \text{ and } j \text{ even} \\ -1 & \text{if } a_j \neq a_{j+1} \text{ and } j \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

Requiring a high $a_k = 0$ reckons high 0-bits above $n$ so that its most significant 1-bit is an 01 transition.

*Proof.* A level $k$+1 curve comprises two level $k$ sub-curves, with the unfold side according to $k$ even or odd

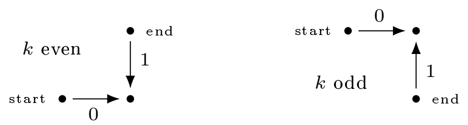



Figure 8

Bit 0 or 1 of $n$ is sub-curve 0 or 1 shown. The direction of the unfolded sub-curve is an extra +1 or −1 according as $k$ even or odd.

The unfolding means sub-curve 1 has segments in reverse order. The sub-curves there are bit 1 for the first or bit 0 for the second. So a state machine on the bits of $n$,

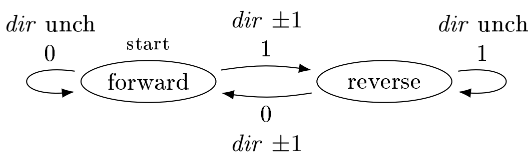



Figure 9:
bits of $n$
high to low

Forward state is always reached by a 0-bit and reverse state by a 1-bit. The direction extra ±1 is accumulated where a different bit switches state.

The initial forward state accumulates ±1 for the high 1-bit. In sum (29), this is achieved by reckoning a high 0-bit (or several such). ☐

Mendès France and Tenenbaum[18] write bit differences in (29) as

$$dir(n) = \sum_{j=0}^{k-1} (-1)^j \; \bigl|a_j - a_{j+1}\bigr|$$

and show for generalized paperfolding curves with specified folding sides $\epsilon_j = \pm 1$ that segment direction is $\sum_{j=0}^{k-1} \epsilon_j \, |a_j - a_{j+1}|$. In the generalized curves, $\epsilon_j$ controls which side each unfold in figure 8 goes (here alternating, or in the dragon curve always left).

The generating function at (28) is cumulative *turn* by factor $1/(1{-}x)$ in the usual way. Its direct interpretation is terms $\pm 1$ for each bit position $k$ in (29). Bit pair 01 or 10 and sign for position $k$ is $(-1)^k \, (x^{2^k} + x^{2.2^k})$. Then a factor $1{+}\cdots{+}x^{2^k-1}$ replicates for arbitrary bits below and a denominator $1 - x^{2^{k+2}}$ replicates for arbitrary bits above, so

$$\begin{aligned} \frac{(-1)^k \, (x^{2^k} + x^{2.2^k}) \, (1 + x + \cdots + x^{2^k-1})}{1 - x^{2^{k+2}}} &= \frac{(-1)^k \, x^{2^k} \, (1 + x^{2^k}) \, (1 - x^{2^k})}{(1-x) \, (1 - x^{2^{k+2}})} \\ &= \frac{(-1)^k \, x^{2^k}}{(1-x) \, (1 + x^{2^{k+1}})} \quad \text{as in (28)} \end{aligned}$$

Davis and Knuth have direction implicit in their location formula (here ahead at (60)). They write $n$ in a "folded" representation where powers of 2 have alternating signs,

$$\begin{aligned} n &= 2^{k_0} + (-1).2^{k_1} + (-1)^2.2^{k_2} + \cdots + (-1)^t.2^{k_t} \\ &k_0 > k_1 > \cdots > k_t \qquad\qquad \text{folded representation of } n \end{aligned} \tag{30}$$

This representation follows unfoldings and it locates bit runs. Term $+1$ is above the high end of each run, and $-1$ is at the low end of each run.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 0 | $\cdots$ | binary |
| $+1$ | 0 | 0 | 0 | $-1$ | 0 | $\cdots$ | folded |
| $k_0$ | | | | $k_1$ | | $\cdots$ | |

Davis and Knuth show every $n \neq 0$ has two folded representations, the lowest term being either $+1$ or $-1$ and which then determines the terms above. The one with an even number of terms, so $t$ odd and lowest term sign $-1$, gives direction

$$dir(n) = [0,1]_n - \sum_{j=0}^{t} (-1)^{k_j} \qquad \text{for } t \text{ odd} \tag{31}$$

Each $k_j$ power is one bit position above the transition in (29), so the sum is $(-1)^{k_j-1} = -(-1)^{k_j}$. If $n$ is odd ($k_t = 0$) then there is no transition in (29) corresponding to the final $k_t$. Adding back 1 when odd adjusts for that.

The folded representation with instead an odd number of $k$ powers, so $t$ even and loest sign $+1$, effectively arrives at a point from the far end of the curve so gives direction of the segment preceding the point

$$dir(n{-}1) = [0,1]_n - \sum_{j=0}^{t} (-1)^{k_j} \qquad \text{for } t \text{ even, } n{\geq}1$$

Binary reflected Gray code locates bit transitions in $n$ too. Its shift and XOR at (33) gives a 1-bit at each bit transition, including highest 1-bit as a transition from 0s above. *dir* is then $\pm 1$ at each such 1-bit, with sign according as bit position odd or even,

$$dir(n) = PmOneBits\big(Gray(n)\big) \tag{32}$$

$$PmOneBits(n) = \sum_{j=0}^{k-1} \begin{cases} 1 & \text{if } a_j = 1 \text{ and } j \text{ even} \\ -1 & \text{if } a_j = 1 \text{ and } j \text{ odd} \end{cases} \quad \text{bits of } n$$

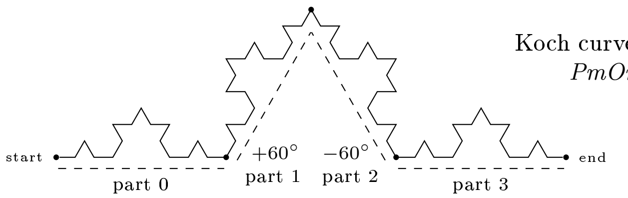
$$= 0, 1, -1, 0, 1, 2, 0, 1, -1, 0, -2, -1, 0, 1, -1, 0, \ldots \qquad \text{A065359}$$

$$Gray(n) = \text{BITXOR}(n, \lfloor n/2 \rfloor) \qquad \text{binary reflected Gray code} \tag{33}$$

$$= 0, 1, 3, 2, 6, 7, 5, 4, 12, 13, 15, 14, 10, 11, 9, 8, \ldots \qquad \text{A003188}$$

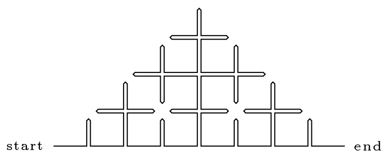
The Gray code is a permutation of the integers in blocks $0 \le n < 2^k$, so the *dir* sequence is such a permutation of *PmOneBits*, and is a permutation which arranges successive steps of *dir* to be $+1$ or $-1$ (the *turn* sequence).

*PmOneBits* is directions in the Koch curve [16]. Koch curve segment $n$ is direction $PmOneBits(n) . 60°$ since base 4 sub-part 1 is $+60°$ and sub-part 2 is $-60°$, the same as the two bits.




Koch curve directions
*PmOneBits*

Koch curve turns are $+1$ or $-2$ according as $CountLowZeros(n)$ even or odd (the "period-doubling" sequence). Closing the curve up with 90° directions makes a triangular shape with left turns and 180° reversals. Segments are double-traversed except on the X axis.




Koch curve 90° turns,
segments double-traversed

The *Gray* permutation applied to these overlapping segment steps gives the non-overlapping alternate paperfolding curve.

Direction in the alternate paperfolding curve can also be expressed by signed product of adjacent bits.

**Theorem 4.** *For $n =$ binary $a_k \ldots a_1 a_0$, with $a_0$ the least significant bit,*

$$dir(n) = a_0 - 2 \,.\, Alt11Pairs(n) \tag{34}$$

$$Alt11Pairs(n) = \sum_{j=0}^{k-1} (-1)^j \, a_j \, a_{j+1}$$

$$= 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 1, \ldots$$

*Proof.* Within a run of 1-bits, successive terms of *Alt11Pairs* cancel. For an odd length run, they cancel entirely to 0, the same as transitions (29). For an even length run, there is a single net $\pm 1$ in *Alt11Pairs* whereas transitions are $\mp 1$ at the start of the run and the same at the end of the run, hence factor $-2$ in (34).

If $n$ is odd then the lowest run of 1-bits ends at the least significant bit and there is no ending transition for it. Adding $a_0$ adjusts for that (in a similar way to the folded (31)). $\square$

Since *Alt11Pairs* terms cancel in odd length runs of 1-bits, an equivalent definition is sum of $(-1)^p$ for bit positions $p$ which are the low end of an even length run of 1-bits in $n$.

Increments of *Alt11Pairs* follow from *turn* which is the increments of *dir*, or from how low 1-bits increment and change the *Alt11Pairs* sum.

$$\begin{aligned} \mathit{dAlt11Pairs}(n) &= \mathit{Alt11Pairs}(n) - \mathit{Alt11Pairs}(n-1) \qquad \text{for } n \geq 1 \\ &= a_0 - \mathit{TurnLpred}(n) \\ &= \begin{cases} -\mathit{TurnLpred}(n) & \text{if } n \text{ even} \\ \phantom{-}\mathit{TurnRpred}(n) & \text{if } n \text{ odd} \end{cases} \\ &= 0, 0, 1, -1, 0, -1, 1, 0, 0, 0, 1, 0, \ldots \end{aligned} \tag{35}$$

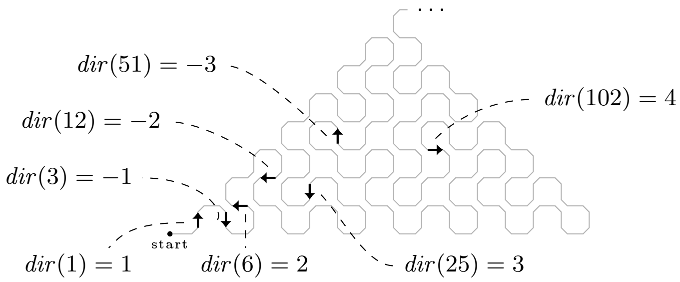
L and R cases in (35) have an attractive symmetry, but actually *TurnRpred* on odd $n$ is just $a_1$, the second least significant bit of $n$.

*dir* is a maximum when the bit transitions sum (29) has a transition for every $+1$ term, and no $-1$ terms. This means a transition at each even position and no transition at each odd position, and so alternating 11 and 00 with the high 1 at an even position. The resulting $n$ is unique.

$$\begin{aligned} \mathit{DirMax}_k &= \max_{n=0}^{2^k-1} \mathit{dir}(n) = \lceil k/2 \rceil = 0, 1, 1, 2, 2, 3, 3, \ldots \\ \mathit{DirMaxN}_k &= \tfrac{1}{5}\Big([2,4].2^k - [2,3,3,2]\Big) \qquad \text{unique } n \\ &= 0, 1, 1, 6, 6, 25, 25, 102, 102, 409, 409, \ldots \qquad \text{dup A037481} \\ &= \text{binary } 0, 1, 1, 110, 110, 11001, 11001, \ldots \\ &= 1100\ 1100 \ldots \text{for odd number of bits } k \text{ or } k{-}1 \end{aligned}$$

Similarly *dir* is a minimum when every $-1$ term and no $+1$ terms, which is 11 and 00 starting at an even position and again a unique $n$.

$$\begin{aligned} \mathit{DirMin}_k &= \min_{n=0}^{2^k-1} \mathit{dir}(n) = -\lfloor k/2 \rfloor = 0, 0, -1, -1, -2, -2, -3, \ldots \\ \mathit{DirMinN}_k &= \tfrac{1}{5}\Big([4,2].2^k - [4,4,1,1]\Big) \qquad \text{unique } n \\ &= \big\lfloor \tfrac{1}{2} \mathit{DirMaxN}_{k+1} \big\rfloor \\ &= 0, 0, 3, 3, 12, 12, 51, 51, 204, 204, 819, \ldots \qquad \text{dup A043291} \\ &= \text{binary } 0, 0, 11, 11, 1100, 1100, 110011, 110011, \ldots \qquad \text{dup A153435} \\ &= 1100\ 1100 \ldots \text{for even number of bits } k \text{ or } k{-}1 \end{aligned}$$




The number of left and right turns from 1 to $n$ inclusive are

$$\begin{aligned} \mathit{TurnsL}(n) &= \sum_{j=1}^{n} \mathit{TurnLpred}(j) \\ &= \tfrac{1}{2}\big(n + \mathit{dir}(n)\big) \qquad (36) \\ &= 1, 1, 1, 2, 3, 4, 4, 4, 5, 5, 5, 5, 6, 7, 7, 8, 9, 9, \ldots \\ \mathit{TurnsR}(n) &= \sum_{j=1}^{n} \mathit{TurnRpred}(j) \\ &= \tfrac{1}{2}\big(n - \mathit{dir}(n)\big) \qquad (37) \\ &= 0, 1, 2, 2, 2, 2, 3, 4, 4, 5, 6, 7, 7, 7, 8, 8, 8, 9, \ldots \end{aligned}$$

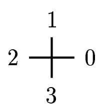
Forms (36),(37) follow since all turns are left or right so total lefts plus rights is simply $n$. Then difference lefts minus rights is net direction $\mathit{dir}$ (its sum (27)). Sum and difference of (38),(39) are then (36),(37).

$$\mathit{TurnsL}(n) + \mathit{TurnsR}(n) = n \qquad (38)$$

$$\mathit{TurnsL}(n) - \mathit{TurnsR}(n) = \mathit{dir}(n) \qquad (39)$$

$\mathit{dir}(n)$ mod 4 is a net segment direction East, North, West or South.

$$\mathit{dir}(n) \bmod 4 \equiv 0, 1, 0, 3, 0, 1, 2, 1, 0, 1, 0, 3, 2, 3, 0, 3, \ldots$$




Arndt [2] gives some bit twiddling for $\mathit{dir}$ mod 4,

$$\mathit{dir}(n) \equiv \mathit{CountOneBits}\big(\mathrm{BITXOR}\big(1010\ldots1010, \mathit{Gray}(n)\big)\big) \bmod 4$$

This is similar to the $\mathit{PmOneBits}(\mathit{Gray}(n))$ form (32). BITXOR leaves even positions unchanged so $+1$ each. The bits at odd positions are flipped by the BITXOR constant and it is a multiple of 8 bits (4 bits flipped in each) so that resulting count of odd positions is negated mod 4.

Back in figure 9, transitions between forward and reverse add $\pm1$ to the direction, so forward is $\mathit{dir}$ even which is horizontal and reverse is $\mathit{dir}$ odd which is vertical. Those states can be split into $\mathit{dir} \equiv 0, 2 \bmod 4$ and $1, 3 \bmod 4$ according to how the same sign $\pm1$ accumulates and different signs cancel. The following state machine is per the morphism expansion given by Arndt [2].

$$dir \bmod 4 = \quad 0 \to 0,1 \quad 1 \to 0,3 \quad 2 \to 2,3 \quad 3 \to 2,1 \quad \text{starting } 0 \tag{40}$$

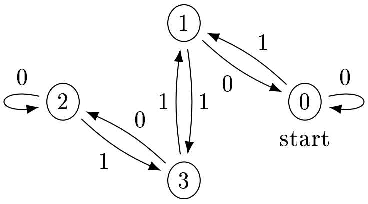
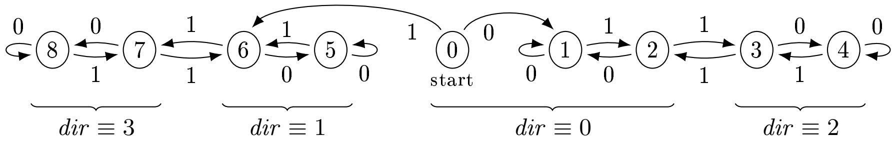


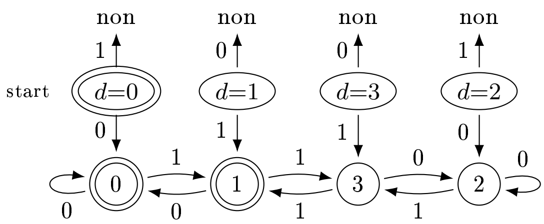
Figure 10:
$dir(n) \bmod 4$
bits of $n$
high to low

Some state machine manipulations or considering the *dir* sum gives the following for bits of $n$ low to high,




At the start state, the low bit is $n$ even or odd so goes to states for horizontal $dir \equiv 0,2$ or vertical $dir \equiv 1,3$ respectively. The states then effectively look for even length runs of 1-bits. These are transitions at high and low positions with the same parity so giving the same $\pm 1$ at each and so direction $+2 \bmod 4$. An odd length run of 1-bits is transitions at different parity bit positions so $+1$ and $-1$ cancel out.

The states each side of the start have the same structure and transitions (and are the same as high to low figure 10). Predicates for those $n$ with $dir(n) = d$ can be formed by bits of $n$ low to high commencing at a suitable place.




predicates
$dir(n) \equiv d \bmod 4$
bits of $n$
low to high

The start state is chosen according to the desired $d$ direction to test. The double-circled accepting states are then those $n$ with $dir(n) \equiv d$.

Reaching "non" is non-accepting. Horizontal $d = 0,2$ are only even $n$ so for them a low 1-bit goes immediately to non. Vertical $d = 1,3$ are only odd $n$ so for them a low 0-bit goes immediately to non.

Starting state $d$=0 is accepting since $n$ of no bits is 0 which is $dir(0) = 0$. Further 0-bits go to state 0 which is also accepting, being $n$=0 represented by multiple 0-bits. The other starting states are non-accepting.

$$\begin{aligned} dir(n) &\equiv 0 \text{ at } n = 0,\ 2,\ 4,\ 8,10,14,16,18,20,\ldots && 2{\times}\text{A203463} \\ &\equiv 1 \text{ at } n = 1,\ 5,\ 7,\ 9,17,21,23,27,29,\ldots \\ &\equiv 2 \text{ at } n = 6,12,22,24,26,30,38,44,48,\ldots && 2{\times}\text{A022155} \end{aligned}$$

$$\equiv 3 \text{ at } n = 3, 11, 13, 15, 19, 25, 35, 43, 45, \ldots$$

Direction gives coordinate steps $dx$ and $dy$,

$$\begin{aligned} dx(n) &= \text{Re } i^{dir(n)} && (41) \\ &= 1, 0, 1, \quad 0,\ 1, 0, -1, 0,\ 1, 0, 1, \quad 0,\ -1, \quad 0, 1, \ldots \\ dy(n) &= \text{Im } i^{dir(n)} && (42) \\ &= 0, 1, 0, -1,\ 0, 1, \quad 0, 1,\ 0, 1, 0, -1, \quad 0, -1, 0, \ldots \end{aligned}$$

The curve turns 90° at every point so $dx$ and $dy$ are alternately zero and non-zero. Combined, they are the Golay-Rudin-Shapiro sequence.

## 3.1 GRS

Shapiro [22] defines a pair of polynomials, which as in Rudin's presentation [20] but without Rudin's extra factor $x$, are given by mutual recurrences

$$\begin{aligned} P_k(x) &= P_{k-1}(x) + x^{2^{k-1}} Q_{k-1}(x) \qquad \text{starting } P_0(x) = Q_0(x) = 1 \qquad (43) \\ Q_k(x) &= P_{k-1}(x) - x^{2^{k-1}} Q_{k-1}(x) \end{aligned}$$

$P_k$ and $Q_k$ have $2^k$ terms each. The coefficients of $P_k$ are a prefix of its next $P_{k+1}$. Continued infinitely, these coefficients are

$$P_\infty(x) = \sum_{n=0}^{\infty} GRS(n) \,.\, x^n$$

where $GRS$ is the Golay-Rudin-Shapiro sequence. Per Shapiro, the coefficients of $P$ replicate with negation in the final quarter,

$$\overset{\text{copy}}{\begin{array}{|l|l|l|} \hline \text{terms } \ 0 \ldots \ 2.2^{k-1}{-}1 & 2.2^{k-1} \ldots & \underbrace{3.2^{k-1} \ldots}_{\text{negate}} \\ \hline \end{array}} \qquad (44)$$

Brillhart and Carlitz [5] show

$$\begin{aligned} GRS(n) &= (-1)^{\mathit{Count11Pairs}(n)} \\ &= 1, 1, 1, -1, 1, 1, -1, 1, 1, 1, 1, -1, -1, -1, 1, -1, \ldots && \text{A020985} \\ \mathit{Count11Pairs}(n) &= \text{count 11 bit pairs in } n \text{, overlapping pairs allowed} \\ &= \sum_{j=0}^{k-1} a_j\, a_{j+1} \qquad \text{where } n = a_k \ldots a_0 \text{ binary} \\ &= 0, 0, 0, 1, 0, 0, 1, 2, 0, 0, 0, 1, 1, 1, 2, 3, \ldots && \text{A014081} \end{aligned}$$

and give an equivalent recurrence, which is often used as a definition in fact,

$$GRS(2n) = GRS(n) \qquad GRS(2n{+}1) = (-1)^n . GRS(n) \qquad (45)$$

The alternating sign form, which is the odd terms, is

$$GRSalt(n) = (-1)^n . GRS(n)$$
$$= 1, -1, 1, 1, 1, -1, -1, -1, 1, -1, \ldots$$

(45) is a new low bit on $n$. $GRS(2n{+}1)$ is a new low 1-bit and $(-1)^n$ flips the sign of $GRS(n)$ when $n$ was odd, since that bit makes a new low 11 bit pair. Shapiro's copying at (44) does this at the high end. The copy is a new high 1-bit and its second quarter is an existing high 1 so together a new high 11 pair and so negate.

**Theorem 5** (Mendès France and Tenenbaum [18]). *The alternate paperfolding curve $dx$ and $dy$ steps are the Golay-Rudin-Shapiro sequence,*

$$GRS(n) = \begin{cases} dx(n) & \text{if } n \text{ even} \\ dy(n) & \text{if } n \text{ odd} \end{cases} \tag{46}$$

$$= dx(n) + dy(n) \;\; = dsum(n) \tag{47}$$

*and consequently its alternating signs form is*

$$GRSalt(n) = \begin{cases} dx(n) & \text{if } n \text{ even} \\ -dy(n) & \text{if } n \text{ odd} \end{cases}$$

$$= dx(n) - dy(n) \;\; = ddiff(n) \tag{48}$$

*Proof.* $dx$ and $dy$ (41),(42) use $dir$ mod 4. In (34) $dir$ by $Alt11Pairs$, $2 \equiv -2$ mod 4 so the signs there can be ignored,

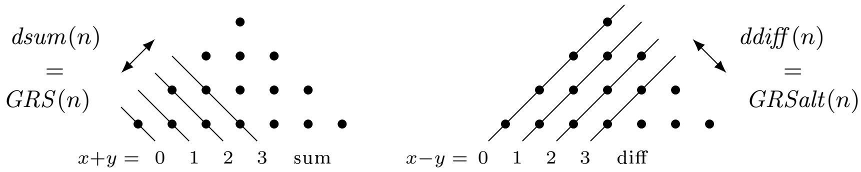
$$dir(n) \equiv a_0 + 2.Count11Pairs(n) \mod 4$$

So even $n$, which is $a_0 = 0$, has $dir \equiv 0$ or 2 according to the parity of $Count11Pairs$ which is the same as $GRS(n)$. And odd $n$, which is $a_0 = 1$, has $dir \equiv 1$ or 3 according to the parity of $Count11Pairs$ again the same as $GRS(n)$.

The curve takes horizontal and vertical steps alternately so one of $dx$ and $dy$ is 0 and the other non-0, hence the sum (47). Difference (48) is since $GRSalt$ negates odd terms, which are the $dy$ terms. □

Mendès France and Tenenbaum show this from the generalized paperfolding curves of Davis and Knuth in which the unfold side for each level is arbitrary (here alternating, or in the dragon curve always left). They form the turn sequence of such a curve, count left and right turns up to $n$, take the difference which is net direction, and form polynomials in the $dx$, $dy$ steps which they show satisfy a general recurrence which is (43) in the alternating case.

The geometric interpretation of $dsum$ is steps between anti-diagonals. The geometric interpretation of $ddiff$ is steps between leading diagonals.




The non-0 terms of $dx$ are the $GRS$ sequence by recurrence (45), and likewise the non-0 terms of $dy$ are the $GRSalt$ sequence.

$$
\begin{aligned}
dx(2n) &= GRS(2n) &&= GRS(n) && (49)\\
dy(2n{+}1) &= GRS(2n{+}1) &&= GRSalt(n) && (50)
\end{aligned}
$$

See section 9 on *GRS* terms summed to give $x, y$ coordinates.
*dir* mod 4 is related to *GRS* by the $dx, dy$ cases,

$$
GRS(n) = \begin{cases} +1 & \text{if } dir(n) \equiv 0 \,\text{or}\, 1 \bmod 4 \\ -1 & \text{if } dir(n) \equiv 2 \,\text{or}\, 3 \bmod 4 \end{cases} \tag{51}
$$

$$
dir(n) \bmod 4 = [1,2]_n - GRS(n)
$$

A variation on *dir* mod 4 is sometimes used for expressing *GRS* by a morphism (for example Allouche[1] with letters A to D),

$$
\begin{aligned}
GRS4(n) &= swap23\big(dir(n) \bmod 4\big)\\
&= 0,1,0,2,0,1,3,1,0,1,0,2,3,2,0,2,\dots \qquad \text{A100260}
\end{aligned}
$$

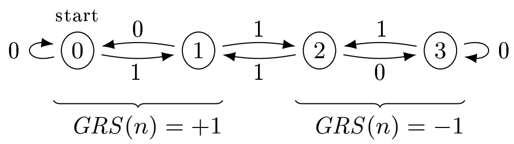
$$
swap23(d) = 0,1,3,2 \quad \text{for } d = 0,1,2,3 \text{ respectively}
$$

$$
GRS4 = \quad 0 \to 0,1 \quad 1 \to 0,2 \quad 2 \to 3,1 \quad 3 \to 3,2 \quad \text{starting } 0 \tag{52}
$$

$$
GRS(n) = \begin{cases} +1 & \text{if } GRS4(n) = 0 \,\text{or}\, 1 \\ -1 & \text{if } GRS4(n) = 2 \,\text{or}\, 3 \end{cases} \tag{53}
$$

Morphism (52) is the same as (40) after swapping $2 \leftrightarrow 3$ throughout. This swap is no change to the cases at (51) so the corresponding (53).

Symbols or values can be chosen arbitrarily for which two are to be $GRS = +1$ and which two $-1$. *GRS4* is the same state machine structure as the *dir* mod 4 state machine in figure 10, but labels are consecutive 0 to 3 across.




*GRS4*(*n*), bits of *n* high to low

*dir* mod 4 values have the attraction of a bitwise interpretation. The low bit is the previously seen bit from index $n$, ready to compare to the next bit for a possible new 11 pair. *GRS* is the second bit and such a 11 pair flips it.

Putting *GRS* in the low (and previous bit in the second) could suit working with *GRS* as $0,1$. This would be a swap $1 \leftrightarrow 2$ of *dir* mod 4.

Those $n$ with $GRS(n) = \pm 1$, and alternating $GRSalt(n) = \pm 1$, are

$$
\begin{aligned}
GRS(n) &= +1 \ \text{ at } n = 0,1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 9,\dots && \text{A203463}\\
GRS(n) &= -1 \ \text{ at } n = 3,6,11,12,13,15,19,22,\dots && \text{A022155}\\
GRSalt(n) &= +1 \ \text{ at } n = 0,2,\ 3,\ 4,\ 8,10,11,13,\dots\\
GRSalt(n) &= -1 \ \text{ at } n = 1,5,\ 6,\ 7,\ 9,12,17,21,\dots
\end{aligned}
$$

*GRSalt* is opposite sign to *GRS* at odd $n$, so its lists of $n$ are the same as the *GRS* lists but the odd terms swap between $+1$ and $-1$.

Those $n$ which are the start of a run of consecutive same *GRS* values can be characterized,

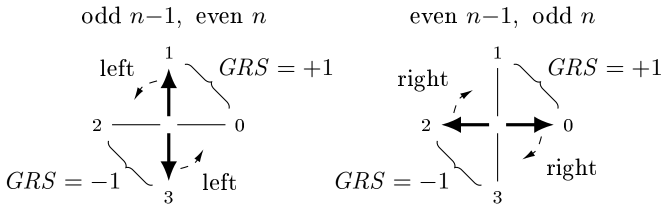
$$GRSrunSpred(n) = \begin{cases} 1 & \text{if } n{=}0 \text{ or } GRS(n) \neq GRS(n{-}1) \\ 0 & \text{otherwise} \end{cases}$$
$$= 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, \ldots$$

**Theorem 6.** *Run starts in the Golay-Rudin-Shapiro sequence are determined by alternate paperfolding curve turns as follows*

$$GRSrunSpred(n) = \begin{cases} 1 & \textit{if } n{=}0 \\ 1 & \textit{if } n{\neq}0 \textit{ even and } TurnLpred(n) \\ 0 & \textit{if } n \quad \textit{odd and } TurnRpred(n) \end{cases} \tag{54}$$

*Proof.* $dir(n{-}1)$ changes from $0$ or $1$ (for $GRS = +1$) and $2$ or $3$ (for $GRS = -1$) or vice versa when a left or right turn at $n$ as follows,

Figure 11:
$dir(n{-}1)$ mod 4,
and turn at $n$
going to $dir(n)$




An odd $n{-}1$ is $dir(n{-}1)$ vertical $1$ or $3$ mod 4 and in both cases a left turn at $n$ increments to the opposite $GRS$. Similarly even $n{-}1$ and a right turn. $\square$

Odd $n$ at (54) is simply

$$GRSrunSpred(n) = 0 \qquad \text{when } n \equiv 1 \bmod 4 \tag{55}$$
$$GRSrunSpred(n) = 1 \qquad \text{when } n \equiv 3 \bmod 4 \tag{56}$$

(55) is binary $n{-}1 = \ldots 00$ incrementing to $n = \ldots 01$ which is no change to its 11 pairs so no change to $GRS$ and so $n$ not the start of a run. (56) is binary $n{-}1 = \ldots 10$ incrementing to $n = \ldots 11$ and the new low 1 makes a new 11 so $GRS$ changes sign and so $n$ is the start of a run.

This alternating odd turns and then paperfolding turns at even $n$ is a generalized paperfolding curve. Allouche[1] uses it as a $z_n$, for $n \geq 1$, summed mod 2 to form the $GRS$ sequence as $0, 1$ (rather than $\pm 1$). The $GRS$ value flips at each run start so sum of starts mod 2 is $GRS$,

$$\sum_{j=1}^{n} GRSrunSpred(j) \equiv 0, 1 \bmod 2 \;\; \text{according as } GRS(n) = +1, -1$$

This pattern of odd terms alternating and even terms as *turn* is also per Dean's $\alpha$ from section 2.1

$$GRSrunSpred(n) = \begin{cases} 1 & \text{if } n{=}0 \\ 1 & \text{if } \alpha_n = 3 \text{ or } 4 \\ 0 & \text{if } \alpha_n = 1 \text{ or } 2 \end{cases} \qquad n \geq 1$$
$$\alpha_n = 2\; GRSrunSpred(n) + [2, 1]$$

For *GRSalt*, the corresponding run start predicate is as follows. *GRSalt* is negated at odd $n$ so a vertical mirror image of figure 11 and so the left and right turns swap.

$$\begin{aligned}
\mathit{GRSrunSpredAlt}(n) &= \begin{cases} 1 & \text{if } n{=}0 \text{ or } \mathit{GRSalt}(n) \neq \mathit{GRSalt}(n{-}1) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } n{=}0 \\ 1 & \text{if } n{\neq}0 \text{ even and } \mathit{TurnRpred}(n) \\ 0 & \text{if } n \quad\ \text{odd and } \mathit{TurnLpred}(n) \end{cases} \\
&= 1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, \ldots
\end{aligned}$$

$$\begin{aligned}
\mathit{GRSrunSpredAlt}(n) &= 1 \qquad \text{when } n \equiv 1 \bmod 4 \\
\mathit{GRSrunSpredAlt}(n) &= 0 \qquad \text{when } n \equiv 3 \bmod 4
\end{aligned}$$

Runs of the same value in *GRS* are at most 4 long since every $n \equiv 3 \bmod 4$ is a run start per (56). All lengths 1 to 4 can be found in initial values of *GRS*, then Shapiro's copying means they all occur infinitely. Similarly *GRSalt*.

$$\begin{aligned}
\mathit{GRSrun} &= 3, 1, 2, 1, 4, 3, 1, 1, 3, 1, 2, 1, \ldots && \text{A203531} \\
\mathit{GRSrunAlt} &= 1, 1, 3, 3, 1, 1, 2, 1, 4, 1, 3, 4, \ldots
\end{aligned}$$

With $dx$ from *GRS* at (46), those $n$ with $\mathit{GRS}(n) = \pm 1$ become $\mathit{dir}(2n) \equiv 0, 2 \bmod 4$. Similarly $\mathit{GRSalt}(n)$ and $dy(2n{+}1)$. So runs of 1 to 4 in *GRS* and *GRSalt* mean steps between those $n$ with $\mathit{dir}(n) \equiv d \bmod 4$ are $2, 4, 6, 8, 10$ and all such steps occur infinitely. For example if $n$ is a segment East then there is another East at one of $n + 2, 4, 6, 8, 10$.

Any consecutive ten $n$ to $n{+}9$ inclusive contains at least one segment of each $\mathit{dir} \bmod 4$. A full set of 10 is needed when $\mathit{dir}(n{+}9) \bmod 4$ does not occur in $n$ through $n{+}8$. Direction parity alternates so only $n{+}1, n{+}3, n{+}5, n{+}7$ might be the same as $n{+}9$. Some state machine manipulations to ask when each possible $d \equiv \mathit{dir} \bmod 4$ occurs at $n{+}9$, but not at $n$ to $n{+}8$, shows those $n$ at the start of such a set of 10 are

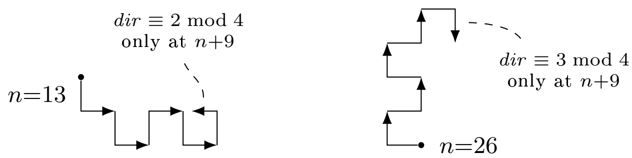
$$\begin{aligned}
&\text{bits} \quad \boxed{\mathit{NextTurnLpred} \mid 1101} \quad \text{or} \quad \boxed{\mathit{NextTurnRpred} \mid 1010} \\
&\quad = 13, 26, 42, 61, 77, 93, 106, 122, 141, 154, \ldots
\end{aligned}$$

where

$$\begin{aligned}
\mathit{NextTurnLpred}(n) &= \mathit{TurnLpred}(n{+}1) \\
\mathit{NextTurnRpred}(n) &= \mathit{TurnRpred}(n{+}1)
\end{aligned}$$

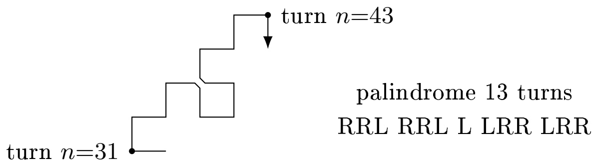
Some adding can push the low bits up to 0000 and make a carry for the "next" so conditions become $n \equiv 13 \bmod 16$ and $\mathit{TurnLpred}(n{+}3)$, or $n \equiv 10 \bmod 16$ and $\mathit{TurnRpred}(n{+}6)$.

In the curve, such a set of 10 is 4 steps $dx$ in the same direction until a step back the other way $-dx$. Similarly $dy$.




Allouche[1] shows the only possible palindrome lengths in the turn sequences of generalized paperfolding curves are $1\ldots7, 9, 11, 13$, and in 2-value directions like $GRS$ only $1\ldots8, 10, 12, 14$.

For a given curve, some state machine manipulations can find the intersection of those $n$ with same value at $n+length-1$, and so on, so forming a palindrome. The first palindrome of length 13 in *turn* is at $n$=31 to $n$=43 inclusive. Segments $n$=30 to $n$=43 inclusive are also the first palindrome of 14 $GRS$ values.

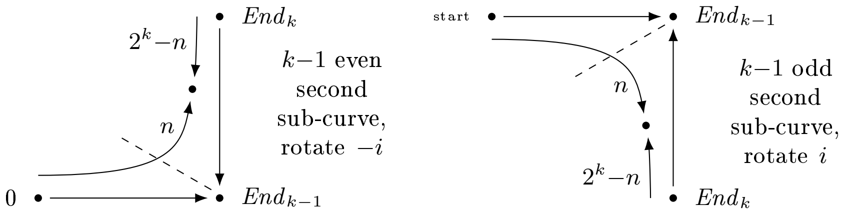



# 4 Coordinates

It's convenient to calculate curve locations in complex numbers, and number points starting $n$=0 at the origin and first segment directed East. The end of the curve unfolds by factor $b = i{+}1$ when $k$ even or $\overline{b}$ when $k$ odd (eg. figure 1)

$$
\begin{aligned}
b &= 1{+}i \qquad \overline{b} = 1{-}i \\
End_k &= b^{\lceil k/2 \rceil} . \overline{b}^{\lfloor k/2 \rfloor} && \text{curve end} && (57)\\
&= i^{-\lfloor k/2 \rfloor} . b^k && && (58)\\
&= [1, b] . 2^{\lfloor k/2 \rfloor} \\
&= 1,\ 1{+}i,\ 2,\ 2{+}2i,\ 4,\ 4{+}4i,\ \ldots && && \text{Re A016116, Im A077957}
\end{aligned}
$$

Davis and Knuth [8] give a coordinate formula using their folded representation (30). An $n$ in the second half of the curve is a point in that sub-curve directed back from $End_k$. The unfold is on the left or right according as $k{-}1$ odd or even.




$$point(n) \;=\; End_k + i.(-1)^k.point(2^k{-}n) \qquad 2^{k-1} \le n \le 2^k \tag{59}$$

They expand (59) repeatedly which is $n$ in folded representation per (30) and give, for an arbitrary unfold angle $\theta$,

$$point(n) = \zeta^{-d_0}(1+\zeta)^{k_0} + \zeta^{-d_1}(1+\zeta)^{k_1} - \cdots + (-1)^t \zeta^{-d_t}(1+\zeta)^{k_t} \tag{60}$$

$$\text{where } \zeta = e^{i(\pi-\theta)} \quad \text{unfold by angle } \theta$$

$$d_j = (-1)^{k_0} + \cdots + (-1)^{k_{j-1}} + \lfloor k_j/2 \rfloor$$

$d_j$ contains sum of $(-1)^k$ which is per the direction *dir* form here at (31).

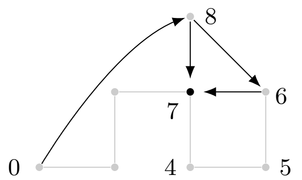
For $\zeta = i$ , the formula simplifies a little using *End* form (58) and reducing exponents $i^{(-1)^k} = i.(-1)^k$.

$$\begin{aligned} point(n) &= End_{k_0} + i.(-1)^{k_0}.End_{k_1} + i^2.(-1)^{k_0+k_1}.End_{k_2} \\ &\qquad + \cdots + i^t.(-1)^{k_0+\cdots+k_{t-1}}.End_{k_t} \\ &= 0,\ 1,\ 1+i,\ 2+i,\ 2,\ 3,\ 3+i,\ 2+i,\ 2+2i, \ldots \qquad \text{undup Re A020986, Im A020990} \end{aligned}$$

In (59), the middle $n = 2^{k-1}$ is the end of the first sub-curve and also the end of the second sub-curve. The location is the same. In folded representation (30) this is either a final $2^{k-1}$ or $2^k - 2^{k-1}$, or negatives of those when odd number of terms above. These are the two possible folded representations of $n$. The resulting *point* is the same since

$$End_{k-1} = End_k + (-i).(-1)^{k-1}.End_{k-1}$$

For odd $n$, or odd part of $n$, the geometric interpretation of these final terms is to arrive at the target $z$ either from the segment before or the segment after, according as $n$ final term $+1$ or $-1$ respectively.

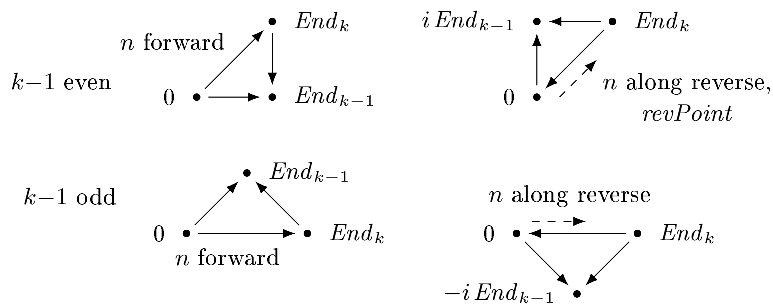



$n{=}7,\ \ point(n){=}2{+}i$

$7 = 8 \quad\ - 1$

$7 = 8 - 2 + 1$

Another approach to curve unfolding is to take $n$ in binary and for the second sub-curve calculate coordinates along a reversed curve.



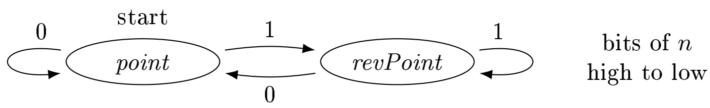

Write $n$ in $k$ many bits with highest bit $a = 0$ or $1$. Then the above expansions become

$$\text{for } n = a.2^{k-1} + n_1 \ \text{ with } n_1 < 2^{k-1}$$

$$
\begin{aligned}
point(n) &= \begin{cases} point(n_1) & \text{if } a=0 \\ End_{k-1} + i.(-1)^{k-1}.revPoint_{k-1}(n_1) & \text{if } a=1 \end{cases} \\
revPoint_k(n) &= End_k - point(2^k - n) \\
&= \begin{cases} i.(-1)^{k-1}.point(n_1) & \text{if } a=0 \\ i.(-1)^{k-1}.End_{k-1} + revPoint(n_1) & \text{if } a=1 \end{cases}
\end{aligned}
$$

$revPoint_k$ is the reverse of a particular expansion level $k$. In general, successive levels taken in reverse are not prefixes of the next, hence particular $k$.

Both forward and reverse descend to *point* or *revPoint* according as $a$=0 or $a$=1 respectively, so bit above determines which state (like *dir* figure 9).



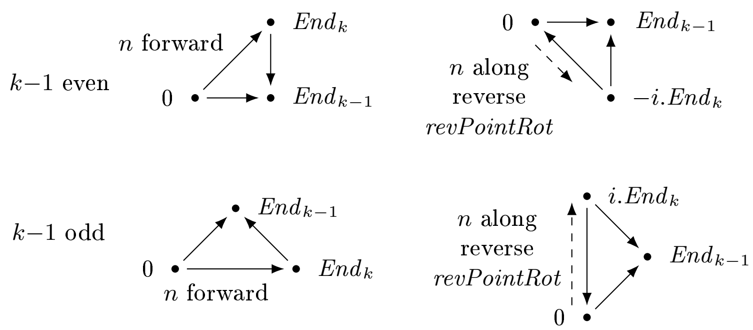

Both forward and reverse add $End_{k-1}$ when $a$=1, but with various factors of $\pm i$. It's convenient to multiply $-i.(-1)^k$ through *revPoint* so its *End* is without further factor.

$$
revPointRot_k(n) = -i.(-1)^{k-1}.revPoint_k(n)
$$

$$
point(n) = \begin{cases} point(n_1) & \text{if } a=0 \\ End_{k-1} + revPointRot(n_1) & \text{if } a=1 \end{cases} \tag{61}
$$

$$
revPointRot(n) = \begin{cases} point(n_1) & \text{if } a=0 \\ End_{k-1} - revPointRot(n_1) & \text{if } a=1 \end{cases} \tag{62}
$$

Geometrically this means taking the reverse curve at +90° when $k-1$ even or −90° when $k-1$ odd. The first halves of both forward and reverse are forward curves to $End_{k-1}$, hence plain $point(n_1)$ in both (61),(62). The second half of reverse $n$ has that sub-part directed 180° from the direction it descends to, hence $-revPointRot$ at (62).




The $-revPointRot$ in (62) is at an $a$=1 bit with a further 1-bit above it. So sign change below each 11 bit pair, including reckoning a triplet as two overlapping pairs, and longer runs likewise.

| | $End_{k-1}$ | $End_{k-2}$ | $End_{k-3}$ | $End_{k-4}$ | | | $End_{l-1}$ | $End_{l-2}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | $\cdots$ |
| | | | $-$ | $-$ | $-$ | | | | $-$ | |

$$\begin{aligned} point(n) &= End_{k-1} + End_{k-2} - End_{k-2} + \cdots && \text{for each 1-bit of } n, \\ &- End_{l-1} \;\; - End_{l-2} \;\; + \cdots && \text{sign change below 11} \\ &+ End_{m-1} + End_{m-2} - \cdots \\ &- \cdots \end{aligned} \tag{63}$$

The expansion of each individual segment also gives a coordinate formula for a new low bit. The expansion shown in figure 2 is a function

$$expand(z) = b.\overline{z} \tag{64}$$

which doubles out points

$$expand(point(n)) = point(2n) \tag{65}$$

Repeated *expand* of a unit length is the curve endpoint, with $expand^k$ meaning apply that function $k$ times.

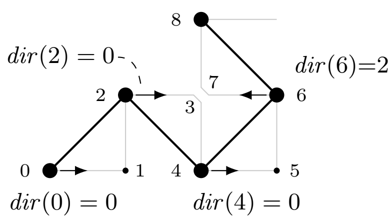
$$\begin{aligned} expand^k(z) = expand(\dots expand(z)) \quad &= \begin{cases} z.End_k & \text{if } k \text{ even} \\ \overline{z}.End_k & \text{if } k \text{ odd} \end{cases} \\ End_k \;= expand^k(1) \end{aligned} \tag{66}$$

The conjugate $\overline{z}$ in (64) means factor $b$ is alternately $b, \overline{b}$, per *End* form (57).

A point $n$ with low bit $a$ and bits $n_1$ above is then

$$point(2n_1 + a) \;=\; expand\big(point(n_1)\big) \;+\; i^{\,dir(2n_1)}.a \tag{67}$$

If $a$=1 then the curve direction at $2n_1 = n - a$ is the direction to go to the new point in between.




Each $dir(2n_1)$ is horizontal 0 or 2 since the curve turns ±90 at each point so an even numbered segment is horizontal, ±1.

$$\begin{aligned} point(2n{+}1) - point(2n) \;\; &= i^{\,dir(2n)} && \text{even segment direction} \\ &= dx(2n) = GRS(n) && \text{as from (41)} \end{aligned}$$

Applying (67) repeatedly is repeated *expand* on these steps, which is factor *End* per (66).

$$n = \text{binary } a_{k-1}a_{k-2}\dots a_1 a_0$$

$$
\begin{aligned}
point(n) = {} & End_{k-1}\, a_{k-1} && \text{high bit} \\
& + End_{k-2}\, a_{k-2}\ GRS(a_{k-1}) \\
& + End_{k-3}\, a_{k-3}\ GRS(a_{k-1} a_{k-2}) \\
& + \cdots \\
& + End_1 \quad a_1 \quad GRS(a_{k-1} a_{k-2} \cdots a_2) \\
& + End_0 \quad a_0 \quad GRS(a_{k-1} a_{k-2} \cdots a_2 a_1) && \text{low bit}
\end{aligned}
$$

These $GRS$ factors are the same as the forward/reverse signs (63). $GRS(n_1)$ changes only when new bit pair 11.

All of the above coordinate formulas are expressed with factors determined by bits of $n$ from high to low. For say computer calculation, the formulas can be applied low to high by assuming lowest $End_0$ term has factor 1 and proceeding upwards from there. The factors on all the powers are then correct relative to each other and if the high $End_{k-1}$ turns out to have factor $-1$ then negate to adjust all.

The various $End$ and signs by $GRS$ or $dir$ are additions and subtractions of powers-of-2 for the $x$ and $y$ coordinates of $point$.

For computer calculation in binary, bits of $x$ and $y$ can be generated from base-4 digits of $n$ using successive sub-curve directions. Sub-curves are within power-of-2 squares and a bit each of $x$ and $y$ goes to a sub-square according to a digit of $n$. Compared to the $point$ forms with negations, the effect is to avoid subtractions by putting a 1 bit only when nothing later will go below it.

**Theorem 7.** *$point(n)$ can be calculated by the following procedure converting base-4 digits of $n$ to bits of $x$ and $y$,*

> **let** base-4 digits of $n = a_k a_{k-1} \dots a_0$ with $a_0$ least signficant
> $d \leftarrow 0$
> **for each** digit position high $j = k$, down to low $j = 0$
>     $d \leftarrow transition(d,\ a_j)$
>     **if** $a_j + d \equiv 2 \bmod 4$ **then** $x_j \leftarrow 1$ **else** $x_j \leftarrow 0$ (68)
>     **if** $a_j - d \equiv 2 \bmod 4$ **then** $y_j \leftarrow 1$ **else** $y_j \leftarrow 0$ (69)
> $x \leftarrow$ bits $x_k x_{k-1} \dots x_0$ and $y \leftarrow$ bits $y_k y_{k-1} \dots y_0$
> **if** $d = 2$ or 3 **then** $x \leftarrow x+1$ and $y \leftarrow y+1$

*where $transition(d, a)$ is in the following state machine*

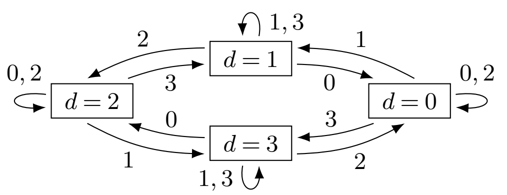
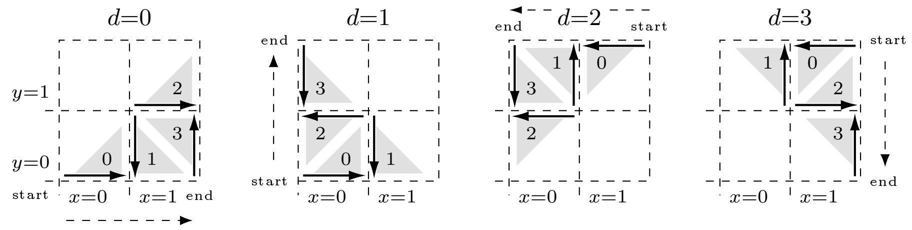



Figure 12
$d = dir(n)$ by
base-4 digits of $n$
high to low

*Proof.* Curve level $2k$ is a triangular half of the square $x$=0 to $x$=$2^k$ and $y$=0 to $y$=$2^k$. Its level $2k-2$ sub-curves are the following $d$=0 case,




In level $2k$ with $d$=0, if the high base-4 digit of $n$ is $1, 2, 3$ then those sub-curves are in the second half of $x$, so a 1 bit for $x$. And if the high base-4 digit of $n$ is 2 then that sub-curve is in the second half of $y$ so a 1 bit for $y$, and otherwise a 0 bit.

Similarly the other curve directions. Thick sub-curve lines are shown in the direction of expansion so sub-curve extent is on the left.

For curve $2k$ in each direction and in the square which is its extents, the direction, digit, and $x, y$ output bit cases are

| | digit $a_j$ = | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| direction | $d$=0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| | $d$=1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| | $d$=2 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| | $d$=3 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| | | $x_j$ output bit | | | | $y_j$ output bit | | | |

This table is based on direction $d$ of curve $2k$, before *transition* to the digit sub-curve. Procedure (68),(69) is after the transition. Working through the combinations of sub-curve $d$ and $x$ or $y$ bit shows the expressions are equivalent to the table.

The expressions work because the output bit depends only on digit $a_j$ and the new $d$, not on the original $d$. For example state $d$=1 in figure 12 is reached by a digit $a_j = 1$ coming from either $d$=0 or a self-loop at $d$=1. In both cases the output bit is $x_j$=1.

As a remark, a table lookup before $d$ transition might give $x$ and $y$ bits one CPU instruction earlier, but otherwise expression or table should be just a matter of convenience.

In directions $d = 0, 1$, curve start is at the bottom left of the square. In directions $d = 2, 3$, curve start is the top right of the square. This is a unit square after all digits of $n$. If the final direction is $d$=2 or 3 then must add 1 to go to the top right. □

## 4.1 Coordinate Norm Increments

Coordinate norm $|z|^2 = x^2 + y^2$ has increment

$$\begin{aligned} dnorm(n) &= \left|point(n+1)\right|^2 - \left|point(n)\right|^2 \\ &= 1, 1, 3, -1, 5, 1, -5, 3, 5, 5, 7, -5, -7, -3, \ldots \end{aligned}$$

The curve goes horizontally or vertically so one of $x$ or $y$ changes by $\pm 1$. An $x$ change is the following increment, and similarly vice-versa $dy$,

$$(x+dx)^2 + y^2 \ - (x^2+y^2) \ = \ 2x.dx + 1$$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 7 | | 9 | | 11 | | 13 | | 15 | |
| | | | | | | 5 | | 5 | | 5 | | 5 | | 5 | | 5 |
| | | | | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |
| | | | | 3 | | 3 | | 3 | | 3 | | 3 | | 3 | | 3 |
| | | | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |
| | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 |
| start | 1 | | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |

*dnorm* for segments, without signs

$x$ changes when $n$ even and $y$ changes when $n$ odd so, and at (71) using *dir* to combine,

$$dnorm(n) = \begin{cases} 2x.dx+1 & \text{if } n \text{ even} \\ 2y.dy+1 & \text{if } n \text{ odd} \end{cases} \tag{70}$$

$$= 2\,\mathrm{Re}\,(-i)^{dir(n)}.point(n) \ + 1 \tag{71}$$

*dnorm* is always odd. Taking half rounded down is

$$\tfrac{1}{2}\big(dnorm(n)-1\big) = \begin{cases} x.dx & \text{if } n \text{ even} \\ y.dy & \text{if } n \text{ odd} \end{cases} \tag{72}$$

$$= \mathrm{Re}\,(-i)^{dir(n)}.point(n)$$

$$= 0, 0, 1, -1, 2, 0, -3, 1, 2, 2, 3, -3, -4, -2, \ldots \qquad \text{abs A068915}$$

An absolute value removes the $dx$ or $dy$ factor from (72) since $x, y \geq 0$

$$\left|\tfrac{1}{2}\big(dnorm(n)-1\big)\right| = XorY(n) = \begin{cases} x & \text{if } n \text{ even} \\ y & \text{if } n \text{ odd} \end{cases}$$

$$= 0, 0, 1, 1, 2, 0, 3, 1, 2, 2, 3, 3, 4, 2, \ldots \qquad n{\geq}1 \text{ A068915}$$

The opposite *YorX* is a shift by 1 index position since $x$ and $y$ values repeat.

$$YorX(n) = \begin{cases} y & \text{if } n \text{ even} \\ x & \text{if } n \text{ odd} \end{cases} = XorY(n{+}1)$$

$$= 0, 1, 1, 2, 0, 3, 1, 2, 2, 3, 3, 4, 2, \ldots \qquad \text{A068915}$$

*YorX* is OEIS sequence A068915 by Aaron K. Johnson. That sequence is defined by a recurrence

$$a(0) = 0 \quad a(1) = 1 \qquad \text{A068915}$$

$$a(2n) \quad = \big|a(n) - a(n{-}1)\big| \tag{73}$$

$$a(2n{+}1) = \ a(n) + a(n{+}1) \tag{74}$$

These recurrences are seen in *YorX* as follows. $2n$ is even so want $y$. An *expand* from (65) is

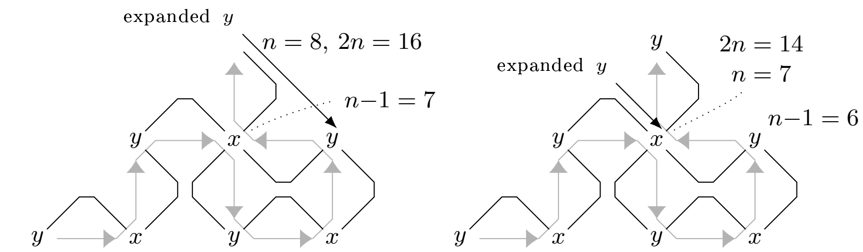
$$YorX(2n) \ = \mathrm{Im}\, point(2n) \ = x(n) - y(n)$$

When $n$ even $\geq 2$, have $y(n) = YorX(n)$ and the preceding segment is vertical so its $x$ is the same as at $n$ so $x(n) = YorX(n{-}1)$. When $n$ odd, the opposite $x(n) = YorX(n)$ and preceding is horizontal so $y(n) = YorX(n{-}1)$,

$$
\begin{aligned}
YorX(2n) &= \begin{cases} YorX(n{-}1) - YorX(n) & \text{if } n \text{ even} \\ YorX(n) - YorX(n{-}1) & \text{if } n \text{ odd} \end{cases} \qquad (75) \\
&= (-1)^n \big( YorX(n{-}1) - YorX(n) \big) \\
&= \big|\, YorX(n) - YorX(n{-}1) \big| \qquad \text{per (73)}
\end{aligned}
$$

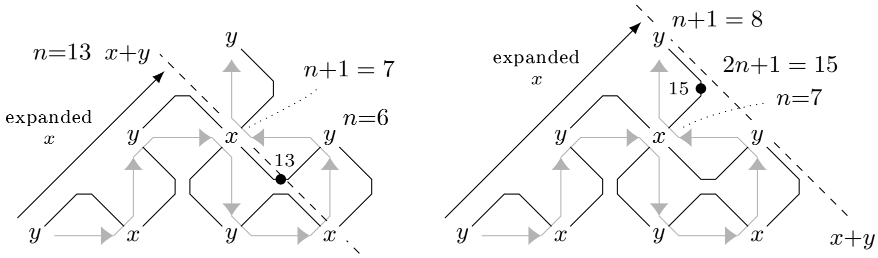
(75) is how the sign removed by the recurrence absolute value alternates according to $n$ odd or even.

The following diagram shows examples $2n = 14, 16$,




For the recurrence odd case, $2n{+}1$ is odd so want $x$ for $YorX$. In the unexpanded coordinates, this is a *Manhattan* sum $x{+}y$, but then also step to the new point in the expanded segment. This is per the *point* low bit formula (67),

$$
YorX(2n{+}1) = \operatorname{Re} point(2n{+}1) = x(n) + y(n) + i^{\,dir(2n)} \qquad (76)
$$




When $n$ even, the segment expands on the left. When $n$ odd, the segment expands on the right (both before mirror image). So the desired $x{+}y$ sum is at the $n{+}1$ location in both cases.

When $n$ even, the segment to $n{+}1$ is horizontal so $y$ is the same at both, hence $YorX(n)$ for $y$ and $YorX(n{+}1)$ for the $x$ at $n{+}1$. When $n$ odd, the segment to $n{+}1$ is vertical so $x$ is the same at both, hence $YorX(n)$ for $x$ and $YorX(n{+}1)$ for the $y$ at $n{+}1$. Thus (74) in both cases.

Algebraically, the same can be seen in (76) by expressing $dir(2n)$ in terms of $dir(n)$. An extra low 0 bit in theorem 3 bit difference sum (29) gives

$$
dir(2n) = -dir(n) + \big(1 \text{ if } n \text{ odd}\big)
$$

$dir(2n)$ is even since the curve is horizontal at even segments, so can negate in the $i$ power at (77) without changing the result. When $n$ even, have $dir$ even so the $i$ power is real and is $dx$ there. When $n$ odd, have $dir$ odd so the $i$ power is imaginary and the effect of "$-1$ when $n$ odd" is to rotate back to get that as a real $dy$,

$$
\begin{aligned}
i^{\,dir(2n)} &= i^{\,dir(n)-(1 \text{ if } n \text{ odd})} \\
&= \begin{cases} \operatorname{Re} i^{\,dir(n)} & \text{if } n \text{ even} \\ \operatorname{Im} i^{\,dir(n)} & \text{if } n \text{ odd} \end{cases} = \begin{cases} dx(n) & \text{if } n \text{ even} \\ dy(n) & \text{if } n \text{ odd} \end{cases}
\end{aligned} \tag{77}
$$

Putting this into (76) is a step of $x$ or $y$ to the next $n$ according as $n$ even or odd. Pair $\mathit{YorX}(n)$ and $\mathit{YorX}(n{+}1)$ is the incremented and un-incremented for all $n$, per (74),

$$
\mathit{YorX}(2n{+}1) = \begin{cases} x+y+dx & \text{if } n \text{ even} \\ x+y+dy & \text{if } n \text{ odd} \end{cases} = \mathit{YorX}(n) + \mathit{YorX}(n{+}1)
$$

The same sort of argument gives recurrences for $\mathit{XorY}$. It can use $n{+}1$ for both sum and difference cases.

$$
\begin{aligned}
\mathit{XorY}(2n) &= \mathit{XorY}(n) + \mathit{XorY}(n{+}1) \\
\mathit{XorY}(2n{+}1) &= (-1)^n \Big(\mathit{XorY}(n) - \mathit{XorY}(n{+}1)\Big)
\end{aligned}
$$

For computer calculation, the base-4 to binary form in theorem 7 can be used, with sign $(-1)^n$ to choose between $x$ or $y$ at its (68),(69).

Half *dnorm* rounded up is similar. It adds 1 to (72),

$$
\begin{aligned}
\tfrac{1}{2}\big(dnorm(n)+1\big) &= \begin{cases} x.dx+1 & \text{if } n \text{ even} \\ y.dy+1 & \text{if } n \text{ odd} \end{cases} \\
&= 1, 1, 2, 0, 3, 1, -2, 2, 3, 3, 4, -2, -3, -1, \ldots
\end{aligned}
$$

Absolute values can be taken by multiplying through $dx$ or $dy$ to cancel those signs, provided $x, y \neq 0$. When $x{=}0$ have $dx = +1$ and when $y{=}0$ have $dy = +1$ so such a multiply holds for those too. The respective resulting $+dx$ or $+dy$ is location $n{+}1$, hence (78).

$$
\begin{aligned}
\Big|\tfrac{1}{2}\big(dnorm(n)+1\big)\Big| &= \begin{cases} x+dx & \text{if } n \text{ even} \\ y+dy & \text{if } n \text{ odd} \end{cases} \\
&= \mathit{YorX}(n{+}1) \;= \mathit{XorY}(n{+}2)
\end{aligned} \tag{78}
$$

A similar multiply through $dx$ or $dy$ cancels the sign in the full *dnorm* from (70). Its resulting $x + (x + dx)$ is like $2\times$ the segment midpoint.

$$
\begin{aligned}
\Big|dnorm(n)\Big| &= \begin{cases} 2x+dx & \text{if } n \text{ even} \\ 2y+dy & \text{if } n \text{ odd} \end{cases} \\
&= 2 \operatorname{Re} (-i)^{dir(n)}.point(n) \;+ 1 \\
&= 1, 1, 3, 1, 5, 1, 5, 3, 5, 5, 7, 5, 7, 3, \ldots
\end{aligned}
$$

# 5 Coordinates to N

$point(n)$ can be inverted low to high to calculate $n$ at a given location $z$. Suppose $z = point(n)$ and that in (63) the total sign changes would leave sign $s$ on terms below the last so $s = GRS(n)$. Then unpoint($z$, $s$) $\qquad$ $z$ = Gaussian integer, $s = \pm 1$

**loop** until $z$=0 or $z = -s$ or $z = -i.s$

  **if** $z \equiv i \bmod b^2$ **then** $s \leftarrow -s$

  $bit$ 0 or 1 $= z \bmod b$ $\qquad$ bits of $n$ low to high

  **if** $bit$=1 **then** $z \leftarrow z - s$ $\qquad$ step to even point

  $z \leftarrow \overline{z/b}$ $\qquad$ unexpand

**end loop**

**if** $z$=0 and $s$=1 **then** $n$ in unrotated curve

**otherwise** rotated or reflected copy

The two $s = \pm 1$ are directions $d = 0, 2$ (horizontal) at an even point and $d = 1, 3$ (vertical) at an odd point, respectively.

$$s = i^{\,dir \,-\, (1 \text{ if } z \text{ odd})}$$

This is $s = i^{dir(2n)} = GRS(n)$ which would be the next sign factor in (63). $z$ odd is when $n$ odd so that $dir(2n)$ has an extra bit transition.

$z \equiv i \bmod b^2$ is when the lowest two bits of $n$ are 11 and so a sign change for all powers below. The sign below is $s$ so change to $-s$ for the present term and above.

For computer calculation everything can be done in Cartesian coordinates $x{+}iy$ without full complex number arithmetic. $bit \equiv z \bmod b$ is simply $x{+}y \equiv 0, 1 \bmod 2$ and division $\overline{z/(i{+}1)}$ is $(x, y) \leftarrow (\frac{x+y}{2}, \frac{x-y}{2})$. The test for $z{\equiv}i \bmod b^2$ is equivalent to $x{\equiv}0 \bmod 2$ and $y{\equiv}1 \bmod 2$ since $z \bmod b^2$ goes in a $2{\times}2$ repeating pattern.

The loop reduces $z$ by dividing $b$ each time, except for the $s$ subtraction. Considering just magnitudes, $|z|$ decreases when

$$\begin{aligned} |z| - \left|\frac{z-s}{b}\right| &\ge |z| - \frac{|z|+1}{\sqrt{2}} = \left(1 - \tfrac{1}{2}\sqrt{2}\right)|z| - \tfrac{1}{2}\sqrt{2} \\ &> 0 \quad \text{when } |z| > 1{+}\sqrt{2} \end{aligned}$$

So $|z|$ decreases until $|z| \le 1{+}\sqrt{2}$ and for points there it can be verified explicitly that all integer $z$ and $s{=}{\pm}1$ reach one of the loop ends.

For a given $n$ let $other(n)$ be the point number which is the other visit to that location. This can be found from $n$ without calculating the location as such.

| high | | flip | | flip | | | low |
|---|---|---|---|---|---|---|---|
| ... | $\neq t$ | $x$ | $t$ | $x$ | $t$ | 1 | 0...0 |
| | | repeat $\ge 0$ times | | | | | |

Figure 13: $other(n)$ bit fields

$$other(n) = 0, -, -, 7, -, -, 14, 3, -, 13, -, 31, 28, 9, 6, \ldots$$

$n$ in binary is divided into the fields shown in figure 13. $t$ is the bit above lowest 1-bit. This is per $turn(n)$ from (3). Each bit $x$ is arbitrary and is flipped until reaching bit above $\neq t$.

High 0-bits are understood on $n$ as necessary to make the fields shown. When $t$=1 and the highest of $n$ is one of the $t$ bits then the $x$=0 above it is flipped.

This happens for points on the join side of the triangle. They are locations within level $k$ which have their second visit in the next level $k{+}1$.

If $t{=}0$ then the pattern might continue infinitely into high 0-bits on $n$. This occurs for points on the $x$ axis and $x{=}y$ diagonal. They have no second visit within the curve.

This *other* bit flip is found by taking bits of $n$ and the forward/reverse *End* terms they imply (63), then apply *unpoint* to those terms with opposite final sign.

$$
\begin{array}{lll}
\text{other}(n) & n \neq 0 & \\
\quad s = 1 & & \text{sign on } n \\
\quad h = -1 & & \text{sign on } \mathit{other}(n)\text{, starting opposite} \\
\quad \delta = 0 & & \\
\quad \textbf{loop} & & \\
\qquad a_0 = \text{low bit of } n, \quad a_1 = \text{second lowest bit of } n & & \\
\qquad \textbf{if } a_1, a_0 = 1,1 \quad \textbf{then } s \leftarrow -s & & \\
\qquad z = b.a_1 + a_0 + \delta & & \\
\qquad c_0 = 0 \,\text{or}\, 1 \ \equiv z \bmod b & & \mathit{other}(n) \text{ bits, low to high} \\
\qquad \textbf{if } z \equiv i \bmod b^2 \quad \textbf{then } h \leftarrow -h & & \\
\qquad \delta \leftarrow \overline{(\delta + a_0.s - c_0.h)/b} & & \\
\qquad \text{drop lowest bit of } n & & \\
\quad \textbf{end loop} & &
\end{array}
$$

$s$ is the sign below the last bit of $n$. If $n$ bits are 11 then it changes to $-s$ for the present term of $n$ and above. $h$ is the sign below the bits of the $\mathit{other}(n)$ being calculated. Taking only the low bits of $n$ and $\mathit{other}(n)$ does not in general give the same location. $\delta$ is the offset from location $n$ to $\mathit{other}(n)$. It changes when the *End* terms in $n$ and $\mathit{other}(n)$ are not the same (different sign, or zero and not zero). Bits of $n$ and $\delta$ then give the other location $\bmod b$ and $b^2$ for bit of $\mathit{other}(n)$ and sign change on $h$.

Following up through possible bits of $n$ gives combinations of $s, h, \delta, \mathit{bit}$ as states of a finite state machine. This state incorporates a "current" *bit* since two bits are required at each step. The next higher bit is taken as input and the output is a bit of $\mathit{other}(n)$ at the "current" position. The higher bit goes into the new state. The initial state is $s{=}1, h{=}{-1}, \delta{=}0$ and $\mathit{bit}$ = low of $n$.

The states and outputs simplify to the bit flips above. $\delta$ takes five possible values $0, \pm 1, \pm b$.

The turn bit $t$ above lowest 1 is unchanged by this *other* going up by states. This is a complicated way to see the turn at first and second visits are the same (see section 12.2).

Each bit $x$ in figure 13 is flipped. Differences BITXOR$(n, \mathit{other}(n))$ which occur are therefore 1-bits at every second bit in a single run.

$$
\text{binary} \quad \boxed{0\ldots0} \boxed{1} \overbrace{\boxed{0}\boxed{1}}^{\text{repeat } \geq 0 \text{ times}} \underbrace{\boxed{0\ldots0}}_{\substack{\geq 2 \\ \text{low 0-bits}}}
$$

Figure 14: *OXpred*

$$
\begin{aligned}
\mathit{OXpred}(c) &= \begin{cases} 1 & \text{if } c = \mathrm{BITXOR}(n, \mathit{other}(n)) \text{ for some } n \\ 0 & \text{otherwise} \end{cases} \\
&= 1 \text{ at } c = 4, 8, 16, 20, 32, 40, 64, 80, 84, 128, 160, \ldots \qquad \text{4×A181666} \\
&= \text{binary } 100, 1000, 10000, 10100, 100000, 101000, \ldots
\end{aligned}
$$

The smallest $n$ where a given $c$ occurs is found from the bit fields figure 13. Take $t{=}1$ so the high $\neq t$ bit is 0, then take all the bits which will flip as 0s. The $t$ bits are $c$ shifted down, and the lowest 1-bit is immediately below them.

$$
\begin{aligned}
\mathit{OXminN}(c) &= \tfrac{1}{2}c + 2^{\mathit{CountLowZeros}(c)-2} \qquad \text{for } c \text{ satisfying } \mathit{OXpred}(c) \\
&= 3, 6, 12, 11, 24, 22, 48, 44, 43, 96, 88, 86, \ldots
\end{aligned}
$$

The number of distinct XOR differences occurring within level $k$ is

$$
\begin{aligned}
\mathit{NumOXpred}_k &= \sum_{c=0}^{2^k-1} \mathit{OXpred}(c) \\
&= \binom{\lfloor k/2 \rfloor}{2} + \binom{\lceil k/2 \rceil}{2} \qquad \text{binomials} \qquad (79) \\
&= \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \left\lceil \frac{k-1}{2} \right\rceil = \left\lfloor \frac{(k-1)^2}{4} \right\rfloor \qquad (80) \\
&= 0, 0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, \ldots \qquad \text{A002620}
\end{aligned}
$$

The binomials (79) are locations of the bit flip run. In the bit fields of figure 14, if an even number of low 0s then $\lfloor k/2 \rfloor - 1$ remaining even positions. If an odd number of low 0s then $\lceil k/2 \rceil - 1$ remaining odd positions. The binomials select two of them in each case to be start and end. The start and end can coincide, so $+1$ on the possible positions. Products (80) follow from these binomials.

Differences $n - \mathit{other}(n)$ have either $+1$ or $-1$ at each bit flip position according to whether the flip is $0{\to}1$ or $1{\to}0$ respectively. The differences which occur are therefore $\pm 1$ at every second bit position in a single run.

$$
\text{binary} \quad \boxed{0\ldots0}\,\overbrace{\boxed{\pm1}\,\boxed{0}}^{\text{repeat } \geq 0 \text{ times}}\,\boxed{\pm1}\,\underbrace{\boxed{0\ldots0}}_{\substack{\geq 2 \\ \text{low 0-bits}}} \qquad d = n - \mathit{other}(n)
$$

$$
\begin{aligned}
\mathit{Opred}(d) &= \begin{cases} 1 & \text{if } d = n - \mathit{other}(n) \text{ for some } n \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } d = \sum \pm 2^{p_j} \text{ where} \\ & \quad \text{each } p_{j+1} = p_j + 2 \text{ and } p_0 \geq 2 \\ 0 & \text{otherwise} \end{cases} \\
&= 1 \text{ at } |d| = 4, 8, 12, 16, 20, 24, 32, 40, 44, 48, 52, \ldots
\end{aligned}
$$

The number of distinct differences $|n - \mathit{other}(n)|$ is

$$NumOpred_k = \left( \sum_{l=1}^{\lfloor k/2 \rfloor - 1} 2^{l-1} \left( \lfloor k/2 \rfloor - l \right) \right) + \left( \sum_{l=1}^{\lceil k/2 \rceil - 1} 2^{l-1} \left( \lceil k/2 \rceil - l \right) \right) \tag{81}$$

$$= [2,3].2^{\lfloor k/2 \rfloor} - k - 2 \tag{82}$$

$$= 0, 0, 0, 1, 2, 5, 8, 15, 22, 37, 52, 83, \ldots \qquad \text{A077866}$$

The sums (81) are over length $l$ many bits of each $\pm 1$. The top-most bit position is $+1$ to get the positive differences, leaving $2^{l-1}$ combinations of $\pm 1$ below. These $l$ bits can be located at any of the remaining $\lceil k/2 \rceil - l$ (or ceil) bit positions. Working through those sums gives powers (82).

The locations of first occurrence of each *Opred* difference follow from the unfolding.

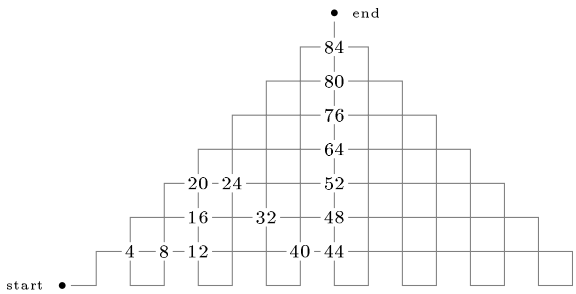



Figure 15: $k$=7 locations of first occurrence of *Opred* differences

On unfolding, the new second half has the same set of differences within it, so new differences are only at the join points. Those points have $n$ decreasing on the first half and increasing on the second half, from the unfolding of the $x$ and $x{=}y$ sides ahead in points theorem 10.

The differences are then a high $+1$ with further $\pm 1$ every second bit position. The smallest is all $-1$ and the biggest is all $+1$. In figure 15 for example the middle column has $44 = 64 - 16 - 4$ up to $84 = 64 + 16 + 4$.

The smallest new difference $+1$ then all $-1$ is bigger than the biggest previous difference of $+1$ at second highest bit then all $-1$. The new smallest is previous biggest $+4$, as for example 40 to 44 above.

The initial difference 4 is a column of one value, and the 8 is an anti-diagonal of one value.

# 6 Segments in Direction

**Theorem 8.** *The number of segments in direction $d = 0, 1, 2, 3 \bmod 4$ of the alternate paperfolding curve level $k$ are*

$$S(k,d) = \begin{cases} 1, 0, 0, 0 \quad \text{for } d \equiv 0 \,\text{to}\, 3 & \text{if } k{=}0 \\ \frac{1}{4}\left(2^k + 2 \operatorname{Re} (-i)^d . End_k\right) & \text{if } k \geq 1 \end{cases}$$

$$S(k,0) = \begin{cases} 1 & \text{if } k{=}0 \\ 2^{k-2} + 2^{\lfloor (k-2)/2 \rfloor} & \text{if } k \geq 1 \end{cases}$$

$$= 1, 1, 2, 3, 6, 10, 20, 36, 72, 136, 272, \ldots \qquad \text{A005418}$$

$$S(k,1) = \begin{cases} 0 & \textit{if } k{=}0 \\ 2^{k-2} & \textit{if } k \ge 1 \ \textit{even} \\ 2^{k-2} + 2^{\lfloor (k-2)/2 \rfloor} & \textit{if } k \ge 1 \ \textit{odd} \end{cases}$$

$$= 0, 1, 1, 3, 4, 10, 16, 36, 64, 136, 256, \ldots \qquad \text{A051437}$$

$$S(k,2) = \begin{cases} 0 & \textit{if } k{=}0 \\ 2^{k-2} - 2^{\lfloor (k-2)/2 \rfloor} & \textit{if } k \ge 1 \end{cases}$$

$$= 0, 0, 0, 1, 2, 6, 12, 28, 56, 120, 240, \ldots \qquad \text{A122746}$$

$$S(k,3) = \begin{cases} 0 & \textit{if } k{=}0 \\ 2^{k-2} & \textit{if } k \ge 1 \ \textit{even} \\ 2^{k-2} - 2^{\lfloor (k-2)/2 \rfloor} & \textit{if } k \ge 1 \ \textit{odd} \end{cases}$$

$$= 0, 0, 1, 1, 4, \ \ 6, 16, 28, 64, 120, 256, \ldots \qquad \text{A007179}$$

*Proof.* Unfolding for the next level $k{+}1$ repeats the curve, with directions turned $\pm 1$, so counts are the original plus $d \mp 1$ of unfold.

$$S(k{+}1, d) = \begin{cases} S(k,d) + S(k,d{-}1) & \text{if } k \text{ even} \\ S(k,d) + S(k,d{+}1) & \text{if } k \text{ odd} \end{cases} \qquad \square$$

The total segments is simply $2^k$. Since the curve always turns $\pm 90°$ the number of verticals and horizontals are the same for $k \ge 1$.

$$\begin{aligned} \text{total} \quad & S(k,0) + S(k,1) + S(k,2) + S(k,3) = 2^k \\ \text{horizontals} \quad & S(k,0) + S(k,2) = \tfrac{1}{2} 2^k \qquad k \ge 1 \\ \text{verticals} \quad & S(k,1) + S(k,3) = \tfrac{1}{2} 2^k \qquad k \ge 1 \end{aligned}$$

**Theorem 9.** *Among the first $n$ segments of the alternate paperfolding curve, the number in direction $d$ mod 4 is*

$$\mathit{SN}(n,d) = \tfrac{1}{4}\Big(n + 2\,\mathrm{Re}\,(-i)^d \mathit{point}(n) \ + \big((-1)^d \ \textit{if } n \ \textit{odd}\big)\Big) \tag{83}$$

$$\begin{aligned} \mathit{SN}(n,0) &= 0, 1, 1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 5, \ldots \\ \mathit{SN}(n,1) &= 0, 0, 1, 1, 1, 1, 2, 2, 3, 3, 4, 4, 4, 4, 4, \ldots \\ \mathit{SN}(n,2) &= 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 2, \ldots \\ \mathit{SN}(n,3) &= 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3, \ldots \end{aligned}$$

*Proof.* Segments alternate horizontal and vertical so total horizontals are $\lceil n/2 \rceil$, which is *SN* directions 0 plus 2. The difference of directions 0 and 2 is the net horizontal position Re *point*,

$$\mathit{SN}(n,0) + \mathit{SN}(n,2) = \lceil n/2 \rceil \tag{84}$$

$$\mathit{SN}(n,0) - \mathit{SN}(n,2) = \mathrm{Re}\,\mathit{point}(n) \tag{85}$$

(84)+(85) and (84)−(85) give

$$\mathit{SN}(n,0) \ = \tfrac{1}{2}\big(\lceil n/2 \rceil + \mathrm{Re}\,\mathit{point}(n)\big)$$

$$SN(n,2) \;= \tfrac{1}{2}\big(\lceil n/2\rceil - \operatorname{Re} point(n)\big)$$

Similarly for the verticals

$$\begin{aligned} SN(n,1) + SN(n,3) &= \lfloor n/2\rfloor \\ SN(n,1) - SN(n,3) &= \operatorname{Im} point(n) \end{aligned}$$

$$\begin{aligned} SN(n,1) \;&= \tfrac{1}{2}\big(\lfloor n/2\rfloor + \operatorname{Im} point(n)\big) \\ SN(n,3) \;&= \tfrac{1}{2}\big(\lfloor n/2\rfloor - \operatorname{Im} point(n)\big) \end{aligned}$$

The $\pm\operatorname{Re},\operatorname{Im}$ parts are selected in (83) by $\operatorname{Re}(-i)^d point$, and the floor or ceil $n/2$ by the $(-1)^d$ offset part. □

# 7 Boundary and Area

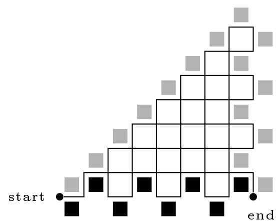
The boundary length of a given level $k$ follows from its triangular shape,

$$\begin{aligned} L_k &= \begin{cases} 1 & \text{if } k=0 \\ [4,2].2^{\lfloor k/2\rfloor} - [4,0] & \text{if } k\ge 1 \end{cases} && \text{left boundary} \\ &= 1, 2, 4, 4, 12, 8, 28, 16, 60, 32, 124, \dots \end{aligned}$$

$$\begin{aligned} R_k &= \begin{cases} 1 & \text{if } k=0 \\ [2,6].2^{\lfloor k/2\rfloor} - [0,4] & \text{if } k\ge 1 \end{cases} && \text{right boundary} \\ &= 1, 2, 4, 8, 8, 20, 16, 44, 32, 92, 64, \dots \end{aligned}$$

$$\begin{aligned} B_k = L_k + R_k \;&= [6,8].2^{\lfloor k/2\rfloor} - 4 && \text{total boundary} \\ &= 2, 4, 8, 12, 20, 28, 44, 60, 92, 124, 188, \dots && 2{\times}\text{A027383} \end{aligned}$$

And likewise the number of unit squares on the boundary




boundary squares $k$=6

$LQ_6 = 15$ left, grey

$RQ_6 = 8$ right, black

$BQ_6 = LQ_6 + RQ_6 = 23$

$$\begin{aligned} LQ_k &= [2,1].2^{\lfloor k/2\rfloor} - [1,0] && \text{left boundary squares} \\ &= 1, 1, 3, 2, 7, 4, 15, 8, 31, 16, 63, \dots \end{aligned}$$

$$\begin{aligned} RQ_k &= [1,3].2^{\lfloor k/2\rfloor} - [0,1] && \text{right boundary squares} \\ &= 1, 2, 2, 5, 4, 11, 8, 23, 16, 47, 32, \dots \end{aligned}$$

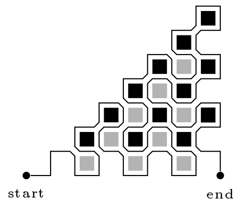
$$\begin{aligned} BQ_k = LQ_k + RQ_k \;&= [3,4].2^{\lfloor k/2\rfloor} - 1 && \text{total boundary squares} \\ &= 2, 3, 5, 7, 11, 15, 23, 31, 47, 63, 95, \dots && \text{A052955} \end{aligned}$$

The unfolding to the respective side means

$$L_k = L_{k-1} + R_{k-1} \;= B_{k-1} \qquad k \text{ even unfold}$$

$$
\begin{aligned}
R_k &= L_{k-1} + R_{k-1} \ = B_{k-1} && k \text{ odd unfold} \\
LQ_k &= LQ_{k-1} + RQ_{k-1} \ = BQ_{k-1} && k \text{ even unfold} && (86) \\
RQ_k &= LQ_{k-1} + RQ_{k-1} \ = BQ_{k-1} && k \text{ odd unfold} && (87)
\end{aligned}
$$

The area enclosed by a given level $k$ follows from its triangular shape too.




area $k$=6

$AL_6 = 9$ left, grey

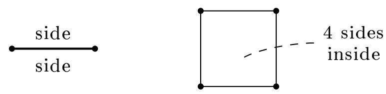
$AR_6 = 12$ right, black

$A_6 = AL_6 + AR_6 = 21$

Since the curve always turns $\pm 90°$ the unit squares enclosed on the left or right side of the curve alternate. Left squares have an even $x+y$ lower left corner. Right squares have an odd $x+y$ lower left corner.

$$
\begin{aligned}
AL_k &= 2^{k-2} - [1, \tfrac{1}{2}].2^{\lfloor k/2 \rfloor} + [1,0] && \text{left area} \\
&= 0, 0, 0, 1, 1, 6, 9, 28, 49, 120, 225, \ldots \\
AR_k &= 2^{k-2} - [\tfrac{1}{2}, \tfrac{3}{2}].2^{\lfloor k/2 \rfloor} + [0,1] && \text{right area} \\
&= 0, 0, 0, 0, 2, 3, 12, 21, 56, 105, 240, \ldots \\
A_k &= AL_k + AR_k && \text{area} \\
&= 2^{k-1} - [\tfrac{3}{2}, 2].2^{\lfloor k/2 \rfloor} + 1 \\
&= \left(2^{\lfloor (k-1)/2 \rfloor} - 1\right)\left(2^{\lceil (k-1)/2 \rceil} - 1\right) \\
&= \tfrac{1}{2}{\times}0,\ 0{\times}0,\ 0{\times}1,\ 1{\times}1,\ 1{\times}3,\ 3{\times}3,\ 3{\times}7,\ 7{\times}7,\ \ldots \\
&= 0, 0, 0, 1, 3, 9, 21, 49, 105, 225, 465, \ldots && \text{A274230}
\end{aligned}
$$

Area and boundary are related by a general rule for non-overlapping curves. Each segment has 2 sides and each enclosed unit square has 4 of the inside, so



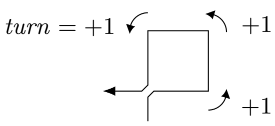

So insides plus outsides is total $4A + B = 2N$. For the alternate paperfolding curve this is

$$4A_k + B_k = 2.2^k$$

The left and right sides separately in a similar way, counting only the left or right side of each segment.

$$4AL_k + L_k = 2^k \qquad\qquad 4AR_k + R_k = 2^k$$

Some enclosed unit squares are formed by 3 consecutive left or right turns.




three consecutive left turns,
left-side enclosed unit square

The number of such runs in curve level $k$ follows from the unfolding. The unfolding duplicates the runs, with lefts and rights swapped in the unfolded copy. New runs might occur at the unfold point. For $k \geq 4$ the unfold point is not such a run as the curve turns away. So successive levels from there on simply sum lefts plus rights, giving

$$Turn3left_k = \begin{cases} 0,0,0,1 & \text{if } k = 0 \text{ to } 3 \\ 2^{k-4} & \text{if } k \geq 4 \end{cases} \tag{88}$$

$$Turn3right_k = \begin{cases} 0 & \text{if } k \leq 3 \\ 2^{k-4} & \text{if } k \geq 4 \end{cases}$$

The proportion of enclosed unit squares arising from such turn runs is then

$$\frac{Turn3left_k}{AL_k} \to \frac{1}{4} \qquad \frac{Turn3right_k}{AR_k} \to \frac{1}{4}$$

Area increases by

$$\begin{aligned} dA_k = A_{k+1} - A_k \;&= \tfrac{1}{2}\Big(2^{\lfloor k/2 \rfloor} - 1\Big).2^{\lceil k/2 \rceil} && \text{area increment} \\ &= 2^{k-1} - 2^{\lfloor (k-1)/2 \rfloor} \\ &= 0,0,1,2,6,12,28,56,120,240,496,992,\ldots && \text{A122746} \end{aligned}$$

The join area between levels is the column or diagonal of unit squares in between the unfolds,

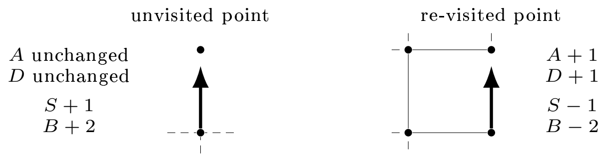
$$\begin{aligned} JA_k &= A_{k+1} - 2A_k && \text{join area} \\ &= 2^{\lfloor k/2 \rfloor} - 1 \\ &= 0,0,1,1,3,3,7,7,15,15,31,31,\ldots && \text{A052551} \end{aligned}$$

# 8 Points

In the triangular shape of each level, the outer points are single visited and the inner ones are double visited, so from the shape

$$\begin{aligned} S_k &= [3,4].2^{\lfloor k/2 \rfloor} - 1 && \text{singles} \\ &= 2,3,5,7,11,15,23,31,47,63,\ldots && \text{A052955} \\ D_k &= \tfrac{1}{2}(2^k + 1 - S_k) \;= A_k && \text{doubles} = \text{area} \qquad \text{A274230} \\ P_k &= S_k + D_k \;= 2^{k-1} + [\tfrac{3}{2},2].2^{\lfloor k/2 \rfloor} && \text{total} \\ &= 2,3,5,8,14,24,44,80,152,288,\ldots && \text{A290075} \end{aligned}$$

Doubles = area holds for any curve where each enclosed unit square has all 4 sides traversed, without overlaps. Each unit square is formed when and only when a segment re-visits a point,




$D = A$ double-visited = area

$S = B/2 + 1$ single-visited and boundary

Total points $P$ and doubles $D$ are also related

$$D_k = \tfrac{1}{2}(2^k + 1 - S_k) \qquad \text{from } S + 2D = 2^k + 1$$
$$P_k = \tfrac{1}{2}(2^k + 1 + S_k)$$

If there were no singles then it would be $D$ doubles $= P$ distinct $= \frac{1}{2}(2^k+1)$ half total points. Every 2 singles reduces the doubles by 1 and increases the distinct points by 1 (as $+2$ singles, $-1$ double).

Total points can have a copy of $D$ added in to make the total $n$ points,

$$P_k + D_k = 2^k + 1$$

With $D = A$, this is Euler's formula for regions of a connected planar graph. Vertices are points $P$, edges are $2^k$ segments, and regions are $A$ enclosed unit squares.

$$\textit{vertices} + \textit{inside regions} = \textit{edges} + 1$$

**Theorem 10.** *Points $n$ of the alternate paperfolding curve on the $x$ axis are characterized by the following Xpred, and obtained by Xnum on an index $m$ starting $m{=}0$ for the first $x$ axis point.*

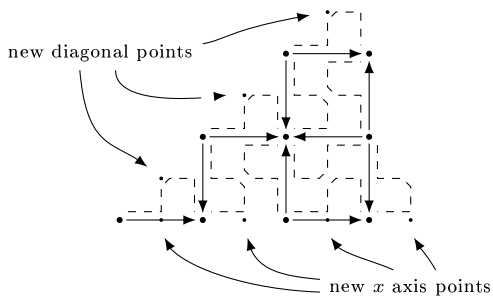
$$\begin{aligned}
\mathit{Xpred}(n) &= \begin{cases} 1 & \textit{if } n \textit{ base-4 digits only } 0 \text{ or } 1 \\ 0 & \textit{otherwise} \end{cases} && x \textit{ axis} \\
&= 1,1,0,0,1,1,0,0,0,0,0,0,\dots && \text{A151666} \\
\mathit{gXpred}(x) &= \prod_{j=0}^{\infty} \left(1 + x^{4^j}\right) && (89) \\
\mathit{Xnum}(m) &= m \textit{ in binary change to base-4 digits 0,1} && m{\ge}0 \\
&= 0,1,4,5,16,17,20,21,64,65,68,69,\dots && \text{A000695}
\end{aligned}$$

*Points $n$ on the $x{=}y$ diagonal are similarly*

$$\begin{aligned}
\mathit{Gpred}(n) &= \begin{cases} 1 & \textit{if } n \textit{ base-4 digits all } 0 \text{ or } 2 \\ 0 & \textit{otherwise} \end{cases} && x{=}y \textit{ diagonal} \\
&= \mathit{Xpred}(2n) && (90) \\
&= 1,0,1,0,0,0,0,0,1,0,1,0,\dots \\
\mathit{gGpred}(x) &= \mathit{gXpred}(x^2) \;= \prod_{j=0}^{\infty} \left(1 + x^{2.4^j}\right) \\
\mathit{Gnum}(m) &= m \textit{ in binary change to base-4 digits 0,2}
\end{aligned}$$

$$= 2\ \mathit{Xnum}(m)$$
$$= 0, 2, 8, 10, 32, 34, 40, 42, 128, 130, 136, 138, \ldots \qquad \text{A062880}$$

*Proof.* The theorem can be verified explicitly for $k \le 2$. Thereafter segments expand twice as




Existing points $n$ become $4n$. Each new $x$ axis point is $+1$ from an existing point there, so base-4 digits 0,1 only. Each new $x{=}y$ diagonal point is $+2$ from an existing point, so base-4 digits 0,2 only.

Diagonals are $\mathit{Xpred}(2n)$ at (90) simply by the digits, or since those diagonals are the $x$ axis points of the previous level in figure 2 and *expand* (64).

$$\mathit{expand}(x + 0i) \ = \ x + xi$$

*gXpred* at (89) is per Neil Sloane in OEIS A000695. It is a usual way to form characteristic sequences of numbers with certain digits. A product of $k$ many terms is all $n$ with up to $k$ many base-4 digits. The next product term $1 + x^{4^k}$ is then 1 to keep existing and $x^{4^k}$ to copy up to those $n$ with a 1-digit at position $k$. Similarly *gGpred*. $\square$

There are $2^k$ points on the $x$ axis so $4^k{-}2^k$ non axis points. These are $n$ with at least one base-4 digit 2 or 3, or equivalently at least one odd position 1-bit. The $m$'th non axis point can be calculated by a recurrence splitting $m$ within $4^k{-}2^k$ levels.

$$\text{for } 4^k{-}2^k \le m < 4^{k+1}{-}2^{k+1}$$
$$\mathit{NonXnum}(m) = \begin{cases} 4^k + \mathit{NonXnum}\big(m - (4^k{-}2^k)\big) & \text{if } m < 2.(4^k{-}2^k) \\ m + 2^{k+1} & \text{if } m \ge 2.(4^k{-}2^k) \end{cases} \tag{91}$$
$$= 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots$$
$$\text{base-4} \ = 2, 3, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, \ldots$$

At (91), the $m{+}2^{k+1}$ case is a high base-4 digit 2 or 3 on the resulting $n$. That digit ensures *NonXpred* and allows the remaining $m$ to run through all digit values below. The descent $4^k + \mathit{NonXnum}$ is a high base-4 digit 1 and so still restricted to *NonXpred* in the digits below.

| | |
|---|---|
| 0, *NonXpred* | $4^k - 2^k$ many |
| 1, *NonXpred* | $4^k - 2^k$ many |
| 2, all digits | $4^k$ many |
| 3, all digits | $4^k$ many |

**Theorem 11.** *$n = NonXnum(m)$ can be calculated by the following bit procedure*

$$k \leftarrow \begin{cases} 1 & \text{if } m = 0 \\ \lfloor \log_4 m \rfloor + 2 & \text{if } m > 0 \end{cases}$$

$$n \leftarrow m + 2^k \tag{92}$$

$$\textbf{loop} \quad k \leftarrow k - 1$$

$$\text{if bit } 2k{+}1 \text{ of } n \text{ is 1 then end loop} \tag{93}$$

$$\text{if bit } 2k \text{ of } n{-}2^k \text{ is 0 then } n \leftarrow n - 2^k \tag{94}$$

*Proof.* This procedure is implicit in recurrence (91). During the loop, $n$ holds the result so far in the high base 4 digits, and an $m + 2.2^k$ in the low digits.

| | high | $k$ low | | |
|---|---|---|---|---|
| $n =$ | result | $m + 2.2^k$ | base 4 digits | Figure 16 |

Offset $+2.2^k$ on $m$ means the second case of (91) becomes

$$m + 2.2^k \ \geq\ 2.(4^k{-}2^k) + 2.2^k \ = 2.4^k$$

which is the bit test at (93). The offset is also already the result $m + 2^{k+1}$.

In recurrence (91), $k$ having $4^k{-}2^k \leq m$ means the first case there is a 1 digit for the result. Its $+4^k$ to the result and index $-(4^k - 2^k)$ moves that digit from $m$ to the result. In the procedure, this is no change to the $n$ since the $2^k$ part of the index change consumes half the $2.2^k$ offset already in $n$, leaving $2.2^{k-1}$ which is the desired offset in the next iteration $k{-}1$.

$m < 4^k{-}2^k$ is a 0 digit in the result. Offset $2.2^k$ must reduce to $2.2^{k-1}$ by subtracting $2^k$. For $m$ with its offset, this range condition and then with the subtraction is

$$\begin{aligned} m + 2.2^k &< 4^k{-}2^k + 2.2^k && m \text{ small so digit } 0 \\ m + 2.2^k - 2^k &< 4^k{-}2^k + 2.2^k - 2^k && = 4^k \end{aligned}$$

If $n{-}2^k$ leaves bit position $2k$ as 0 then it's smaller than $4^k$ and is the digit 0 case, and this subtraction is the new $n$, as at (94).

It can be noted $n - 2^k$ never borrows above the low digits 0 to $k$ inclusive, since offset $m + 2.2^k$ means the digits there are $\geq 2.2^k$.

At $k$=0, the single digit remaining in $n$ is $m + 2.2^0 = 2 \text{ or } 3$ so always end loop at (93). This is when all other digits are 0 or 1 and just lowest digit 2 or 3 ensures *NonXpred*. ☐

For computer calculation on big integers, $n - 2^k$ could be done in-place and if the bit test says not wanted then add back $+2^k$ to undo. If preferred, some bit scanning can determine when the subtract is wanted. If bit $2k$ is already 0, or it is 1 but all 0s from $k$ to $2k-1$ inclusive, so borrow will change it to 0, then the subtract is wanted. On some CPUs, doing the arithmetic might be just as fast as bit scanning. Input values of suitably uniform distribution should keep the subtract within a single machine word most of the time anyway.

Initial $k$ at (92) can be any value large enough that $m < 4^{k+2} - 2^{k+2}$, so after the first decrement $m < 4^{k+1} - 2^{k+1}$ the same as in recurrence (91). If the bound was just $4^{k+1}$ then initial $k = \lfloor \log_4 m \rfloor + 1$ would suffice, but $-2^{k+1}$ means sometimes 1 bigger is needed, hence $+2$.

Each *Gnum* point is on the $x{=}y$ diagonal and is also the first visit by the curve to a given $y$ horizontal. That holds in $k{=}0$ and in two unfolds like figure 3 the new part 2 likewise.

The $m$'th *NonGnum* point, being *NonGpred*, has a similar recurrence to *NonXnum*. For *NonGnum*, digits 1 and 3 have all digits below, and digit 2 is restricted to *NonGpred* below.

$$\text{for } 4^k{-}2^k \le m < 4^{k+1}{-}2^{k+1}$$

$$\mathit{NonGnum}(m) = \begin{cases} m + 2^k & \text{if } m < 2.4^k{-}2^k \\ 2.4^k + \mathit{NonGnum}\big(m - (2.4^k{-}2^k)\big) & \\ & \text{if } 2.4^k{-}2^k \le m < 3.4^k{-}2.2^k \\ m + 2^{k+1} & \text{if } m \ge 3.4^k{-}2.2^k \end{cases} \tag{95}$$

$$= 1, 3, 4, 5, 6, 7, 9, 11, 12, 13, 14, 15, \ldots$$

$$\text{base-4} \;= 1, 3, 10, 11, 12, 13, 21, 23, 30, 31, 32, 33, \ldots$$

**Theorem 12.** *$n = \mathit{NonGnum}(m)$ can be calculated by the following base-4 digit procedure*

$$k \leftarrow \begin{cases} 1 & \text{if } m{=}0 \\ \lfloor \log_4 m \rfloor + 2 & \text{if } m{>}0 \end{cases}$$

$$n \leftarrow m + 2^k$$

$$\begin{aligned} \textbf{loop}\;\; & k \leftarrow k-1 \\ & \text{if digit } k \text{ of } n \text{ is 3 then result } n & (96) \\ & r \leftarrow n - 2^k \\ & \text{if digit } k \text{ of } r \text{ is 1 then result } r & (97) \\ & \text{if digit } k \text{ of } r \text{ is 0 then } n \leftarrow r \end{aligned}$$

*Proof.* Like *NonXnum* figure 16, $n$ holds the result so far in the high base-4 digits and an $m + 2.2^k$ in the low digits.

Offset $+2.2^k$ on $m$ means the third case of (95) becomes

$$m + 2.2^k \;\ge\; 3.4^k{-}2^k - 2.2^k + 2.2^k \;= 3.4^k$$

which is test for digit 3 at (96). The offset is already the result $m + 2^{k+1}$.

Reduced $r = n - 2^k$ is ready for similar comparisons for the first and second cases in (95). For digit 2, the index change in the recurrence is no change to $n$.

For digit 1, the reduced offset $+2^k$ in $r$ is the result $m + 2^k$.

For digit 0, the offset in $n$ must reduce to $2.2^{k-1}$ for the next loop, and that new $n$ is $r$.

At $k$=0, the single digit remaining is $m + 2.2^0 = 2 \,\text{or}\, 3$ and (97),(96) give result 1 or 3 respectively. This is when all other digits are 0 or 2 and just the lowest digit 1 or 3 ensures *NonGpred*. □

For bignum computer calculation, $r$ can be an in-place $n - 2^k$. This subtract is wanted when digit 1 (end of procedure), or digit 0 (next iteration), and otherwise an add back can undo for digit 2 next iteration.

From the triangular shape, single-visited points in the curve continued infinitely are the $x$ axis and the $x$=$y$ diagonal.

$$\begin{aligned}
\mathit{Spred}_\infty(n) &= \mathit{Xpred}(n) \,\text{or}\, \mathit{Gpred}(n) \\
&= \begin{cases} 1 & \text{if } n \text{ base-4 digits only } 0,1 \text{ or only } 0,2 \\ 0 & \text{otherwise} \end{cases} \\
&= 1,1,1,0,1,1,0,0,1,0,1,0,0,0,0,0,1,1,0,0,\ldots \\
\mathit{Snum}_\infty(m) &= 0 \text{ if } m{=}0, \text{ and otherwise:} \\
&\quad\; m{+}1 \text{ in binary, remove second highest bit } a \qquad (98) \\
&\quad\; \text{interpret rest as base-4 digits, multiply } 1{+}a \\
&= 0,1,2,4,5,8,10,16,17,20,21,32,\ldots \qquad \text{A126684} \\
\text{base-4} &= 0,1,2,10,11,20,22,100,101,110,111,200,\ldots
\end{aligned}$$

$\mathit{Snum}_\infty$ bit form (98) works by having second highest bit $a$ select whether to be 0,1s or 0,2s. Using the second highest bit makes alternating runs of $k$ many of each. Taking the bits of $m$+1 leaves a single initial $m$=0 for the $n$=0 which is common to the $x$ axis and $x$=$y$ diagonal.

The double-visited points are those neither $x$ axis nor $x$=$y$.

$$\begin{aligned}
\mathit{Dpred}_\infty(n) &= \begin{cases} 1 & \text{base-4 digits any digit 3 or both 1,2} \\ 0 & \text{otherwise} \end{cases} \\
&= \text{binary 1-bit at both odd and even positions} \\
&= 0,0,0,1,0,0,1,1,0,1,0,1,1,1,1,1,0,0,1,1,\ldots \\
=1 \text{ at } \; n &= 3,6,7,9,11,12,13,14,15,18,19,22,23,24,\ldots \qquad \text{A176237} \\
\text{base-4} &= 3,12,13,21,23,30,31,32,33,102,103,112,\ldots
\end{aligned}$$

The $\mathit{other}(n)$ procedure of section 5 also identifies single-visited points. An $n$ which has no $\mathit{other}(n)$ is $\mathit{Spred}_\infty$. In the bit fields of figure 13 this is $t$=0 and then every second bit also 0 so that there is no $\neq t$ bit. When $t$ and these other 0s fall at odd positions they give base-4 digits 0,1 and when at even positions base-4 digits 0,2.

$\mathit{Spred}_\infty$ has runs of at most 2 consecutive single-visited points after the initial 3 of $n$=0 to 2.

Within a given expansion level $k$, the points at the end of the triangle are single-visited too. They are either unfolded $x$ points or $x$=$y$ points according as $k$ odd or even.

$$\begin{aligned}
\mathit{Spred}_k(n) &= \begin{cases} 1 & \text{if } \mathit{other}(n) \text{ not within } k \text{ bits} \\ 0 & \text{otherwise} \end{cases} \\
&= \mathit{Spred}_\infty(n) \text{ or } \begin{cases} \mathit{Xpred}(2^k - n) & \text{if } k \text{ even} \\ \mathit{Gpred}(2^k - n) & \text{if } k \text{ odd} \end{cases} \\
\mathit{Dpred}_k(n) &= \begin{cases} 1 & \text{if } \mathit{other}(n) \text{ within } k \text{ bits} \\ 0 & \text{otherwise} \end{cases} \\
&= 1 - \mathit{Spred}_k(n)
\end{aligned}$$

$\mathit{Dpred}$ is 1 at both $n$ and $\mathit{other}(n)$, so half is count $D$,

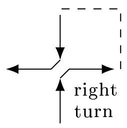
$$S_k = \sum_{n=0}^{2^k} \mathit{Spred}_k(n) \qquad D_k = \tfrac{1}{2} \sum_{n=0}^{2^k} \mathit{Dpred}_k(n) \tag{99}$$

These sums can be calculated from the bit fields of $\mathit{other}(n)$ per figure 13. This is more complicated than singles and doubles by the triangular shape, but gives a combinatorial interpretation to the number of such points.

At each double-visited point, the curve turns either left or right.

$$\begin{aligned}
\mathit{DpredLeft}_k(n) &= \mathit{Dpred}_k(n) \text{ and } \mathit{turn}(n) = +1 \\
\mathit{DpredRight}_k(n) &= \mathit{Dpred}_k(n) \text{ and } \mathit{turn}(n) = -1
\end{aligned}$$

A double-visited point with right turn encloses area on the left of the curve since the curve must eventually curl around to revisit the point and the triangular shape of the curve does not encircle the curve origin. Similarly a double with a left turn encloses area on the right of the curve.




double-visited point
right turn encloses
square on left of curve

Each such double corresponds to an enclosed unit square, so similar to (99)

$$AR_k = \tfrac{1}{2} \sum_{n=0}^{2^k} \mathit{DpredLeft}_k(n) \qquad AL_k = \tfrac{1}{2} \sum_{n=0}^{2^k} \mathit{DpredRight}_k(n)$$

For the curve continued infinitely the left and right doubles are

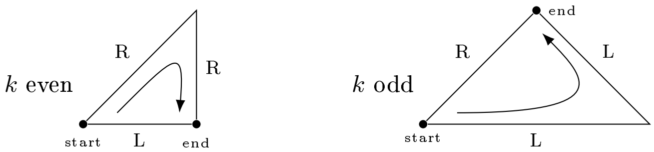
$$\begin{aligned}
\mathit{DpredLeft}_\infty(n) &= \mathit{Dpred}_\infty(n) \text{ and } \mathit{turn}(n) = +1 \\
&= 0,0,0,0,0,0,1,0,0,1,0,0,\ldots \\
=1 \text{ at } n &= 6,9,13,14,22,24,25,29,30,33,36,37,\ldots \\
=0 \text{ at } n &= 0,1,2,3,4,5,7,8,10,11,12,15,\ldots \\
\mathit{DpredRight}_\infty(n) &= \mathit{Dpred}_\infty(n) \text{ and } \mathit{turn}(n) = -1 \\
&= 0,0,0,1,0,0,0,1,0,0,0,1,\ldots \\
=1 \text{ at } \ n &= 3,7,11,12,15,18,19,23,26,27,28,31,\ldots \\
=0 \text{ at } \ n &= 0,1,2,4,5,6,8,9,10,13,14,16,\ldots
\end{aligned}$$

These predicates are cross related,

$$\begin{aligned} \mathit{DpredLeft}_\infty(n) &= \mathit{DpredRight}_\infty(2n) \\ \mathit{DpredRight}_\infty(n) &= \mathit{DpredLeft}_\infty(2n) \end{aligned}$$

since expanding to point $2n$ is the same single or double visited nature of $n$, and flips the turn left/right per the *turn* recurrence (1).

Single visited points are all on the boundary. They are left turns on the right boundary and conversely right turns on the left.




The counts of such points follow from the triangular shape. There is a single-visit of the respective turn between each boundary square. The start and end points have no turn so +2 in the total (100).

$$\begin{aligned} \mathit{Sleft}_k &= RQ_k - 1 \quad = [1,3].2^{\lfloor k/2 \rfloor} - [1,2] \\ &= 0, 1, 1, 4, 3, 10, 7, 22, 15, 46, \dots \\ \mathit{Sright}_k &= LQ_k - 1 \quad = [2,1]\,.\big(2^{\lfloor k/2 \rfloor} - 1\big) \\ &= 0, 0, 2, 1, 6, 3, 14, 7, 30, 15, \dots \\ S_k &= \mathit{Sleft}_k + \mathit{Sright}_k + 2 \end{aligned} \tag{100}$$

Single-visited points with left turns in the curve continued infinitely are simply the $x$ axis points *Xpred* except for $n{=}0$ where there is no turn. Similarly single-visited points with right turns are the $x{=}y$ diagonal *Gpred* except for $n{=}0$.

## 8.1 Boundary Segment Numbers

Segments on the left boundary of the curve continued infinitely are the diagonal stair-step. They are the segments before and after each *Gpred* point,

$$\mathit{Lpred}_\infty(n) = \mathit{Gpred}_\infty(n) \text{ or } \mathit{Gpred}_\infty(n{+}1) \tag{101}$$

$$= \text{base-4 digits } 0, 2 \text{ and optional low } 13\dots33 \tag{102}$$

$$= 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, \dots \qquad n{\ge}1 \text{ A270803}$$

$$g\mathit{Lpred}_\infty(x) = \Big(1{+}\frac{1}{x}\Big)\, g\mathit{Gpred}(x) \; - \frac{1}{x}$$

$$\mathit{Lnum}_\infty(m) = \mathit{Gnum}\Big(\Big\lfloor \frac{m{+}1}{2} \Big\rfloor\Big) - (m \bmod 2) \qquad m{\ge}0 \tag{103}$$

$$= 0, 1, 2, 7, 8, 9, 10, 31, 32, 33, 34, 39, \dots \qquad \text{A270804}$$

(103) uses the low bit of $m$ to select $-1, 0$ for the $n$ and $n{+}1$ cases at (101). There is no segment preceding point 0, hence $m{+}1$ to skip that.

Gawron and Ulas [10] reach $\mathit{Lpred}_\infty$ as compositional formal inverse of the Thue-Morse sequence. They give digit form (102) for that result and note also that $Lnum$ has runs of 4 consecutive integers so a low 2-bits can be taken from $m$ to select those runs as

$$Lnum_\infty(4m{+}r) = 4\,Gnum(m) + r \qquad \text{for } r = -1, 0, 1, 2 \tag{104}$$

Their inverse is for generating functions with terms taken mod 2 which when composed (either way) cancel to just $x$. $gLpred_\infty$ is without its constant 1 term,

$$\begin{aligned} &h(x) = gLpred_\infty(x) - 1 && \text{so constant term 0} && (105)\\ &gThueMorse\big(h(x)\big) \;=\; h\big(gThueMorse(x)\big) \;=\; x && \text{coeffs mod 2} \end{aligned}$$

$$\begin{aligned} ThueMorse(n) &= 0 \text{ or } 1 \;\equiv CountOneBits(n) \bmod 2\\ &= 0,1,1,0,1,0,0,1,1,0,\ldots && \text{A010060}\\ gThueMorse(x) &= x + x^2 + x^4 + x^7 + \cdots \end{aligned}$$

The inverse is unique since it is successive powers $gThueMorse(x)^j$, which have low term $x^j$, summed so as to cancel successive terms other than the low $x$. Gawron and Ulas reach (105) by substituting into a suitable identity on $gThueMorse$.

A similar but easier inverse is $gXpred$ without its low 1 term.

$$\begin{aligned} &h(x) = gXpred(x) - 1 && \text{so constant term 0}\\ &gXpred(x) = (1{+}x)\,gXpred(x^4) && \text{identity, base-4 new low 0 or 1} && (106)\\ &h(x) + 1 = (1{+}x)\,(h(x^4) + 1)\\ &h(x) = (1{+}x)\,h(x^4) + x && && (107) \end{aligned}$$

Inverse $v(x)$ is to satisfy $h(v(x)) = x$. Substitute $x \to v(x)$ into (107), and use

$$g(x^2) = g(x)^2 \tag{108}$$

which is true of any polynomial with coefficients mod 2, so

$$\begin{aligned} &h(v(x)) = \big(1 + v(x)\big)\,h(v(x))^2 - v(x)\\ &x = \big(1 + v(x)\big)\,x^2 - v(x)\\ &v(x) = -1 + \frac{1+x}{1-x^4} && \text{compositional inverse}\\ &0,1,0,0,\ 1,1,0,0,\ 1,1,0,0,\ldots && \text{repeating 1,1,0,0 except first term} \end{aligned}$$

Taking a compositional inverse like this requires constant term 0. For $gXpred$, that can also be arranged by a shift up rather than omitting the constant term. The inverse of this is the Baum-Sweet sequence [3], also shifted. Merta [19] shows this from the other way, starting at shifted Baum-Sweet and finding its inverse is shifted $Xpred$.

$$\begin{aligned} BaumSweet(n) &= \begin{cases} 1 & \text{if } n \text{ in binary all runs of 0-bits are even length}\\ 0 & \text{if } n \text{ in binary any run of 0-bits is odd length} \end{cases}\\ &= 1,1,0,1,1,0,0,1,0,1,0,0,1,\ldots, && n{\ge}1 \text{ A086747}\\ gBaumSweet(x) \;&=\; x\,.\,gBaumSweet(x^2) + gBaumSweet(x^4) && (109) \end{aligned}$$

Identity (109) is copy the sequence for either a new low 1-bit or new low 00 pair, both of which preserve all runs of 0s having even length. The rest, low 10, is zeros in the sequence since the low 0 there is an odd length run. So, and using (108) again to move powers inward or outward,

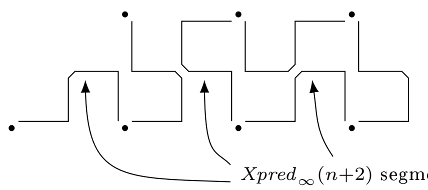
$$\begin{aligned}
&h(x) = x\,.\,gXpred(x) && \text{so constant term 0} \\
&\frac{1}{x}\,h(x) = (1{+}x)\,\frac{1}{x^4}\,h(x^4) && \text{as at (106)} \\
&\frac{1}{v(x)}\,h(v(x)) = \big(1+v(x)\big)\,\frac{1}{v(x)^4}\,h(v(x))^4 \\
&\frac{1}{v(x)}\,x = \big(1+v(x)\big)\,\frac{1}{v(x)^4}x^4 \\
&\frac{1}{x^4}\,v(x)^4 = \frac{1}{x}v(x) + x.\frac{1}{x^2}v(x)^2 && \text{by multiplying } v(x)^5/x^5 \qquad (110)
\end{aligned}$$

(110) mod 2 is per (109) with $v(x) = x.gBaumSweet(x)$. A search of the OEIS for sample values of $v$ suggested Baum-Sweet, but it's not too hard to recognise (110) powers $v(x^2)$ and $v(x^4)$ are spreads for something bit-wise, then try factors of $x$ each side.

Segments on the right boundary of the curve continued infinitely are two segments before and one after the *Xpred* points,

$$\begin{aligned}
Rpred_\infty(n) &= Xpred_\infty(n \text{ or } n{+}1 \text{ or } n{+}2) \\
&= \text{base-4 optional low } 03\ldots32 \text{ or } 03\ldots33 \text{ then } 0,1 \text{ above} \\
&= 1,1,1,1,1,1,0,0,0,0,0,0,\ldots
\end{aligned}$$

Segment $n$ has point $n$ at its start, so in the following diagram $n{+}2$ goes forward from the segments marked to the $x$ axis points shown with dots. From the segment expansions, all of the axis is of these forms.

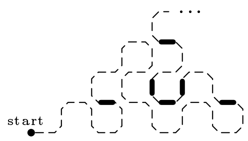



The $m$'th right boundary segment can be written in terms of the *Xpred* base conversion. An even and odd pair of $x$ axis points have total 4 segments before and after, so *Xnum* of $m/2$ and adjust to take those segments. There are no segments before the initial $x{=}0,1$ pair, hence $+1$ and the offsets rotated.

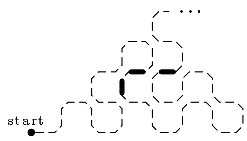
$$\begin{aligned}
Rnum_\infty(m) &= Xnum(\lfloor m/2\rfloor{+}1) - [1,0,2,1] \\
&= 0,1,2,3,4,5,14,15,16,17,18,19,\ldots \\
\text{base-4 } &= 0,1,2,3,10,11,32,33,100,101,102,103,\ldots
\end{aligned}$$

## 8.2 Enclosure Sequence

As each segment is successively appended to the curve, it may enclose a new unit square on the right or left of the curve, or not.




Left enclosures




Right enclosures

A new enclosed unit square is formed when a point is re-visited. So a segment enclosing a unit square has the second-visit of a double-visited point at its end. In the *other*$(n)$ bit fields (figure 13), a second visit is where the highest bit to flip is a 1, so that *other*$(n)$ becomes smaller. Any further bits flipped are arbitrary.

$$
\begin{aligned}
\mathit{DpredFirst}_k(n) &= \mathit{Dpred}_k(n) \text{ and } n < \mathit{other}(n) \\
\mathit{DpredFirst}_\infty(n) &= \mathit{Dpred}_\infty(n) \text{ and } n < \mathit{other}(n) \\
&= 0,0,0,1,\ 0,0,1,0,\ 0,1,0,1,\ 1,0,0,1,0,0,1,1,\dots \\
&= 1 \text{ at } n = 3,6,9,11,12,15,18,19,22,24,25,\dots \\
\mathit{DpredSecond}(n) &= \mathit{Dpred}_\infty(n) \text{ and } n > \mathit{other}(n) \\
&= 0,0,0,0,\ 0,0,0,1,\ 0,0,0,0,\ 0,1,1,0,\ 0,0,0,0,\dots \\
&= 1 \text{ at } n = 7,13,14,23,26,27,28,29,31,39,45,\dots
\end{aligned}
$$

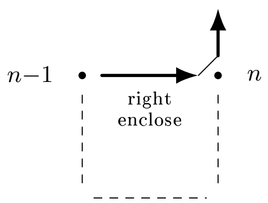
$\mathit{DpredFirst}_k$ is first visit to a point which will be re-visited within level $k$. $\mathit{DpredFirst}_\infty$ is the first visit to a point which will eventually be re-visited by the curve continued infinitely, which means a revisit either in $k$, or in $k+1$ across the join.

For *DpredSecond* no distinction is needed between a level $k$ and continuing infinitely since the other visit precedes $n$.

Totals through to $2^k$ are the number of double-visited points $D$,

$$
D_k = \sum_{n=0}^{2^k} \mathit{DpredFirst}_k(n) \quad = \sum_{n=0}^{2^k} \mathit{DpredSecond}(n)
$$

At each second-visit the curve turns either left or right. When it turns left it is away from the unit square just enclosed on the right. When it turns right it is away from the unit square just enclosed on the left. The turn is never to the same side as the square as that would overlap a side of that square.




turn left at right side enclosure
or would overlap segment
of square just enclosed

Taking the second-visit predicate with turns is

$$
\begin{aligned}
\mathit{DpredSecondL}(n) &= \mathit{DpredSecond}(n) \text{ and } \mathit{TurnLpred}(n) \\
&\quad \text{turn to left, encloses on right} \\
&= 1 \text{ at } n = 13,14,29,45,46,49,52,53,54,56,61,\dots
\end{aligned}
$$

$$DpredSecondR(n) = DpredSecond(n) \text{ and } TurnRpred(n)$$

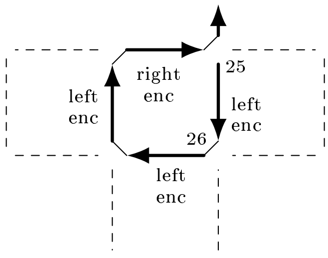
turn to right, encloses on left

$=1$ at $n = 7, 23, 26, 27, 28, 31, 39, 55, 58, 71, 87, \ldots$

The totals of these are then left and right side areas

$$AL_k = \sum_{n=0}^{2^k} DpredSecondR(n) \qquad AR_k = \sum_{n=0}^{2^k} DpredSecondL(n)$$

Up to 3 unit squares can be enclosed consecutively on a given side. The next segment encloses the square between those 3 on the opposite side.




3 left enclosures
are 3 right turns
so next segment
is a right enclosure

This is a run of 3 turns same direction all of which are second visits to double-visited points. The first such run for left enclosures occurs at points $n$=26, 27, 28 which are binary 11010, 11011, 11100.

There cannot be 4 or more consecutive same-side enclosures or that would be 4 turns and the segments would overlap.

Runs of right and left enclosures can occur. For example point $n$=106 has a run of 8 consecutive enclosures. The following diagram shows how this run falls within the preceding segments.

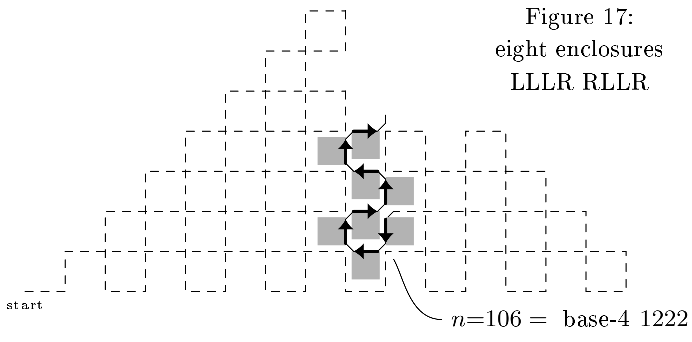



Figure 17:
eight enclosures
LLLR RLLR

A run of 8 is the longest which occurs. That can be seen by expressing *DpredSecond* as a state machine on the bits of $n$ and applying some state machine manipulations to make tests of $n$+1, $n$+2, etc. The intersection of *DpredSecond* of 9 terms $n$ through $n$+8 inclusive is empty.

State machine manipulations on the 8 intersection shows the first of each *DpredSecond* run of 8 has base-4 digit pattern

| | high | | | | | low | |
|---|---|---|---|---|---|---|---|
| *DpredSecondEight* = | ...any ... | 1 | $\underbrace{\dots 2\,\text{or}\,3 \dots}_{\text{repeat} \geq 0}$ | 2 | $\underbrace{\dots 3 \dots}_{\text{repeat} \geq 0}$ | 22 | base-4 |

The last segment of a level $k$ is non-enclosing so a run is entirely within a single level. The number of runs within level $k$ follows from some linear algebra on the state machine state counts, or the digit pattern and summing over lengths of the repeats,

$$EncEight_k = \begin{cases} 0 & \text{if } k \leq 3 \\ \frac{2}{3}2^{k-6} - [1,0].2^{\left\lfloor \frac{k-6}{2} \right\rfloor} + [\frac{1}{3}, -\frac{1}{3}] & \text{if } k \geq 4 \\ \quad = \frac{1}{3}\left(2^{\left\lfloor \frac{k-5}{2} \right\rfloor} - 1\right)\left(2^{\left\lceil \frac{k-5}{2} \right\rceil} - (-1)^k\right) & \end{cases}$$

$$= 0, 0, 0, 0, 0, 0, 0, 1, 1, 5, 7, 21, 35, 85, 155, \dots \qquad \text{A097038}$$

Runs of 8 all have the same enclosure side sequence shown in figure 17. This can be seen from $turn(n)$ which is opposite to the enclosed side. It is bit above lowest 1 bit and its position, on base-4 low digits 222 through 301, and is the same when some 3s for 23...322.

# 9 Cumulative GRS

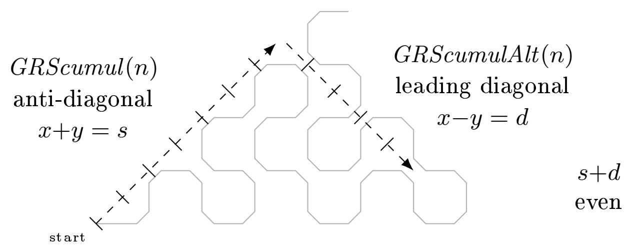
Brillhart and Morton [6] consider cumulative *GRS* (their $s$), and cumulative alternating signs *GRS* (their $t$),

$$GRScumul(n) = \sum_{j=0}^{n} GRS(j) = 1,2,3,2,3,4,3,4,5,6,\dots \qquad \text{A020986}$$

$$GRScumulAlt(n) = \sum_{j=0}^{n} GRSalt(j) = 1,0,1,2,3,2,1,0,1,0,\dots \qquad \text{A020990}$$

Per *dsum* and *ddiff* (47),(48), these correspond to coordinates in the alternate paperfolding curve. Most of the formulas by Brillhart and Morton have corresponding geometric interpretations.

$$\begin{aligned} GRScumul(n) &= Manhattan\big(point(n{+}1)\big) \\ GRScumulAlt(n) &= Leading\big(point(n{+}1)\big) \\ Manhattan(z) &= |\operatorname{Re} z| + |\operatorname{Im} z| \\ Leading(z) &= |\operatorname{Re} z| - |\operatorname{Im} z| \end{aligned}$$



Or with one *expand* and per $dx, dy$ at (49),(50),

$$\begin{aligned} GRScumul(n) &= \operatorname{Re} point\big(2(n{+}1)\big) \\ GRScumulAlt(n) &= \operatorname{Im} point\big(2(n{+}1)\big) \end{aligned}$$

These forms and the *point* procedure from theorem 7 can be used for bitwise computer calculation of one or both *GRScumul* and *GRScumulAlt*.

Blecksmith and Laud [4] calculate *GRScumul* by a chain of probability matrices on the bits of the target $n$.

Brillhart and Morton (Satz 3 and 4) give recurrences which can be written, for $r = 0$ to 3,

$$GRScumul(4n{+}r) = 2\, GRScumul(n) + \begin{cases} -GRS(n) & \text{if } r{=}0 \\ +GRSalt(n) & \text{if } r{=}2 \end{cases}$$

$$GRScumulAlt(4n{+}r) = 2\, GRScumulAlt(n) + \begin{cases} [-1,3].\,GRS(n) & \text{if } r{=}0 \\ -2\, GRSalt(n) & \text{if } r{=}1 \\ -GRSalt(n) & \text{if } r{=}2 \end{cases}$$

Both of these forms are base-4 digits of $n$ becoming "bit" values variously $\pm 1, \pm 2, \pm 3$ according as *GRS* or *GRSalt* of the digits of $n$ above.

Brillhart and Morton show $GRScumulAlt(n) = 0$ when $n{+}1$ written in binary has 0-bits at even positions. This is $Gpred(n{+}1)$ since $x{-}y = 0$ is the $x{=}y$ diagonal. They show $GRScumul(n) = GRScumulAlt(n)$ when $n{+}1$ has 0-bits at odd positions. This is $Xpred(n{+}1)$ since point $x{+}y = x{-}y$ if and only if $y{=}0$ so the X axis. They show too that for given $s$,

$$GRScumul(n) = s \qquad \text{has exactly } s \text{ many solutions } n \tag{111}$$

This corresponds to visits to an $s = x{+}y$ anti-diagonal in the curve. From the triangular shape, there are $\lfloor s/2 \rfloor{+}1$ points on such a diagonal and the curve makes 2 visits to each, except the $x$ axis only 1 visit, and when $s$ even the $x{=}y$ diagonal point only 1 visit, for total $s$ except at $s{=}0$ which has 1 visit.

Among the $s$ many solutions of (111), there is a first and last. Brillhart and Morton establish the last one *GRScumulLastN* (their formula here ahead at (120)), for use in their lower bound on $GRScumul(n)/\sqrt{n}$.

$$\begin{aligned} GRScumulFirstN(s) &= \text{minimum } n \text{ for which } GRScumul(n) = s \\ &= 0, 1, 2, 5, 8, 9, 10, 21, 32, 33, \dots \qquad s \geq 1 \qquad \text{A212591} \\ \text{base-4 } &= 0, 1, 2, 11, 20, 21, 22, 111, 200, 201, \dots \\ GRScumulLastN(s) &= \text{maximum } n \text{ for which } GRScumul(n) = s \\ &= 0, 3, 6, 15, 26, 27, 30, 63, 106, 107, \dots \qquad s \geq 1 \qquad \text{A020991} \\ \text{base-4 } &= 0, 3, 12, 33, 122, 123, 132, 333, 1222, 1223, \dots \end{aligned}$$

These correspond to first and last visits to a given $s = x{+}y$ anti-diagonal of the curve,

$$\begin{aligned} MfirstN(s) &= \text{minimum } n \text{ for which } Manhattan\big(point(n)\big) = s \\ &= GRScumulFirstN(s) + 1 \qquad \text{for } s \geq 1 \end{aligned}$$

$$
\begin{aligned}
&= 0, 1, 2, 3, 6, 9, 10, 11, 22, 33, \ldots \\
\text{base-4} &= 0, 1, 2, 3, 12, 21, 22, 23, 112, 201, \ldots \\
\mathit{MlastN}(s) &= \text{maximum } n \text{ for which } \mathit{Manhattan}\big(\mathit{point}(n)\big) = s \\
&= \mathit{GRScumulLastN}(s) + 1 \qquad\qquad \text{for } s \ge 1 \\
&= 0, 1, 4, 7, 16, 27, 28, 31, 64, 107, \ldots \\
\text{base-4} &= 0, 1, 10, 13, 100, 123, 130, 133, 1000, 1223, \ldots
\end{aligned}
$$

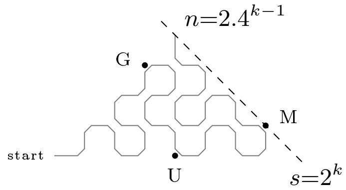
**Theorem 13.** *In the alternate paperfolding curve, the $n$ which is the first visit to anti-diagonal $s = x{+}y$ is given by*

$$
\begin{aligned}
&\mathit{MfirstN}(s) = 0 \textit{ if } s{=}0, \textit{ and otherwise:} \\
&\quad = \mathit{Gnum}(h).4^k + \mathit{Xnum}(2^k{-}1) + 1 && (112) \\
&\qquad \textit{where } s = h.2^{k+1} + 2^k && (113) \\
&\quad = 1 + \begin{cases} \lfloor s/2 \rfloor \textit{ in binary modified by:} \\ \textit{if } s \textit{ even change low } 10\ldots00 \textit{ to base-4 digits } 11\ldots11, \\ \textit{all other bits change to base-4 digits 0,2} \end{cases} && (114) \\
&\quad = \mathit{Xnum}\Big(\Big\lfloor \frac{s{-}1}{2} \Big\rfloor\Big) + \mathit{Xnum}\Big(\Big\lceil \frac{s{-}1}{2} \Big\rceil\Big) + 1 && (115)
\end{aligned}
$$

In (113), $2^k$ is the least significant 1-bit of $s$ and $h$ is everything above.

In (114), the low 1000 can have zero or more low 0-bits. The 1 and those 0s become base-4 digit 1s, and are not subject to the $0{,}1 \to 0{,}2$ of all other bits.

*Proof.* $s = 0, 1$ are respectively $n = 0, 1$ only. $s{=}2^k$ for $k{\ge}1$ can be illustrated

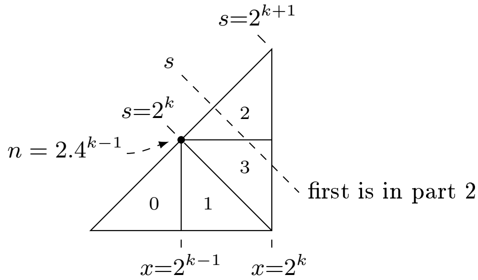



This anti-diagonal $s$ is an unfold of the $x{=}y$ leading diagonal, pivoting at point $x{=}2^{k-1}$ marked U. The first point M on $s$ is the last point G on $x{=}y$. The points on $x{=}y$ are *Gnum* from theorem 10 which are increasing so G is at $m = 2^{k-1}{-}1$ which becomes point M on $s$ so that

$$
\begin{aligned}
\mathit{MfirstN}(2^k) &= 2.4^{k-1} - \mathit{Gnum}(2^{k-1}{-}1) \ = \tfrac{1}{3}(4^k + 2) \\
&= \mathit{Xnum}(2^k{-}1) + 1 && (116) \\
&= 1, 2, 6, 22, 86, 342, 1366, 5462, \ldots && \text{A047849}
\end{aligned}
$$

*Xnum* form (116) is simply that M is one segment above the $x$ axis $x{=}2^k{-}1$. It and the $\frac{1}{3}(4^k{+}2)$ form hold for $k{=}0$ too.

For $2^k < s < 2^{k+1}$, the diagonal passes through the following curve sub-parts




Sub-curve 2 is the first on the diagonal. It begins at the middle $n = 2.4^{k-1}$ shown so

$$\mathit{MfirstN}(2^k{+}s) = 2.4^{k-1} + \mathit{MfirstN}(s) \qquad 2^k < s < 2^{k+1}$$

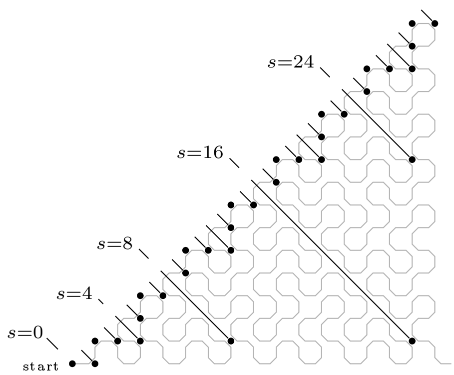
Each $2.4^{k-1}$ here is a base-4 digit 2 corresponding to a bit of $s$. Each digit is at $k{-}1$ so one position below the bit of $s$ and thus giving $\mathit{Gnum}(h).4^k$ of (112).

Digits (114) are the *Gnum* and *Xnum* combination.

For the pair of *Xnum* at (115), when $s$ odd the two are the same so doubling to *Gnum* base-4 digits 0,2. When $s$ even, the low 1000 of $s$ is flipped in the floor thus giving low base-4 digits 1111 and above that the same in both. □

The location in the curve of the first visit follows from M on $s{=}2^k$ for $k{\geq}1$ then replications of that for other $s$, so the lowest 1-bit of $s$. For $s{=}1$ and all its replications as $s$ odd, the point is $\frac{1}{2}$ from the diagonal. In (117) that is handled by the absolute value.

$$\begin{aligned} \mathit{MfirstZ}(s) &= \mathit{point}\big(\mathit{MfirstN}(s)\big) \\ &= s.\tfrac{1+i}{2} + \left|2^{\mathit{CountLowZeros}(s)} - 2\right|.\tfrac{1-i}{2} \qquad s \geq 1 \qquad (117) \\ &= \mathit{unexpand}\big(s + \left|2^{\mathit{CountLowZeros}(s)} {-} 2\right|.i\big) \\ &= 0,\ 1,\ 1{+}i,\ 2{+}i,\ 3{+}i,\ 3{+}2i,\ 3{+}3i,\ 4{+}3i,\ 7{+}i, \ldots \end{aligned}$$



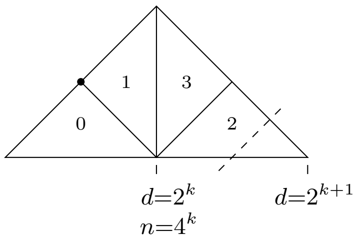

*MfirstZ*
locations of
first visit to
anti-diagonals

Another geometric interpretation of location M, for even $s$, is to follow down through gaps when corners of the curve are chamfered off, until reaching the last such gap. On reaching either a non-gap or the $x$ axis the preceding point is M.

These gaps are all left turns in the curve too, since they are unfolds of the part 0 diagonal $x{=}y$ which are right turns. So the M location is the last double-visited left turn of the run of such points starting from the top of the diagonal, if any.

**Theorem 14.** *The $n$ which is the first visit to leading diagonal $d = x{-}y$ is the X axis $n = \mathit{Xnum}(d)$.*

*Proof.* $d{=}0$ is the $x{=}y$ diagonal and $n{=}0$ at the origin is the first visit. Then $d$ in the range $2^k \le d < 2^{k-1}$ is




Diagonal $d = 2^k$ is not visited by parts 0 and 1 since only $x = 2^k$ is within their triangular shape but it is $n = 4^k$ which is the first point in part 2.

Visits to $d$ in the range $2^k < d < 2^{k-}1$ are in part 2. That part is the same structure as part 0 by unfolding, so reduce to $d - 2^k$ and consider the first visit in sub-parts there. This eventually reduces to a $d = 2^r$ at the lowest 1-bit of $d$, which has first visit $x = 2^r$, and so the first visit to $d$ is on the X axis at $x = d$. ☐

**Theorem 15** (variant of Brillhart and Morton)**.** *The $n$ which is the last visit to anti-diagonal $s = x{+}y$ is given by*

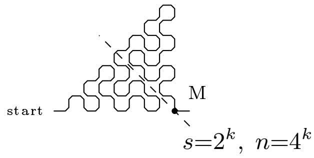
$$\mathit{MlastN}(s) = 0 \text{ if } s{=}0, \text{ and otherwise:}$$

$$= 2^{2k+1} - \mathit{Xnum}(2^{k+1}{-}s) \qquad \text{where } 2^k \le s < 2^{k+1} \tag{118}$$

$$= 1 + \begin{cases} s{-}1 \text{ in binary, change to base-4 digits } 2,3, \\ \text{highest 1-bit change to 3 if } s{=}2^k, \text{ unchanged 1 if not} \end{cases} \tag{119}$$

The $s = 2^k$ case is when $s{-}1$ is all 1-bits. So the high digit of $s{-}1$ changes to 3 when that makes all digits 3, and otherwise it is 1.

*Proof.* For $s{=}2^k$, the triangular shape means the last point $n = 4^k$ of level $2k$ is on the diagonal and is the last visit.




For $2^k < s < 2^{k+1}$, the diagonal passes through the following curve sub-parts

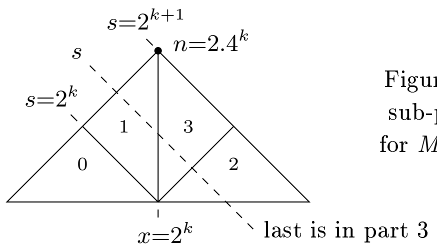



Figure 18: sub-parts for *MlastN*

Sub-part 3 is the maximum. It is a reverse curve going down from the top $n{=}2.4^k$, so seek the first $n$ on a leading diagonal $d = x{-}y = 2^{k+1} - s$ down from there. Per theorem 14, this is $Xnum(d)$ and so (118).

Digits (119) follow from the *Xnum* subtraction. Bits of $2^{k+1}{-}s$ are bits of $s{-}1$ flipped $0{\leftrightarrow}1$ then subtraction from $2^{2k+1}$ flips the sense again so base-4 digits 2,3 except the highest stays as 1. When $s{=}2^k$ changing the highest to 3 too gives the $n{=}4^k$ result (which is a high digit 1 too). $\square$

Brillhart and Morton [6, Satz 18] expand recurrences for *GRScumulLastN* to reach

$$GRScumulLastN(s) = s - 1 + \tfrac{2}{3}(4^r{-}1) + 2\sum_{j=0}^{r-1} \left\lfloor \frac{s-1}{2^{j+1}} \right\rfloor . 4^j \qquad (120)$$
$$\text{where } 2^r \le s < 2^{r+1}$$

This is the digits form (119) of $MlastN(s{-}1)$, without 1+. The sum part of (120) is base conversion binary to base-4 digits 0,1,

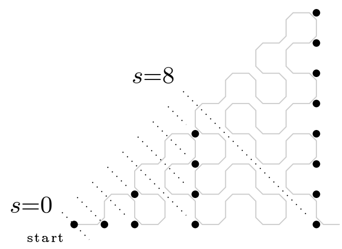
$$t + 2\sum_{j=0}^{r-1} \left\lfloor \frac{t}{2^{j+1}} \right\rfloor . 4^j \ = t \text{ in binary change to base-4 digits 0,1}$$

At $j{=}0$, the sum term is bits of $t$ above the lowest and this is added to $t$ at bit position 1 so doubling those bits, moving them to bit position 2. At $j{=}1$, the term does similar to bits above the two lowest, and so on spreading bits out to base-4.

$\frac{2}{3}(4^r{-}1)$ is base-4 all digit 2s so adds to give base-4 digits 2,3. The selection of $r$ in (120) handles the different cases $s{=}2^k$ or $\neq 2^k$ at (119). Selecting $r$ from $s$, whereas the bits from $s{-}1$, means for $s{=}2^k$ the resulting $r$ is 1 bigger and $\frac{2}{3}(4^r{-}1)$ has an extra high 2 to add, so high digit 3.

In theorem 15, the locations of *MlastN* are the vertical side of sub-part 3 of figure 18. So for $s{\ge}1$, the real part is the high bit of $s$ and the imaginary part is the remainder.

$$\begin{aligned} MlastZ(s) &= point\big(MlastN(s)\big) \\ &= 2^k + r.i \qquad \text{where } s = 2^k{+}r \text{ with } 0{<}r{<}2^k \text{, for } s{\ge}1 \\ &= 0,\ 1,\ 2,\ 2{+}i,\ 4,\ 4{+}i,\ 4{+}2i,\ 4{+}3i,\ 8, \dots \qquad \text{Re A053644, Im A053645} \end{aligned}$$




*MlastZ*
locations of
last visit to each
anti-diagonal

The distance of each *MlastZ* from the $x{=}y$ diagonal, measuring along the anti-diagonal, is $2^k$ at $s{=}2^k$ then decreasing down to 1, so high bit of $s$ subtract remaining bits. Brillhart and Morton[6, Zusatz page 141] have this as

$$\begin{aligned} &\mathit{GRScumulAlt}(\mathit{GRScumulLastN}(s)) = 2^k - r \\ &\qquad \text{where } s = 2^k{+}r \text{ with } 0 \le r < 2^k \\ &\qquad\qquad = 0,1,2,1,4,3,2,1,8,7,6,5,4,3,2,1,16,\ldots \qquad \text{A080079} \end{aligned}$$

which corresponds here to

$$\mathit{Leading}(\mathit{MlastZ}(s)) = \begin{cases} 0 & \text{if } s{=}0 \\ 2^k - r & \text{if } s \ge 1 \end{cases}$$

Visits to columns of given $x$ follow from the anti-diagonals. On *expand*, the anti-diagonals become columns. There is an extra point preceding the first, so $-1$ in (121).

$$\begin{aligned} \mathit{VfirstN}(x) &= \text{minimum } n \text{ for which } \operatorname{Re} \mathit{point}(n) = x \\ &= 2\,\mathit{MfirstN}(x) - 1 \qquad\qquad x{\ge}1 \qquad\qquad (121) \\ &= 0,1,3,5,11,17,19,21,43,65,\ldots \\ \mathit{VlastN}(x) &= \text{maximum } n \text{ for which } \operatorname{Re} \mathit{point}(n) = x \\ &= 2\,\mathit{MlastN}(x) \\ &= 0,2,8,14,32,54,56,62,128,214,\ldots \end{aligned}$$

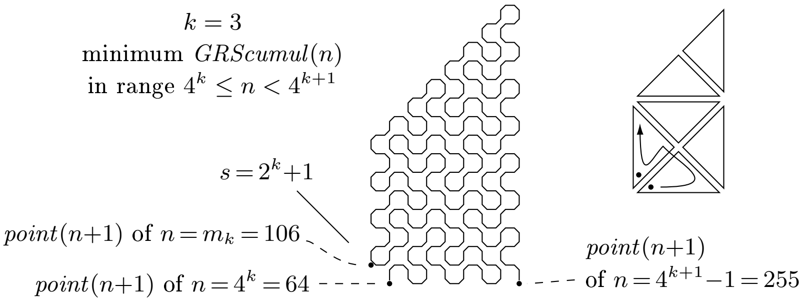
On further expansion, columns become anti-diagonals again, but only the even anti-diagonals. All points on such an even diagonal are from the column so direct relations

$$\begin{aligned} \mathit{MfirstN}(2x) &= 2\ \mathit{VfirstN}(x) \\ \mathit{MlastN}(2x) &= 2\ \mathit{VlastN}(x) \end{aligned}$$

Brillhart and Morton find which $n$ in the range $4^k \le n < 4^{k+1}$ have minimum $s = \mathit{GRScumul}(n)$. They show there are two $n$ with equal minimum $s = 2^k{+}1$. One is the range start $n = 4^k$. The other, for $k \ge 1$, is their $m_k$. (At $k{=}0$ have $m_k = 1$ which is the range start $4^k$ and the other equal minimum is instead $n{=}3$.)

$$\begin{aligned} m_k &= \tfrac{1}{3}(5.4^k - 2) \qquad\qquad (122) \\ &= 1,6,26,106,426,1706,6826,\ldots \quad k \ge 0 \qquad \text{A020989} \\ &= \mathit{GRScumulLastN}(2^k{+}1) \qquad \text{for } k \ge 1 \end{aligned}$$

With $GRScumul(n) = Manhattan(point(n{+}1))$, the two $n$ can be illustrated in the curve,




Other visits to this anti-diagonal $s = 2^k{+}1$ are in earlier parts of the curve, so that (and which also follows from the bit patterns for $GRScumulLastN$),

$$m_k = GRScumulLastN(2^k{+}1) \qquad \text{for } k \ge 1 \tag{123}$$

Brillhart and Morton note $GRScumul(n)$ grows as $\sqrt{n}$ and show ratios

$$\sqrt{\frac{3}{5}} < \frac{GRScumul(n)}{\sqrt{n}} < \sqrt{6} \tag{124}$$

They establish the upper bound by an considering ranges of $n$ and some induction for possible $GRScumul$ in those ranges.

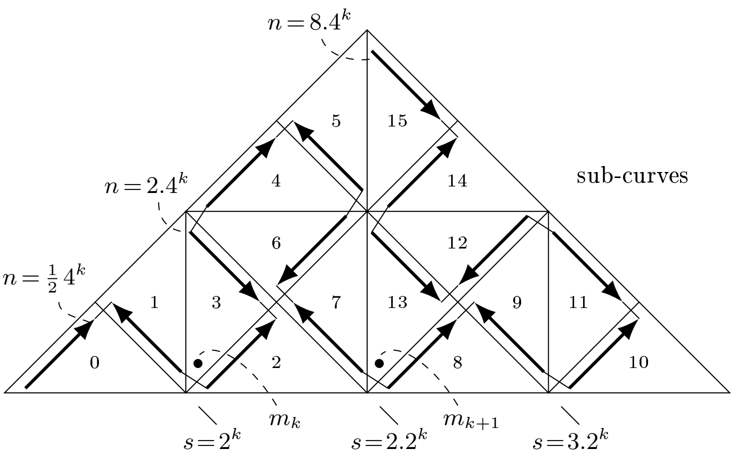
They establish the lower bound by showing $s/\sqrt{GRScumulLastN(s)} > \sqrt{\frac{3}{5}}$ for all $s$, ie. the largest denominator for a given numerator, and note the lower bound is approached for $s = 2^k{+}1$ per (123),(122).

Some ranges for the lower bound are possible too. Sub-curves help to see which ranges to use.

**Theorem 16** (Brillhart and Morton)**.**

$$\sqrt{\frac{3}{5}} < \frac{GRScumul(n)}{\sqrt{n}}$$

*Proof.* The theorem can be verified explicitly for $n < 8$. Suppose then it is true up to $n < 2.4^k$ for some $k \ge 1$. The following diagram shows sub-curves up to $n = 8.4^k$. Each sub-curve is length $n = \frac{1}{2}4^k$.




The end of sub-curve 3 is $n = 2.4^k$. All $n$ above there have

$$s = \mathit{GRScumul}(n) = \mathit{Manhattan}(\mathit{point}(n{+}1)) \geq 2.2^k \qquad \text{for } n \geq 2.4^k$$

$n = 2.4^k$ to $m_{k+1}$ inclusive can use this minimum $s$ for

$$\frac{\mathit{GRScumul}(n)}{\sqrt{n}} \geq \frac{2.2^k}{\sqrt{m_{k+1}}} > \frac{2.2^k}{\sqrt{\frac{1}{3}.20.4^k}} = \sqrt{\frac{3}{5}}$$

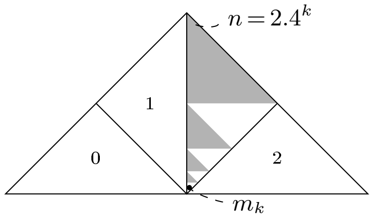
Sub-curves 14 and 15 are $n = 7.4^k$ to $8.4^k$ and they have $s \geq 3.2^k$ so for them

$$\frac{\mathit{GRScumul}(n)}{\sqrt{n}} > \frac{3.2^k}{\sqrt{8.4^k}} = \frac{3}{\sqrt{8}} > \sqrt{\frac{3}{5}}$$

Sub-curve 13 is a copy of sub-curve 3 shifted by $+2^k$ horizontally and $n$ offset $+5.4^k$. Take an $n + 5.4^k > m_{k+1}$ in sub-curve 13. This is $n > m_k$ in sub-curve 3 since $m_{k+1} = m_k + 5.4^k$. Then

$$\begin{aligned}
\frac{\mathit{GRScumul}(n + 5.4^k)}{\sqrt{n + 5.4^k}} &= \frac{\mathit{GRScumul}(n) + 2^k}{\sqrt{n + 5.4^k}} > \frac{\sqrt{\frac{3}{5}n} + 2^k}{\sqrt{n + 5.4^k}} \quad \text{induction} \\
&= \sqrt{\frac{\left(\sqrt{\frac{3}{5}n} + 2^k\right)^2}{n + 5.4^k}} = \sqrt{\frac{3}{5} + \frac{2\sqrt{\frac{3}{5}n}\,.2^k - 2.4^k}{n + 5.4^k}} \\
&> \sqrt{\frac{3}{5} + \frac{2\sqrt{\frac{3}{5}\frac{5}{3}4^k}\,.2^k - 2.4^k}{n + 5.4^k}} = \sqrt{\frac{3}{5}} \qquad \square
\end{aligned}$$

Sub-curve 13 after $m_{k+1}$ has bigger $n$ so it's not enough to use just $s \geq 2.2^k$, it must be shown $s$ becomes bigger too. Appealing to sub-curve 3 handles that by equivalents of sub-curves 14 and 15 within the sub-part, recursively down.




successive
sub-curves
like 14 and 15
after $m_k$
within part 3

Ratios $s = \mathit{GRScumul}(n)/\sqrt{n} = x{+}y$ and $t = \mathit{GRScumulAlt}(n)/\sqrt{n} = x{-}y$ can be illustrated by plotting the paperfolding curve as $\mathit{point}(n{+}1)/\sqrt{n}$. The following diagram is an approximation to the resulting limit set as $n{\to}\infty$.

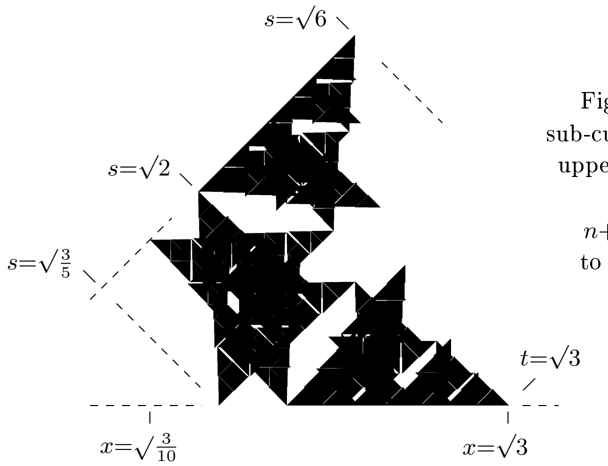
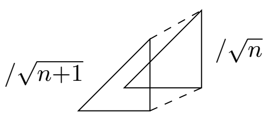



Figure 19:
sub-curve filling,
upper bounds,
$k{=}4$
$n{+}1 = 4^k$
to $4^{k+1}{-}1$

This approximation is made by taking a little triangle on the side of each segment $n{+}1$. Points of the sub-curve there will be shrunk (towards the origin) by between $1/\sqrt{n}$ and $1/\sqrt{n{+}1}$. The convex hull around the triangle corners shrunk by each is an upper bound on where points in the sub-curve might fall. The orientation of the triangle is from the segment direction.




So indentations and holes shown in figure 19 are definitely empty, but the solid areas are only upper bounds and may have more holes or indentations. Many of the holes are small and can be seen only at high resolution.

Brillhart, Erdős and Morton [7] show limits $s/\sqrt{n}$ and $t/\sqrt{n}$ are continuous, which means the shape in figure 19 is connected. Geometrically, this is the curve path variously shrunk by $/\sqrt{n}$ but remaining continuous. They draw plots of those coordinate functions against fractional $n$ too.

Brillhart and Morton [6] had also shown all values $\sqrt{\frac{3}{5}}$ to $\sqrt{6}$ in (124) occur as limits $s = x{+}y$. The geometric interpretation is that a projection of the shape onto the $x{=}y$ diagonal fills that extent. Similarly their $t{=}0$ to $\sqrt{3}$ is a projection onto an $x{=}{-}y$ anti-diagonal.

The portion of the $x{=}y$ diagonal between $\sqrt{2}$ and $\sqrt{6}$ is in fact filled already by *Gnum* points on that diagonal, as shown by Gawron and Ulas [10, theorem

3.8] on squared reciprocal ratios. Their theorem is for $Lnum_\infty$ which is a superset of *Gnum*, but their proof uses only *Gnum* (the $Lnum_\infty$ points are adjacent to *Gnum* points per (104)).

**Theorem 17** (Gawron and Ulas)**.** *Those $n$ with $GRScumulAlt(n)=0$, being $n = Gnum(s) - 1$, have ratios $GRScumul(n)/\sqrt{n}$ which are dense in the range $\sqrt{2}$ to $\sqrt{6}$.*

*Proof.* $n = Gnum(s) - 1$ gives $GRScumul(n) = 2s$ so ratios to be considered are $2s/\sqrt{Gnum(s) - 1}$.

The lower bound is approached by $s = 2^k$ which has $n = Gnum(2^k) - 1 = 2.4^k - 1$.

The upper bound is approached by $s = 2.2^k - 1$ which per Brillhart and Morton is $Gnum(s) = \frac{2}{3}(4.4^k - 1)$.

For $s$ in between, the ratios are dense if squares of the ratios are dense. The increment from one squared ratio to the next is, for $k > 3$,

$$\frac{4(s+1)^2}{Gnum(s+1)-1} - \frac{4s^2}{Gnum(s)-1} < \frac{8s+4}{Gnum(s)-1} < \frac{16.2^k}{2.2^{2k}-1} < \frac{1}{2^{k-3}-1} \tag{125}$$

These increments run from the lower bound to the upper bound and are made arbitrarily small by choosing $k$ big enough. ☐

Gawron and Ulas show density by constructing a sequence of $s$ which converges to any desired ratio $q$ by taking a new low 1-bit on $s$ whenever doing so remains $\geq q$ (corresponding to $\leq$ the ratio here). They show such a sequence always has infinitely many 0-bits, and therefore is arbitrarily close to $q$ because the step for an ever-smaller 1-bit would go past it.

The increments used at (125) can be negative, but the upper bound ensures they cover the range as they go low to high. Increments are negative when enough low 1-bits of $s$.

**Theorem 18.** *Step $2s/\sqrt{Gnum(s) - 1}$ to $2(s+1)/\sqrt{Gnum(s+1) - 1}$ is an increase or decrease according to, and where $k = \lfloor \log_2 s \rfloor$ so $2^k \leq s < 2^{k+1}$,*

$$\begin{array}{ll} \textit{increase} & \textit{if } CountLowOnes(s) \leq \lceil k/2 \rceil \\ \textit{decrease} & \textit{if } CountLowOnes(s) > \lceil k/2 \rceil \\ \textit{unchanged} & \textit{never} \end{array} \tag{126}$$

*decrease* $s = 1, 3, 7, 15, 23, 31, 47, 63, 79, 95, 111, 127, 159, \ldots$
*binary* $1, 11, 111, 1111, 10111, 11111, 101111, 111111, 1001111, \ldots$

*Proof.* Let *dGnum* be *Gnum* increment.

$$\begin{aligned} dGnum(s) &= Gnum(s+1) - Gnum(s) \\ &= \tfrac{1}{3}\left(4^{CountLowOnes(s)+1} + 2\right) \qquad (127) \\ &= 2, 6, 2, 22, 2, 6, 2, 86, 2, 6, 2, 22, \ldots \qquad 2\times\text{A276391} \end{aligned}$$

(127) is since $s+1$ changes $s$ low bits 0111 to 1000, giving corresponding changes in the bits of *Gnum*.

*GRS* ratio squared steps are

$$
\begin{aligned}
&\frac{4(s{+}1)^2}{Gnum(s{+}1) - 1} \;-\; \frac{4s^2}{Gnum(s){-}1} \\
&= \frac{(8s{+}4)\big(Gnum(s){-}1\big) \;-\; 4s^2\big(Gnum(s{+}1) - Gnum(s)\big)}{\big(Gnum(s) - 1\big)\big(Gnum(s{+}1) - 1\big)}
\end{aligned}
$$

The numerator is

$$
step \;=\; (8s{+}4)\big(Gnum(s){-}1\big) \;-4s^2\, dGnum(s)
$$

$step{\neq}0$ since *Gnum* and *dGnum* are both even so first term $\equiv 4 \bmod 8$ and second term $\equiv 0 \bmod 8$.

To find where *step* is an increase, make a lower bound for $Gnum(s)$ by comparing $3Gnum(s)$ and $s2^{k+2}$. They have high 1-bit at the same place, but $3Gnum(s)$ is each bit of $s$ repeated so the highest 0-bit of $s2^{k+2}$ has a 1-bit in $3Gnum(s)$. For example,

$$
\begin{array}{|cc|cc|cc|cc|cc|c|l}
\hline
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & \;3\,Gnum(s) \\
\hline
\end{array}
$$

$$
\begin{array}{|c|c|c|c|c|c|c|l}
\hline
1 & 1 & 0 & 1 & 1 & \; & \qquad\qquad & \;s\,2^{k+2} \\
\hline
\end{array}
$$

high 0 of $s$ $\qquad$ $2^{k+1}$

$2^{k+1}$ is immediately below the bits of $s$ in $s2^{k+2}$, and it can be added without exceeding $3\,Gnum(s)$. If $3\,Gnum(s)$ has a 0-bit there, as shown in the example, then that comes from a 0-bit in $s2^{k+2}$ at some higher position. The highest 0-bit of $s2^{k+2}$ has a 1-bit in $3\,Gnum(s)$ so bigger. Thus, and equality at $s{=}1$,

$$
3\,Gnum(s) \;\ge\; s2^{k+2} + 2^{k+1}
$$

Using this, writing $l = CountLowOnes(s)$, and at (128) using $s{+}1 \le 2^{k+1}$,

$$
\begin{aligned}
step &\ge (8s{+}4)\big(\tfrac{1}{3}s2^{k+2} + \tfrac{1}{3}2^{k+1} - 1\big) - 4s^2\,\tfrac{1}{3}\big(2^{2l+2} + 2\big) \\
&= \tfrac{4}{3}\Big(s^2\big(2^{k+3} - 2^{2l+2}\big) \;+ (4s{+}1)2^{k+1} - (2s{+}4)(s{+}1) + 1\Big) \\
&\ge \tfrac{4}{3}\Big(s^2\big(2^{k+3} - 2^{2l+2}\big) \;+ (4s{+}1)2^{k+1} - (2s{+}4)2^{k+1} \;\; + 1\Big) && (128) \\
&= \tfrac{4}{3}\Big(s^2\big(2^{k+3} - 2^{2l+2}\big) \;+ (2s{-}3)2^{k+1} + 1\Big) \\
&> \tfrac{4}{3}s^2\big(2^{k+3} - 2^{2l+2}\big) \qquad\qquad \text{for } s{\ge}2 && (129) \\
&\ge 0 \;\text{ when } k{+}3 \ge 2l{+}2 \;\text{ so } l \le \lceil k/2 \rceil \text{ per } (126)
\end{aligned}
$$

Case $s{=}1$ is not covered by (129) but can be verified explicitly. It is not an increase and does not have $l \le \lceil k/2 \rceil$.

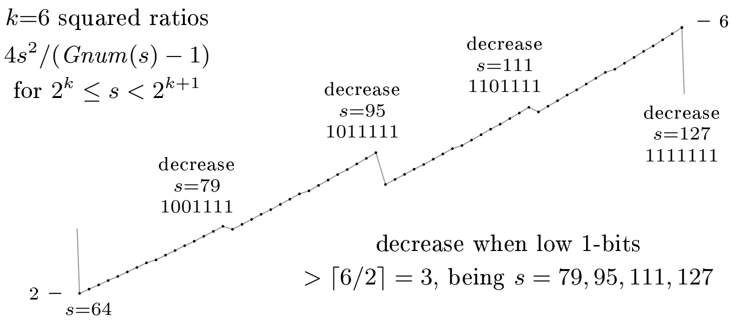
To find where *step* is a decrease, make a simple upper bound for $Gnum(s)$ by shifting $s$ up to the same high bit position,

$$
Gnum(s) \;\le\; s\,2^{k+1}
$$

*Gnum* has $s$ bits spread down with 0s between, so the second-highest 1-bit of $s2^{k+1}$ has a 0-bit at corresponding position in *Gnum*. Equality is when $s{=}2^k$ which has no further 1-bit. So

$$
\begin{aligned}
step &\le (8s{+}4)\left(s2^{k+1}-1\right) - 4s^2\,\tfrac{1}{3}\left(2^{2l+2}+2\right) \\
&= \tfrac{4}{3}s^2\left(3.2^{k+2}-4.2^{2l}+4\right) \;+4s2^{k+1}-8s^2-8s-4 \\
&< \tfrac{4}{3}s^2\left(3.2^{k+2}-4.2^{2l}+4\right) \;-8s-4 \qquad \text{using } s\ge 2^k \qquad (130)\\
&\le 0 \;\text{ when } 2l\ge k{+}2 \;\text{ so } l>\lceil k/2\rceil \;\text{ converse of (126)}
\end{aligned}
$$

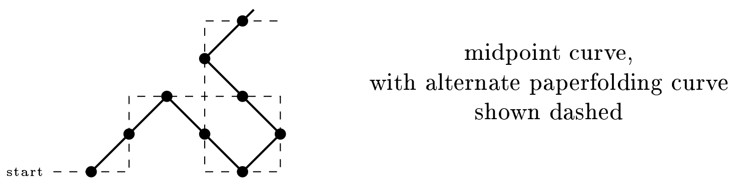
At (130), the $+4$ is overcome when $2l \ge k{+}2$ since difference $3.2^{k+2}-4.2^{2l} < 3.2^{k+2}-4.2^{k+2} = -2^{k+2} \le -4$. □

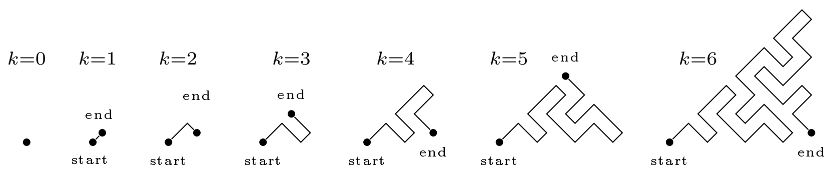



# 10 Midpoint Curve

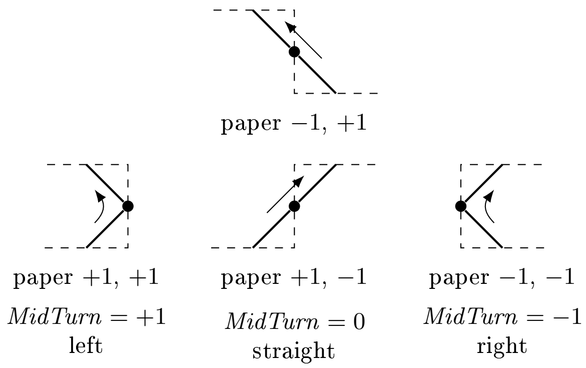
A midpoint curve can be made by connecting the midpoints of each segment of the alternate paperfolding curve.

$$
midpoint(n) = \tfrac{1}{2}\left(point(n) + point(n{+}1)\right)
$$




The alternate paperfolding curve turns $\pm 90°$ so the midpoint curve goes by diagonals. At each midpoint the midpoint curve can turn $+90°, 0°$ or $-90°$ according to the paperfolding curve *turn* (section 2) before and after that midpoint.




Counting the first midpoint point as $n$=0, the first midpoint curve turn is at $n$=1. The alternate paperfolding curve vertices before and after midpoint $n$ are $turn(n)$ and $turn(n{+}1)$. The midpoint turn sequence is then

$$\begin{aligned} MidTurn(n) &= \begin{cases} turn(n) & \text{if } turn(n) = \phantom{-}turn(n{+}1) \\ 0 & \text{if } turn(n) = -turn(n{+}1) \end{cases} \qquad n \geq 1 \\ &= \tfrac{1}{2}\, sturn(n) \qquad \text{from (9)} \\ &= 0, -1, 0, 1, 1, 0, -1, 0, 0, -1, -1, 0, 1, 0, 0, 1, \dots \end{aligned}$$

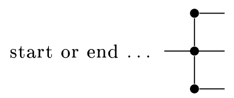
$$\begin{aligned} MidDir(n) &= \sum_{j=1}^{n} MidTurn(j) \\ &= 0, 0, -1, -1, 0, 1, 1, 0, 0, 0, -1, -2, -2, -1, -1, -1, 0, 0, \dots \\ gMidDir(x) &= \frac{1}{1-x}\, gMidTurn(x) \;= \tfrac{1}{2}\frac{1}{1-x}\, gsturn(x) \end{aligned}$$

# 11 Graph

The alternate paperfolding curve as a graph is, by its construction, a planar unit distance graph and has an Euler path from start to end (traverse all edges once). It is bipartite like any graph on a square grid since vertices can be separated into those with coordinates $x{+}y$ odd or even and edges are only between odd and even.

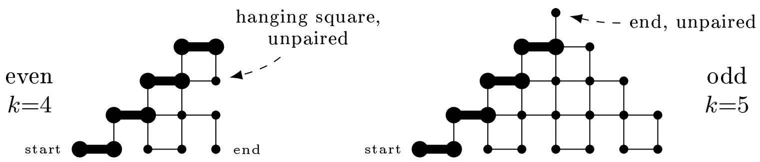
The curve has no Hamiltonian path start to end (visit all vertices once) for $k \geq 3$ since the curve start and end are degree-1, and the other corner of the triangle shape has a hanging square. All three of these would have to be ends of a Hamiltonian path.

If the hanging square is removed then for $k \geq 4$ there is still no Hamiltonian path since start and end lead to vertices with two degree-2 neighbours. A path entering or leaving at the centre vertex shown cannot visit both the upper and lower.



An *independent edge set* in a graph is a set of edges with no end vertices in common, also called a *matching* since it is vertices in pairs with edge between. A *perfect matching* is all vertices in such pairs. A perfect matching is possible only for an even number of vertices.

The alternate paperfolding curve points $P_k$ is even for $k$=0 or $k \geq 3$ but there is no perfect matching except the single pair $k$=0. For $k \geq 3$ the start vertex must pair with the vertex to its right, which leaves the vertex at $1+i$ only able to pair with the vertex to its right, and so on up the $x$=$y$ diagonal. For $k$ even this leaves the hanging square at the top only able to pair 3 of its 4 vertices. For $k$ odd this leaves the end vertex at the top unpaired.



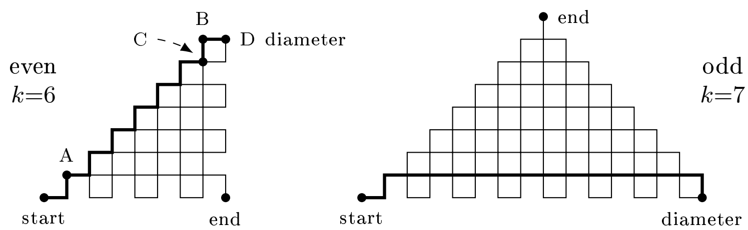

## 11.1 Diameter and Wiener Index

Shortest paths in the graph are by stair step. From the triangular shape of the expansions the diameter of the graph is

$$Diameter_k = \begin{cases} 1, 2, 4 & \text{if } k \leq 2 \\ 2.2^{\lfloor k/2 \rfloor} + [-1, 1] & \text{if } k \geq 3 \end{cases}$$

$$= 1, 2, 4, 5, 7, 9, 15, 17, 31, 33, \ldots \qquad k{\geq}3 \text{ A086341}$$

The diameter endpoints are unique for all $k$. In $k \leq 2$ they are path start to end. For $k \geq 3$ they are curve start to the far corner away from start and end.




For $k$ odd $\geq 3$ the diameter path is unique to the hanging square then 2 choices there. For $k$ even the path up the triangle can take horizontal and vertical steps in any order not going above the diagonal. The number of such paths A to B is the Catalan numbers (one of their many interpretations).

$$Catalan(n) = \frac{1}{n+1} \binom{2n}{n} = 1, 1, 2, 5, 14, 42, 132, \ldots \qquad \text{A000108}$$

The hanging square at D has 2 paths C to D whereas C–B just one in the triangle A–B, so count 2×. The curve height then gives

$$
\mathit{DiameterCount}_k = \begin{cases} 1 & \text{if } k \le 2 \\ 2 & \text{if } k \text{ odd} \ge 3 \\ \dfrac{2}{h-1}\dbinom{2h-4}{h-2} & \text{if } k \text{ even} \ge 4, \text{ with } h = 2^{k/2} \end{cases}
$$
$$
= 1, 1, 1, 2, 4, 2, 264, 2, 5348880, \ldots
$$

The Wiener index is a measure of total distance between pairs of vertices in a graph.

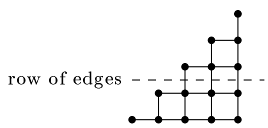
$$
\mathit{Wiener\ index} \;=\; \tfrac{1}{2} \sum_{\text{vertices } u,v} \mathit{distance}(u,v) \tag{131}
$$

Factor $\frac{1}{2}$ has the effect of taking distance between a pair $u, v$ in just one direction, not also its reverse $v, u$.

**Theorem 19.** *The Wiener index of the alternate paperfolding curve $k$ graph is*

$$
\begin{aligned}
W_k &= [\tfrac{1}{15}, \tfrac{11}{120}]\, 4^k . 2^{\lfloor k/2 \rfloor} \;+ [\tfrac{1}{2}, \tfrac{11}{24}]\, 4^k \\
&\quad + [\tfrac{5}{6}, 1]\, 2^k . 2^{\lfloor k/2 \rfloor} \;+ [2, \tfrac{25}{12}]\, 2^k \;- [\tfrac{12}{5}, \tfrac{131}{30}]\, 2^{\lfloor k/2 \rfloor} \\
&= 1, 4, 20, 65, 272, 1022, 4768, 20780, \ldots
\end{aligned}
$$

*Proof.* For even $k$, it's convenient to start from a whole triangle and adjust for absent parts.




triangle grid, height $n$=5 vertices

Shortest paths going between vertices within this triangle are horizontal and vertical stair-step so are $x$ plus $y$ offsets.

A row of edges is crossed by paths between vertices above and below. The number of vertices above is a triangular number, and the rest below. The number of paths crossing the row is the product. Columns of edges the same by symmetry.

$$
\begin{aligned}
T(n) &= \sum_{j=1}^{n} j \;= \tfrac{1}{2} n\,(n+1) \qquad \text{triangular numbers} \qquad (132) \\
&= 0, 1, 3, 6, 10, 15, 21, \ldots \qquad \text{A000217}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Wtriangle}(n) &= 2 \sum_{j=1}^{n-1} T(j) . \big(T(n) - T(j)\big) \\
&= \tfrac{1}{30} (n-1)\, n\, (n+1)\, (n+2)\, (2n+1) \\
&= 0, 0, 4, 28, 108, 308, 728, 1512, \ldots \qquad n \ge 2 \;\; 2 \times \text{A067056}
\end{aligned}
$$

The paperfolding curve does not have the top-most vertex. The total path lengths to it to be subtracted are distance down to each row then lengths along are the triangular numbers sums.

$$\mathit{WtriangleTop}(n) = \sum_{j=2}^{n} j\,(j{-}1) + T(j{-}1) \quad = \tfrac{1}{2}(n{-}1)\,n\,(n{+}1)$$
$$= 0, 0, 3, 12, 30, 60, 105, 168, \ldots \qquad \text{A027480}$$

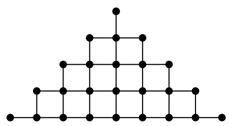
Along the $x$ axis every second edge is absent in the paperfolding curve. Affected paths are between vertices on the axis and each must be 2 longer to go up and along the $y{=}1$ row. Affected paths are those across an odd to even $x$. So for $n$ vertices

$$\mathit{Wextra}(n) = \sum_{x=0}^{n-2} \sum_{x_2=x+2-[0,1]_x}^{n-1} 2 \quad = (n{-}1)^2 - [1,0]$$
$$= 0, 0, 0, 4, 8, 16, 24, 36, \ldots \qquad \text{A137932}$$

Every second edge of the right-most vertical is absent too. The net Wiener index of the even case curve height $n$ is then

$$\begin{aligned}
\mathit{Weven}(n) &= \mathit{Wtriangle}(n) - \mathit{WtriangleTop}(n) \\
&\quad + \mathit{Wextra}(n) + \mathit{Wextra}(n{-}1) \qquad \text{for } n \ge 1 \\
&= \tfrac{1}{30}\,n\,(2n^4 + 7n^3 - 8n^2 + 47n - 120) \\
&= 0, 1, 20, 90, 272, 663, 1404, \ldots \quad n{\ge}1
\end{aligned}$$

For odd $k$ a similar calculation can be made starting from a pyramid of height $n$. Its total number of vertices is $n^2$. Rows of edges have a square number of vertices above and the rest below. Columns up to the middle have a triangular number of vertices to the left and the rest to the right. By symmetry the columns after the middle are the same. The paperfolding curve has the right-most $x$ axis vertex absent, and every second edge of the $x$ axis absent.



pyramid
height $n{=}5$

$$\begin{aligned}
\mathit{Wpyramid}(n) &= \sum_{j=1}^{n-1} j^2\,(n^2 - j^2) \quad + 2\sum_{j=1}^{n-1} T(j).(n^2 - T(j)) \\
&= \tfrac{1}{30}\,(n{-}1)\,n\,(n{+}1)\,(11n^2{+}1) \\
&= 0, 0, 9, 80, 354, 1104, 2779, 6048, \ldots
\end{aligned}$$

$$\begin{aligned}
\mathit{WpyramidEnd}(n) &= \sum_{j=2}^{n} j\,(j{-}1) + T(j{-}1) \\
&\quad + \sum_{j=1}^{n-1} j\,(2n{-}1{-}j) + T(j{-}1) \\
&= \tfrac{1}{6}\,(n{-}1)\,n\,(8n{-}1) \\
&= 0, 0, 5, 23, 62, 130, 235, 385, \ldots
\end{aligned}$$

$$\begin{aligned}
\mathit{Wodd}(n) = {} & \mathit{Wpyramid}(n) - \mathit{WpyramidEnd}(n) \quad n \ge 1 \\
& + \mathit{Wextra}(2n{-}2)
\end{aligned}$$

$$= \tfrac{1}{30}(n-1)(11n^4+11n^3-39n^2+126n-240)$$
$$= 0, 4, 65, 316, 1022, 2624, 5783, \ldots \qquad n\ge 1$$

With height of the paperfolding curve $2^{\lfloor k/2\rfloor}+1$ many vertices,

$$W_k = \big[\,\mathit{Weven}(2^{\lfloor k/2\rfloor}+1),\ \mathit{Wodd}(2^{\lfloor k/2\rfloor}+1)\big] \qquad \square$$

The Wiener index can be used for mean path length between pairs of vertices. Such a mean is usually taken over vertex pairs in one direction (like the Wiener index) and excluding a vertex to itself, so pairs are binomial

$$\mathit{Pairs}_k = \binom{P_k}{2} = 1, 3, 10, 28, 91, 276, 946, \ldots$$
$$\frac{W_k}{\mathit{Pairs}_k} = 1,\ \tfrac{4}{3},\ 2,\ \tfrac{65}{28},\ \tfrac{272}{91},\ \tfrac{511}{138}, \ldots$$

This mean path length can be expressed as a fraction of *Diameter* which is the longest path.

$$\frac{W_k}{\mathit{Pairs}_k.\mathit{Diameter}_k} \to \frac{4}{15} = .2666\ldots \text{ if } k \text{ even} \tag{133}$$
$$\to \frac{11}{30} = .3666\ldots \text{ if } k \text{ odd} \qquad \text{A040006}$$

These are the same means as in the whole triangle or whole pyramid graphs respectively, essentially since the modifications made for the alternate paperfolding are only linear out of quadratic total paths.

A geometric distance calculation can be made to give $x$ and $y$ distance between two points $n$ and $m$ in curve $k$ (all of 0 to $2^k$, not just the distinct locations) . Like the Wiener index, the sum here is distance $n$ to $m$ and not also back the other way. So for example $WV_0 = 1$ is the single segment $z{=}0$ to $z{=}1$.

$$WV_k = \sum_{n=0}^{2^k}\ \sum_{m=n+1}^{2^k} \mathit{ReImDiff}\big(\mathit{point}(n), \mathit{point}(m)\big) \tag{134}$$
$$\mathit{ReImDiff}(z_1, z_1) = \big|\mathrm{Re}(z_1-z_2)\big| + \big|\mathrm{Im}(z_1-z_2)\big|i$$
$$= \Big(\big[\tfrac{2}{15}+\tfrac{2}{15}i,\ \tfrac{7}{30}+\tfrac{2}{15}i\big]4^k + \big[\tfrac{2}{3}+\tfrac{1}{3}i,\ \tfrac{2}{3}+\tfrac{2}{3}i\big]2^k + \big[\tfrac{1}{5}-\tfrac{7}{15}i,\ -\tfrac{4}{15}+\tfrac{2}{15}i\big]\Big)\,2^{\lfloor k/2\rfloor} \tag{135}$$
$$= 1,\ 2{+}2i,\ 10{+}6i,\ 40{+}28i,\ 180{+}156i,\ 1040{+}632i, \ldots$$

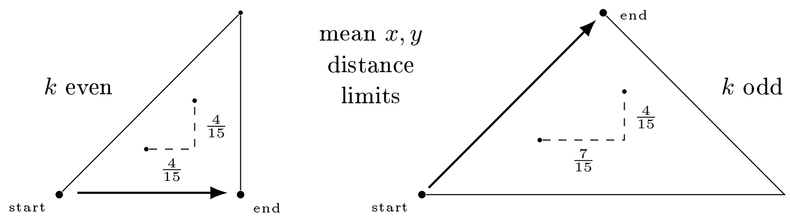
The sum at (134) is calculated using the triangular shape of level $k$. Points on the boundary lines are single-visited and points inside are double-visited for total $2^k{+}1$. Working through those sums and locations gives the power form (135). (It can be convenient to start with a triangle or pyramid of height $n$ like above then put in $n = 2^{\lfloor k/2\rfloor}$.)

The limit mean $x, y$ distances between points is different in $k$ even or odd. Using height $2^{\lfloor k/2\rfloor}$ as a scale factor,

$$WVpairs_k = \binom{2^k+1}{2} = 1, 3, 10, 36, 136, 528, 2080, \ldots \qquad \text{A007582}$$

$$\frac{WV_k}{WVpairs_k . 2^{\lfloor k/2 \rfloor}} \to \frac{4}{15} + \frac{4}{15}i \ = .2666\ldots + .2666\ldots i \quad \text{if } k \text{ even}$$

$$\to \frac{7}{15} + \frac{4}{15}i \ = .4666\ldots + .2666\ldots i \quad \text{if } k \text{ odd}$$




These limits are the same as two points chosen in the respective triangle or pyramid, ignoring single or double visited. The single-visited points in the curve are only $2^{k/2}$ out of the $2^k$ total, so do not affect the limit.

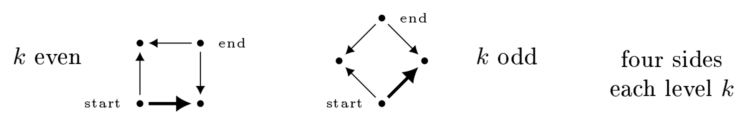
The $k$ even Re and Im limits are the same by symmetry and per the coefficient of the high $4^k$ term in (135). The other terms are different Re, Im since there is no top-most point $2^{k/2}.(1+i)$. With *Leading* for Re − Im difference,

$$\begin{aligned} Leading(WV_k) &= (\tfrac{1}{3}2^k + \tfrac{2}{3})\,2^{k/2} && k \text{ even} \\ &= -\sum_{n=0}^{2^k} Leading\Big(2^{k/2}\,(1+i) \ - point(n)\Big) \\ &= 1, 4, 24, 176, 1376, 10944, \ldots \quad k \text{ even} && \text{A103334} \end{aligned}$$

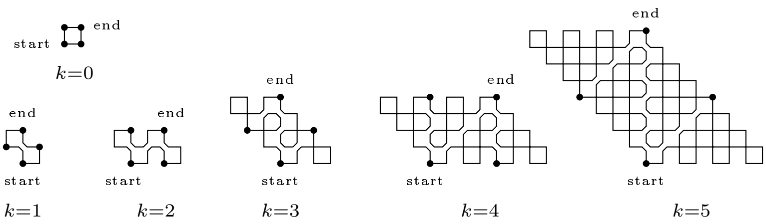
# 12 Twin Alternate

Two copies of the alternate paperfolding curve can be placed back to back, start to end. Call this a twin alternate. The sides touching are either the $x$ axis or the $x{=}y$ diagonal according as even or odd level. In both cases they mesh perfectly.

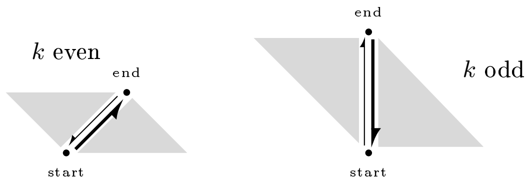
It's convenient to number twin alternates starting $k{=}0$ as a unit square, so that level $k$ is four curves level $k$, which is two curves $k{+}1$. This numbering gives $2^k$ unit squares inside, and all expansions have non-overlapping segments.




The sub-curve shown thick is the plain curve in its normal direction, first segment East. The initial levels are then




In terms of two back to back curves the shape is




The expand rule from figure 2 holds for the twin alternate, with suitable rotation. However the mirror image in that rule means whereas the inside was on the left of the segments going anti-clockwise, after that mirror image the same segments go clockwise and the inside is on the right.

Each twin alternate level is a subset of the preceding. This can be seen in sub-curves of the sides

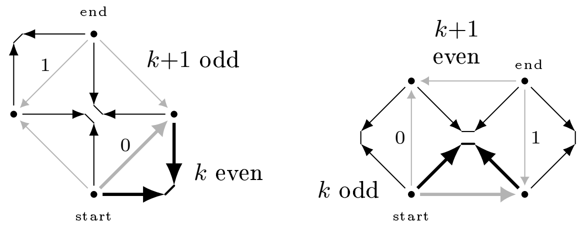



Figure 20: $k$+1 curves and $k$ sub-curves

For $k$+1 odd its sub-curves are two $k$ even twin alternates. The second copy attaches up at the North West corner. Similarly $k$+1 even has sub-curves two $k$ odd. The second copy attaches on the right at the East corner.

The unit squares inside the twin alternate can be numbered according to these copies. A given $k$ is copied either North West or East according as $k$ even or odd. The result is a kind of Z-order replication progressing away from the initial unit square at the origin. Each alternate bit of $n$ goes either $i{-}1$ North West or 2 East. The bottom left corner of each unit square is then

$$
\begin{aligned}
\mathit{TSquare}(n) &= (i{-}1)x + 2y \quad = 2y{-}x + xi \\
&= 0,\ -1{+}i,\ 2,\ 1{+}i,\ -2{+}2i,\ -3{+}3i,\ 2i,\ -1{+}3i, \ldots \\
x &= \text{even position bits of } n \qquad (136) \\
&= 0,1,0,1,2,3,2,3,0,\ldots \qquad \text{A059905} \\
y &= \text{odd position bits of } n \\
&= 0,0,1,1,0,0,1,1,2,\ldots \qquad \text{A059906}
\end{aligned}
$$

The boundary follows from the 4 sub-curves in a square. Two of each side left and right are the twin alternate boundary so

$$TB_k = 2B_k \quad = 4, 8, 16, 24, 40, 56, 88, \ldots \qquad 4\times \text{A027383}$$
$$TBQ_k = 2BQ_k = 4, 6, 10, 14, 22, 30, 46, \ldots \qquad \text{A027383}$$

The number of distinct points is the parallelogram shape extents, less one at each far corner

$$\begin{aligned} TP_k &= (2^{\lfloor \frac{k+1}{2} \rfloor}+1)(2^{\lfloor \frac{k+2}{2} \rfloor}+1) \ - 2 \qquad (137) \\ &= 2.2^k + [3,4]2^{\lfloor k/2 \rfloor} - 1 \\ &= 4, 7, 13, 23, 43, 79, 151, 287, \ldots \qquad \tfrac{1}{2}\,\text{A183977} \end{aligned}$$

**Theorem 20.** *The diameter of twin alternate $k$ as a graph is*

$$\begin{aligned} Tdiameter_k &= [3,4]\, 2^{\lceil k/2 \rceil} - 2 \\ &= \tfrac{1}{2} B_{k+1} \quad = BQ_{k+1} - 1 \qquad (138) \\ &= 2, 4, 6, 10, 14, 22, 30, \ldots \qquad \text{A027383} \end{aligned}$$

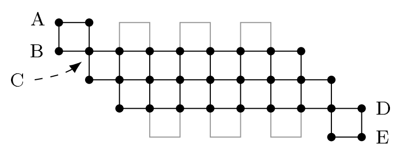
*which is attained between the top left and bottom right corners only. The number of paths of this length between those vertices is binomials,*

$$TdiameterCount_k = \begin{cases} 2 & \textit{if } k{=}0 \\ 4\dbinom{4h{-}4}{h{-}2}\dfrac{4h^2+4h+6}{3h\,(3h{-}1)} & \textit{if } k \textit{ even} \geq 2 \\ 4\dbinom{3h{-}4}{h{-}2}\dfrac{h^2+h+6}{2h\,(2h{-}1)} & \textit{if } k \textit{ odd} \end{cases}$$
$$\textit{where } h = 2^{\lceil k/2 \rceil}$$
$$= 2, 4, 4, 52, 172, 50388, 802620, \ldots$$

*Proof.* Since the grid is convex, distances between vertices are the geometric stair-step, except within the top or bottom rows where there are absent edges. Those absent edges add +2 to relevant paths but those distances are still smaller than top left to bottom right.

The correspondence to boundary length at (138) is since the lengths across and diagonally up go as the boundaries of the component paperfolding curves.

Diameter paths go A–E through a grid like




It's convenient to count paths across the parallelogram grid B to D. The hanging square at A has 2 paths A–C whereas 1 path B–C, so 2× the parallelogram paths. Likewise at E for total 4×.

Number the parallelogram rows 0 to $n$ and columns likewise 0 to $n$ plus possible $w$ many additional columns up to the edges after $w$ (drawn thick).

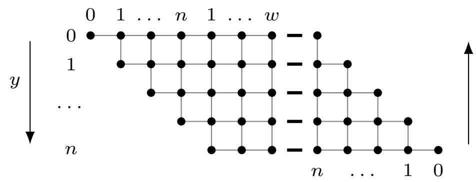



Figure 21: parallelogram numbering for paths

Paths can be counted by the crossings of the edges after $w$. The number of paths to the vertices in column $w$ is given by entries of the Catalan triangle. Likewise from the end to the column after $w$. For distance $x$ horizontal to a column and $y$ down (or up), the Catalan triangle is

$$cat(x,y) = \binom{x+y}{y}\,\frac{x-y+1}{x+1} \qquad \text{A009766}$$

Total paths are then products of the counts to each side of each thick edge

$$\begin{aligned} C(n,w) &= \sum_{y=0}^{n} \; cat(n+w,y)\,.\,cat(n,u) \qquad \text{where } u=n-y \\ &= \sum_{y=0}^{n} \binom{n+w+y}{y}\frac{n+w-y+1}{n+w+1}\binom{n+u}{u}\frac{n-u+1}{n+1} \\ &= \sum_{y=0}^{n} \binom{n+w+y}{y}\frac{u+w+1}{n+w+1}\binom{n+u}{u}\frac{y+1}{n+1} \end{aligned} \tag{139}$$

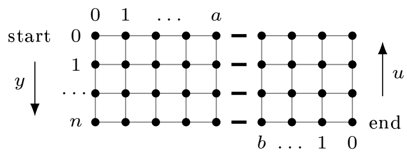
At (139) the *cat* factor part $n-y$ becomes $u$ and conversely $n-u$ becomes $y$. The resulting factors can be applied into the opposite binomial. $y/(n+1)$ reduces the $y$ binomial to $y-1$, and $u/(n+w+1)$ reduces the $u$ to $u-1$. The factors are $y+1$ and $u+w+1$ so leaves 1 or $w+1$ on the original binomial,

$$\begin{aligned} C(n,w) = \sum_{y=0}^{n} \quad & \left(\binom{n+w+y}{y-1} + \binom{n+w+y}{y}\frac{1}{n+w+1}\right) \\ & \cdot \left(\binom{n+u}{u-1} + \binom{n+u}{u}\frac{w+1}{n+1}\right) \end{aligned}$$

When factor $y{=}0$ is taken out, the resulting binomial has $y-1$ negative which is understood as binomial $=0$. Likewise $u{=}0$. If preferred those terms can be taken separately with indices running from $y{=}1$ and/or to $y=n-1$ as necessary. In any case the factors depend only on $n,w$ and can be taken out to leave sums of binomial products of the form,

$$\sum_{y=0}^{n}\binom{a+y}{y}\binom{b+u}{u} \;=\; \binom{a+b+n+1}{n}$$

This identity is crossings like figure 21 but of a full rectangle, so that the number of paths to the crossing column is binomial of distance $a$ left, $b$ right, and rows 0 to $n$. The binomials are how many ways to arrange the vertical steps among the total steps in each case.




paths across rectangular grid

Working through the terms and bringing them to common base $n$ in the binomial gives

$$C(n,w) = \binom{3n+w+1}{n} \frac{(n+w+2)(n+w+3) - 2n}{(2n+w+2)(2n+w+3)}$$

The binomial numerator runs $3n+w+1$ down to $2n+w+2$. The smallest two can cancel with the denominator of the factor if preferred, and which shows the result is of course an integer.

For the twin alternate, the relevant height and width are

$$n = h-2, \quad w = \begin{cases} h+1 & \text{if } k \text{ even} \\ 1 & \text{if } k \text{ odd} \end{cases} \quad \text{where } h = 2^{\lceil k/2 \rceil}$$

$$\mathit{TdiameterCount}_k = 4\, C(n,w) \qquad k \geq 1 \qquad \square$$

As a remark, the parallelogram cross-products above are conceived for $w \geq 0$ but some similar calculation shows the formula holds for $w = -1, -2$ too, which is rows shortened, and Catalan triangle entries taken as 0 outside the triangle.

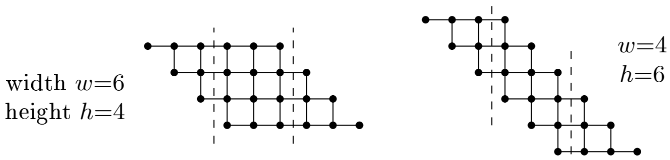
$w = -2$ gives factor $n\,(n-1)/(2n\,(2n+1))$ which can be incorporated into a reduced binomial,

$$C(n,-2) = \binom{3n-1}{n-2} = 1, 8, 55, 364, 2380, \dots \quad n \geq 2 \qquad \text{A013698}$$

**Theorem 21.** *The Wiener index of twin alternate $k$ as a graph is*

$$\begin{aligned} TW_k &= [\tfrac{43}{20}, \tfrac{16}{5}]\, 4^k.2^{\lfloor k/2 \rfloor} + 8.4^k + \tfrac{1}{12}[31, 32]\, 2^k.2^{\lfloor k/2 \rfloor} \\ &\quad + [2, -2]\, 2^k - \tfrac{1}{15}[101, 92]\, 2^{\lfloor k/2 \rfloor} \\ &= 8, 40, 212, 936, 4420, 21552, 104616, \dots \end{aligned}$$

*Proof.* Similar to the curve Wiener index in theorem 19, it's convenient to start from an equivalent size full grid and subtract. For the twin alternate this is a parallelogram grid. Consider width $w$ and height $h$ many vertices.




All shortest paths go by stair steps. Total paths can be calculated by crossings of vertical and horizontal edges, with triangular numbers $T$ from (132).

$$\begin{aligned}
Wpar(w,h) = {} & \sum_{y=1}^{h-1} yw\,(h-y)w && \text{rows of vertical edges} && (140)\\
& + 2\sum_{x=1}^{m-1} T(x)\big(wh - T(x)\big) && \text{columns slope} && (141)\\
& + \sum_{x=1}^{M-m-1} \big(T(m)+xm\big)\big(wh - (T(m)+xm)\big) && \text{middle} && (142)\\
& \text{where } m = \min(w,h),\ M = \max(w,h)
\end{aligned}$$

Vertical edges in a row are crossed by paths going between the vertices above and the vertices below. For $y$ vertex rows below there are $yw$ vertices below and $(h-y)w$ above, per (140).

Similarly horizontal edges in columns. Columns in the sloping part have a triangular number of vertices to the left and the rest to the right. The parallelogram is symmetric so the same in the sloping part on the right, for (141). The middle columns have the triangle sloping part plus some full columns of $h$ vertices to the left, for (142). The size of the sloping part is $\min(w,h)$, and the middle extends to $\max(w,h)$.

Working through the sums, and taking $w{=}1$ to mean no edges,

$$Wpar(w,h) = \tfrac{1}{60}\begin{cases} 0 & \text{if } w{\le}1\\ 5whm\Big(2M(M{+}h) + m^2 - 3\Big) - 10w^2h \\ \quad - (m{-}2)(m{-}1)m(m{+}1)(m{+}2) & \text{if } w{\ge}2 \end{cases}$$
$$\text{where } m = \min(w,h),\ M = \max(w,h)$$

$$\begin{aligned}
h{=}1 &= 0,\ 1,\ 4,\ 10,\ 20,\ 35,\ 56,\ldots && \text{path A000292}\\
h{=}2 &= 0,10,28,\ 60,110,182,280,\ldots && w{\ge}2 \text{ A006331}\\
h{=}3 &= 0,35,88,176,308,493,740,\ldots
\end{aligned}$$

The twin alternate does not have the degree-1 bottom right vertex. Its contribution to *Wpar* is distance $2y$ to get to each row which sums to a triangular number, times the $w$ vertices in each row. Then across each row a triangular number sum of distances.

$$\begin{aligned}
WparEnd(w,h) &= 2wT(h{-}1) + hT(w{-}1) \qquad w{\ge}2\\
&= \tfrac{1}{2}wh(w + 2h - 3)
\end{aligned}$$

The twin alternate also does not have the degree-1 top left vertex. By symmetry its contribution is the same *WparEnd*. Subtracting both removes the path between them twice. This is length $(w{-}1) + 2(h{-}1)$ so add that back.

The twin alternate also has every second edge on the bottom row absent. Extra distance caused by this is *Wextra* from theorem 19, provided there is a full row above to go along, which means $h \ge 3$. Likewise the top row edges.

$$\begin{aligned}
TWpar(w,h) = {} & Wpar(w,h)\ + 2\,Wextra(w{-}1) \qquad w{\ge}2,\ h{\ge}3 && (143)\\
& - 2\,WparEnd(w,h) + (w{-}1) + 2(h{-}1)\\
TW_k = {} & TWpar\Big(2^{\left\lceil\frac{k+1}{2}\right\rceil}{+}1,\ 2^{\left\lfloor\frac{k+1}{2}\right\rfloor}{+}1\Big) && (144)
\end{aligned}$$

$k{=}0$ at (144) is $w{=}3, h{=}2$. It does not have $h \geq 3$ but its $Wextra(w{-}1) = 0$ so formula (143) holds there too. $\square$

The twin alternate has $TP$ many vertices (137) and the number of pairs of distinct vertices is a binomial.

$$Tpairs_k = \binom{TP_k}{2} = 6, 21, 78, 253, 903, 3081, \ldots$$

Mean path length between such a pair is then

$$\frac{TW_k}{Tpairs_k} = \frac{4}{3}, \frac{40}{21}, \frac{106}{39}, \frac{936}{253}, \frac{4420}{903}, \ldots$$

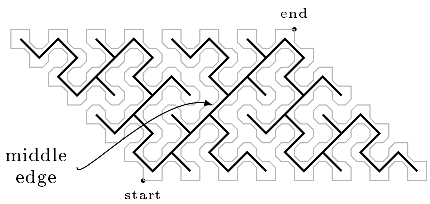
This mean path length can be expressed as a fraction of $Tdiameter$ which is the longest path. Limits follow from the coefficients of the highest powers in each term.

$$\begin{aligned} \frac{TW_k}{Tpairs_k . Tdiameter_k} &\to \frac{43}{160} = \frac{129}{480} = .26875 \quad \text{if } k \text{ even} \\ &\to \frac{4}{15} = \frac{128}{480} = .2666\ldots \quad \text{if } k \text{ odd} \end{aligned} \tag{145}$$

The odd limit is the same as the even limit of the plain curve at (133). Two even curves back-to-back make an odd twin alternate.

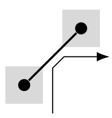
## 12.1 Twin Alternate Area Tree

When the corners of the twin alternate curve are chamfered off, the unit squares enclosed inside the curve are connected through the resulting gaps. Call this an area tree.




twin alternate $k{=}6$
area tree

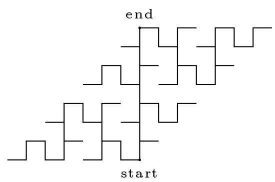
An equivalent definition is to connect unit squares which are on the left of consecutive curve segments. When the curve turns to the right the unit squares on the left of the segments are distinct. A turn is always left or right (never straight ahead) so those connections are at corners of the squares



edge between unit squares
on left sides of consecutive
curve segments

Mandelbrot [17] conceives these area connections as rivers. The curve is the riverbank going upstream until reaching a source and then back down along the other side of the river and tributaries. For a closed curve like the twin alternate the squares inside the curve form entirely inland waterways. For area enclosed on the outside of a curve (or any unclosed curve), the rivers flow eventually to the “sea” outside.

It’s convenient to draw the tree turned $+45°$ using factor $b/2$. The start of the curve at the origin can be taken as the root. Successive levels extend by copying.

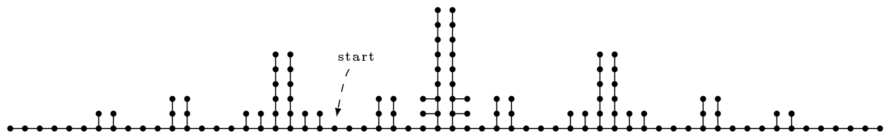



$k$=6 twin alternate area tree

$$\begin{aligned} \mathit{TAVertexToZ}(n) &= \mathit{TSquare}(n)\,.\,\tfrac{b}{2} \\ &= y{-}x + yi \qquad\qquad \text{Z-order } x,y \text{ per (136)} \end{aligned} \tag{146}$$

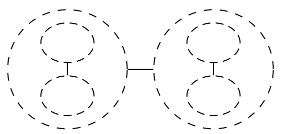
In this layout, edges are horizontal and vertical. From the Z-order point numbering, the negative $x$ axis is points $n = \mathit{Xnum}(x)$. The leading diagonal $x{=}y$ North East is points $n = \mathit{Gnum}(y)$ (the same as the curve in fact).

The area tree is quite sparse when straightened to a line of its diameter.




$k$=7 twin alternate area tree straightened to branches off the diameter

The twin alternate curve is symmetric in 180° rotation so the squares connected by the new middle edge are at equivalent positions in each half. So the middle edge is the centre and the two halves are isomorphic. likewise each half has isomorphic halves across its centre edge, etc, all the way down to a single vertex.

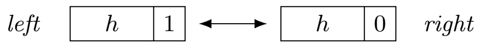


halves identical across centre edge,
each half likewise identical across centre edge

A tree with this recursive isomorphic halves property always has $2^k$ vertices. Various such trees can be made by choosing which vertex of each half to connect. The connection can be between the same vertex in each half, like the twin alternate area tree has, or between any two of equal eccentricity.

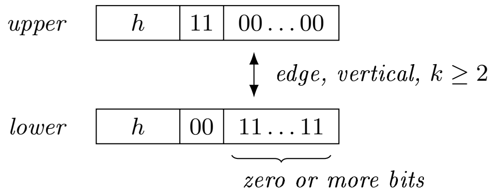
A straight-line path of $2^k$ vertices is trivially such a tree and is the only such tree for $k \le 2$. For $k$=3 there is the 8-path and one non-path. The twin alternate area tree is the 8-path. For $k \ge 4$ the twin alternate area tree is one among several trees.

**Theorem 22.** *Label vertices of twin alternate area tree $k$ with point numbers $n$ per TAVertexToZ at* (146). *A horizontal edge is between a given $n$ and least significant bit toggled,*




*edge, horizontal, $k \geq 1$*

*A vertical edge is between two $n$ points of the following form*




*Proof.* In twin alternate $k$ suppose the unit squares at corners are point numbers $s_k$ start, $e_k$ end, and $a_k, c_k$ opposite corners. Corner $c$ is at the connection to the copy for the next level and $a$ is the other corner. Twin alternate curve $k+1$ consists of two copies of $k$

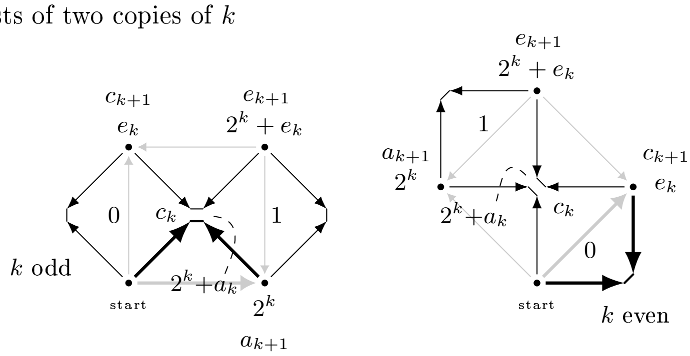



Figure 22: corner vertex numbers

The vertex numbers in each part 1 have $2^k$ added which is a high 1-bit. The start is $s_k = 0$ always. The end point $e$ is in the 1-part each time so add $2^k$ always for all 1-bits.

Corner $a_{k+1}$ is the start in part 1 each time. Corner $c$ is $e$ in part 0 each time. So

$$e_k = 2^k - 1 \qquad k \text{ many 1-bits}$$

$$a_k = \begin{cases} 0 & \text{if } k{=}0 \\ 2^{k-1} & \text{if } k \geq 1 \end{cases}$$

$$c_k = \begin{cases} 0 & \text{if } k{=}0 \\ e_{k-1} & \text{if } k \geq 1 \end{cases} = \begin{cases} 0 & \text{if } k{=}0 \\ 2^{k-1} - 1 & \text{if } k \geq 1 \end{cases}$$

The connection in level $k+1$ is from $c$ in part 0 to $a$ in part 1 which is $c_k$ to $2^k + a_k$. For $k{=}0$ this is $c_0 = 0$ to $a_0 + 2^0 = 1$. For $k \geq 1$ it is $2^{k-1} - 1$ to $3.2^{k-1}$ per the theorem. Expansions replicate the connections of previous levels.

The direction of the edges follows by $k{=}0$ explicitly, and then $k \geq 1$ by considering the direction of the last segment of the sub-curves, which by the unfoldings is down for $k$ even and up for $k$ odd.

Or the bit patterns give the direction too. $k{=}0$ is a change of the low bit $x$ only so horizontal. $k \geq 1$ increments both $x$ and $y$ going *lower* to *upper*, so that *TAVertexToZ* has real part $y{-}x$ unchanged so vertical. □

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1111 & — & 1110 & & 1011 & — & 1010 \\
 & & & & & & | & & | & & | & & \\
 & & & & 1101 & — & 1100 & & 1001 & — & 1000 & & \\
 & & & & & & | & & & & & & \\
 & & 0111 & — & 0110 & & 0011 & — & 0010 & & & & \\
 & & | & & | & & | & & & & & & \\
0101 & — & 0100 & & 0001 & — & 0000 & & & & & & \\
 & & & & & & \text{start} & & & & & &
\end{array}
$$

$k$=4 twin alternate area tree
Z-order vertex bit patterns

These bit patterns can be used to construct the tree for computer calculation, including drawing it in sheared Z-order shape by following the edge directions. For a whole tree it's probably most efficient to make edges *upper* to *lower* by a loop over bit patterns. If going by $n$ or an isolated part of the tree then horizontal edges are always simply the low bit toggled, and some bit-twiddling on $n$ can identify when $n$ has an edge to an upper and/or lower vertex,

TAVertexToLower($n$) $\quad n \geq 0$
 $\quad mask = \mathrm{BITXOR}(n, n{-}1)$ — lowest 1 and all bits below
 $\quad$**if** $\mathrm{BITAND}(n, mask{+}1) \neq 0$ — bit above lowest 1
 $\quad$**then** $n$ is an *upper* and has edge downwards to
 $\qquad lower = n - mask - 2$

TAVertexToUpper($n$)
 $\quad mask = \mathrm{BITXOR}(n, n{+}1)$ — lowest 0 and all bits below
 $\quad$**if** $\mathrm{BITAND}(n, mask{+}1) = 0$ — bit above lowest 0
 $\quad$**then** $n$ is a *lower* and has edge upwards to
 $\qquad upper = n + mask + 2$

Direction upper or lower from $n$ can also be a parameter $0$ = go down, or $1$ = go up. A possible low run of that bit is skipped and the next run (opposite bit) must be more than a single bit long. *mask* is the same as above but applied with an XOR to toggle the low run and next two bits.

TAVertexToOther($n$, $direction = 0$ down or $1$ up)
 $\quad transitions = \mathrm{BITXOR}(n,\ 2n{+}direction)$
 $\quad mask = \mathrm{BITXOR}(transitions,\ transitions{-}1)$
 $\quad$**if** $\mathrm{BITAND}(transitions,\ mask{+}1) = 0$
 $\quad$**then** there is an edge to
 $\qquad \mathrm{BITXOR}(n,\ 2\,mask + 1)$

Predicates for when a vertex $n$ has an upper or lower neighbour follow from the bit patterns. They are simply a test of bit above lowest 1-bit or 0-bit.

$$
\begin{aligned}
&\mathit{TAVertexToUpperPred}_{\infty}(n) = \big(\mathit{BitAboveLowestZero}(n) = 0\big) && (147)\\
&\qquad = 1,1,0,1,1,0,0,1,1,1,0,\ldots && \text{A014577}\\
&\mathit{TAVertexToLowerPred}(n) = \big(n{\geq}1 \text{ and } \mathit{BitAboveLowestOne}(n) = 1\big)
\end{aligned}
$$

Going to lower always reduces $n$ so the same in a tree level $k$ or tree continued infinitely. Going to upper increases $n$ and (147) is for the tree continued infinitely.

**Theorem 23.** *The number of vertices of degree 0, 1, 2 or 3 in twin alternate area tree k are*

$$TADegCount(k,0) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k \geq 1 \end{cases} \qquad \text{A000007}$$

$$TADegCount(k,1) = \begin{cases} 0,2,2 & \text{if } k = 0 \text{ to } 2 \\ 2^{k-2} & \text{if } k \geq 3 \end{cases}$$
$$= 0,2,2,2,4,8,16,32,64,128,\ldots$$

$$TADegCount(k,2) = \begin{cases} 0,0,2 & \text{if } k = 0 \text{ to } 2 \\ 2^{k-1}+2 & \text{if } k \geq 3 \end{cases}$$
$$= 0,0,2,6,10,18,34,66,130,258,\ldots \qquad k{\geq}3 \text{ A052548}$$

$$TADegCount(k,3) = \begin{cases} 0 & \text{if } k \leq 2 \\ 2^{k-2}-2 & \text{if } k \geq 3 \end{cases}$$
$$= 0,0,0,0,2,6,14,30,62,126,\ldots \qquad k{\geq}3 \text{ A000918}$$

*Proof.* The degree of the connecting vertices $c_k$ and $a_k$ are the same by symmetry. The degree follows either from the curve ends which meet there, or from edges by the bit patterns. In either case for $k \geq 3$ they are degree 2.

The connection increases them to degree 3 and leaves other vertices replicated. So

$$\begin{aligned} TADegCount(k{+}1,1) &= 2\,TADegCount(k,1) \qquad k \geq 3 \\ TADegCount(k{+}1,2) &= 2\,TADegCount(k,2) - 2 \\ TADegCount(k{+}1,3) &= 2\,TADegCount(k,3) + 2 \end{aligned}$$

□

*Second Proof of Theorem 23.* Vertex degrees can also be counted from the bit patterns of theorem 22.

For $k \geq 1$ every vertex can toggle its low bit for the left to right edge so degree $\geq 1$. $n$ is only one of these left or right so degree cannot be 4, only at most 3.

Degree-1 vertices those $n$ which are neither *upper* nor *lower* forms. For $k \geq 3$, an $n$ with one or more trailing 1-bits is not *lower* when entirely 11…11 or 011…11, but they are both *upper* so not degree 1. The other not *lower* are …1011…11 and must have only one trailing 1-bit to avoid being *upper*, thus $n \equiv 5 \bmod 8$. Similarly $n$ with trailing 0-bits giving only $n \equiv 2 \bmod 8$.

$$\begin{aligned} TADegCount(k,1) &= \text{count } n \equiv 2,5 \bmod 8 \\ &= 2.2^{k-3} \qquad k \geq 3 \end{aligned}$$

Degree-2 vertices are *upper* but not *lower* or vice-versa. For $k \geq 3$, an $n =$ …0011…11 with $\geq 1$ trailing 1-bits is a *lower* and to avoid being an upper must be just one trailing 1-bit so …001. Conversely, an entire 11…11 or 011…11 is not *lower* but is *upper*. Other non-*lower* are …1011…11 is and if $\geq 2$ trailing 1-bits then it is an *upper*. With $h$ many high bits for the latter, and $n$ with trailing 0 bits treated the same flipped,

$$TADegCount(k,2) = 2\left(2^{k-3} + 2 + \sum_{h=0}^{k-4} 2^h\right) \qquad k \geq 3$$

Degree-3 vertices are those $n$ which are both *upper* and *lower* forms. Even $n = 1100\ldots00$ is an *upper*. It is also a *lower* only in no trailing 1s form 00, so must have $\geq 2$ trailing 0-bits. Similarly odd $n = 0011\ldots11$ must have $\geq 2$ trailing 1-bits.

$$\textit{DegCount3pred}_k(n) \text{ when } n = \begin{array}{|c|c|c|} \hline \cdots & \begin{array}{c}11\\00\end{array} & \underbrace{\begin{array}{c}00\cdots00\\11\cdots11\end{array}}_{\geq 2 \text{ bits}} \\ \hline \end{array}$$

The bits above these endings are arbitrary so with $h$ many such bits,

$$TADegCount(k,3) = 2\sum_{h=0}^{k-4} 2^h \qquad \square$$

A yet further approach for $TADegCount(k,1)$ is that a degree-1 vertex has all sides of the twin alternate unit square consecutive and so is on the left of a sequence of 3 left turns (88). There are $\textit{Turn3left}_{k+1}$ of these in each of the two curves making up a twin alternate. For $k\geq 2$ the start and end turn away so they do not form any further 3-left, so that

$$TADegCount(k,1) = 2\,\textit{Turn3left}_{k+1} \qquad k \geq 2$$

The total of all degrees is twice total edges in the usual way for any graph. The twin alternate has $2^k$ unit squares inside so the area tree has $2^k-1$ edges.

$$2^k-1 \;=\; \tfrac{1}{2}\sum_{d=0}^{3} d.\,TADegCount(k,d)$$

The degree of a given vertex $n$ follows from the bit patterns. An *upper* is a 1-bit above lowest 0. A *lower* is a 0-bit above lowest 1. But in both cases the bit above cannot be outside $k$ bits for a level $k$ tree.

$$\begin{aligned} TADegree_k(n) &= \big(1 \text{ if } k \geq 1\big) \\ &\qquad + TAVertexToLowerPred(n) \\ &\qquad + TAVertexToUpperPred_k(n) \\ TAVertexToUpperPred_k(n) &= TAVertexToUpperPred_\infty(n) \\ &\qquad \text{and } n \neq 2^k-1, 2^{k-1}-1 \end{aligned}$$

For the tree continued infinitely there is no restriction,

$$\begin{aligned} TADegree_\infty(n) &= \begin{cases} 2 & \text{if } n=0 \\ 2 - BitAboveLowestZero(n) \\ \quad + BitAboveLowestOne(n) & \text{if } n \geq 1 \end{cases} \\ &= 2,2,1,3,2,1,2,3,2,2,1,2,3,1,2,3,2,2,1,\ldots \end{aligned}$$

Using $BitAboveLowestZero(n) = BitAboveLowestOne(n{+}1)$ per the bit patterns in figure 4, this can be written as increment

$$TADegree_\infty(n) = \begin{cases} 2 & \text{if } n = 0 \\ 2 - dBitAboveLowestOne(n) & \text{if } n \geq 1 \end{cases}$$

where, for $n \geq 1$,

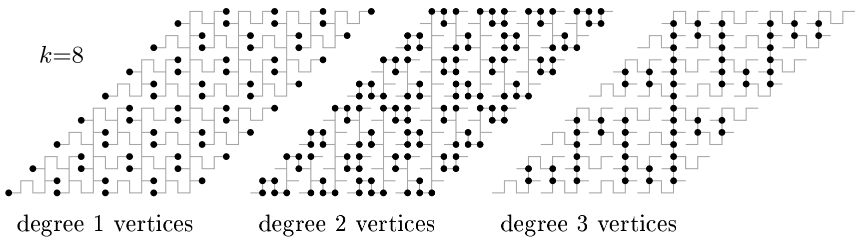
$$\begin{aligned} dBitAboveLowestOne(n) &= BitAboveLowestZero(n) - BitAboveLowestOne(n) \\ &= BitAboveLowestOne(n{+}1) - BitAboveLowestOne(n) \\ &= 0, 1, -1, 0, 1, 0, -1, 0, 0, 1, 0, -1, 1, 0, -1, 0, \dots \qquad n{\geq}1 \end{aligned}$$

Each 1 in $dBitAboveLowestOne$ is at $n \equiv 2 \,\text{or}\, 5 \bmod 8$ and are the degree-1 vertices from the second proof above. In between them is exactly one $-1$ for a degree-3 vertex. This can be seen firstly by $n \equiv 1 =$ binary 001 and $n \equiv 6 =$ binary 110 are always $dBitAboveLowestOne(n) = 0$. Then at $n \equiv 3 \bmod 4$ further bits to the lowest zero are some $x011\dots11$. This increments to $n \equiv 4 \bmod 4 = x100\dots00$. They have $dBitAboveLowestOne = x{-}1$ and $-x$ respectively so one is 0 and the other $-1$, hence exactly one $-1$.

The cases can be written out

$$dBitAboveLowestOne(n) = \begin{cases} -BitAboveLowestOne(n) & n{\equiv}0 \bmod 4 \\ BitAboveLowestOne(n{+}1) & n{\equiv}1 \bmod 4 \\ 1 - BitAboveLowestOne(n) & n{\equiv}2 \bmod 4 \\ BitAboveLowestOne(n{+}1) - 1 & n{\equiv}3 \bmod 4 \end{cases}$$

Locations of the various degree vertices in the tree can be illustrated,




The degree-3 locations are pairs of vertices in the same layout as the whole twin alternate area tree. That can be seen in an initial level such as a single pair in $k$=4, then replication of the tree replicates the pairs (and makes the degree-2 connection vertices into degree-3).

The connection argument for the degree counts above also gives counts of edges which have vertices of degree 1, 2 or 3 at each end. Twin alternate $k \geq 4$ has corner square degree-2 as above, and also the squares connected to that corner are degrees 2 and 3. When the degree-2 of each half are linked their adjacent edges change from 2,2 and 2,3 to 2,3 and 3,3 and the new edge is 3,3 also. So

$$\begin{aligned} TAEdgeCount(k, 2{,}2) &= 2\,TAEdgeCount(k{-}1, 2{,}2) - 2 \qquad k \geq 6 \\ TAEdgeCount(k, 3{,}3) &= 2\,TAEdgeCount(k{-}1, 3{,}3) + 3 \end{aligned}$$

$$TAEdgeCount(k,\, other) \;=\; 2\, TAEdgeCount(k{-}1,\, other)$$

With initial counts calculated explicitly,

$$TAEdgeCount(k,\, 1{,}1) = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases}$$

$$TAEdgeCount(k,\, 1{,}2) = \begin{cases} 0,0,2,2 & \text{if } k = 0 \text{ to } 3 \\ 2^{k-3} & \text{if } k \ge 4 \end{cases}$$
$$= 0,0,2,2,2,4,8,16,32,\ldots$$

$$TAEdgeCount(k,\, 2{,}2) = \begin{cases} 0,0,1,5,8 & \text{if } k = 0 \text{ to } 4 \\ 5.2^{k-4}+2 & \text{if } k \ge 5 \end{cases}$$
$$= 0,0,1,5,8,12,22,42,82,\ldots$$

$$TAEdgeCount(k,\, 1{,}3) = \begin{cases} 0,0,0,0 & \text{if } k = 0 \text{ to } 3 \\ 2^{k-3} & \text{if } k \ge 4 \end{cases}$$
$$= 0,0,0,0,2,4,8,16,32,\ldots$$

$$TAEdgeCount(k,\, 2{,}3) = \begin{cases} 0,0,0,0,2 & \text{if } k = 0 \text{ to } 4 \\ 2^{k-2} & \text{if } k \ge 5 \end{cases}$$
$$= 0,0,0,0,2,8,16,32,64,\ldots$$

$$TAEdgeCount(k,\, 3{,}3) = \begin{cases} 0,0,0,0,1 & \text{if } k = 0 \text{ to } 4 \\ 3.2^{k-4}-3 & \text{if } k \ge 5 \end{cases}$$
$$= 0,0,0,0,1,3,9,21,45,\ldots$$ $k{\ge}4$ A068156

Various graph-theoretic topological indices are based on sums over edges and their vertex degrees. Notice all the edge types (except the solitary 1,1 in $k$=1) go as a power $2^k$ so all contribute to a limit if taking a mean index over number of edges.

As an example, the second Zagreb index $M_2$ of Gutman and Trinajstić[12] is product of vertex degrees at the ends of each edge.

$$ZagrebM2(graph) = \sum_{edges} degree_1 \,.\, degree_2$$

$$TAZagrebM2_k = \sum_{d_1,d_2=1,2,3} d_1.d_2.\, TAEdgeCount(k,\, d_1,d_2)$$
$$= \begin{cases} 0,1,8,24,63 & \text{if } k = 0 \text{ to } 4 \\ 81\,.\,2^{k-4}-19 & \text{if } k \ge 5 \end{cases}$$
$$= 0,1,8,24,63,143,305,629,1277,\ldots$$

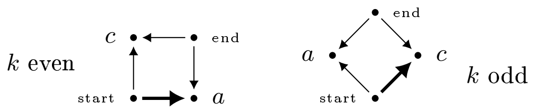
#### 12.1.1 Twin Alternate Area Tree Diameter, Wiener Index

**Theorem 24.** *The diameter of twin alternate area tree $k$ is*

$$TAdiameter_k = [7,10].2^{\lfloor k/2 \rfloor} - 2k - 7 \tag{148}$$
$$= 0,1,3,7,13,23,37,59,89,\ldots$$ A053599

*This is uniquely attained between the geometrically most distant vertices of the parallelogram shape.*

*Proof.* The twin alternate curve has connection corner $c$ and other corner $a$ on alternating sides of start and end in the manner of figure 22.




twin alternate $k$,
four sides
each level $k$

Let $TAcornerEcc_k$ be the eccentricity of vertex $a$ or $c$. The twin alternate is symmetric in 180° rotation so they have the same eccentricity. The claim will be this eccentricity is

$$\begin{aligned} TAcornerEcc_k &= [5,7].2^k - k - 5 \\ &= 0, 1, 3, 6, 11, 18, 29, 44, 67, \ldots \end{aligned} \tag{149}$$

Formulas (148) and (149) hold trivially for $k$=0 which is a single vertex so that $TAdiameter_0 = TAcornerEcc_0 = 0$.

Tree $k$+1 comprises two sub-trees as per figure 20. A path which goes between the two halves has length which is the eccentricity of the corner on both sides and an edge between,

$$TAdiameter_{k+1} = 2\,TAcornerEcc_k + 1$$

which is the theorem (148). Also per that formula this distance is greater than $TAdiameter_k$ which would be the maximum staying only in one half of the tree.

For $TAcornerEcc_{k+1}$, new corner $C$ in $k$+1 is shown in the following diagram.

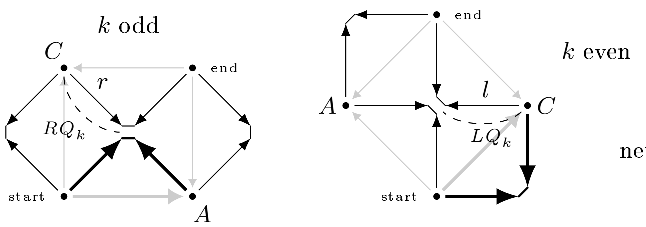



Figure 23:
new corner vertex
eccentricity

The longest path going from C to anything in the other tree half is distance to the middle connection vertex, the edge across, and eccentricity of the corner in the other half. Going to the middle requires following unit squares on the boundary of the curve sides marked $r$ or $l$ for the odd or even cases respectively. There are $RQ_k$ or $LQ_k$ boundary squares, so that many vertices, less 1 to count edges between them, plus 1 for the middle edge between the tree halves,

$$LRQ_k = \begin{cases} LQ_k & \text{if } k \text{ even} \\ RQ_k & \text{if } k \text{ odd} \end{cases} \tag{150}$$

$$= BQ_{k-1} \quad \text{for } k \ge 1 \tag{151}$$

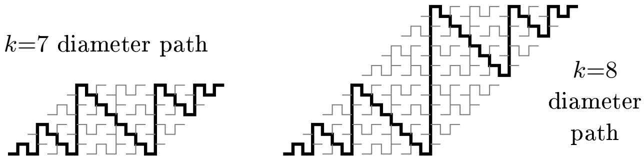
$$TAcornerEcc_{k+1} = TAcornerEcc_k + LRQ_k \tag{152}$$

$$TAcornerEcc_k = \sum_{j=0}^{k-1} LRQ_j$$

This is per (149), and comparing to the diameter formula is greater than $TAdiameter_k$ which is an upper bound on any path from $C$ to a vertex in the first tree half.

The corner eccentricity construction repeatedly goes to the far half sub-tree. Working through the expansions this is the far corner of the parallelogram shape.

As a remark, in figure 23 it can be seen the new $A$ corners are the same right or left side to the middle so that their eccentricity is the same as C, per the 180° symmetry noted above. ☐




The $LRQ$ count of boundary squares on alternating sides at (150) occurs below too. Form (151) as $BQ$ is the curve unfolding from (86),(87).

**Theorem 25.** *The height of twin alternate area tree $k$ (eccentricity of its start) is*

$$\begin{aligned} TAheight_k &= [4,6].2^{\lfloor k/2 \rfloor} - k - 4 && (153)\\ &= NumOpred_{k+2} \\ &= 0, 1, 2, 5, 8, 15, 22, 37, 52, \ldots && \text{A077866} \end{aligned}$$

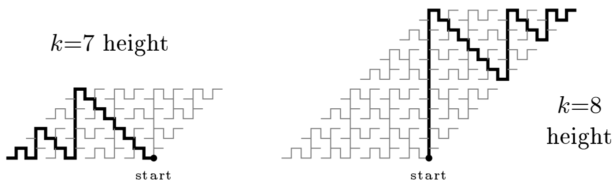
*Proof.* Tree $k$=0 is a single vertex so height 0. For $k \geq 1$, in figure 23 suppose the eccentricity of the start for tree $k$+1 is attained by going into the second tree half. Per figure 23 it goes around the right or left side boundary squares to the midpoint then corner eccentricity $k$ at the middle.

$$RLQ = \begin{cases} RQ_k & \text{if } k \text{ even} \\ LQ_k & \text{if } k \text{ odd} \end{cases} = 2^{\lfloor k/2 \rfloor}$$

$$\begin{aligned} TAheight_{k+1} &= TAcornerEcc_k + RLQ_k && (154)\\ &= TAcornerEcc_k + 2^{\lfloor k/2 \rfloor} \end{aligned}$$

This is the theorem (153), and working through the formulas shows it is greater than $TAheight_k$ which would be the height staying only in the first half.

Notice at (154) the left/right sides are swapped from the corresponding corner eccentricity (152), so $RLQ$ here instead of $LRQ$. ☐



**Theorem 26.** *The Wiener index* (131) *of twin alternate area tree $k$ is*

$$TAW_k = [\tfrac{15}{14}, \tfrac{43}{28}].4^k.2^{\lfloor k/2 \rfloor} - (\tfrac{1}{4}k{+}1)4^k - \tfrac{1}{14}2^k \tag{155}$$
$$= 0,\ 1,\ 10,\ 84,\ 584,\ 3984,\ 24864, \dots$$

*Proof.* As in figure 22, the tree comprises two halves connected across middle edge $c, a$.

Let $TAwS$ be the sum of distances from start vertex $s$ to all other vertices, and let $TAwC$ be the sum of distances from corner connection vertex $c$ to all other vertices.

$$TAwS_k = \sum\nolimits_v distance(s,v) \qquad TAwC_k = \sum\nolimits_v distance(c,v)$$
$$= 0, 1, 4, 18, 56, 200, \dots \qquad = 0, 1, 6, 22, 80, 248, \dots$$

$TAwS$ can be calculated from the two $k{-}1$ sub-trees. For $s$ to vertices in the lower half the total distance is $TAwS_{k-1}$. For $s$ to vertices in the upper half take first the distance from $s$ to $c$, which is $RLQ_{k-1}$ as from the diameter in theorem 24. There are $2^{k-1}$ vertices in the upper half, so that factor on this distance. Then $a$ is the same as $c$ by symmetry so $TAwC_{k-1}$ from $a$ to the upper vertices.

Similarly $TAwC$, except the distance $e$ to $c$ is $LRQ_{k-1}$

$$TAwS_k = TAwS_{k-1} + 2^{k-1}RLQ_{k-1} + TAwC_{k-1} \tag{156}$$
$$TAwC_k = TAwS_{k-1} + 2^{k-1}LRQ_{k-1} + TAwC_{k-1} \tag{157}$$
$$\text{starting } TAtwS_0 = 0,\ \ TAtwC_0 = 0$$

(156)+(157) is a recurrence for sum $TAwSC = TAwS + TAwC$

$$TAwC_k = TAwSC_{k-1} + 2^{k-1}LRQ_{k-1}$$
$$TAwSC_k = 2\,TAwSC_{k-1} + 2^{k-1}BQ_{k-1} \ \ = 2^{k-1}\sum_{j=0}^{k-1} BQ_j$$

The Wiener index can then be calculated from $TAwC$ of the two tree halves. Distance between vertex pairs both in the upper half is Wiener $TAW_{k-1}$, and the same for both in the lower half. For one vertex in the lower half and one in the upper there is distance $TAwC_{k-1}$ to go from lower vertices to $c$, multiplied by $2^{k-1}$ upper vertices which each one then goes to. The same upper vertices to $a$. Then add $4^{k-1}$ total paths going across edge $c, a$.

$$TAW_k = 2\,TAW_{k-1} + 2.2^{k-1}\,TAwC_{k-1} + 4^{k-1}$$
$$= 2^{k-1}(2^k{-}1) + 2^k\sum_{j=0}^{k-1} TAwC_{k-1}$$

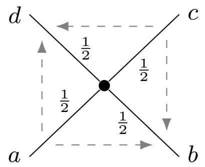
The result is sums and sums of sums of powers of 2 which can be worked through for (155). ☐

*Second Proof of Theorem 26.* The Wiener index can also be calculated bottom-up by considering traversals of edges.

Take each of the $2^k$ vertices in tree $k$ and possible edges in directions $a, b, c, d$

to adjacent unit squares. Let $a, b, c, d$ be the number of vertices in the sub-tree on the other side of each such edge respectively. (Vertices are at most degree-3 so at least one of these counts is 0 for no other vertices and no edge there.)

$k$
vertex

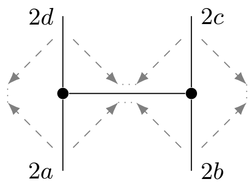


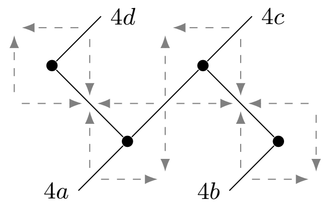

$$TAW_k = \tfrac{1}{2} \sum_{\text{vertices}} \; \sum_{t=s,a,c,e} t\,(2^k - t) \tag{158}$$

The Wiener index is sum of crossings of each edge. The number of paths crossing an edge is product of number of vertices on each side. For example $e$ on one side and everything else $2^k - e$ on the other. Summing over all edges at each vertex counts edges twice (the vertex at each end) so $\frac{1}{2}$ at (158).

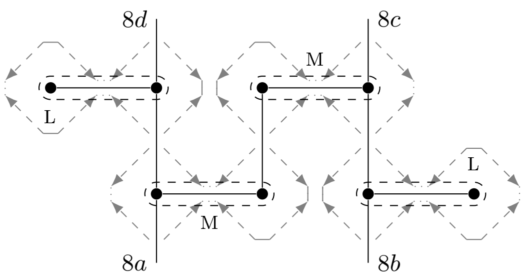
Each vertex expands 1, 2 and 3 times per the following diagrams. A little care is needed for which original edge goes to which new vertex. It's convenient to use the definition of the tree as unit squares inside curves. Each segment expands to 2 segments and the edges remain between the original segment ends. Segments are drawn here expanding on the left to $k$+1, then on the right to $k$+2, then on the left to $k$+3. Done this way the $a, b, c, d$ corners are fixed locations. By symmetry a right, left, right alternating expansion is the same final result.

$k$+1




$k$+2




$k$+3

In $k$+3, the horizontal pairs of vertices shown encircled are the expansion of each vertex in $k$+2. Crossings of the edges from one pair to another and from a pair to the outside are the same as $k$+2 but with $2\times$ vertices each side so $4\,TAW_{k+2}$.

The two L vertices are leaves so their edges are crossed 1 for each of the $2^{k+3} - 1$ vertices on the other side.

The two M edge crossings by vertices on each side are

$$\begin{aligned} &(8a{+}8d + 3)\,(8b{+}8c + 5) \;+\; (8a{+}8d + 5)\,(8b{+}8c + 3) \\ &= 32\,(2a{+}2d + 1)\,(2b{+}2c + 1) \;-\; 2 \end{aligned}$$

Product $(2a{+}2d + 1)\,(2b{+}2c + 1)$ is crossings of the middle edge in $k{+}1$. $TAW_{k+1}$ also counts crossings of its outer edges. They are $4\,TAW_k$ since $k$ is entirely outer edges. So net for $k{+}3$ is

$$\begin{aligned} TAW_{k+3} = {} & 4\,TAW_{k+2} && \text{between pairs} \\ & + 2\,(2^{k+3}-1).2^k && \text{L pairs} \\ & + 32\left(TAW_{k+1} - 4\,TAW_k\right) - 2.2^k && \text{M pairs} \end{aligned} \qquad \square$$

The Wiener index divided by number of vertex pairs is a mean distance between vertices. Such a mean is usually taken over vertex pairs in one direction (like the Wiener index) and excluding a vertex to itself, so number of pairs is binomial $\binom{2^k}{2} = \frac{1}{2}(4^k - 2^k)$. This mean can be expressed as a fraction of $TAdiameter$. The limit of that fraction as $k{\to}\infty$ follows from coefficients of the highest powers in each term.

$$\begin{aligned} \frac{TAW_k}{\frac{1}{2}(4^k-2^k).\,TAdiameter_k} & \to \frac{15}{49} = \frac{300}{980} = 0.306122\ldots && k \text{ even} \\ & \to \frac{43}{140} = \frac{301}{980} = 0.307142\ldots && k \text{ odd} \end{aligned}$$

Like the mean in the whole twin alternate graph at (145), the odd and even cases are not the same but differ by just 1.

Gutman, Furtula and Petrović [11] consider a terminal Wiener index which is distances between pairs of terminal vertices (ie. leaf nodes, degree 1).

**Theorem 27.** *The terminal Wiener index of twin alternate area tree $k$ is, in terms of the full Wiener index,*

$$\begin{aligned} TATW_k &= \begin{cases} 0, 1, 3, 7 & \text{if } k = 0 \text{ to } 3 \\ \frac{1}{16}\,TAW_k + \frac{1}{16}4^k - \frac{13}{32}2^k & \text{if } k \geq 4 \end{cases} \qquad (159) \\ &= 0, 1, 3, 7, 46, 300, 1784, 10736, \ldots \end{aligned}$$

*Proof.* Make a calculation similar to $TAW$ theorem 26 above. $c$ and $a$ are non-terminal vertices for $k{-}1 \geq 3$ and remain so on joining. So the calculation simply replaces vertex count $2^k$ with $TADegCount(k,1)$.

$$TAtwS_k = \sum_{\text{leaf } v} distance(S,v) \qquad TAtwC_k = \sum_{\text{leaf } v} distance(C,v)$$
$$= 0, 1, 3, 7, 18, 58, \ldots \qquad\qquad = 0, 1, 3, 7, 24, 70, \ldots$$

$$\begin{aligned} TAtwS_k = {} & TAtwS_{k-1} && \text{lower} \quad k{\geq}4 \\ & + TADegCount(k{-}1, 1).RLQ_{k-1} && \text{s to } c \\ & + TAtwC_{k-1} && a \text{ into upper} \\ TAtwC_k = {} & TAtwS_{k-1} && \text{upper} \quad k{\geq}4 \\ & + TADegCount(k{-}1, 1).LRQ_{k-1} && e \text{ to } a \\ & + TAtwC_{k-1} && c \text{ into lower} \end{aligned}$$

starting $TAtwS_3 = 7, \;\; TAtwC_3 = 7$

$$\begin{aligned}
TAtwSC_k &= TAtwS_k + TAtwC_k \\
&= 2\,TAtwSC_{k-1} + 2^{k-3}BQ_{k-1} && k \ge 4 \\
&= 14.2^{k-3} \; + 2^{k-3}\sum_{j=3}^{k-1} 2BQ_j \\
TAtwC_k &= TAtwSC_{k-1} + 2^{k-3}LRQ_{k-1} && k \ge 4 \\
TATW_k &= 2\,TATW_{k-1} && \text{halves} \\
&\quad + 2\,TADegCount(k,1)\,.\,TAtwC_{k-1} && c, a \text{ into halves} \\
&\quad + \;TADegCount(k{-}1,1)^2 && \text{across } c \text{ to } a \\
&= 2.4^{k-3} + 5.2^{k-3} + 2^{k-2}\sum_{j=3}^{k-1} TAtwC_{k-1} && k \ge 4 \qquad \square
\end{aligned}$$

$TATW$ term $\frac{1}{16}TAW$ in (159) arises essentially from the number of terminal vertices $TADegCount(k,1)$ being $\frac{1}{4}$ of the total $2^k$ (for $k{\ge}3$).

The mean distance between distinct pairs of terminal vertices as a fraction of the diameter has the same limit as the full $TAW$.

$$\frac{TATW_k}{\binom{TADegCount(k,1)}{2}\,.\,TAdiameter_k} \to \left[\frac{15}{49}, \frac{43}{140}\right] \quad \text{same as } TAW$$

### 12.1.2 Twin Alternate Area Tree Parent, Depth, Width

**Theorem 28.** *Label the vertices of twin alternate area tree $k$ with point numbers $n$ and layout per TAVertexToZ at* (146)*. The parent of vertex $n \ge 1$ is in the direction given by the following state machine on bits of $n$ high to low.*

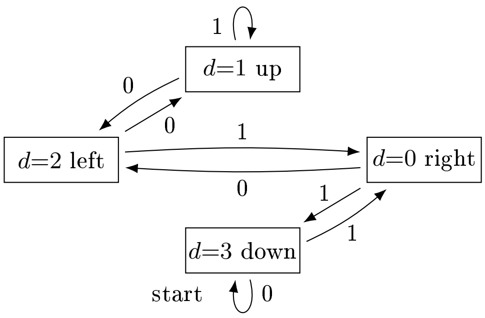



Figure 24: $TAparentDir(n)$, bits of $n$ high to low

$$\begin{aligned}
TAparentDir(n) &= \text{final state of bits high to low} \\
&= 0,2,3,1,0,3,0,2,1,2,3,3,0,2,3,1,0,\ldots \qquad n{\ge}1
\end{aligned}$$

$$TAstepDir(n,d) = \begin{cases} TAVertexUpper(n) & \text{if } d = 1 \\ TAVertexLower(n) & \text{if } d = 3 \\ \mathrm{BITXOR}(n,1) & \text{if } d = 0 \text{ or } 2 \end{cases}$$

$$\begin{aligned}
TAparent(n) &= TAstepDir(n,\, TAparentDir(n)) \\
&= 0,3,0,7,4,1,6,9,14,11,8,3,12,15,12,19,16,\ldots \qquad n{\ge}1
\end{aligned}$$

The initial state is $d$=3. Since $n \geq 1$ there is always a high 1-bit so always a transition from there to $d$=0 (right). If $n$=1 then that is the only transition.

*Proof.* In the corners of theorem 22, the parent node is towards the start $s$. The expansion of figure 22 shows how aiming towards a given corner in $k$+1 becomes some corner of $k$ according to the high bit of $n$.

For example when aiming for $s$, if the high bit of $n$ is 1 then must go towards the 0–1 connection at $2^k$+$a$, across that edge, and from there to the start. For finding the parent it's enough to know the new state is "to $a$ or if already there then across".

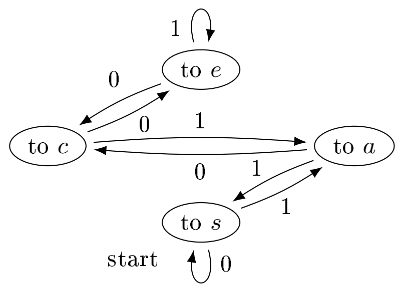



Figure 25: *TAparentDir* aiming-for

For the direction, if $k$=1 with $k$=0 sub-parts then "to $a$" is edge across horizontal to the right (the $k$ odd case in figure 22, turned +45°). Similarly "to $c$" is horizontal to the left.

The "to $s$" and "to $e$" cases occur when "to $a$" or "to $c$" was $k \geq 1$ and therefore is an edge down or up. $\square$

The *TAparentDir* state machine in figure 24 has the same structure as the *dir* mod 4 state machine in figure 10, but directions +1 mod 4 there, and an 0↔1 bit flip for the transitions out of states 1, 2 there, which are states 0, 1 here.

The effect of these transition bit changes is that the runs of 1-bits which *dir* identifies become like

| | | | | | |
|---|---|---|---|---|---|
| *dir*($n$) | 1111111 | 00000 | 111111 | 00000 | 111111 |
| *TAparentDir*($n$) | 1010101 | 10000 | 101010 | 01111 | 010101 |

A high run of 1-bits in *dir* becomes alternating 1010 in *TAparentDir*. That alternating run ends with a bit $a$. The following run of 0-bits in *dir* becomes either 1000 or 0111, whichever repeats bit $a$ instead of alternates. The next run of 1-bits in *dir* is again alternating bits in *TAparentDir*, beginning with $a$.

The bit changes from *dir* to *TAparentDir* can be expressed by bit pairs as

$$\mathit{Flip1110}(n) = \begin{cases} \text{at 11 pair in } n\text{, output flip pair low and all below} \\ \text{at 10 pair in } n\text{, output flip pair low} \end{cases}$$

$$= 0, 1, 3, 2, 6, 7, 4, 5, 12, 13, 15, 14, 9, 8, 11, 10, 24, \ldots$$

$$\text{binary} = 0, 1, 11, 10, 110, 111, 100, 101, 1100, 1101, \ldots$$

$$\mathit{TAparentDir}\big(\mathit{Flip1110}(n)\big) + 1 \equiv \mathit{dir}(n) \mod 4$$

Pairs 11 or 10 are found in $n$ without any flips. The output begins as $n$ and is modified by $0 \leftrightarrow 1$ flips. The flip at a pair is the lower bit of that pair, and for 11 also all bits below it. These flips are cumulative, so some will cancel.

An even length run of 1-bits has an odd number of 11 pairs, thereby giving the different styles 1010 or 0101 for run of 1-bits below. An odd length run of 1-bits has an even number of 11 pairs so net unchanged below. The 10 flip is at the bottom of each run of 1-bits and gives 1000 or 0111 for the 0-bit runs.

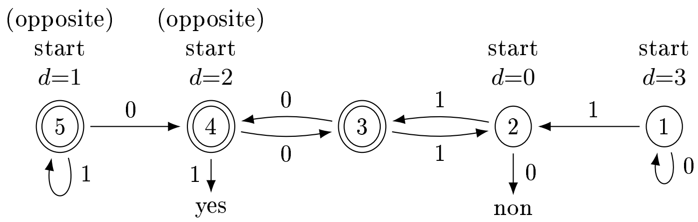
The inverse, from *TAparentDir* runs to those of *dir* is a flip the other way around. 10 is low and all below, and 11 just the low.

$$\begin{aligned}
\mathit{UnFlip1110}(n) &= \begin{cases} \text{at 10 pair in } n \text{, output flip low and all below} \\ \text{at 11 pair in } n \text{, output flip low} \end{cases} \\
&= 0, 1, 3, 2, 6, 7, 4, 5, 13, 12, 15, 14, 8, 9, 11, 10, 26, \ldots \\
\text{binary} &= 0, 1, 11, 10, 110, 111, 100, 101, 1101, 1100, \ldots
\end{aligned}$$

$$\mathit{UnFlip1110}\big(\mathit{Flip1110}(n)\big) = n \qquad \text{inverse}$$

$$\mathit{dir}\big(\mathit{UnFlip1110}(n)\big) - 1 \equiv \mathit{TAparentDir}(n) \mod 4$$

The state machine of figure 24 is bits high to low. Some usual state machine manipulations can take bits low to high instead.




$\mathit{TAparentDir}(n) = d$ bits of $n$ low to high

The start state is 2 to test for $\mathit{TAparentDir}(n) = 0$ (right), or start state 1 to test for $\mathit{TAparentDir}(n) = 3$ (down). In both cases an $n$ is accepted by ever reaching "yes", or ending in the double-circle accepting states. Reaching "non" or ending in a non-accepting is an $n$ not of the respective parent direction.

The start state is 4 or 5 to test for $\mathit{TAparentDir}(n) = 2$ or 1 respectively. For these the sense of accepting or not accepting is opposite.

State 1 is never a final state since $n \geq 1$ has at least one 1-bit. So the accepting-ness of that state does not matter. It is reckoned non-accepting for a little symmetry.

For $d=0$, an even $n$ goes immediately to "non". This is simply that even $n$ has no edge to the right at all (its horizontal edge is to the left). For odd $n$ the low 1 goes to state 3. From there base-4 digits 0 or 3 return to state 3 each time. This can be written as

$$\begin{aligned}
\mathit{TAparentDir}(n) = 0 \ \text{ iff } \ n \text{ odd and } \lfloor n/2 \rfloor \text{ base-4 entirely 0,3,} \\
\text{or lowest non-0,3 is 2}
\end{aligned}$$

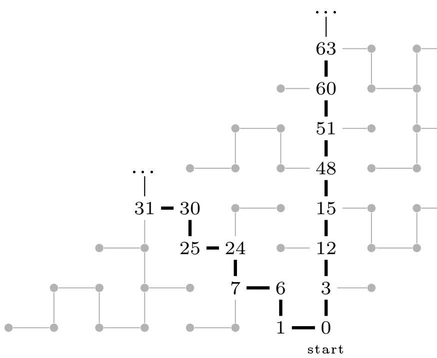
For $d=3$, state 1 skips low bits $100\ldots00$ before this test.

A state machine low to high with a single start and a direction result according to final state is possible, by what is effectively simultaneous transitions from the various starts. Written out as a full DFA it becomes a little complicated.

High 0-bits on $n$ do not change the state machine results. In all states a run of 0s remains the same accepting-ness. Geometrically these 0s are simply vertices $n$ in the first half, quarter, etc, within a bigger tree.

The way the tree is constructed extending at corner $c_k$ means there are two spines continuing infinitely. A given $k$ is extended at $c_k = 2^{k-1}-1$. The next expansion is at $c_{k+1} = 2^k-1 = e_k$. Ie. the further copy is from a vertex $e_k$ in the original level $k$.

One spine is the verticals upward from the root. The other spine is stair-step North West.




$k$=6
twin alternate area tree
two spines continuing infinitely

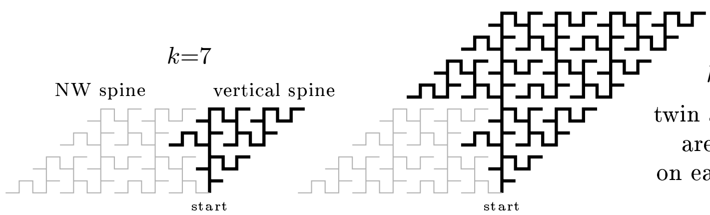
From the $c_k$ connection replication, these are vertices

$$\begin{aligned}
\mathit{TAspineV}(m) &= 3\,\mathit{Xnum}(m) && \text{vertical} \\
&= 0, 3, 12, 15, 48, 51, 60, 63, 192, 195, \ldots && \text{A001196} \\
\mathit{TAspineNW}(m) &= \left\lfloor \tfrac{3}{2}\,\mathit{Xnum}(m) \right\rfloor && \text{stair-step} \qquad (160) \\
&= 0, 1, 6, 7, 24, 25, 30, 31, 96, 97, \ldots
\end{aligned}$$

The North West $x=-y$ line is $6\,\mathit{Xnum}$ then each following horizontal is $+1$. The floor at (160) combines these.

Also from the tree construction, the vertices in these two halves are those $n$ with odd or even length in binary.

$$\begin{aligned}
\text{vertical spine part vertices} &= 0, 2, 3, 8, 9, 10, 11, 12, 13, 14, \ldots && \text{A053754} \\
\text{NW spine part vertices} &= 1, 4, 5, 6, 7, 16, 17, 18, 19, 20, \ldots && \text{A053738}
\end{aligned}$$




$k$=8
twin alternate
area tree
on each spine

The "aiming for" procedure of *TAparentDir* can be applied to go towards end $e$. Starting from $n$=0 or $n$=1 this steps along the two infinite spines. Starting from other $n$ goes first to the spine of its respective half then descends that spine.

$$\begin{aligned}
\mathit{TAtospineDir}(n) &= \text{final state of figure 24 starting from } d{=}1 \\
&= 1, 1, 2, 1, 1, 0, 2, 1, 2, 1, 2, 3, 1, 0, \ldots \\
\mathit{TAtospine}(n) &= \mathit{TAstepDir}(n,\, \mathit{TAtospineDir}(n)) \\
&= 3, 6, 3, 12, 7, 4, 7, 24, 9, 14, 11, 8, 15, 12, \ldots \\
\mathit{TAspine}(m, n) &= \mathit{TAtospine}(\mathit{TAtospine}(\ldots(n))) \qquad m \text{ times} \\
\mathit{TAspine}(m, 0) &= \mathit{TAspineV}(m) \\
\mathit{TAspine}(m, 1) &= \mathit{TAspineNW}(m{+}1)
\end{aligned} \tag{161}$$

$n{=}0$ has $\mathit{TAtospine}(0) = 3$ so that under the state machine rule it goes up the vertical spine. Starting from $n{=}1$ at (161) goes up the stair-step.

The depth of a vertex is its distance to the root. The root itself is depth 0. The aiming-for corner procedure for parent direction gives the depth of vertex $n$ by summing distances across preceding trees.

**Theorem 29.** *The depth of vertex $n$ in the twin alternate area tree is given by sums RLQ followed by run LRQ according to bit runs in $n$,*

$$n = \begin{array}{|c|c|c|c|c} \multicolumn{1}{l}{\text{high}} & & & & \multicolumn{1}{r}{\text{low}} \\ \hline \underbrace{101\ldots01} & 100\ldots00 & \underbrace{10\ldots10} & 011\ldots11 & \underbrace{0101\ldots} \\ \hline \end{array} \tag{162}$$

$$\begin{aligned}
\mathit{TAdepth}(n) = \mathit{RLQ}_k + \mathit{LRQ}_{k-1} \quad &+ \mathit{RLQ}_p + \mathit{LRQ}_{p-1} \quad + \cdots \\
+\cdots \qquad\quad & \qquad +\cdots \\
+\mathit{LRQ}_{k-l} \quad & \qquad +\mathit{LRQ}_{p-q}
\end{aligned} \tag{163}$$

$$= 0, 1, 2, 1, 4, 5, 2, 3, 6, 5, 8, 7, 2, 3, 4, 3, 10, \ldots$$

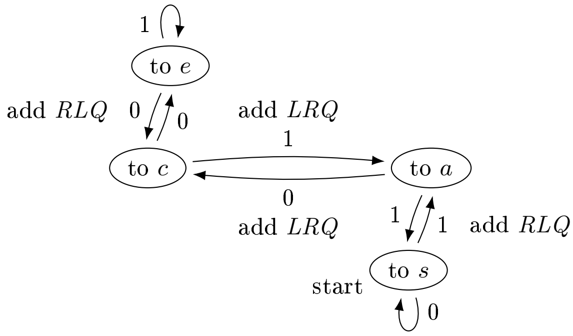
*These are the runs of 1-bits in $\mathit{UnFlip1110}(n)$,*

$$\mathit{UnFlip1110}(n) = \begin{array}{|c|c|c|c|c} \hline \underbrace{111\ldots11} & 000\ldots00 & \underbrace{111\ldots11} & 000\ldots00 & \underbrace{11\ldots} \\ \hline \end{array}$$

The bit runs in $n$ at (162) are alternating $1, 0$. Between each is a run either 1000 or 0111. It starts with a repeat bit $a$, ie. not alternate, and has zero or more opposite bits $1{-}a$. The next alternating run starts with $a$.

The indices $k$ etc for $\mathit{RLQ}$ and $\mathit{LRQ}$ terms are the bit positions of all alternating $1, 0$ run bits. Bit positions are counted starting 0 for the least significant bit as usual.

*Proof.* In the manner of *TAparent*, the distance to the start $s$ follows by the state machine of figure 25.




On expansion, when the target corner is in the opposite half of the tree the distance across that other half is added. In the manner of *TAdiameter* theorem 24 this is either $RLQ$ or $LRQ$ following the boundary squares on the right or left side of the sub-curve.

The positions where $RLQ$ or $LRQ$ distances are added are then the $n$ bit runs of the theorem, and per the *dir* to *TAparentDir* correspondence these runs are the bits of *UnFlip1110*. □

*UnFlip1110*$(n)$ itself has a geometric interpretation as the total sizes of all power-of-2 sub-trees traversed to reach $n$.

Let $\mathit{WidthS}(k,d)$ be the number of vertices at depth $d$ from the tree start. Let $\mathit{WidthC}(k,d)$ be the number of vertices at depth $d$ from the corner connection $c$, in the manner of *TAcornerEcc* from theorem 24. Mutual recurrences follow by the tree as two $k{-}1$ halves,

$$\mathit{WidthS}(k,d) = \mathit{WidthS}(k{-}1,d) + \mathit{WidthC}(k{-}1,\ d{-}\mathit{RLQ}_{k-1}) \tag{164}$$
$$\mathit{WidthC}(k,d) = \mathit{WidthS}(k{-}1,d) + \mathit{WidthC}(k{-}1,\ d{-}\mathit{LRQ}_{k-1}) \tag{165}$$

starting

$$\begin{aligned} \mathit{WidthS}(k,0) &= \mathit{WidthC}(k,0) = 1 \\ \mathit{WidthS}(k,d) &= \mathit{WidthC}(k,d) = 0 \quad \text{if } d<0 \end{aligned}$$

so, depths $d=0$ to $\mathit{TAheight}_k$,

$$\begin{aligned} &\mathit{WidthS}(0,d) = 1 \\ &\mathit{WidthS}(1,d) = 1,1 \\ &\mathit{WidthS}(2,d) = 1,2,1 \\ &\mathit{WidthS}(3,d) = 1,2,2,1,1,1 \\ &\mathit{WidthS}(4,d) = 1,2,3,3,2,2,1,1,1 \\ &\mathit{WidthS}(5,d) = 1,2,3,3,3,4,3,2,2,2,2,1,1,1,1,1 \end{aligned}$$

The two terms of (164),(165) are vertices from the first and second $k{-}1$ sub-parts. For the second sub-part the depth $d$ is reduced by the distance to the connection point and is then *WidthC*.

The sum of widths at all depths is the total $2^k$ vertices

$$2^k \;=\; \sum_{d=0}^{TAheight_k} WidthS(k,d) \;=\; \sum_{d=0}^{TAcornerEcc_k} WidthC(k,d)$$

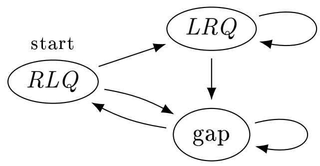
The maximum width is unbounded with increasing $k$ since there are $2^k$ vertices within $TAheight_k$ and the latter grows only as $2^{\lfloor k/2 \rfloor}$.

The width at $d$ is the number of solutions to $TAdepth(n) = d$ so from (163)

$$d = \underset{k-1 \geq w}{RLQ_w} + \overbrace{LRQ_{w-1} + \cdots + LRQ_x}^{\text{zero or more } LRQ} \quad \underset{\text{index gap} \geq 1}{\uparrow} \quad + RLQ_y + LRQ_{y-1} + \cdots + LRQ_z \quad \underset{\text{index gap} \geq 1}{\uparrow} \quad + \cdots$$

These runs are in the $WidthS$ recurrence (164) too. An $RLQ$ subtraction from $d$ goes to C and $WidthC$ can stay there for a run of $LRQ$ subtractions.

So a combinatorial interpretation of $WidthS$ is the number of ways to write $d$ as sums of $RLQ$ and $LRQ$ terms in such runs.




The index positions are significant. The $RLQ$ values repeat, and values 1,2 repeat $LRQ$ at the low end too. These become distinct ways to make $d$, where runs and gaps permit.

$$\begin{aligned}
&WidthS(4,3) = 3 \text{ ways} \\
&\quad 3 = RLQ_3 + RLQ_1 \quad = 2 + \text{gap} + 1 + \text{gap} \\
&\quad 3 = RLQ_3 + RLQ_0 \quad = 2 + \text{gap} + \text{gap} + 1 \\
&\quad 3 = RLQ_2 + RLQ_1 \quad = \text{gap} + 2 + \text{gap} + 1
\end{aligned}$$

$RLQ_0 = LRQ_0 = 1$ are the same but the runs and gaps mean they never make distinct forms. $RLQ_0$ only occurs when the position above it (index 1) is a gap, whereas $LRQ_0$ only occurs when not a gap.

As a remark, all of $RLQ$ and $LRQ$ are distinct except for 1 and 2 noted and $RLQ$ repeat pairs. From the power formulas two $LRQ$ fall between each $RLQ_k = 2^{\lfloor k/2 \rfloor}$ pair,

$$RLQ_{2k+3} = RLQ_{2k+2} > \; LRQ_{2k} > LRQ_{2k-1} \; > RLQ_{2k+1} = RLQ_{2k} \qquad k \geq 2$$

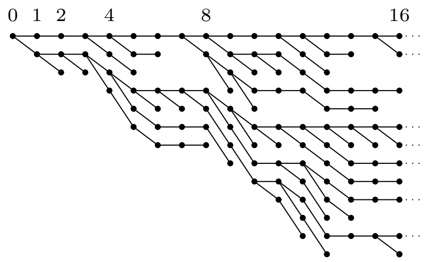
The same run forms apply for $WidthC$ except it starts in an $LRQ$ run already. So start $LRQ_{k-1}$ and further $LRQ$ terms then a gap etc, or gap immediately with no high $LRQ_{k-1}$ at all.

Repeatedly expanding (164) is $WidthS$ as sum of $WidthC$, where depth $< 0$ is taken to have width 0.

$$WidthS(k,d) = \sum_{j=0}^{k-1} WidthC(j,\ d-RLQ_j) \qquad d \ge 1$$

For a given $d$ these $WidthC$ terms are 0 when $j$ big enough that $d-RLQ_j < 0$. So when $k$ is big enough $WidthS(k,d)$ does not change with further increases in $k$. This is the width of a twin alternate area tree continued infinitely. ($WidthC$ treated similarly would be the same as $WidthS$ since (165) becomes only its $WidthS$ term when $k$ big enough that $d-LRQ_{k-1} < 0$.)

$$\begin{aligned} WidthS(\infty,d) &= WidthS(k,d) \text{ for } k \text{ where } RLQ_k > d \\ &= 1,2,3,3,4,6,6,5,6,8,9,9,11,13,11,9,10,12,12,12,\ldots \end{aligned}$$




twin alternate area tree continued infinitely

### 12.1.3 Twin Alternate Area Tree Independence and Domination

The twin alternate area tree has a perfect matching (section 11) by horizontal pairs of vertices. This is the expansion of each $k-1$ vertex to 2 adjacent vertices in $k$ (the low bit toggle in the numbering of theorem 22). Or top-down $k$ is two copies of $k-1$ starting from perfect matching of the 2 vertices in $k$=1.

An *independent set* in a graph is a set of vertices which have no edges between them, so no adjacent vertices in the set. The *independence number* is the size of the largest independent set of the graph. The *independence ratio* is the proportion of this to the number of vertices.

Any tree with a perfect matching has independence ratio $\frac{1}{2}$. An independent set can have at most one vertex of each pair, and a set of that size can be constructed working outwards taking neighbours opposite present/absent. So for twin alternate area tree $k$,

$$TAindnum_k = \begin{cases} 1 & \text{if } k{=}0 \\ 2^{k-1} & \text{if } k \ge 1 \end{cases} \qquad TAindRatio_k = \begin{cases} 1 & \text{if } k{=}0 \\ \frac{1}{2} & \text{if } k \ge 1 \end{cases}$$

Taking neighbours alternately present/absent is unique up to complement, but there are various other sets attaining $TAindnum_k$ too. At each vertex absent from the set its neighbours in other pairs can be either present or absent.

A *dominating set* in a graph is a set of vertices which has every vertex of the graph either in the set or adjacent to one or more of the set. The *domination number* is the size of the smallest set which dominates in the graph.

**Theorem 30.** *The domination number of twin alternate area tree $k$ is*

$$TAdomnum_k = \begin{cases} 1, 1, 2 & \textit{if } k = 0, 1, 2 \\ 3 \,.\, 2^{k-3} & \textit{if } k \geq 3 \end{cases}$$

$$= 1, 1, 2, 3, 6, 12, 24, 48, 96, 192, \ldots \qquad \text{A098011}$$

*The number of dominating sets of this size is*

$$TAdomnumCount_k = \begin{cases} 1, 2, 4 & \textit{if } k = 0, 1, 2 \\ 2^{2^{k-2}} & \textit{if } k \geq 3 \end{cases}$$

$$= 1, 2, 4, 4, 16, 256, 65536, 4294967296, \ldots$$

*Proof.* The domination number for $k \leq 3$ can be verified explicitly. In $k$=3 it can be noted that if any combination of the start, end and connection vertices are optionally allowed to be undominated then the size of the smallest dominating set is still *TAdomnum*.

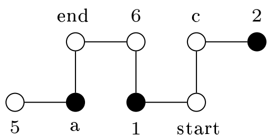



Figure 26:
$k$=3 twin alternate area tree
$TAdomnum_3 = 3$,
unchanged by optional undominated

In figure 26, the optional undominating would mean only vertices 2, 5 and the middle 1,6 need be dominated. Their separation means $TAdomnum_3 = 3$ vertices are still required to do so.

Suppose the theorem and optional undominating is true of some $k{-}1 \geq 3$. When two copies of $k{-}1$ join, there could be a cross-domination allowing the connection vertex in one half to be undominated. But that does not reduce $TAdomnum_{k-1}$ in that half, so the two halves

$$TAdomnum_k = 2\; TAdomnum_{k-1} \qquad k \geq 4$$

In $k$ the optional undominated vertices are some of the start, end and connection vertices of the two $k{-}1$, so that their optional undominating in $k$ again still gives $TAdomnum_k$ there.

The count of dominating sets can be verified explicitly for $k \leq 3$. Thereafter the sets are all those in each half, so product

$$TAdomnumCount_k = TAdomnumCount_{k-1}^2 \qquad k \geq 4$$

□

The *domination ratio* is the ratio of domination number to number of vertices in a graph. For the twin alternate area tree this is

$$TAdomRatio_k = \frac{TAdomnum_k}{2^k} = \begin{cases} 1, \frac{1}{2}, \frac{1}{2} & \text{if } k = 0, 1, 2 \\ \frac{3}{8} & \text{if } k \geq 3 \end{cases}$$

An *independent dominating set* in a graph is a set of vertices which is both independent and dominating. This is equivalent to being a maximal independent set. A *maximal independent set* is an independent set to which no further vertex can be added and still be an independent set. This means dominating since any undominated vertex would have no neighbour and so could be added and still be independent.

The *independent domination number* of a graph is the size of the smallest independent dominating set. Or equivalently, the size of the smallest maximal independent set and as such also called the *lower independence number*.

The independent domination number is always $\geq$ the domination number, since requiring independence restricts the dominating sets considered. The two are equal for the twin alternate area tree, but a smaller count of sets.

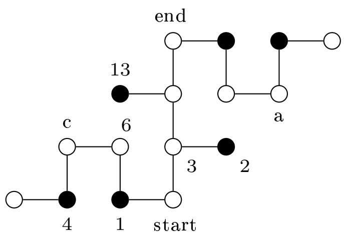
**Theorem 31.** *The independent domination number of twin alternate area tree $k$ is equal to the domination number*

$$TAindomnum_k = TAdomnum_k$$

*The number of independent dominating sets of this size is*

$$\begin{aligned} TAindomnumCount_k &= \begin{cases} 1, 2, 3, 4 & \text{if } k = 0 \text{ to } 3 \\ 7^{2^{k-4}} & \text{if } k \geq 4 \end{cases} \\ &= 1, 2, 3, 4, 7, 49, 2401, 5764801, \ldots \qquad {\scriptstyle k\geq 4 \text{ A165425}} \end{aligned}$$

*Proof.* The theorem can be verified explicitly for $k \leq 4$. Then for $k$=4,




$k$=4 twin alternate area tree
$TAindomnum_4 = 6$,
unchanged by optional undominated

Similar to theorem 30, for $k$=4 if any combination of the start, end and connection vertices are optionally allowed to be undominated then the size of the smallest independent dominating set is still $TAindomnum_4 = TAdomnum_4 = 6$. Those vertices undominated separate the rest into 5 parts. The middle is a path-4 requiring 2 vertices for domination.

The number of independent dominating sets in $k$=4 can be seen by considering how the vertices shown in figure 27 might be rearranged. 1, 4 dominate as many vertices as possible in their tail. But 2 can come inwards to 3. Doing so allows 1 to move up to 6, and when that happens 4 can then move outwards too. These moves are 3 sets. Likewise by symmetry 13 etc in the upper half. But 3 and 13 cannot both move inwards or not an independent set. So $TAindomnumCount_4 = 1+3+3 = 7$.

All these $k$=4 independent dominating sets have all of start, end and connection vertices absent. So on joining all combinations of sets in $k-1$ remain independent in $k$, giving

$$TAindomnumCount_k = TAindomnumCount_{k-1}^2 \qquad k \geq 5 \qquad \square$$

A *total dominating set* in a graph is a set of vertices for which all graph vertices are adjacent to one or more in the set. This differs from an ordinary dominating set in that a vertex in the set does not dominate itself, it must have some neighbour.

**Theorem 32.** *The number of total dominating sets in twin alternate area tree $k$ is*

$$TAtotdomsets_k = \begin{cases} 0, 1, 4, 25 & \text{if } k = 0 \text{ to } 3 \\ 30^{2^{k-3}} & \text{if } k \geq 4 \end{cases}$$
$$= 0, 1, 4, 25,\ 900,\ 810000,\ 656100000000, \ldots$$

*Proof.* The theorem can be verified explicitly for $k \leq 4$. Then for $k$=4,

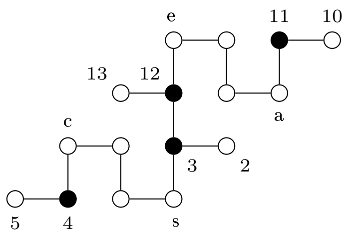



Figure 27:
$k$=4 twin alternate area tree
part of total domination,
$s, a, c, e$ always dominated

Vertex 3 must be present to dominate leaf vertex 2, and 3 then also dominates the start vertex $s$. Similarly $4, 11, 12$ required and dominate $a, c, e$.

All higher $k$ are formed by connections across those $s, a, c, e$. Since they are already dominated in their $k$=4 parts there are no additional sets formed by cross-domination at those connections, only the sets formed within $k$=4. It can be verified explicitly that $k$=4 has $30^2 = 900$ sets. Further $k$ squares that successively. ☐

The *total domination number* is the size of the smallest total dominating set of a graph. Similar to theorem 32, from no cross-domination the total domination number and count of sets of that size are

$$TAtotdomnum_k = \begin{cases} \text{none} & \text{if } k{=}0 \\ 2 & \text{if } k{=}1 \\ 2^{k-1} & \text{if } k \geq 2 \end{cases}$$
$$= \text{none}, 2, 2, 4, 8, 16, 32, \ldots$$

$$TAtotdomnumCount_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \text{ to } 3 \\ 2^{2^{k-3}} & \text{if } k \geq 4 \end{cases}$$
$$= 0, 1, 1, 1, 4, 16, 256, 65536, 4294967296, \ldots$$

The *total domination polynomial* of a graph has terms $c_n\, x^n$ where $c_n$ is the number of total dominating sets of $n$ vertices. Again similar to theorem 32, from no cross-domination after $k$=4 this is a power

$$TAtotdompoly(x) = \begin{cases} 0 & \text{if } k=0 \\ x^2 & \text{if } k=1 \\ x^2\,(x{+}1)^2 & \text{if } k=2 \\ x^2\,(x^2{+}3x{+}1)^2 & \text{if } k=3 \\ \left(x^4\,(x{+}1)\,(x{+}2)\,(x^2{+}3x{+}1)\right)^{2^{k-3}} & \text{if } k\ge 4 \end{cases}$$

A *semi-total dominating set* in a graph is a set of vertices where each not in the set has a neighbour in the set, and each in the set has a neighbour or distance 2 away in the set (or both). Semi-total is similar to total domination, but relaxes to allow set members dominated up to distance 2 away. It is still a plain dominating set and so falls between the conditions of total and plain dominating.

The *semi-total domination number* of a graph is the size of the smallest semi-total dominating set.

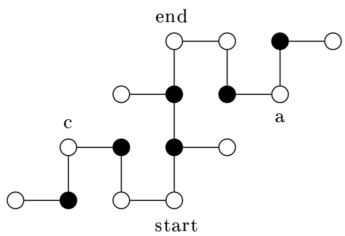
**Theorem 33.** *The semi-total domination number of twin alternate area tree k is the same as the domination number for* $k \ge 4$,

$$TAsemitotdomnum_k = \begin{cases} none, 2, 2, 4 & \text{if } k \le 3 \\ TAdomnum_k & \text{if } k \ge 4 \end{cases}$$

*The number of semi-total dominating sets of this size is*

$$TAsemitotdomnumCount_k = \begin{cases} 0, 1, 3, 11 & \text{if } k = 0 \text{ to } 3 \\ 1 & \text{if } k \ge 4 \end{cases}$$

*Proof.* The theorem can be verified explicitly for $k \le 3$. For $k{=}4$ the unique minimum semi-total dominating set is




$k{=}4$ twin alternate area tree
$TAsemitotdomnum_4 = 6$
unique such set

Similar to theorem 30, if any or all of the start, end and connection vertices are optionally allowed to be undominated, and/or their neighbours allowed to be present and undominated, then the size of the smallest set is still this sole $TAsemitottdomnum_4 = 6$.

So on connecting to $k \ge 5$ by two $k{-}1$ halves, any cross-domination to one of the halves doesn't reduce the size there, and hence $TAsemitottdomnum_k = 2\,TAsemitottdomnum_{k-1}$ is the best, and attained by the single set each half. □

A *perfect dominating set* in a graph is a dominating set where each vertex not in the set is dominated by just one from the set. The *perfect domination number* of a graph is the size of the smallest perfect dominating set.

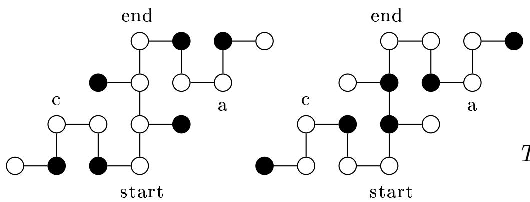
**Theorem 34.** *The perfect domination number of twin alternate area tree k is the same as the domination number.*

$$TAperfdomnum_k = TAdomnum_k$$

*The number of perfect dominating sets of this size is*

$$TAperfdomnumCount_k = \begin{cases} 1,2,2,2 & \text{if } k = 0 \text{ to } 3 \\ 2^{2^{k-4}} & \text{if } k \geq 4 \end{cases}$$
$$= 1,2,2,2,2,4,16,256,65536,\ldots$$

*Proof.* The theorem can be verified explicitly for $k \leq 4$. For $k$=4 the two perfect dominating sets are




$k$=4 two sets
$TAperfdomnum_4 = 6$
$TAperfdomnumCount_4 = 2$

Both sets have all start, end and connections absent so on joining for $k$=5 there is no cross-domination and they remain perfect dominating. The new start, end and connections are likewise absent so likewise perfect dominating in bigger $k$ too. The number of sets formed this way is product in each half so

$$TAperfdomnumCount_k = 2\, TAperfdomnumCount_{k-1}^2$$

Suppose there are no other such sets in some $k-1 \geq 3$. This is so for $k$=4 above. If the connection C is in the set then this is not of these forms and therefore is at least 1 vertex bigger. In the other half its connection would be undominated. From *TAdomnum* theorem 30, such an undominated does not reduce the size of any dominating set there, so total in $k$ is too big. ☐

The *disjoint domination number* of a graph is the smallest combined size of two disjoint dominating sets. In the twin alternate area tree, the two sets can both be the minimum *TAdomnum* size.

**Theorem 35.** *The disjoint domination number of twin alternate area tree k is*

$$TAdisdomnum_k = \begin{cases} none & \text{if } k{=}0 \\ 2\, TAdomnum_k & \text{if } k \geq 1 \end{cases}$$
$$= 2,4,6,12,24,48,96,\ldots \quad k{\geq}1 \qquad k{\geq}1 \text{ A058764}$$

*The number of pairs of such sets is*

$$TAdisdomnumCount_k = \begin{cases} 0,1,2 & \text{if } k = 0,1,2 \\ 2^{2^{k-3}-1} & \text{if } k \geq 3 \end{cases}$$
$$= 0,1,2,1,2,8,128,32768,\ldots \qquad k{\geq}3 \text{ A058891}$$

*Proof.* For $k<3$ the theorem can be verified explicitly. For $k$=3 the tree is a path and figure 26 shows a $TAdomnum_3 = 3$ dominating set. Reversing it along the path, so $2, 6, c$, is a further dominating set of 3 vertices and is disjoint.

For $k \geq 4$, the replications in theorem 30 constructing $TAdomnum$ mean the two sets remain disjoint, thus giving two sets each $TAdomnum$, and no extra sets by cross-domination. The $k$=3 pair is the only pair in $k$=3. Further $k$ is product counts in each half, and 2× since can also flip the pairings in the second half relative to the first.

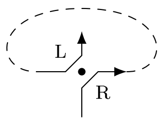
$$TAdisdomnumCount_k = 2\, TAdisdomnumCount_{k-1}^2 \qquad k \geq 4 \qquad \square$$

The independent and dominating set counts grow as various powers $c^n$ where $n = 2^k$ is the number of vertices. $c$=1 would be a single set (each vertex only 1 choice), and $c$=2 would be all sets (each vertex 2 choices present or absent), or $c$=3 for set pairs of $TAdisdomnumCount$ the (each vertex in set A,B, or none). The counts can be compared by their base $c$.

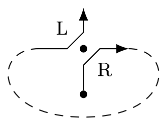
$$\begin{aligned}
\sqrt[2^k]{TAtotdomsets_k} &= 30^{1/8} = 1.529819\ldots && k{\geq}4 \\
\sqrt[2^k]{TAdomnumCount_k} &= 2^{1/4} = 1.189207\ldots && k{\geq}3 && \text{A010767} \\
\sqrt[2^k]{TAindomnumCount_k} &= 7^{1/16} = 1.129324\ldots && k{\geq}4 && \text{A011240} \\
\sqrt[2^k]{TAtotdomnumCount_k} &= 2^{1/8} = 1.090507\ldots && k{\geq}4 && \text{A010770} \\
\sqrt[2^k]{TAdisdomnumCount_k} &\to 2^{1/8} \\
\sqrt[2^k]{TAperfdomnumCount_k} &= 2^{1/16} = 1.044273\ldots && k{\geq}4 && \text{A010778} \\
\sqrt[2^k]{TAsemitotdomnumCount_k} &= 1 \qquad k{\geq}4
\end{aligned}$$

## 12.2 Twin Alternate Turn Tree

For any non-crossing closed curve or curve continuing infinitely and not encircling its start on a square grid, the turn at revisited points is the same for each visit. An opposite turn would either enclose either the end or the start.

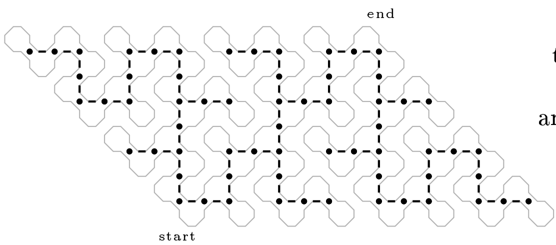



opposite turns would enclose curve end




opposite turns would enclose curve start

The twin alternate is a closed curve of this type. Some of its points are right turns. Those points and the segments between them form a tree.




twin alternate $k$=6, right turn points and segments between

This is a subdivision of twin alternate area tree $k-1$, ie. an extra vertex inserted in each edge. That follows from the same sort of connection arguments used in that tree. The connection $c$ between the two twin alternate halves is on the boundary so is a left turn. The connection changes it to a right turn and the rest of the halves are copies of the previous level.

As from the *turn* recurrence (2), the turn at odd $n$ is alternately L,R at $n \equiv 1,3 \bmod 4$ respectively. Since the curve turns either left or right at every point, this gives the odd turns at every second point in a $2{\times}2$ grid.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| + | R | + | R | + | R | + | R | + | R | + |
| L | + | L | + | L | + | L | + | L | + | L |
| + | R | + | R | + | R | + | R | + | R | + |
| L | + | L | + | L | ∘ | L | + | L | + | L |
| + | R | + | R | + | R | + | R | + | R | + |
| L | + | L | + | L | + | L | + | L | + | L |
| + | R | + | R | + | R | + | R | + | R | + |

turns at odd locations

The R turns are at $z \equiv i \bmod b^2$ in the pattern. The turns with one trailing 0-bit, so $n \equiv 2,6 \bmod 8$, are then a copy at 45° and opposite R,L, and so on.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L | R | R | R | L | R | R | R | L | R | R | R | L | R | R |
| L | · | L | R | L | · | L | R | L | · | L | R | L | · | L |
| R | R | L | R | R | R | L | R | R | R | L | R | R | R | L |
| L | L | L | · | L | L | L | ∘ | L | L | L | · | L | L | L |
| L | R | R | R | L | R | R | R | L | R | R | R | L | R | R |
| L | · | L | R | L | · | L | R | L | · | L | R | L | · | L |
| R | R | L | R | R | R | L | R | R | R | L | R | R | R | L |

turns at locations 0 to 2 trailing 0-bits

The effect is to make twin alternate curves tiling the plane. The left turns likewise, turned 90°.

## 13 Fractional Locations

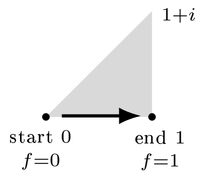
A fractional point $0 \le f \le 1$ in the alternate paperfolding curve fractal is a limit

$$\mathit{fpoint}(f) = \lim_{k\to\infty} \mathit{unexpand}^k\Big(\mathit{point}\big(\lfloor f.2^k \rfloor\big)\Big) \qquad \text{fractional point}$$

$$\mathit{unexpand}^k(z) = \mathit{expand}^{-k}(z) = \begin{cases} z/\mathit{End}_k & \text{if } k \text{ even} \\ \overline{z}/\mathit{End}_k & \text{if } k \text{ odd} \end{cases}$$

$n = \lfloor f.2^k \rfloor$ is the first $k$ bits below the binary point of $f$. Stopping there means an $f$ somewhere in a sub-curve $k$. That sub-curve has a fixed finite extent which decreases as $1/\sqrt{2}^k$ so *fpoint* converges on some $z$.

The effect of repeated unexpands this way is that $f = 0 \text{ to } 1$ is a triangular region on the left of the endpoints. This is an even curve level unexpanded an even number of times. Such an unexpand is simply scale by $1/2^{k/2}$. For an odd level pyramid with end at the top, the unexpands are mirror image and scale giving the same result.




As from *point* in section 4, *fpoint* is a change of $f$ bits $2^{-j}$ to terms $\pm End_{-j}$. The $\pm$ signs are per the rule at (63), flip below each 11 bit pair.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $fpoint(f) =$ | $+End_{-1}$ | $+End_{-2}$ | $-End_{-3}$ | $+End_{-4}$ | | $-End_{-6}$ | $-End_{-7}$ | | |
| $f = .$ | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | $\cdots$ |
| | | | $-$ | $-$ | $-$ | | | $-$ | |

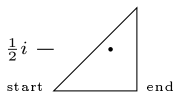
When $f$ is rational, its bits are an initial fixed part then a repeating periodic part (of length at most denominator $-$ 1). The *End* terms and sign changes are then likewise periodic and give a location as some $x{+}iy$ with rational $x, y$.

If the periodic part of $f$ has an odd number of 11 bit pairs then that is a net negative on the resulting *End* terms. This can be accounted for in the calculation, or doubling the length of the periodic part ensures an even number of sign changes for purely periodic in *End* terms.

When $f$ is irrational, it might give a rational Re or Im. The simplest is when there are 1-bits only at even positions so *End* terms all real and Im$\,=0$. An example eventually all even positions is the Kempner-Mahler number, a sum of powers of powers of 2 of a type Kempner [14] showed is transcendental.

$$\begin{aligned} KM &= \sum_{j=0}^{\infty} \frac{1}{2^{2^j}} \quad = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{65536} \cdots \\ &= 0.81642150\ldots && \text{A007404} \\ &= 0.11010001\ldots \text{ binary} && \text{A036987} \end{aligned}$$

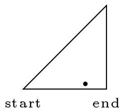
After the initial $\frac{1}{2}$ all 1-bits are at even positions. The *End* terms effectively halve the number of 0s between so with the *KM* bit pattern this is $j{-}1$ in each term, so limit




$$\begin{aligned} fpoint(KM) &= \tfrac{3}{2} - KM \ + \tfrac{1}{2}i \\ &= 0.683578\ldots + 0.5i \\ \text{binary } &= .1010111011111\ldots && \text{A267442} \end{aligned}$$

For the alternating signs sum $C(2,\infty)$ of Shallit at (16), the alternating signs in the sum give 1-bits in runs. Working through their positions is

$$fpoint\big(C(2,k)\big) \ = \tfrac{1}{2}{+}\tfrac{1}{2}i \ + \sum_{j=0}^{k-1} \frac{i^{j-1}}{2^{2^j}} \tag{166}$$




$$= \left( \tfrac{1}{2} + \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \frac{(-1)^j}{4^{4^j}} \right) + \left( \tfrac{1}{2} - \sum_{j=0}^{\lceil k/2 \rceil - 1} \frac{(-1)^j}{2^{4^j}} \right) i \qquad (167)$$

$$\to 0.746093\ldots + 0.062484\ldots i$$

The $i$ power in (166) is a kind of rotating sum with terms $1, i, -1, -i$ instead of alternating $\pm 1$. The real and imaginary parts are separated at (167), with $2j$ or $2j{+}1$ worked into the denominator. The resulting $4^j$ powers in each are variants Kempner also noted have transcendental limits.

**Theorem 36.** *Fractional points $f$ on the $x$ axis (right boundary) are*

$$\mathit{fXpred}(f) = \mathit{1} \text{ if } f = \begin{cases} 1 \\ \text{or base-4 digits only 0,1} \\ \text{or } (n+\frac{2}{3})/4^k \text{ where } n \text{ odd integer} \\ \quad\quad \text{of } k \text{ many base-4 digits 0,1} \end{cases}$$

*Fractional points $f$ on the $x{=}y$ diagonal are*

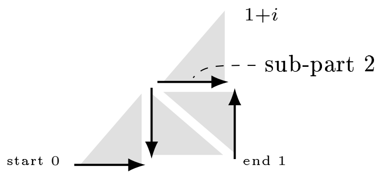
$$\mathit{fGpred}(f) = \mathit{1} \text{ if } f = \begin{cases} \text{base-4 digits only 0,2} \\ \text{or } (n+\frac{5}{6})/4^k \text{ where } n \text{ integer} \\ \quad\quad \text{of } k \text{ many base-4 digits 0,2} \end{cases}$$

*Fractional points $f$ on the vertical at $x{=}1$ are*

$$\mathit{fVpred}(f) = \mathit{1} \text{ if } f = \begin{cases} \text{base-4 digits only 2,3} \\ \text{or } (n+\frac{1}{3})/4^k \text{ where } n \text{ even integer} \\ \quad\quad \text{of } k \text{ many base-4 digits 2,3} \end{cases}$$

*fGpred and fVpred both allow $k{=}0$ for no digits just $f{=}\frac{5}{6}$ or $f{=}\frac{1}{3}$ respectively.*

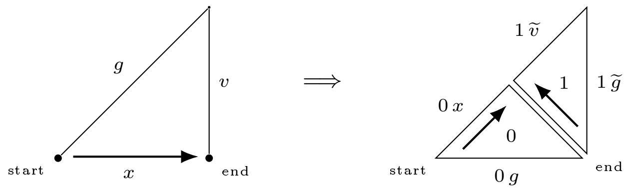
*Proof.* $f$ at the top corner $z = 1{+}i$ is found by considering two expansions,




Only sub-curve 2 touches the top corner point $z = 1{+}i$. Likewise in sub-curves of it so that $f = .222\ldots$ base-4 $= \frac{2}{3}$ is the only $f$ there.

$f$ at the start of the curve is only $f{=}0$ by a similar argument. Only sub-curve 0 touches the start so $f = .000\ldots$ base-4.

For the theorem, take the curve sides $x$ axis, $g$ diagonal, and $v$ vertical. The curve comprises self-similar halves,




The respective sides are then

$$x = 0g \text{ or } 1111\ldots \qquad g = 0x \text{ or } 1\widetilde{v} \qquad v = 0\widetilde{g} \text{ or } 0101\ldots$$

Side $x$ is $g$ of part 0, so a 0 bit followed by $g$. Also the start of the unfolded part 1, which is $f$=0 unfolded to $f$=1 and represented as binary .111....

Side $g$ is $x$ of part 0, and $v$ of part 1. The unfolding means the latter is reversed $1-v$ which is indicated by $\widetilde{v}$. The effect of that negation is to flip bits 0↔1.

Side $v$ is the top corner of part 0 which is $f$=$\frac{2}{3}$ so bit 0 followed by 1010.... Also $g$ of part 1, in reverse which is bit flipped.

These descents, with sides forward and reversed, are then a state machine

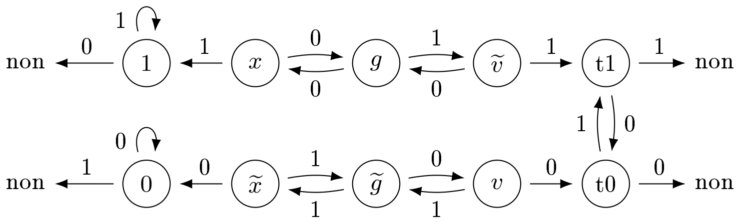



Figure 28: boundary $f$ bits

The starting state is the desired $x, g, v$ side. An $f$ with bits remaining always in the state machine is on the boundary. If it ever reaches "non" then $f$ is non-boundary.

The base-4 digit conditions and offsets of the theorem can be expressed as state machines and are the same as figure 28.

The *fXpred* case of digits all 0,1 includes exact fractions $n/4^k$ when infinite trailing 0 digits. The state machine in figure 28 matches such fractions ending zero bits 1000... and also ending infinite 1-bits 0111.... Similarly *fGpred* case digits 0,2.

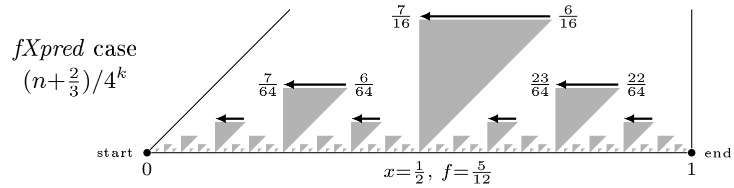
The *fVpred* case of digits all 2,3 includes exact fractions $n/4^k$ when all trailing 3s. The state machine in figure 28 matches those both as trailing 1-bits and trailing 0-bits. □

When considering whether a given $f$ is on the boundary it might be known or proved $f$ is not a 3rd or 6th and so not subject to the $+\frac{2}{3}$ etc cases. For example any irrational $f$ is not a 3rd. The conditions then become, reckoning the first bit below the binary point as position $-1$ so odd,

$$\begin{aligned} \mathit{fXpred} &= \text{ 0s at all odd bit positions} \\ \mathit{fGpred} &= \text{ 0s at all even bit positions} \\ \mathit{fVpred} &= \text{ 1s at all odd bit positions} \end{aligned}$$

The remaining combination would be 1s at all even bit positions. This is an *fMpred* on the middle anti-diagonal line $x+y=1$ between the two half sub-curves. Points there are $0\,v$ or $1\,\widetilde{x}$, both of which are 1s at even positions.

The *fXpred* case $n+\frac{2}{3}$ is the top point of sub-curves directed West. In the finite iterations these are in enclosed unit squares on the $x$ axis. The first such is $n$=6 to 7 which is the biggest such sub-curve in the fractionals and goes $\frac{6}{16}$ to $\frac{7}{16}$ with top point $f=(6+\frac{2}{3})/16=\frac{5}{12}$ at $x$=$\frac{1}{2}$.




**Theorem 37.** *Fractional locations $z$ in the alternate paperfolding fractal are visited 1, 2, 3 or 6 times each.*

*Curve start $z$=0 and the top $z=1+i$ have 1 visit each.*

*Curve end $z$=1 has 2 visits.*

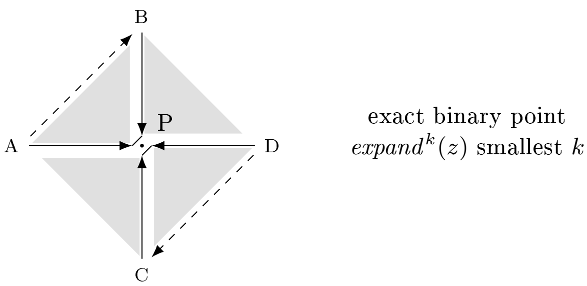
*Other exact binary locations $z=(x+iy)/2^k$ for integer $x,y,k$ have 3 visits when on the boundary or 6 visits when inside.*

*Other locations straight or 45° diagonal between exact binary points have 1 visit when on the boundary or 2 visits when inside.*

*Other locations have 1 visit.*

*Proof.* As in the proof of theorem 36, curve start and top are 1 visit and the curve end is 2 visits (top of part 0 and start of part 1).

Successive expansions put sub-curve ends at new exact binary locations between existing ones. At an exact binary $z$ this is as follows. The dashed sub-curves have expanded to A,P,B and C,P,D.




Sub-curve A end has 2 visits, being $f$=$\frac{1}{3}$ and $f$=1 in that sub-curve. Likewise B, but its $f$=1 is the same as in A so 3 visits. If $z$ is on the boundary (the boundary being A across to D) then these are the only visits. If $z$ is not on the boundary then sub-curves C and D are 3 more visits.

A location $z$ straight or 45° diagonal between exact binary points is between sub-curves

Since $z$ here is not an exact binary fraction, subsequent expansions have the location always between two such sub-curves this way. So only ever 2 bit patterns of $f$ adjacent to the point and so 2 visits when inside the curve or 1 visit when the boundary.

Otherwise $z$ is inside a sub-curve and remains inside on every expansion so is always a finite distance away from anything except its contained sub-curve and so just 1 visit. □

**Theorem 38.** *The number of visits for a point $f$ in the alternate paperfolding fractal are*

$$
\begin{aligned}
O &= f \textit{ odd position bits eventually all 0s or all 1s} \\
E &= f \textit{ even position bits eventually all 0s or all 1s}
\end{aligned}
$$

$$
\mathit{fvisits}(f) = \begin{cases}
1 & \textit{if } f = 0 \text{ or } \frac{2}{3} \\
2 & \textit{if } f = 1 \text{ or } \frac{1}{3} \\
 & \textit{and otherwise} \\
1 & \textit{if neither O,E} \\
1 & \textit{if one of O,E and } f \textit{ on the boundary} \\
2 & \textit{if one of O,E and } f \textit{ not on the boundary} \\
3 & \textit{if both O,E and } f \textit{ on the boundary} \\
6 & \textit{if both O,E and } f \textit{ not on the boundary}
\end{cases}
$$

*Proof.* An $f$ which is eventually O or E is one of *fXpred*, *fGpred* or *fVpred* for those bits, so on the boundary of some sub-curve.

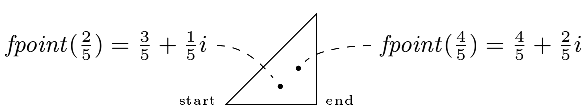
When both O and E, $f$ is eventually 0 or 1 when O and E both 0s or both 1s, or eventually $\frac{1}{3}$ when they are opposites. These are at a corner of the sub-curve which is an exact binary $z$ so 3 or 6 visits.

When just one O or E, the location is on a sub-curve boundary but never an exact binary and so its $z$ is straight or 45° between exact binary and so 1 or 2 visits.

Neither O nor E means never on the boundary of a sub-curve so $z$ always within a sub-curve and so 1 visit. □

The case of one of O,E can be $f$ either rational or irrational. When rational it is an $f$ with an eventually repeating pattern of base-4 digits. A denominator for the repeating part is some $4^h-1$, but not 3 since that is an exact binary location. Such a denominator has a factor 3, but also other factors. For example $\mathit{fpoint}(\frac{7}{15}) = \frac{2}{3}+\frac{1}{3}i$ is on the $x+y=1$ anti-diagonal and not an exact binary.

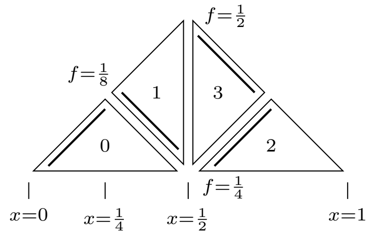
The case of neither O,E can be $f$ either rational or irrational. A rational example is $\mathit{fpoint}(\frac{2}{5}) = \frac{3}{5}+\frac{1}{5}i$. Its expansions go in a cycle of $f$ within sub-curves $\frac{2}{5} \to \frac{4}{5} \to \frac{2}{5}$. Both it and $\mathit{fpoint}(\frac{4}{5})$ are non-boundary in their sub-curves.



**Theorem 39.** *The only fixed points fpoint$(f)=f$ of the alternate paperfolding curve are $f=0$ and $f=1$.*

*Proof.* The fixed point requires $y=0$, so an *fXpred* boundary point. From the *fXpred* cases or bit patterns, the maximum $f$, other than $f=1$, is $f = .011010\ldots = \frac{5}{12}$. So there are no fixed points in the range $\frac{5}{12} < f < 1$.

The fixed point requires $x-f = 0$ but it can be sees $x > f$ in the range $0 < f < \frac{1}{2}$ by considering sub-curves in that range.




$f = 0$ to $\frac{1}{2}$
sub-curves

Parts 2 and 3 are $f = \frac{1}{4}$ to $\frac{1}{2}$ and $x = \frac{1}{2}$ to 1. So $x > f$ other than $f=\frac{1}{2}$.

Part 1 is $f = \frac{1}{8}$ to $\frac{1}{4}$ and $x = \frac{1}{4}$ to $\frac{1}{2}$. Only $f=\frac{1}{4}$ is common to these and that point is $x=\frac{1}{2}$, so $x > f$.

Part 0 can be taken in further sub-curves. They scale $x/2$ and $f/4$ so that $f$ is yet smaller, giving $x > f$ other than at $f=0$. □

$x-f$ and $y$ can be illustrated by plotting as functions of $f$.

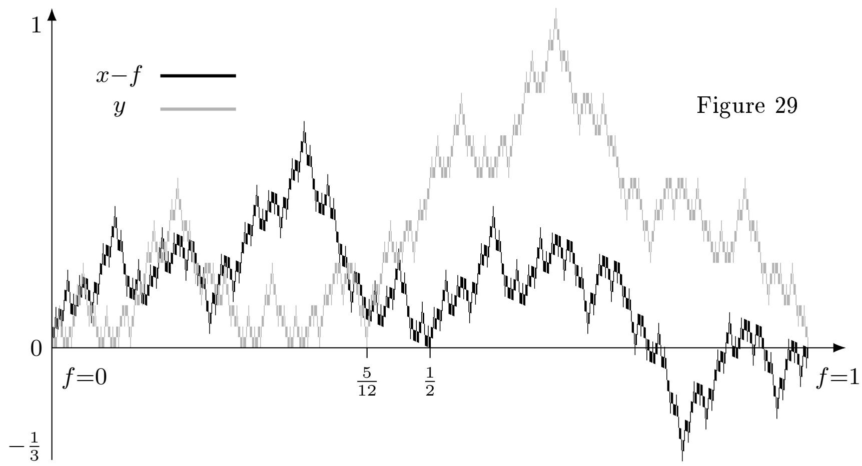



Figure 29

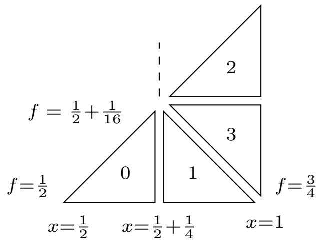
A fixed point would be an $f$ axis touch by both $x-f$ and $y$ at the same place. But $x-f$ touches only at and above $\frac{1}{2}$, and $y$ touches only at and below $\frac{5}{12}$, other than $f=0$ and $f=1$.

**Theorem 40.** *The diagonal fixed points fpoint$(f) = (1+i).f$ of the alternate paperfolding curve are curve start $f=0$ and the middle $f=\frac{1}{2}$ at $z=\frac{1}{2}+\frac{1}{2}i$.*

*Proof.* A diagonal fixed point requires both $x=f$ and $y=f$. As from theorem 39, $x > f$ for $0 < f < \frac{1}{2}$ so there are no diagonal fixed points in that range.

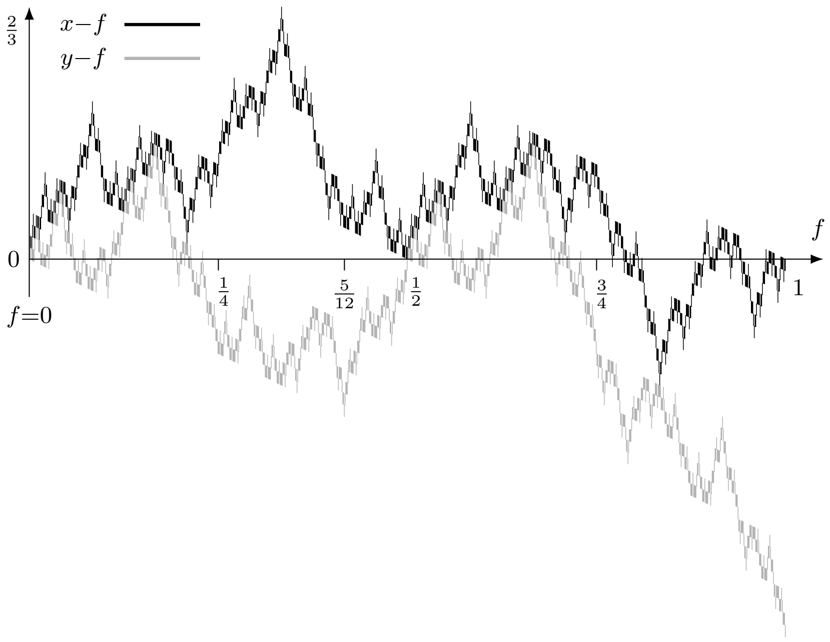
For $\frac{3}{4} \le f \le 1$ have $y < \frac{1}{2}$, so $f > y$ and no diagonal fixed points in that range.

For $\frac{1}{2} < f < \frac{3}{4}$, consider 4 sub-parts of it as follows,




Parts 1,2,3 are at least $+\frac{1}{4}$ from the start at $x=\frac{1}{2}$ so $x \ge \frac{3}{4}$. But those parts have $f$ at most $+\frac{1}{4}$ from the start $f=\frac{1}{2}$ so $f < \frac{3}{4}$ and $x > f$. In part 0, a corresponding argument applies but with $x$ offset halved and $f$ offset quartered, so again $x > f$, and so on in successive sub-parts. $\square$

The conditions for a diagonal fixed point can be illustrated by plotting $x-f$ and $y-f$ as functions of $f$.




A diagonal fixed point would be an $f$ axis touch by $x-f$ and $y-f$ at the same place. $x-f$ is in black the same as figure 29. $y-f$ is in grey. It descends to $y-f = -1$ which is $y=0$ at $f=1$.

For $f > \frac{3}{4}$, $y-f$ is all negative with no axis touches so no diagonal fixed points. There are $x-f$ axis touches in that range, but no $y-f$.

Conversely, in $0 < f \le \frac{3}{4}$ have various $y-f$ axis touches, but $x-f$ is positive except at $f=\frac{1}{2}$.

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