PAPERS ON SPACE ANALYSIS

MACFARLANE

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BY

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CONTENTS:

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PRINCIPLES OF THE ALGEBRA OF PHYSICS. By Prof. A. MACFARLANE,
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[This paper was read before a joint session of Sections A and B on August 21.]

La seule manière de bien traiter les élémens d'une science exacte et rigoureuse, c'est d'y mettre toute la rigueur et l'exactitude possible.

D'ALEMBERT.

The question as to the possibility of representing areas and solids by means of the apparent multiplication of the symbols for lines has always appeared to me to be one of great difficulty in the application of algebra to geometry; nor has the difficulty, I think, been properly met in works on the subject.

D. F. GREGORY.

Tant que l'algèbre et la géometrie ont été sépareés, leur progrès ont été lents et leurs usages bornés, mais lorsque ces deux sciences se sont réunies, elles se sont prêtcés des forces mutuelles, et ont marché ensemble d'un pas rapide vers la perfection.

LAGRANGE.

IN the preface to the new edition of the *Treatise on Quaternions* Professor Tait says, "It is disappointing to find how little progress has recently been made with the development of Quaternions. One cause, which has been specially active in France, is that workers at the subject have been more intent on modifying the notation, or the mode of presentation of the fundamental principles, than on extending the applications of the Calculus." At the end of the preface he quotes a few words from a letter which he received long ago from Hamilton—"Could anything be simpler or more satisfactory? Don't you feel, as well as think, that we are on the right track, and shall be thanked hereafter? Never mind when." I had the high privilege of studying under Professor Tait, and know well his single-minded devotion to exact science. I have always felt that Quaternions is on the right track, and that Hamilton and Tait deserve and will receive more and more as time goes on thanks of the highest order. But at the same time I am convinced that the notation can be improved; that the principles require to be corrected and extended; that there is a more complete algebra which unifies Quaternions, Grassmann's method and Determinants, and applies to physical quantities in space. The guiding idea in this paper is generalization. What is sought for is an algebra which will apply directly to physical quantities, will include and unify the several branches of analysis, and when specialized will become ordinary algebra. That the time is opportune for a discussion of this problem is shown by the recent discussion between Professors Tait and Gibbs in the columns of *Nature* on the merits of Quaternions, Vector Analysis, and Grassmann's method; and also by the discussion in the same Journal of the meaning of algebraic symbols in applied mathematics.

A student of physics finds a difficulty in the principle of Quaternions which makes the square of a vector negative. Hamilton says, *Lectures*, page 53, "Every line in tri-dimensional space has its square equal to a negative number, which is one of the most novel but essential elements of the whole quaternion theory." Now, a physicist is accustomed to consider the square of a vector quantity as essentially positive, for example, the expression $\frac{1}{2}mv^2$. In that expression $\frac{1}{2}m$ is positive, and as the whole is positive, v^2 must be positive; but v denotes the velocity, which is a directed quantity. If this is a matter of convention merely, then the convention in quaternions ought to conform with the established convention of analysis; if it is a matter of truth, which is true?

The question is part of the wider question—Is it necessary to take, as is done in quaternions, the scalar part of the product of two vectors negatively? I find that not only can problems, involving products of vectors, be worked out without the minus, but that the expressions so obtained are more consistent with those of algebra. Let, for example (fig. 1), **A** denote a vector of length a and direction α, and **B** another vector of length b and direction β, their sum is **A** + **B**, and the square of their sum I take to be $a^2 + 2ab \cos \alpha\beta + b^2$, where $\cos \alpha\beta$ denotes the cosine of the angle between the directions α and β. Suppose **B** to change until its direction is the same as that of **A**, the above expression becomes $a^2 + 2ab + b^2$, which agrees with the expression in algebra. But the quaternion method makes it $-(a^2 + 2ab + b^2)$. The sum of **A** and the opposite of **B** is **A**—**B**; its square is $a^2 - 2ab \cos \alpha\beta + b^2$ which becomes $a^2 - 2ab + b^2$, when **A** and **B** have the same direction, but according to quaternions it is $-a^2 + 2ab - b^2$.

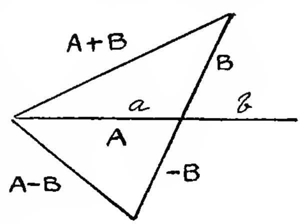

FIG. 1.

In ordinary algebra there are two kinds of quantity, the arithmetical or signless quantity, and what is called the algebraic quantity. The former (fig. 2), can be adequately represented on a straight line produced indefinitely in one direction from a fixed point. All the additive quantities are laid off end to end, and from the final point the subtractive quantities are laid off end to end, but in the opposite direction. The final point must stop short of the origin, in order that the result may be possible, under the supposition that the quantity is signless. But the algebraic quantity requires for its representation (fig. 3), a straight line produced

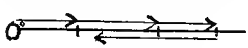

FIG. 2.

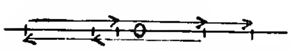

FIG. 3.

indefinitely in either direction from the fixed point. It is a directed quantity, which may have one or other of two directions. But though this quantity has a sign, its square is signless, or essentially positive. Hence only a positive quantity has a square root, and that root is ambiguous, on account of the two directions which the algebraic quantity may have. The generalization of this for space is that the square of any directed quantity is essentially positive, and that the square root of a signless quantity is entirely ambiguous as regards direction.

There is a want of harmony between the notation of Quaternions and that of Determinants. Let, as usual,

$\alpha = xi + yj + zk, \qquad \beta = x'i + y'j + z'k, \qquad \gamma = x''i + y''j + z''k,$

then

$$S\alpha\beta\gamma = -\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

Would it not be simpler, if the scalar of α, β, γ, which has the same geometrical meaning as the determinant, had also the same sign? The inconsistency in sign arises from taking the scalar negatively and from taking the positive order of the vectors in the product to be from right to left. The positive order ought to be that of the natural order in writing, namely, from left to right, and from top to bottom. And why is it that only this determinant appears in quaternions while the whole series appears in the *Ausdehnungslehre?*

Another associated question is — Why is ∇^2 equal not to Laplace's operator but to the negative of it? Given the definition of ∇ as meaning $\frac{d}{dx}\,i + \frac{d}{dy}\,j + \frac{d}{dz}\,k$, does it follow necessarily that $\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$? I have nowhere seen a reason given for the introduction of the minus, excepting one drawn from the analogy to the supposed negative square of a vector. If it is neither untrue, nor inconsistent, it would certainly be simpler to have

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}.$$

In Thomson and Tait's *Natural Philosophy*, vol. I, p. 178, ∇^2 is defined as equal to Laplace's operator; also in other works, as Lamb's *Hydrodynamics*, and Ibbetson's *Mathematical Theory of Elasticity*. In Clerk Maxwell's *Electricity and Magnetism*, vol. II, p. 237, it is defined negatively; but in a footnote it is stated that the negative sign is employed to make the expressions consistent with those in which quaternions are employed. As further examples of this anomaly I may instance $du = -Sd\rho\,\nabla u$, and $\nabla\rho = -3$.

The investigation of this question means the investigation of the fundamental rules of quaternions. These we find in the rules for the combination of the symbols i, j, and k, namely:

$$jk = i \qquad ki = j \qquad ij = k$$
$$kj = -i \qquad ik = -j \qquad ji = -k$$
$$i^2 = -1 \qquad j^2 = -1 \qquad k^2 = -1$$

In the preface to his *Lectures* Hamilton narrates how, in his search for the extension to space of the imaginary algebra of the plane, he arrived at these rules, and how having formulated and partly tested them he felt that the new instrument for applying calculation to geometry had been attained. How are these rules established, not as properties of symbols, but as truths in geometry and physics? Writers on quaternions illustrate them by two different things—the summing of angles in space, and the rotation of a line about an axis. Let (fig. 4) i, j, k, denote three mutually perpendicular axes which are usually designated as the axes of x, y and z. In order to distinguish clearly between an axis and a quadrant of rotation about the axis, let $i^{\frac{\pi}{2}}$, $j^{\frac{\pi}{2}}$, $k^{\frac{\pi}{2}}$ denote quadrants of rotation in the positive direction about the respective axes. The directions of positive rotation are indicated by the arrows. Now in quaternions by $k^{\frac{\pi}{2}} j^{\frac{\pi}{2}}$ is meant (the $\frac{\pi}{2}$ is not expressed explicitly) a quadrant of the great circle round j followed by a quadrant of the great circle round k; the sum of these is the quadrant from k to j, which is the negative of a quadrant round i or $i^{\frac{3\pi}{2}}$; or it may be considered as a quadrant round $-i$, and therefore denoted by $-i^{\frac{\pi}{2}}$. Hence, supposing the order of the summing to be from right to left,

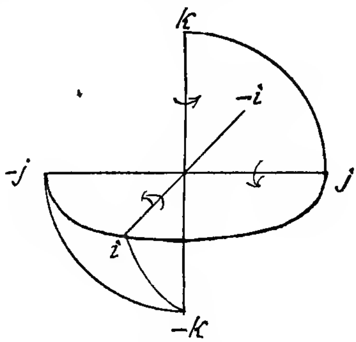

FIG. 4.

$$k^{\frac{\pi}{2}} j^{\frac{\pi}{2}} = -i^{\frac{\pi}{2}}, \qquad i^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = -j^{\frac{\pi}{2}}, \qquad j^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = -k^{\frac{\pi}{2}}.$$

Again (see fig. 4) by $j^{\frac{\pi}{2}} k^{\frac{\pi}{2}}$ is meant a quadrant of the great circle round k followed by a quadrant of the great circle round j; this is equivalent to the quadrant from $-j$ to $-k$, which is a quadrant of the great circle round i and in the positive direction; hence, $j^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = i^{\frac{\pi}{2}}$ and similarly $k^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = j^{\frac{\pi}{2}}$ and $i^{\frac{\pi}{2}} j^{\frac{\pi}{2}} = k^{\frac{\pi}{2}}$.

By $i^{\frac{\pi}{2}} i^{\frac{\pi}{2}}$ is meant the sum of two quadrants of the great circle round i; it is equivalent to a semicircle round i; hence $i^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = i^{\pi}$. But as any one of the semicircles round i (for example, the one from j to $-j$ in the positive direction), is equivalent to any one of the semicircles joining the same points, the axis of $i^{\frac{\pi}{2}} i^{\frac{\pi}{2}}$ is not restricted to i, but may be any axis whatever. Let such indefinite axis be denoted by ξ; then $i^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = \xi^{\pi}$ and this is what is denoted by -1. Similarly $j^{\frac{\pi}{2}} j^{\frac{\pi}{2}} = \xi^{\pi} = -1$; and $k^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = \xi^{\pi} = -1$. Thus the minus comes in from the repetition of a quadrantal versor, and it is itself a versor with an indefinite axis. If the order of the versors and the order of writing are identified, the rules are

$$j^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = -i^{\frac{\pi}{2}} \qquad k^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = -j^{\frac{\pi}{2}} \qquad i^{\frac{\pi}{2}} j^{\frac{\pi}{2}} = -k^{\frac{\pi}{2}}$$

$$k^{\frac{\pi}{2}} j^{\frac{\pi}{2}} = i^{\frac{\pi}{2}} \qquad i^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = j^{\frac{\pi}{2}} \qquad j^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = k^{\frac{\pi}{2}}$$

$$i^{\frac{\pi}{2}} i^{\frac{\pi}{2}} = \xi^{\pi} = -1; \quad j^{\frac{\pi}{2}} j^{\frac{\pi}{2}} = \xi^{\pi} = -1 \qquad k^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = \xi^{\pi} = -1$$

If the process of finding the sum of two arcs of great circles is distributive, then by the application of the above rules, we can find the sum of any two quadrantal arcs. Let $li + mj + nk$ denote one axis, and $l'i + m'j + n'k$ another; then $(li + mj + nk)^{\frac{\pi}{2}}$ denotes a quadrant of the great circle round the former, and $(l'i + m'j + n'k)^{\frac{\pi}{2}}$ a quadrant of the great circle round the latter. The sum of the former and the latter in the order named is denoted by $(li + mj + nk)^{\frac{\pi}{2}} (l'i + m'j + n'k)^{\frac{\pi}{2}}$. If the rule of distribution holds, the sum is equal to

$$(li^{\frac{\pi}{2}} + mj^{\frac{\pi}{2}} + nk^{\frac{\pi}{2}})(l'i^{\frac{\pi}{2}} + m'j^{\frac{\pi}{2}} + n'k^{\frac{\pi}{2}})$$

$$= -(ll' + mm' + nn') + (mn' - nm')i^{\frac{\pi}{2}} + (nl' - ln')j^{\frac{\pi}{2}} + (lm' - ml')k^{\frac{\pi}{2}}$$

and by applying the rule of distribution in the reverse order

$$= -(ll' + mm' + nn') + \left\{ (mn' - nm')i + (nl' - ln')j + (lm' - ml')k \right\}^{\frac{\pi}{2}}.$$

The first term has any axis, an angle π, and a multiplier $ll' + mm' + nn'$. The second term has an axis

$$\frac{(mn' - nm')i + (nl' - ln')j + (lm' - ml')k}{\sqrt{(mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2}}$$

an angle $\frac{\pi}{2}$ and a multiplier

$$\sqrt{(mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2}$$

These two terms together denote the arc of a great circle which is the sum of the two given arcs, its axis being the axis specified and its angle such that $-(ll' + mm' + nn')$ is its cosine.

We have next to consider the other meaning which is given to the fundamental rules: that they express the effect of a rotation on a line. Let $i^{\frac{\pi}{2}} j$ denote the turning by a quadrant round i of a line initially along j; and here I introduce the $\frac{\pi}{2}$ to denote explicitly what is meant by the first symbol. Hamilton obtains the same elementary rules as before, namely,

$$j^{\frac{\pi}{2}} k = i \qquad k^{\frac{\pi}{2}} i = j \qquad i^{\frac{\pi}{2}} j = k$$
$$k^{\frac{\pi}{2}} j = -i \qquad i^{\frac{\pi}{2}} k = -j \qquad j^{\frac{\pi}{2}} i = -k$$
$$i^{\frac{\pi}{2}} i = -1 \qquad j^{\frac{\pi}{2}} j = -1 \qquad k^{\frac{\pi}{2}} k = -1$$

or, to speak more correctly, the first six are obtained, while the remaining three are assumed. A quadrant rotation round j (see fig. 4) changes a line originally along k to a line along i; hence the direction denoted by $j^{\frac{\pi}{2}} k$ is identical with the direction i. Similarly, for the other two equations of the first set. A quadrant rotation in the positive direction round k turns a line originally along j to a line in the direction opposite to i; hence $k^{\frac{\pi}{2}} j = -i$. Similarly for the other two equations of the second set. If we keep to the same meaning of the symbols as before, $i^{\frac{\pi}{2}} i$ ought to mean the effect of a quadrant rotation round i upon a line in the direction of i; and as that produces no change, we ought to have $i^{\frac{\pi}{2}} i = i$. Similarly $j^{\frac{\pi}{2}} j = j$ and $k^{\frac{\pi}{2}} k = k$. It follows that the true meaning of the rules lies in the summing of versors or arcs of great circles, and not in the rotation of a line.

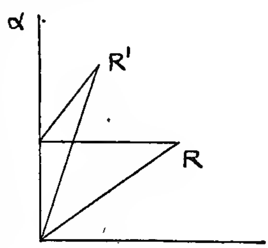

FIG. 5.

This will be seen more clearly when we attempt to form the product of a quadrantal rotation round any axis and any line. Let $li + mj + nk$ denote the axis a (fig. 5), round which there is a quadrant of rotation, and $xi + yj + zk$ the line **R** which is turned. If the distributive rule applies, we get the result by decomposing the quadrant rotation round the given axis into the sum of three component rotations

$$li^{\frac{\pi}{2}} + mj^{\frac{\pi}{2}} + nk^{\frac{\pi}{2}}$$

and finding their several effects on the several components of the line $xi+yj+zk$. According to the quaternion rules we obtain — $(lx+my+nz)$ $+(mz-ny)\,i+(nx-lz)\,j+(ly-mx)\,k$. Now this expression is not the expression for the resulting line, or for any line, unless $lx+my+nz=0$. What is the true expression? It is $(lx+my+nz)\,(li+mj+nk)$ which is the component along the axis, and $(mz-ny)\,i+(nx-lz)\,j+(ly-mx)\,k$ is the expression for the other component, which is perpendicular to the axis and the initial line. The argument here is, of course, not so much about the proper expression for the result of the rotation, as about the meaning of the fundamental rules.

To make the rules which are true for versors applicable to vectors, it is necessary to identify a vector of unit length with a quadrantal versor having the same axis. In the new edition of his *Elements*, p. 46, Prof. Tait makes the transition from versors to vectors thus "One most important step remains to be made. We have treated i, j, k simply as quadrantal versors, and **i, j, k** as unit-vectors at right angles to each other, and coinciding with the axes of rotation of these versors. But if we collate and compare the equations just proved, we have $i^2=-1$, $\mathbf{i}^2=-1$, § 9; $ij=k$ and $i\mathbf{j}=\mathbf{k}$; $ji=-k$ and $j\mathbf{i}=-\mathbf{k}$. Now the meanings we have assigned to i, j, k are quite independent of, and not inconsistent with, those assigned to **i, j, k**. And it is superfluous to use two sets of characters when one will suffice. Hence it appears that i, j, k may be substituted for **i, j, k**; in other words, *a unit-vector when employed as a factor may be considered as a quadrantal versor whose plane is perpendicular to the vector.* Of course, it follows that every vector can be treated as the product of a number and a quadrantal versor. This is one of the main elements of the singular simplicity of the quaternion calculus."

By i is here meant what we have designated by $i^{\frac{\pi}{2}}$ and by **i** a unit-vector along the axis of i. We have already seen one difficulty opposing the identification, namely, taking as a principle that $i^{\frac{\pi}{2}}\,i=-1$. But waiving that insuperable objection, there still remains for consideration the case of the combination of two vectors. This kind of product, in which both factors are vectors, has in recent times been generally neglected. This is evident from what is said by Clifford (Mathematical papers, p. 386) "In every equation we must regard the *last* symbol in every term as either a vector or an operation; but all the others must be regarded as operations." This view does not explain the product of physical quantities.

Let xi, yj, zk denote line-vectors along the axes of i, j, k respectively; then according to the principles of quaternions

$$(yj)\,(zk)=yzi \qquad (zk)\,(xi)=zxj \qquad (xi)\,(yj)=xyk$$
$$(zk)\,(yj)=-yzi \qquad (xi)\,(zk)=-zxj \qquad (yj)\,(xi)=-xyk$$
$$(xi)\,(xi)=-x^2 \qquad (yj)\,(yj)=-y^2 \qquad (zk)\,(zk)=-z^2.$$

As the distributive principle is to be applied, the meaning of these partial products must be such that the product of any two vectors is obtained by taking the products of the several components of the one with the several components of the other.

Let $yjzk$ denote or be represented (fig. 6) by the rectangle included between yj and zk; its magnitude is yz and its orientation is defined by jk. But in space of three dimensions the aspect or orientation jk may be represented so far as direction is concerned by the complementary axis i. Hence we may write $yjzk = yzjk = yzi$. Similarly, $zkxi = zxki = zxj$ and $xiyj = xyij = xyk$. The expression $zykj$ denotes the same area in magnitude and plane as $yzjk$, but is taken the opposite way round; the complementary axis is $-i$. In the same sense $ik = -j$ and $ji = -k$. So far, the quaternion rules appear to hold good but even here, a difficulty appears on closer consideration. We have taken the vectors in the order of writing and obtain $jk = i$; if, as was pointed out, we take the versors also in the order of writing we obtain $j^{\frac{\pi}{2}} k^{\frac{\pi}{2}} = -i^{\frac{\pi}{2}}$.

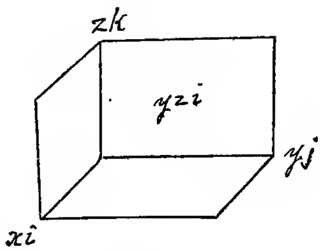

FIG. 6.

The question remains: What consistent meaning must be attached to $xixi$ and $yjyj$ and $zkzk$ in order that when they are taken along with the other partial products we may obtain a complete distributive product? The view which I have arrived at is that the expression $xixi = x^2i^2$ means the area of the square which is formed by the projection of xi on its own direction; and that it is essentially positive. Similarly y^2j^2 and z^2k^2 are essentially positive, and the three terms are to be combined by arithmetical addition. Individually they have no direction, whether their sum has or not. Hence I take the rules to be $ii = +$, $jj = +$, and $kk = +$.

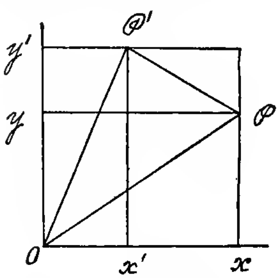

FIG. 7.

Let $\mathbf{R} = xi + yj + zk$ and $\mathbf{R}' = x'i + y'j + z'k$ be any two line-vectors. By applying the above rules distributively we obtain:

$$\mathbf{RR}' = (xi + yj + zk)\ (x'i + y'j + z'k)$$
$$= xx' + yy' + zz' + (yz' - zy')\, i + (zx' - xz')\, j + (xy' - yx')\, k.$$

Let OP and OP' be the projections of $\mathbf{R}$ and $\mathbf{R}'$ on the plane of i and j. Then from the figure (fig. 7) it is evident that the area of the triangle $OPP' = xy' - \frac{1}{2}xy - \frac{1}{2}x'y' - \frac{1}{2}(x - x')\ (y' - y) = \frac{1}{2}(xy' - yx')$

Thus $(xy' - yx')\,k$ denotes the magnitude and orientation of the parallelogram formed by the projections of $\mathbf{R}$ and $\mathbf{R}'$ on the plane of i and j. Similarly $(yz' - zy')\,i$ denotes the oriented area formed by the projections of

R and **R**$'$ on the plane of j and k, and $(zx'-xz')\,j$ that for the plane of k and i. The geometrical sum of these areas is equal in magnitude and orientation to the area of the parallelogram formed in space by **R** and **R**$'$, or rather the area formed by **R** and the component of **R**$'$ which is perpendicular to **R**.

The expression $xx'+yy'+zz'$ is the area formed by **R** and the projection of **R**$'$ upon **R**. For (fig. 8) the projection of **R**$'$ is equal to ON, which is equal to $OL+LM+MN$, the sum of the projections on **R** of $x'i$, $y'j$ and $z'k$ respectively. Hence the product of **R** and the projection of **R**$'$ is

$$r\left(x'\frac{x}{r}+y'\frac{y}{r}+z'\frac{z}{r}\right)=xx'+yy'+zz'$$

Hence by the complete product **RR**$'$ we mean the product of **R** and the component of **R**$'$ which is parallel to **R**, together with the product of **R** and the component of **R**$'$ which is perpendicular to **R**. This product is distributive, that is, we get the same result whether we take the product directly, or take the several products of the components of **R** and **R**$'$ and add them together, the non-directed products by ordinary addition, the directed products by geometrical addition. The expression $xx'+yy'+zz'$ is one of the fundamental expressions of the Cartesian analysis; the other term is expressed by the square root of the sum of the squares of its components, namely,

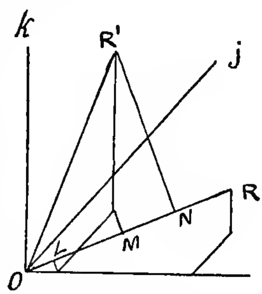

FIG. 8.

$$\sqrt{(yz'-zy')^2+(zx'-xz')^2+(xy'-yx')^2}$$

because that analysis does not provide an explicit notation for direction.

What reason do writers on quaternions give for taking $xx'+yy'+zz'$ negatively in the case of the product of two vectors? In the passage quoted above Professor Tait refers to section 9 of his *Treatise* for the proof that the square of a unit vector is -1. There we find: "It may be interesting, at this stage, to anticipate so far as to remark that in the theory of quaternions the analogue of $\cos\theta+\sqrt{-1}\,\sin\theta$ is $\cos\theta+\omega\,\sin\theta$, where $\omega^2=-1$. Here, however, ω is not the algebraic $\sqrt{-1}$, but is any directed unit-line whatever in space."

In the above expression ω really means the versor $\omega^{\frac{\pi}{2}}$. The algebraic imaginary $\sqrt{-1}$ means, as is well known, a turning of $\frac{\pi}{2}$; what is indefinite about it is that the axis is not specified; and it must be supposed constant, if the rules about the manipulation of $\sqrt{-1}$ are to hold good.

The true reason for taking the expression negatively is to satisfy the rule of association. In the preface to his *Lectures*, p. 53, Hamilton shows that if the product

$$(xi+yj+zk)\ (x'i+y'j+z'k)\ (x''i+y''j+z''k)$$

is to satisfy the associative rule, as well as the distributive, and if the scalar part already obtained in the multiplication is to be treated as a mere number, then we must have

$$xix'i=-xx' \qquad yj\,y'j=-yy' \qquad zkz'k=-zz'$$

"On this plan every line in tridimensional space has its square equal to a negative number."

But what quantity in space possesses such associative and distributive properties? It is proved to be true of the summing of versors, that is, of arcs of great circles on a sphere, when the portion of the arc designated by the versor may be taken anywhere on the great circle (fig. 9). As any two great circles have a common line of intersection, the arcs may be moved along until the second starts from the end of the first, as AB and BC. The sum of AB and BC, denoted by (AB) (BC) is equal to AC, the arc of the great circle which joins A and C. A third versor, as DE, will not in general pass through A or C, but it will meet the great circle AC in some point as D. Shift AC back to FD; then the versor FE is the sum of FD and DE, and therefore the sum of AB, BC, DE. The associative property means, that if BC and DE are first summed and then AB with the result, the arc of the great circle so obtained will be equal in magnitude and on the same circle as the arc obtained by the former mode of procedure. The proof of the theorem is not simple; in Tait's *Elementary Treatise* it is accomplished by the help of the fundamental properties of the curves known as *Spherical Conics*, discovered only in recent times by Magnus and Chasles. Doubtless many a one has been discouraged from the study of quaternions by the abstruse nature of the fundamental principles.

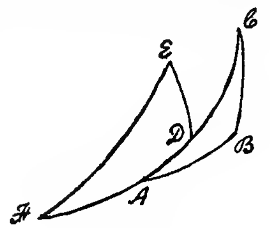

FIG. 9.

It is clear from the figure that the summing of versors cannot be adequately represented by a versor rotating a line at right angles to its axis. The versor AB followed by the versor BC may rotate a line non-conically from A to C, but the subsequent versor DE cannot in general operate in the same way upon the line at C. To do so, the great circle of DE must intersect the great circle of BC in the point C.

As the result of the investigation we conclude,

First, That the product of two vectors or directed magnitudes,

$$\mathbf{T}=pi+qj+rk \text{ and } \mathbf{S}=ui+vj+wk, \text{ is}$$

$$\mathbf{TS}=pu+qv+rw+(qw-rv)i+(ru-pw)j+(pv-qu)k.$$

Hence that there is a generalized product which includes the product of real quantities, such geometric products as are sometimes used in proving the propositions of the second book of Euclid, the products of Grassmann's *Ausdehnungslehre*, determinants, and generally the products of physical magnitudes. By a physical magnitude I mean a symbol which represents not only ratio and direction but these combined with the physical unit. The corresponding generalized algebra forms a large complement of the algebra of physics.

Second, That the product of two quadrantal versors or geometric ratios $r = (xi + yj + zk)^{\frac{\pi}{2}}$ and $r' = (x'i + y'j + z'k)^{\frac{\pi}{2}}$ is

$$rr' = -(xx' + yy' + zz') - \left\{ (yz' - zy')i + (zx' - xz')j + (xy' - yx')k \right\}^{\frac{\pi}{2}}$$

Hence that there is a generalized product which includes the product of ratios, and the product of complexes, and which is the special subject of analytical trigonometry, spherical trigonometry and the method of quaternions.

Third, The effect of a quadrantal rotation $(li + mj + nk)^{\frac{\pi}{2}}$ upon a line $xi + yj + zk$ is

$$(lx + my + nz)(li + mj + nk) + (mz - ny)i + (nx - lz)j + (ly - mx)k.$$

The subject of rotation and the effect of rotation on a line may be considered as belonging to the versor part of the algebra of space. The effect of a rotation of any angle upon a line is still more complex, and does not answer to the definition of a product as a distributive function.

Before the time of DesCartes, an algebraic quantity was represented by a line, the product of two quantities by the rectangle formed by the lines, the product of a quantity by itself as the square formed by the line, the product of three quantities by the right solid formed by the lines, which when the lines were equal, became the cube. Each term of a cubic equation was interpreted as denoting a solid, and the equation was actually solved by cutting up a cube. In order to explain higher powers than the cube, space of four or any adequate number of dimensions was imagined. This concrete view of a product corresponds to the vector part of generalized algebra.

The doctrine of DesCartes was that the algebraic symbol did not represent a concrete magnitude, but a mere number or ratio, expressing the relation of the magnitude to some unit. Hence that the product of two quantities is the product of ratios, and instead of being represented by a rectangle may be represented in the same way as either factor; that the powers of a quantity are ratios like the quantity itself, and therefore there is no need of imagining space of more than one dimension. This view of a product corresponds to the versor part of the generalized algebra.

The theory here advanced will be elaborated and developed in the pages which follow; but before proceeding to that development, I propose to consider several other objections which have been or may be made against the various methods of extending algebra to quantities in space, with the view of discussing their validity; and, if they appear to be valid, whether they are removed by the theory advanced.

Some mathematicians have objected to the negative character of the scalar in the product of two vectors. In the recent discussion in the columns of *Nature* (Vol. XLIII, p. 511), Professor Gibbs says, "When we come to functions having an analogy to multiplication, the product of the lengths of two vectors and the cosine of the angle which they include, from any point of view except that of the quaternionist, seems more simple than the same quantity taken negatively. Therefore we want a notation for what is expressed by $-S\alpha\beta$ rather than $S\alpha\beta$ in quaternions." This agrees with the theory here advanced. But I do not look upon the product of two vectors as merely having an analogy to muliplication, but as multiplication itself generalized.

It has been objected that while the scalar product and the vector product are each of primary importance, the quaternion proper which is their sum, is of very secondary importance. Thus, Professor Hyde, in a paper on the "Calculus of Direction and Position" (*Amer. Journ. of Math.*, Vol. VI, p. 3), says, "The combination of these different functions in the vector renders the product of two vectors which are neither parallel nor perpendicular to each other necessarily a complex quantity, having a scalar and a vector part corresponding to the real and imaginary parts of the ordinary complex $a+b\sqrt{-1}$, thus making a thing which should be simple just the opposite. It seems to me that quaternions proper, *i. e.*, these complex quantities, are practically of little use. In nearly all the applications to geometry and mechanics, scalars and vectors are used separately. For the special cases to which the complex $a+b\sqrt{-1}$ is put, the directed quantity is not needed."

In reply it may be said that the works of Hamilton and Tait make it abundantly evident that the quaternion idea is essential to the algebraic treatment of spherical trigonometry and of rotations. As regards the use of the complex $a+b\sqrt{-1}$, it is indefinite, unless restricted to a plane. It is shown in the development which follows that when the axis is introduced, many of the known theorems in trigonometry can be greatly extended, and that the entire meaning of the formulæ becomes evident as truths in geometry, not mere consequences from the conventional use of symbols.

In the letter to *Nature* quoted above, Professor Gibbs urges the same objection. "The question arises whether the quaternionic product can claim a prominent and fundamental place in a system of vector analysis. It certainly does not hold any such place among the fundamental geometrical conceptions as the geometrical sum, the scalar product, or the vector product. The geometrical sum $\alpha+\beta$ represents the third side of a triangle as determined by the sides α and β. $V\alpha\beta$ represents in magnitude the area of the parallelogram determined by the sides α and β, and in direction the normal to the plane of the parallelogram. $S\gamma V\alpha\beta$ represents the volume of the parallelopiped determined by the edges α, β and γ. These conceptions are the very foundations of geometry. We may arrive at the same conclusion from a somewhat narrower but very practical point of view. It will hardly be denied that sines and cosines play the leading parts in trigonometry. Now, the notations $V\alpha\beta$ and $S\alpha\beta$ represent the sine and cosine of the angle included between α and β combined in each case with certain other simple notions. But the sine and cosine combined with these auxiliary notions are incomparably more amenable to analytical transformation than the simple sine and cosine of trigonometry, exactly as numerical quantities combined (as in algebra) with the notion of positive or negative quality are incomparably more amenable to analytical transformation than the simple numerical quantities of arithmetic. I do not know of anything which can be urged in favor of the quaternionic product of two vectors as a *fundamental* notion in vector analysis, which does not appear trivial or artificial in comparison with the above considerations. The same is true of the quaternionic quotient and of the quaternion in general."

It may be observed that Professor Gibbs does not give the geometrical meaning of $S\alpha\beta$ but that of $S\alpha V\beta\gamma$. The geometrical meaning given to the latter cannot be transferred to the former. They may have a common meaning when α, β, γ denote quadrantal versors, but the common meaning is not so evident when α, β, γ denote vectors. The meaning which I attach to $V\alpha\beta$ is not, strictly speaking, the area of the parallelogram determined by the sides α and β, for then from the symmetry of the idea there would be nothing to determine the positive sign; it rather is the area formed by α and the component of β which is perpendicular to α; and as a complement we have the area formed by α and the component of β which is parallel to α. If α and β are both of unit length or, rather, if we consider their direction apart from their physical magnitude, $V\alpha\beta$ expresses the sine and $S\alpha\beta$ the cosine of the angle between the directions α and β; and in this case the product $\alpha\beta$ denotes the angle between α and β. But it is of the greatest importance that the angle should be treated as a whole, not merely the sine part separately and the cosine part separately. Thus, the argument from trigonometry leads to the opposite conclusion to that at which Professor Gibbs arrives.

It seems to me that the essence of a product is that it is a distributive function of the factors. Thus in ordinary algebra $(a+b+c)(a'+b'+c') = aa' + bb' + cc' + bc' + cb' + ca' + ac' + ab' + ba'$. We have nine partial products, and in my view the product of two quantities, each consisting of three parts, is not complete, unless it contains the nine partial products; otherwise, the product is not a generalization of the product of ordinary algebra. As a consequence of not treating together the two complementary parts of the product of two vectors, Grassmann and his followers have restricted their attention to associative products and treat of these only in a detached manner. In treating of the product of a number of vectors, that is a very arbitrary principle which holds that all the terms into which two similar directions enter must vanish; but that is a principle of the *Ausdehnungslehre* and of determinants.

Are the principles of the method of quaternions consistent with the theory of dimensions which has played so important a part in mathematical physics since the time of Fourier? Do they remove Gregory's difficulty as to how areas and solids can be represented by the apparent multiplication of lines? Professor Hyde, in the preface to the *Directional Calculus*, a valuable text-book on Grassmann's method, states that Grassmann's system is founded on and absolutely consistent with the idea of geometric dimensions, while Hamilton's is not. We find this objection amplified in the paper referred to, *Am. Jour. Math.*, Vol. VI, p. 3. "From this assumption it follows as above, that $ij = k$ and also that $i/j = -ij = -k$, *i. e.*, the ratio of two quantities is the same thing as their product except as to sign. To be sure we may say that these are units, and we have the analogy that $1/1 = 1 \times 1$; but they, *i. e.*, vectors, are geometric and directed units, and such a relation appears to me to upset all one's preconceived ideas of geometric quantities without any corresponding advantage. If, in the equation $1/1 = 1 \times 1$, 1 be taken as the unit of length, then the members of the equation have evidently not the same meaning, 1/1 being merely a numerical quantity, while 1×1 is a unit of area, it being a fundamental geometrical conception that the product of a length by a length is an area, that of a length by an area a volume, while the ratio of two quantities of the same order as that of a length to a length is a mere number of the order zero. In quaternions, however, we have the remarkable result that the product of a length by a length is not merely represented by, but actually equal to a length perpendicular to the plane of the two."

This objection is not valid against the method of quaternions as the algebra of versors or directed quotients, that is, geometric ratios; but it is valid against it as claiming to be the algebra of vectors or physical magnitudes. The primary definition of the quaternion is the quotient, not the product, of two directed lines. "From the purely geometrical point of view, a quaternion may be regarded as the quotient of two directed lines in space, or what comes to the same thing as the factor or operator which changes one directed line into another," *Ency. Brit.*, Art. *Quaternions*. The latter definition, as we have seen, is not exactly the same thing as the former; the former is the primary and true definition. The product of two vectors is derived analytically from the quotient of two vectors; no geometric meaning is attached to it as a whole, but it is interpreted as a quaternion. Thus, Hamilton, *Elements*, p. 303: "We proceed to consider, in the following section some of the general consequences of this definition, or interpretation of a product of two vectors, as being equal to a certain quotient or quaternion."

If the product of two vectors is a quaternion, then the definition of a quaternion as the quotient of two lines is not correct. But this confusion vanishes when the product of two vectors is perceived to be distinct from and independent of that of two versors. The directed part of a versor, or of any number of versors is not a line in the sense of involving the unit of length; it is of zero dimensions like the ordinary sine of trigonometry. A directed term in the product of vectors may be of one, two, three or any number of dimensions in length. A quantity having three dimensions in length is not necessarily a scalar, nor is it true that a directed quantity is necessarily of one dimension in length. The idea of an axis is different from the idea of a directed line of unit length. I look upon the symbols i, j, k as denoting not a unit-vector, but direction simply, the idea contained in the word *axis*. In writing $ij = k$, we do not equate a product of lines to a line, but the axis denoted by ij to the axis k. In space of four dimensions this equation is not true; it depends for its truth on the tridimensional character of space. In such an expression as xi it is more philosophical and correct to consider the x as embodying the unit, while i denotes simply the axis. I look upon the magnitude as containing the physical unit, to be arithmetical ratio and unit combined; and different vectors have different physical units. A line is a vector which has length for unit; a linear velocity involves length directly and time inversely; momentum involves mass and length directly and time inversely. An axis is not a physical quantity, but merely a direction. It follows from the theory of vector-algebra here advanced that the reciprocal of a vector has the same axis as the vector but the reciprocal magnitude. As the dimensions depend on the magnitude not on the axis, it follows that

$$ij = \frac{1}{i}\,j = i\frac{1}{j} = \frac{1}{i}\,\frac{1}{j} = k;$$

that is, the axis of the term which involves i and j, or of the term which involves one directly and one reciprocally, or of the term which involves both reciprocally is k.

It appears to me that this same principle of dimensions is not observed strictly in Grassmann's method or in the "Directional Calculus." We meet such an equation as $p_2 = p_1 + \varepsilon$ where p_1 and p_2 denote points and ε denotes a vector. Notwithstanding that a point is of zero dimensions and ε is used to denote a line-vector, we have a point equated to the sum of a point and a line. That ε is of one dimension in length is evident, for the expression $\varepsilon_1\varepsilon_2$ denotes the area of a parallelogram, and $\varepsilon_1\varepsilon_2\varepsilon_3$ denotes the volume of a solid, while $\varepsilon\mathfrak{f}$ denotes the moment of a force. It appears that either the equation is heterogeneous, or else p_1 and p_2 must be understood as denoting vectors from some common point; if the latter view is correct, the point-analysis reduces to a vector-analysis. From the physical point of view it is more correct to treat of a mass-vector than of a point having weight; for the differential coëfficient with respect to time of a mass-vector is the momentum, which is itself a mass-vector. If the latter is of one dimension in length, so is the former. The product of a point and a mass is not a physical idea.

Professor Hyde indicates another element in which Grassmann's method appears superior to Hamilton's. "Now quaternions deal only with the vector or line direction and the scalar — for a quaternion is only the sum of these two; it knows nothing of a vector having a definite position, which is the complete representation of the space qualities of a force." This is the distinction which Clifford emphasized between a vector which may be anywhere and one which is restricted to a definite line; to distinguish the latter from the former he introduced the word *rotor*, short for rotator, the velocity of rotation being a typical localized vector. The contrast between vector and rotor is of great importance, and it is convenient to have a notation which specifies a rotor completely as depending on two vectors. In the works of Hamilton and Tait a force is specified by two vectors, as α and ρ, the former denoting the magnitude and direction of the force, the latter the vector from an origin to the point of application. That which is denoted in quaternions by ρ is denoted in Grassmann's method by p, and it appears that p is equivalent to the vector from an origin.

The method of Grassmann is applicable, so far as it goes, to space of n dimensions, while the method of Hamilton appears to be restricted to space of three dimensions. How is it possible to unify the two and develop an algebra not only of three dimensional space but of four dimensional space? Professor Hyde, in his preface, says, "As the great generality of Grassmann's processes—all results being obtained for n-dimensional space—has been one of the main hindrances to the general cultivation of his system, it has been thought best to restrict the discussion to space of two or three dimensions . . . It seems scarcely possible that any method can ever be devised, comparable with this, for investigating n-dimensional space."

On this subject Professor Gibbs says, *Nature*, Vol. XLIV, p. 82, "Such a comparison (of Hamilton's and of Grassmann's systems) I have endeavored to make, or rather to indicate the basis on which it may be made, so far as systems of geometrical algebra are concerned. As a contribution to analysis in general, I suppose that there is no question that Grassmann's system is of indefinitely greater extension, having no limitation to any particular number of dimensions." Also in *Nature*, Vol. XLIII, p. 512, "How much more deeply rooted in the nature of things are the functions $S\alpha\beta$ and $V\alpha\beta$ than any which depend on the definition of a quaternion, will appear in a strong light, if we try to extend our formulæ to space of four or more dimensions. It will not be claimed that the notions of quaternions will apply to such a space, except indeed in such a limited and artificial manner as to rob them of their value in a system of geometrical algebra. But vectors exist in such a space, and there must be a vector analysis for such a space." In reply Professor Tait said, "It is singular that one of Professor Gibbs' objections to quaternions should be precisely what I have always considered (after perfect inartificiality) their chief merit, viz., that they are *uniquely adapted to Euclidian space*, and therefore specially useful in some of the most important branches of physical science. What have students of physics, as such, to do with space of more than three dimensions?"

The view which I have arrived at, unifying Hamilton and Grassmann and developing a more comprehensive algebra is: That $i^2 = +$ $j^2 = +$ $k^2 = +$ do not involve the condition of three dimensions, being true for space of any number of dimensions, while $ij = k$ $jk = i$ $ki = j$ do involve and indeed express the condition of three dimensions. The rules $ij = -ji$ $jk = -kj$ $ki = -ik$ are also true generally. In space of four dimensions we require four mutually rectangular axes; let the fourth be denoted by u. Then it is not true that $ij = k$; but it is true that $ijk = u$, $jku = -i$, $kui = j$, $uij = -k$.

A difficulty has been felt in the apparent heterogeneity of a sum of scalar and vector terms. Hamilton was never quite satisfied, and speculated on an extraspatial unit. Now, the heterogeneity is not in dimensions, for all the terms have the same number of dimensions with respect to each unit involved in the units of the factor-vectors. The theory of axes here advanced and the extension of algebra to space of four dimensions show that all the terms are homogeneous in the sense of having an axis, but that for some terms it may be any axis; for others, the fourth axis in a space of four dimensions.

DEFINITIONS AND NOTATION.

I propose to use a notation which shall conform as far as possible with the notation of algebra, the Cartesian analysis, quaternions, etc., but shall at the same time embody what I conceive to be the principles of the algebra of physics. The most logical procedure is to generalize as far as possible the notation of algebra.

By an *arithmetical* quantity is meant an essentially positive or signless quantity; it has no direction or any direction. For example, the mass of a body, or its kinetic energy.

By a *scalar* is meant a quantity which has magnitude, and may be positive or negative, but is destitute of a definite axis; or it is the element of a physical quantity which is independent of the axis. It is equivalent to the ordinary algebraic quantity, and is denoted, as usual, by an Italic letter as a, b, x, X, etc. The work done by or against a force, and the volume of a geometric figure are examples. These quantities, though both scalar, differ in dimensions, and they are scalars for different reasons.

By a *vector* is meant a quantity which has magnitude and an axis. It requires three numbers to specify it completely. The simplest example is the displacement of a point, represented by a straight line drawn from its original to its final position. Other examples are a linear velocity, an area in a plane, and a current of fluid. These several quantities differ in dimensions and in the nature of the physical unit; and there are vectors which have the same dimensions in length, yet have different kinds of axes. What they have in common is a want of symmetry in space.

A vector is denoted by a black capital letter as $\mathbf{A}$, its magnitude by a and its axis by α. Thus $\mathbf{A} = a\alpha$, $\mathbf{B} = b\beta$, $\mathbf{R} = r\rho$. Sometimes it is necessary to introduce a dot to separate the expression for the magnitude from the expression for the direction; but when the two symbols are single, as in $a\alpha$, the dot may be omitted. The difference of type shows that a denotes the algebraic magnitude and α merely its axis, not another algebraic magnitude. In Clerk-Maxwell's *Electricity and Magnetism*, German capitals are used to denote vectors, but these are difficult to make, and plain black letters have already been used for the purpose, as by Fleming in his book on *Alternate Current Transformers*. The simple a and α are more commodious than Ta and Ua as used in works on quaternions, and the notation is also more in harmony with the Cartesian analysis. What is done is merely to introduce α to specify the axis in space, leaving the expression for the scalar part of the magnitude the same as before. In the case of mutually rectangular components, i, j and k are used to denote the axes.

Vector quantities may be classified according to the nature of the axis. By a line-vector is meant one which has a simple axis of direction,—a vector in the primary meaning of the word as used by Hamilton. It is of one dimension in length.

By the *pole* of two axes is meant the axis which is perpendicular to both. The pole of α and β is denoted by $\overline{\alpha\beta}$; the pole of $\overline{\alpha\beta}$ and γ is denoted by $\overline{\overline{\alpha\beta}\gamma}$; that of α and $\overline{\beta\gamma}$ by $\overline{\alpha\overline{\beta\gamma}}$ and so on. An axis which is perpendicular to α but otherwise indefinite, may be denoted by $\overline{\alpha}$. This notation enables us to express explicitly three mutually rectangular axes. Let α and β be any two independent axes; then, α and $\overline{\alpha\beta}$ and $\overline{\overline{\alpha\beta}\alpha}$ denote three mutually rectangular axes. In the works on quaternions, there is no systematic notation for direction; consequently to specify the axis which is perpendicular to two given axes, it is necessary to use a special non-systematic symbol.

By a *tensor* is meant an arithmetical ratio or quantity destitute of dimensions and of axis. This is the primary meaning of the word as used by Hamilton; it is primarily used to denote the magnitude of the quaternion quotient defined as a ratio of two lines in space. To conceive a, b, x, X, etc., as tensors, is to suppose the unit thrown into the symbols i, j, k. It is certainly not convenient to regard i, j, k as denoting directed physical units; it is more philosophical, more practical, and more in harmony with mathematical analysis to regard them as axes, and a, b, x, X, etc., as magnitudes, not mere tensors.

By a *vector-scalar* is meant a scalar quantity which has position in space; for example, the physical quantity which Clerk Maxwell calls a *mass-vector*; it is proportional to the mass and to the vector from an origin to the mass. Such a quantity may be denoted by $\mathbf{A} \cdot m$, where the Italic letter denotes the scalar or signless quantity, and $\mathbf{A}$ denotes the vector from an origin to the position of the quantity. This idea corresponds to the weighted point of the *Ausdehnungslehre*.

By a *rotor* is meant a localized vector, or a vector-vector; it has magnitude, direction and position; for example, a force or a rotational velocity. It may be denoted by such a symbol as $\mathbf{A} \cdot \mathbf{F}$ where $\mathbf{A}$ denotes the vector from an origin to the point of application, and $\mathbf{F}$ denotes the vector quantity.

By a *versor* is meant an amount of arc of a great circle on the sphere; it has an axis and an amount of angle. A versor, as a whole, may be denoted by a small black letter as $\mathbf{a}$, and analytically by α^A, where α denotes its axis, and A the amount of its angle in circular measure. Thus $\alpha^{\frac{\pi}{2}}$ is the imaginary $\sqrt{-1}$ for the axis a; while $a^{2\pi}$ is equivalent to the trigonometrical +, provided that in this case a denote any axis. I consider that it is more convenient, and more in harmony with trigonometry and the law of indices to consider $\frac{\pi}{2}$, not 1, as the index of a quadrantal versor.

By a *quaternion* is meant a geometric ratio; it is an ordinary arithmetical ratio, or tensor, combined with a versor. It is denoted by $a\alpha^{A}$, where a denotes the ratio and α^{A} the versor. The ratio and axis may be expressed synthetically as a vectŏr-ratio **A**, giving the expression $\mathbf{A}^{A}$ for the quaternion.

By a *dyad* is meant a physical ratio, or the rate connecting two vector quantities, and these may involve different physical units. Let **S** denote the dependent vector, **R** the independent; if the former is directly proportional to the latter, the dependence is expressed by the rate $\mathbf{R}^{-1}\,\mathbf{S}$. Professor Gibbs in his *Vector Analysis* bases the treatment of vectors largely on the conception of a dyad; and the word, I believe, is due to him. The dyad is in a certain sense a localized quaternion; it has an axis and an angle, but the angle is localized, that is, it must start from a specific direction. There is also this difference, that the dyad generally has dimensions in its magnitude, while the quaternion quotient has not.

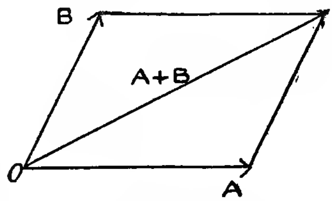

FIG. 10.

By a *matrix* is meant the sum of the rates connecting a vector quantity with the three independent components of another vector quantity. In its simplest form it is equivalent to a homogeneous strain or linear-vector operator. As it is a sum of dyads, Professor Gibbs calls it a *dyadic*. The synthetic symbol used to denote a matrix is a Greek capital letter as Φ.

ADDITION AND SUBTRACTION OF VECTORS.

Addition.—By adding two quantities of the same kind of vector quantity is meant finding their geometric resultant, or what is called in mechanics compounding them. This process is called addition, because when the vectors have a common axis, the process reduces to ordinary algebraic addition. Suppose two quantities of a vector **A** and **B** to have a common point of application O (fig. 10), their resultant or sum is the diagonal of the parallelogram of which **A** and **B** are the sides. The principle of the parallelogram of forces is thus one of the fundamental principles of the algebra of physics.

Subtraction.—By subtracting one quantity of a vector from another quantity is meant finding the quantity which added to the former produces the latter. Let **A** (fig. 11) be the quantity to be subtracted, and **B** the quantity to be subtracted from; the remainder is the vector from the end of **A** to the end of **B**, the cross-diagonal of the parallelogram formed by **A** and **B**, and taken in the direction from **A** to **B**.

To subtract a quantity of a vector is equivalent to reversing the axis and then adding. In the figure (fig. 11) $-\mathbf{A}$ is the opposite of **A** in direction; and the diagonal from the corner of the parallelogram formed by $-\mathbf{A}$ and **B** is equal to the cross-diagonal of the parallelogram formed by **A** and **B**. To define subtraction as addition after reversal seems to me less accurate than to recognize the two processes of composition and resolution of vector quantities. Let a small minus before the **A** denote reversal of axis, while a large minus denotes subtraction, then we have the theorem or principle

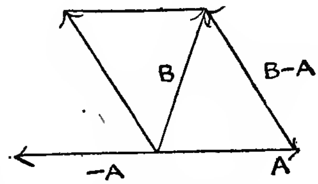

FIG. 11.

$\mathbf{B} - \mathbf{A} = \mathbf{B} + {}^{-}\mathbf{A}$. Hence we have the rules $- \mathbf{A} = + {}^{-}\mathbf{A}$ and $+ \mathbf{A} = - {}^{-}\mathbf{A}$, which mean respectively: to subtract a quantity is equivalent to adding the opposite quantity; and to add a quantity is equivalent to subtracting the opposite quantity.

Commutative Rule.—When the point of application of a vector is indefinite, the sum of two quantities of it as **A** and **B** is the same, whether they are applied simultaneously, or **A** first and then **B**, or **B** first and then **A**. Hence the commutative rule in adding and subtracting quautities of a vector

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

Associative Rule.—It follows from the commutative property that if a third quantity **C** is to be compounded, it is immaterial whether the sum of **A** and **B** be added to **C**, or **A** be added to the sum of **B** and **C**. Hence the associative rule in adding and subtracting quantities of a vector

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

It follows that the rules for the transformation of equations between quantities of a vector by adding or subtracting equal terms on the two sides are the same as those in ordinary algebra, where the axis of all the terms is constant.

Given the magnitude and axis of each of the components; to find the magnitude and axis of the sum.

$$\text{Given } \mathbf{A} = a\alpha, \qquad \text{and} \qquad \mathbf{B} = b\beta;$$

$$\text{then } \mathbf{A} + \mathbf{B} = a\alpha + b\beta$$

$$= \sqrt{a^2 + b^2 + 2ab\cos\alpha\beta} \cdot \overline{\alpha\beta}\ \alpha^{\tan^{-1}\frac{b \sin \alpha\beta}{a + b \cos \alpha\beta}}$$

Here $\sqrt{a^2 + b^2 + 2ab\cos\alpha\beta}$ gives the magnitude of the sum, while the rest of the expression denotes its axis in terms of the given quantities. In that expression $\overline{\alpha\beta}$ denotes the axis, and $\tan.^{-1}\frac{b \sin \alpha\beta}{a + b \cos \alpha\beta}$ the angle of the versor which changes α into the direction of the sum.

For the generalized addition which applies to quantities of a scalar situated at different points or to quantities of a vector applied at different points, see the end of the paper.

PRODUCT OF TWO VECTORS.

Different forms of the product.—Let

$$\mathbf{A} = a_1 i + a_2 j + a_3 k$$

and

$$\mathbf{B} = b_1 i + b_2 j + b_3 k$$

be any two vector quantities, not necessarily of the same kind. Their product, according to the rules (p. 72), is

$$\begin{aligned}\mathbf{AB} &= (a_1 i + a_2 j + a_3 k)\,(b_1 i + b_2 j + b_3 k) \\ &= a_1 b_1 ii + a_2 b_2 jj + a_3 b_3 kk + a_2 b_3 jk + a_3 b_2 kj + a_3 b_1 ki + a_1 b_3 ik \\ &\quad + a_1 b_2 ij + a_2 b_1 ji; \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 + (a_2 b_3 - a_3 b_2)\, i + (a_3 b_1 - a_1 b_3)\, j \\ &\quad + (a_1 b_2 - a_2 b_1)\, k; \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}\end{aligned}$$

Here the vector part is written in the form of a determinant. In the Cartesian analysis this vector determinant is imperfectly expressed by means of the composite determinant

$$\left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right\|$$

Let $\mathbf{A}$ and $\mathbf{B}$ be given in the form $a\alpha$ and $b\beta$ respectively; then it is evident(from p. 72) that

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = ab \cos \alpha\beta;$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix} = ab \sin \alpha\beta \cdot \overline{\alpha\beta}$$

where $\overline{\alpha\beta}$ is used to denote the axis which is perpendicular to α and β.

Hence

$$\begin{aligned}\mathbf{AB} &= ab \cos \alpha\beta \cdot \alpha\alpha + ab \sin \alpha\beta \cdot \overline{\alpha\beta} \\ &= ab\,(\cos \alpha\beta + \sin \alpha\beta \cdot \overline{\alpha\beta}).\end{aligned}$$

Notation for the two parts of the product.—In quaternions the negative of $a_1 b_1 + a_2 b_2 + a_3 b_3$ is called the *scalar* of **AB** and is denoted by *S***AB**, while the other term is called the vector of **AB** and is denoted by *V***AB**. The objection to this notation is the association of the negative sign with the word scalar, and the want of a convenient notation for the magnitude of the vector part. As they are not linked to anything in ordinary algebra, they make the connection obscure and the transition difficult from ordinary algebra to the algebra of space.

I have found it convenient to use for this purpose the functional expressions *cos* and *Sin*. They possess all the advantage of a logical generalization; for when abstraction is made of the magnitude of the product they then have their trigonometrical meaning. They make the formulæ much more self-interpreting. Thus, we write

$$\mathbf{AB} = \cos \mathbf{AB} + \mathrm{Sin}\, \mathbf{AB},$$

Sin with a capital denoting the complete vector quantity, while *sin* denotes its magnitude irrespective of axis.

The product of two vectors is not, in general, commutative.—For

$$\mathbf{AB} = a_1b_1 + a_2b_2 + a_3b_3 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}$$

and

$$\mathbf{BA} = b_1a_1 + b_2a_2 + b_3a_3 + \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ i & j & k \end{vmatrix}$$

$$= a_1b_1 + a_2b_2 + a_3b_3 - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}$$

Hence, it is commutative only if $Sin\ \mathbf{AB} = 0$, that is if $\beta = \alpha$. This condition is satisfied by the quantities of ordinary algebra, but not by quantities in a plane.

Square of a vector.—Let $\mathbf{B} = \mathbf{A}$,

$$\text{then } \mathbf{A}^2 = a_1{}^2 + a_2{}^2 + a_3{}^2 = a^2.$$

The square of a vector has no axis, or, what is probably more correct to say, it has any axis. To find a vector from its square is an entirely indeterminate problem, when the vector is in space. If the vector is restricted to one straight line, there still is an ambiguity of forwards or backwards. Hence the double sign for the square root. Again, since the square of any vector is positive, a negative scalar cannot be the square of a vector. In the algebra of vectors the square root of a negative scalar is not only imaginary, it is impossible.

Reciprocal of a vector.—By the reciprocal of a vector is meant the vector which combined as a factor with the original vector produces the product +1. Since

$$\mathbf{AB} = ab\ (cos\ \alpha\beta + sin\ \alpha\beta \cdot \overline{\alpha\beta})$$

in order that the product may be 1, b must equal a^{-1} and β be identical with α. Thus, $\mathbf{A}^{-1} = a^{-1}\,\alpha$. It follows that

$$\mathbf{A}^{-1} = \frac{a\alpha}{a^2} = \frac{\mathbf{A}}{\mathbf{A}^2} = \frac{a_1 i + a_2 j + a_3 k}{a_1{}^2 + a_2{}^2 + a_3{}^2};$$

and that $\mathbf{A}^{-1}\mathbf{B} = \frac{b}{a}\,(cos\ \alpha\beta + sin\ \alpha\beta \cdot \overline{\alpha\beta})$.

The expression inside the parenthesis depending on the axes is the same for $\mathbf{AB}$, $\mathbf{A}^{-1}\mathbf{B}$, $\mathbf{AB}^{-1}$, $\mathbf{A}^{-1}\mathbf{B}^{-1}$.

In quaternions the reciprocal of a vector has the opposite axis to that of the vector, but this arises from treating a vector as a quadrantal versor. The reciprocal, as above defined, corresponds to the inverse of a line in geometry, when the constant quantity is 1. Curvature, denoted by $\frac{d^2\mathbf{R}}{ds^2}$, is a directed quantity; its reciprocal, denoted by $\left(\frac{d^2\mathbf{R}}{ds^2}\right)^{-1}$, is the radius of curvature; they have the same axis, but reciprocal magnitudes.

The reciprocal, as above defined, is a true generalization of the reciprocal of algebra; the axis being no longer constant is expressed by α.

It explains why the rule of signs for a quotient is the same as the rule of signs for a product. For example, $\frac{a}{-b}=\frac{-a}{b}$, which means that it is immaterial to the result whether the minus sign occurs in the numerator or the denominator. This view of the generalized reciprocal also explains the change of signs of the trigonometrical functions in the several quadrants.

Generalized trigonometrical functions.—The other trigonometrical functions may be defined in terms of the generalized cosine and sine. Thus,

$$Tan\,\mathbf{AB}=\frac{Sin\,\mathbf{AB}}{cos\,\mathbf{AB}}=\frac{(a_2b_3-a_3b_2)i+(a_3b_1-a_1b_3)j+(a_1b_2-a_2b_1)k}{a_1b_1+a_2b_2+a_3b_3}$$

$$Cot\,\mathbf{AB}=\frac{cos\,\mathbf{AB}}{Sin\,\mathbf{AB}}=\frac{cos\,\mathbf{AB}\;Sin\,\mathbf{AB}}{sin^2\,\mathbf{AB}}$$

$$sec\,\mathbf{AB}=\frac{1}{cos\,\mathbf{AB}}=\frac{1}{a_1b_1+a_2b_2+a_3b_3}$$

$$Cosec\,\mathbf{AB}=\frac{1}{Sin\,\mathbf{AB}}=\frac{Sin\,\mathbf{AB}}{sin^2\,\mathbf{AB}}$$

While *Tan* **AB** denotes both the magnitude and the axis, *tan* **AB** may be used to denote the magnitude apart from the axis. Whatever the dimensions of **A** and of **B**, *Tan* **AB** has its simple trigonometrical meaning, only it has an axis in space. For

$$Tan\,\mathbf{AB}=\frac{ab\;Sin\,a\beta}{ab\cos a\beta}=Tan\;a\beta.$$

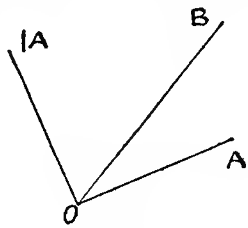

FIG. 12.

Complementary vector.—By the complementary vector (fig. 12) of **A** with respect to **B**, Grassmann means the vector which has the same magnitude as **A** and is drawn perpendicular to **A** in the plane of **A** and **B**.

Thus $|\mathbf{A}=a\cdot\overline{\overline{a\beta}a}$

The product $\mathbf{B}\,|\,\mathbf{A}=cos\,\mathbf{B}\;|\,\mathbf{A}+Sin\,\mathbf{B}\,|\,\mathbf{A}$

$=sin\,\mathbf{AB}+Cos\,\mathbf{AB}$

where $sin\,\mathbf{AB}=\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}$

and $Cos\,\mathbf{AB}=(a_1b_1+a_2b_2+a_3b_3)\;\overline{a\beta}$.

PRODUCT OF THREE VECTORS.

Different forms of the product.—Let $\mathbf{A}=a_1i+a_2j+a_3k$, $\mathbf{B}=b_1i+b_2j+b_3k$, and $\mathbf{C}=c_1i+c_2j+c_3k$ denote any three vectors, not necessarily of the same kind; by their product is meant the product of the product of **A** and **B** with **C**, according to the rules for vectors. Thus

$$\mathbf{ABC} = (a_1b_1 + a_2b_2 + a_3b_3)\,(c_1i + c_2j + c_3k)$$
$$+ \left\{(a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k\right\}(c_1i + c_2j + c_3k)$$
$$= (a_1b_1 + a_2b_2 + a_3b_3)\,(c_1i + c_2j + c_3k) + \begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \\ i & j & k \end{vmatrix}$$
$$+ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The second term may be written

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_1 & c_2 \\ i & j \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \begin{vmatrix} c_2 & c_3 \\ j & k \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \begin{vmatrix} c_3 & c_1 \\ k & i \end{vmatrix}$$

If we write $\mathbf{A} = a\alpha$, $\mathbf{B} = b\beta$, $\mathbf{C} = c\gamma$, then

$$\mathbf{ABC} = abc\,(\cos \alpha\beta + \sin \alpha\beta \cdot \overline{\alpha\beta})\,\gamma$$
$$= abc\left\{\cos \alpha\beta \cdot \gamma + \sin \alpha\beta \sin \overline{\alpha\beta}\gamma \cdot \overline{\overline{\alpha\beta}\gamma} + \sin \alpha\beta \cos \overline{\alpha\beta}\gamma\right\}$$

where $\cos \overline{\alpha\beta}\gamma$ denotes the *cosine* between the axis $\overline{\alpha\beta}$ and γ, and $\overline{\overline{\alpha\beta}\gamma}$ denotes the axis perpendicular to $\overline{\alpha\beta}$ and γ.

The parts of the product may be expressed more synthetically by means of the generalized cosine and Sine. Thus

$$\mathbf{ABC} = \cos \mathbf{AB} \cdot \mathbf{C} + Sin\,(Sin\,\mathbf{AB})\mathbf{C} + \cos\,(Sin\,\mathbf{AB})\,\mathbf{C};$$

where the dot is used as a separatrix, to separate the expression for the cosine from the vector.

Thus,

$$\cos \mathbf{AB} \cdot \mathbf{C} = abc \cos \alpha\beta \cdot \gamma = a_1b_1 + a_2b_2 + a_3b_3;$$

$$Sin\,(Sin\,\mathbf{AB})\,\mathbf{C} = abc \sin \alpha\beta \sin \overline{\alpha\beta}\gamma \cdot \overline{\overline{\alpha\beta}\gamma}$$
$$= \begin{vmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \\ i & j & k \end{vmatrix}$$

and $\cos\,(Sin\,\mathbf{AB})\,\mathbf{C} = abc \sin \alpha\beta \cos \overline{\alpha\beta}\gamma = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Axis of the third term.—The first term of the product has the axis γ, and the second term the axis perpendicular to $\overline{\alpha\beta}$ and γ; the question arises whether the axis of the third term is implicitly given as the third mutual perpendicular, namely, $\overline{\overline{\gamma\,\overline{\alpha\beta}\gamma}}$. It can be shown that it is not so; the term is a scalar for space of three dimensions, but has an axis in space of four dimensions. If the vectors **A**, **B**, **C** are each of one dimension in length, each of the terms of the product is of three dimensions in length. The third term involves the three axes of space symmetrically, hence has no axes. It is a scalar, but not of the same kind as *cos* **AB**. This view of the term becomes clearer, when the product of three line-vectors in space of four dimensions is considered.

To express the second term as the difference of two terms similar to the first.—The second term *Sin* (*Sin* **AB**) **C** expressed in terms of i, j, k is

$$\left\{-(b_2c_2+b_3c_3)\,a_1+(c_2a_2+c_3a_3)\,b_1\right\}i$$
$$+\left\{-(b_3c_3+b_1c_1)\,a_2+(c_3a_3+c_1a_1)\,b_2\right\}j$$
$$+\left\{-(b_1c_1+b_2c_2)\,a_3+(c_1a_1+c_2a_2)\,b_3\right\}k$$

By adding the null term $(b_1c_1a_1 - c_1a_1b_1)\,i$ to the i term, we get

$$-\cos\mathbf{BC}\,a_1i+\cos\mathbf{CA}\,b_1i.$$

By treating similarly the other two components and adding the results, we obtain

$$Sin\,(Sin\,\mathbf{AB})\,\mathbf{C}=-\cos\mathbf{BC}\cdot\mathbf{A}+\cos\mathbf{CA}\cdot\mathbf{B}.$$

Hence,

$$\mathbf{ABC}=\cos\mathbf{AB}\cdot\mathbf{C}-\cos\mathbf{BC}\cdot\mathbf{A}+\cos\mathbf{CA}\cdot\mathbf{B}+\cos\,(Sin\,\mathbf{AB})\,\mathbf{C}.$$

The vector which is the sum of all the vector terms may be called the total vector.

The product of three vectors is not indifferent as regards association.—The expression **ABC**, without any parenthesis, means that the association of the factors begins at the left, while **A**(**BC**) denotes that the association begins at the right. By applying the rules of multiplication we get

$$\mathbf{A}(\mathbf{BC})=\mathbf{A}\cos\mathbf{BC}+Sin\,\mathbf{A}\,(Sin\,\mathbf{BC})+\cos\mathbf{A}\,(Sin\,\mathbf{BC}).$$

On comparing these terms with those of **ABC**, it will be seen, by a well-known property of the determinant, that the third terms are equal. But

$$Sin\,\mathbf{A}\,(Sin\,\mathbf{BC})=-Sin\,(Sin\,\mathbf{BC})\,\mathbf{A}=\cos\mathbf{CA}\cdot\mathbf{B}-\cos\mathbf{AB}\cdot\mathbf{C};$$

Hence the total vector of **A**(**BC**) is

$$\cos\mathbf{BC}\cdot\mathbf{A}+\cos\mathbf{CA}\cdot\mathbf{B}-\cos\mathbf{AB}\cdot\mathbf{C},$$

which is equal in magnitude to the total vector of **ABC**, but does not have the same direction. The condition which must obtain for the rule of association to be applied is

$$\cos\mathbf{AB}\cdot\mathbf{C}=\cos\mathbf{BC}\cdot\mathbf{A},$$

that is, **C** and **A** must have the same direction.

When the three vectors are coplanar, the middle vector and the non-associated vector may be interchanged.—For then

$$(\mathbf{AB})\mathbf{C}=\cos\mathbf{AB}\cdot\mathbf{C}-\cos\mathbf{BC}\cdot\mathbf{A}+\cos\mathbf{CA}\cdot\mathbf{B},$$

and

$$(\mathbf{AC})\mathbf{B}=\cos\mathbf{AC}\cdot\mathbf{B}-\cos\mathbf{CB}\cdot\mathbf{A}+\cos\mathbf{BA}\cdot\mathbf{C}.$$

Hence,

$$\mathbf{A}^3=a^2\mathbf{A}=\mathbf{A}a^2;$$
$$(\mathbf{AB})\mathbf{A}=(\mathbf{AA})\mathbf{B}=a^2\mathbf{B};$$
$$(\mathbf{AB})\mathbf{A}^{-1}=(\mathbf{AA}^{-1})\mathbf{B}=\mathbf{B}.$$

But $(\mathbf{BA})\mathbf{A}^{-1}=(\mathbf{BA}^{-1})\mathbf{A}$ and is not $=\mathbf{B}$.

It is evident (fig. 13) that $\mathbf{BAA}^{-1}$ is the vector which is the reflection of **B** in **A**.

Cyclical products.—The three products of **A**, **B**, **C** obtained by taking the factors in cyclical order, and so changing the mode of association are,

$$\mathbf{ABC} = cos\,\mathbf{AB}\cdot\mathbf{C} + \left\{-cos\,\mathbf{BC}\cdot\mathbf{A} + cos\,\mathbf{CA}\cdot\mathbf{B}\right\} + cos\,(Sin\,\mathbf{AB})\,\mathbf{C},$$

$$\mathbf{BCA} = cos\,\mathbf{BC}\cdot\mathbf{A} + \left\{-cos\,\mathbf{CA}\cdot\mathbf{B} + cos\,\mathbf{AB}\cdot\mathbf{C}\right\} + cos\,(Sin\,\mathbf{BC})\,\mathbf{A},$$

$$\mathbf{CAB} = cos\,\mathbf{CA}\cdot\mathbf{B} + \left\{-cos\,\mathbf{AB}\cdot\mathbf{C} + cos\,\mathbf{BC}\cdot\mathbf{A}\right\} + cos\,(Sin\,\mathbf{CA})\,\mathbf{B}.$$

The last term has the same value in the three products; it expresses the volume of the parallelopiped formed by the three vectors and may be denoted by *vol* **ABC**. The sum of the three products is

$$\mathbf{ABC} + \mathbf{BCA} + \mathbf{CAB} = cos\,\mathbf{AB}\cdot\mathbf{C} + cos\,\mathbf{BC}\cdot\mathbf{A} + cos\,\mathbf{CA}\cdot\mathbf{B} + 3\,vol\,\mathbf{ABC}.$$

By abstracting the common magnitude *abc* of the total vectors, the following ratio-vectors are obtained:

$$cos\,\alpha\beta\cdot\gamma - cos\,\beta\gamma\cdot\alpha + cos\,\gamma\alpha\cdot\beta \qquad (1)$$

$$cos\,\beta\gamma\cdot\alpha - cos\,\gamma\alpha\cdot\beta + cos\,\alpha\beta\cdot\gamma \qquad (2)$$

$$cos\,\gamma\alpha\cdot\beta - cos\,\alpha\beta\cdot\gamma + cos\,\beta\gamma\cdot\alpha \qquad (3)$$

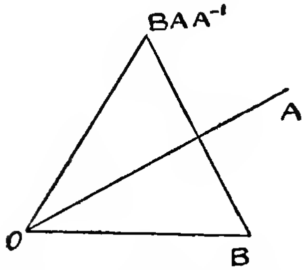

FIG. 13.

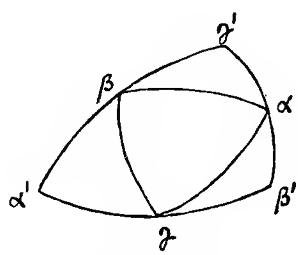

FIG. 14.

In quaternions these expressions are obtained from $\beta\alpha^{-1}\gamma$, $\gamma\beta^{-1}\alpha$, $\alpha\gamma^{-1}\beta$, but here we are led to them directly by varying the product so as to get the three modes of association. Let α, β, γ (fig. 14) be the extremities of the axes on the unit-sphere. As the vector (1) has a negative component along α, it will be on the opposite side of the arc $\beta\gamma$ from α; let α' be its axis. Similarly β' denotes the axis of (2) and γ' that of (3).

Since (1) + (2) = $2\,cos\,\alpha\beta\cdot\gamma$ and (2) + (3) = $2\,cos\,\beta\gamma\cdot\alpha$ and (3) + (1) = $2\,cos\,\gamma\alpha\cdot\beta$; the axes α', β', γ' are such that the triangle $\alpha'\beta'\gamma$ has its sides bisected by the triangle $\alpha\beta\gamma$.

Notation.—The square of each of the vectors (1), (2), (3) is

$$cos^2\,\alpha\beta + cos^2\,\beta\gamma + cos^2\,\gamma\alpha - 2cos\,\alpha\beta\,cos\,\beta\gamma\,cos\,\gamma\alpha,$$

which is the complement to one of $vol^2\, \alpha\beta\gamma$. In spherical trigonometry $vol\ \alpha\beta\gamma$ is denoted by $2n$, which is equal to

$$2\left\{ \sin s \sin (s-a) \sin (s-b) \sin (s-c) \right\}^{\frac{1}{2}}$$

and the need of a name for the function has been felt. It has been called by some the "sine of the trihedral angle" formed by α, β, γ; by others the "Staudtian" (Casey, *Spherical Trigonometry*, p. 22). The notation $(\alpha\beta\gamma)$ is used by Lagrange in the *Mecanique Analytique;* in quaternions it is denoted by $-S\alpha\beta\gamma$ and the total vector by $V\alpha\beta\gamma$.

PRODUCT OF FOUR VECTORS.

Different ways of association.—A product of four vectors may be formed in five different ways, according to the nature of the association, namely, $((\mathbf{AB})\,\mathbf{C})\,\mathbf{D}$, $(\mathbf{A}\,(\mathbf{BC}))\,\mathbf{D}$, $\mathbf{A}\,((\mathbf{BC})\,\mathbf{D})$, $\mathbf{A}\,(\mathbf{B}\,(\mathbf{CD}))$, $(\mathbf{AB})\,(\mathbf{CD})$, of which the first and last are the most important. When no parenthesis is used, the first form is understood.

The first form of the product.—Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be expressed as before in terms of i, j, k and let $\mathbf{D} = d_1 i + d_2 j + d_3 k$. Then

$$\mathbf{ABCD} = (a_1 b_1 + a_2 b_2 + a_3 b_3)\,(c_1 d_1 + c_2 d_2 + c_3 d_3) \quad (1)$$

$$+ \begin{Vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{Vmatrix} \quad (2)$$

$$+ (a_1 b_1 + a_2 b_2 + a_3 b_3) \begin{vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ i & j & k \end{vmatrix} \quad (3)$$

$$+ \begin{Vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \\ \begin{vmatrix} d_2 & d_3 \\ j & k \end{vmatrix} & \begin{vmatrix} d_3 & d_1 \\ k & i \end{vmatrix} & \begin{vmatrix} d_1 & d_2 \\ i & j \end{vmatrix} \end{Vmatrix} \quad (4)$$

$$+ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} (d_1 i + d_2 j + d_3 k) \quad (5)$$

The term (2) is equal to

$$-(b_1 c_1 + b_2 c_2 + b_3 c_3)\,(a_1 d_1 + a_2 d_2 + a_3 d_3)$$
$$+ (c_1 a_1 + c_2 a_2 + c_3 a_3)\,(b_1 d_1 + b_2 d_2 + b_3 d_3),$$

and the term (4) is equal to

$$-(b_1 c_1 + b_2 c_2 + b_3 c_3) \begin{vmatrix} a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \\ i & j & k \end{vmatrix} + (c_1 a_1 + c_2 a_2 + c_3 a_3) \begin{vmatrix} b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \\ i & j & k \end{vmatrix}.$$

Let $\mathbf{A} = a\alpha$, $\mathbf{B} = b\beta$, $\mathbf{C} = c\gamma$, $\mathbf{D} = d\delta$; then

$$\mathbf{ABCD} = abcd \left\{ \cos \alpha\beta \cos \gamma\delta + \sin \alpha\beta \sin \overline{\alpha\beta}\gamma \cos \overline{\overline{\alpha\beta}\gamma}\delta \right.$$
$$+ \cos \alpha\beta \sin \gamma\delta \cdot \overline{\gamma\delta} + \sin \alpha\beta \sin \overline{\alpha\beta}\gamma \sin \overline{\overline{\alpha\beta}\gamma}\delta \cdot \overline{\overline{\overline{\alpha\beta}\gamma}\delta}$$
$$\left. + \sin \alpha\beta \cos \overline{\alpha\beta}\gamma \cdot \delta \right\}.$$

These five terms are equal in order to (1), (2), (3), (4), (5) respectively. By expanding the second and fourth terms,

$$\mathbf{ABCD} = abcd\Big\{\cos\alpha\beta\cos\gamma\delta - \cos\beta\gamma\cos\alpha\delta + \cos\gamma\alpha\cos\beta\delta + \cos\alpha\beta\sin\gamma\delta\cdot\overline{\gamma\delta} - \cos\beta\gamma\sin\alpha\delta\cdot\overline{\alpha\delta} + \cos\gamma\alpha\sin\beta\delta\cdot\overline{\beta\delta} + \sin\alpha\beta\cos\overline{\alpha\beta}\gamma\cdot\delta\Big\}.$$

The product may be expressed more synthetically by

$$\mathbf{ABCD} = \cos\mathbf{AB}\cos\mathbf{CD} + \cos(Sin\,(Sin\,\mathbf{AB})\,\mathbf{C})\,\mathbf{D} + \cos\mathbf{AB}\,Sin\mathbf{CD} + Sin\Big\{Sin\,(Sin\mathbf{AB})\,\mathbf{C}\Big\}\mathbf{D} + \cos\,(Sin\mathbf{AB})\,\mathbf{C}\cdot\mathbf{D}.$$

The symmetrical product. — By the symmetrical product is meant (**AB**) (**CD**).

Since $\mathbf{AB} = ab\,(\cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta})$

and $\mathbf{CD} = cd\,(\cos\gamma\delta + \sin\gamma\delta\cdot\overline{\gamma\delta})$

$$(\mathbf{AB})\,(\mathbf{CD}) = abcd\Big\{\cos\alpha\beta\cos\gamma\delta + \cos\alpha\beta\sin\gamma\delta\cdot\overline{\gamma\delta} + \cos\gamma\delta\sin\alpha\beta\cdot\overline{\alpha\beta} + \sin\alpha\beta\sin\gamma\delta\cos\overline{\alpha\beta}\,\overline{\gamma\delta} + \sin\alpha\beta\sin\gamma\delta\sin\overline{\alpha\beta}\,\overline{\gamma\delta}\cdot\overline{\overline{\alpha\beta}\,\overline{\gamma\delta}}\Big\}$$

This differs essentially from the product of two quaternions, for in it the last two terms are negative. How then can it satisfy the law of the norms? By considering the five terms to be independent of one another.

COMPOUND AXES.

By an axis of the *first degree* is meant the direction of a line; it is denoted by an elementary symbol such as α.

By an axis of the *second degree* is meant the product of two elementary axes, denoted in general by $\alpha\beta$.

Now,

$$\alpha\beta = \cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta};$$

hence, $\alpha^2 = +$ and when β is perpendicular to α, the axis reduces to $\overline{\alpha\beta}$. Also $\overline{\beta\alpha} = -\overline{\alpha\beta}$.

By an axis of the *third degree* is meant the product of three elementary axes, denoted in general by $\alpha\beta\gamma$. We have seen that

$$\alpha\beta\gamma = \cos\alpha\beta\cdot\gamma - \cos\beta\gamma\cdot\alpha + \cos\gamma\alpha\cdot\beta + \sin\alpha\beta\cos\overline{\alpha\beta}\gamma\cdot\overline{\overline{\alpha\beta}\gamma},$$

where $\overline{\overline{\alpha\beta}\gamma}$ denotes the axis of the third term.

Let $\gamma = \alpha$; then the axis reduces to $\alpha\beta\alpha$, that is β.

Let $\gamma = \beta$; then the axis reduces to $\alpha\beta\beta$, which is equal to

$$2\cos\alpha\beta\cdot\beta - \alpha.$$

Hence, if α and β are at right angles, $\alpha\beta\beta$ reduces to $-\alpha$.

If α, β and γ are mutually rectangular, the general axis $\alpha\beta\gamma$ reduces to $\overline{\alpha\beta\gamma}$, which therefore is an axis in a space of four dimensions. In such a space, volume has an axis. It is such that

$$\overline{\alpha\beta\gamma} = \overline{\beta\gamma\alpha} = \overline{\gamma\alpha\beta} = -\overline{\gamma\beta\alpha} = -\overline{\beta\alpha\gamma} = -\overline{\alpha\gamma\beta}.$$

The rule of signs for a determinant of the third order is the rule for the direction along this axis. In a space of three dimensions when α, β, γ are mutually rectangular $\overline{\alpha\beta\gamma}$ is the only extraspatial axis, and may be denoted in a certain sense by 1; and $\overline{\alpha\beta}$ is equivalent to the complementary axis γ. Thus, $ij = k$ introduces the condition of three dimensions.

By an axis of the *fourth degree* is meant the product of four elementary axes; it is denoted in general by $\alpha\beta\gamma\delta$, and we have shown that

$$\alpha\dot{\beta}\gamma\delta = \cos\alpha\beta\cos\gamma\delta - \cos\beta\gamma\cos\alpha\delta + \cos\gamma\alpha\cos\beta\delta$$

$$+ \cos\alpha\beta\sin\gamma\delta\cdot\overline{\gamma\delta} - \cos\beta\gamma\cos\alpha\delta\cdot\overline{\alpha\delta} + \cos\gamma\alpha\sin\beta\delta\cdot\overline{\beta\delta}$$

$$+ \sin\alpha\beta\cos\overline{\alpha\beta\gamma}\cdot\overline{\alpha\beta\gamma}\,\delta.$$

If α, β and γ are mutually rectangular, the axis reduces to $\overline{\alpha\beta\gamma}\,\delta$. If $\delta = \alpha$, the axis has the same direction as $\overline{\beta\gamma}$, but the sign remains to be determined. As in space of three dimensions $\overline{\beta\gamma} = \alpha$ and $\overline{\alpha\beta\gamma} = 1$, the sign is $+$. Hence, $\overline{\alpha\beta\gamma}\,\alpha = \overline{\beta\gamma}$ in general. Let $\delta = \beta$; then since $\overline{\alpha\beta\gamma} = -\overline{\beta\alpha\gamma}$, it follows that $\overline{\alpha\beta\gamma}\,\beta = -\overline{\alpha\gamma}$. Similarly, $\overline{\alpha\beta\gamma}\,\gamma = \alpha\beta$.

If, in addition δ is at right angles to α, β and γ, we have a new axis $\overline{\alpha\beta\gamma\delta}$, which is transformed according to the rules for a determinant of the fourth order, namely, $\overline{\alpha\beta\delta\gamma} = -\overline{\beta\gamma\delta\alpha} = \overline{\gamma\delta\alpha\beta} = -\overline{\delta\alpha\beta\gamma}$, etc.

The following table contains the different types of axes for the first four degrees, with their reduced equivalents. It is supposed that i, j, k, u are mutually rectangular.

DEGREE.	TYPE.	GENERAL REDUCED AXIS.	REDUCED AXIS IN SPACE OF THREE DIMENSIONS.
First.	i		
Second.	i^2	$+$	$+$
	ij		k
Third.	i^3	i	i
	i^2j	j	j
	iji	j	j
	ijj	$-i$	$-i$
	ijk		$+$
Fourth.	i^4	$+$	$+$
	i^3j	ij	k
	i^2ji	$-ij$	$-k$
	i^2jj	$+$	$+$

DEGREE.	TYPE.	GENERAL REDUCED AXIS.	REDUCED AXIS IN SPACE OF THREE DIMENSIONS.
Fourth.	i^2jk	jk	i
	$ijii$	$-ij$	$-k$
	$ijij$	$+$	$+$
	$ijik$	jk	i
	$ijji$	$-$	$-$
	$ijjj$	$-ij$	$-k$
	$ijjk$	$-ik$	j
	$ijki$	jk	i
	$ijkj$	$-ik$	j
	$ijkk$	ij	k
	$ijku$		Non-existent.

These principles suffice to reduce an axis of any degree.

General product of two vectors.—Let

$$\mathbf{R} = x\alpha + y\beta + z\gamma + w\delta + \text{etc.}$$
$$\mathbf{R}' = x'\alpha + y'\beta + z'\gamma + w'\delta + \text{etc.};$$

then

$$\mathbf{RR}' = \Sigma xx' + \Sigma\,(xy' + yx')\cos\alpha\beta + \Sigma\,(xy' - yx')\sin\alpha\beta\cdot\overline{\alpha\beta}.$$

Thus,

$$\cos\mathbf{RR}' = \Sigma xx' + \Sigma(xy' + yx')\cos\alpha\beta$$

and

$$\mathit{Sin}\,\mathbf{RR}' = \Sigma(xy' - yx')\sin\alpha\beta\cdot\overline{\alpha\beta}.$$

In a space of four dimensions α, β, γ, δ may be independent; and $\frac{\cos\mathbf{RR}'}{rr'}$ expresses the cosine of the angle between the vectors, and $\frac{\mathit{Sin}\,\mathbf{RR}'}{rr'}$ expresses the directed sine. In a space of three dimensions, these expressions still have the same meaning, although only three of the axes can be independent. In a space of two dimensions the component areas all have the same direction but may differ in sign. For three components,

$$\mathit{Sin}\,\mathbf{RR}' = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ \sin\beta\gamma & \sin\gamma\alpha & \sin\alpha\beta \end{vmatrix}$$

where the sines are algebraic quantities, that is, have a common direction but may be positive or negative. In a space of one dimension

$$\mathbf{RR}' = \Sigma xx' + \Sigma(xy' + yx')$$

which agrees with ordinary algebra. Whatever the space,

$$\mathbf{R}^2 = \Sigma x^2 + 2\Sigma xy\cos\alpha\beta.$$

Product of three vectors in space of four dimensions.—Let

$$\mathbf{A} = a_1 i + a_2 j + a_3 k + a_4 u$$
$$\mathbf{B} = b_1 i + b_2 j + b_3 k + b_4 u$$
$$\mathbf{C} = c_1 i + c_2 j + c_3 k + c_4 u.$$

Then $$\mathbf{ABC} = \Sigma i^3 + \Sigma i^2 j + \Sigma iji + \Sigma ijj + \Sigma ijk.$$

$$\Sigma i^3 = a_1 b_1 c_1 i + a_2 b_2 c_2 j + a_3 b_3 c_3 k + a_4 b_4 c_4 u.$$

$$\Sigma i^2 j = (a_2 b_2 + a_3 b_3 + a_4 b_4)\, c_1 i + (a_3 b_3 + a_4 b_4 + a_1 b_1)\, c_2 j. + (a_4 b_4 + a_1 b_1 + a_2 b_2)\, c_3 k + (a_1 b_1 + a_2 b_2 + a_3 b_3)\, c_4 u.$$

$$\Sigma iji = (a_2 c_2 + a_3 c_3 + a_4 c_4)\, b_1 i + (a_3 c_3 + a_4 c_4 + a_1 c_1)\, b_2 j + (a_4 c_4 + a_1 c_1 + a_2 c_2)\, b_3 k + (a_1 c_1 + a_2 c_2 + a_3 c_3)\, b_4 u.$$

$$\Sigma ijj = -(b_2 c_2 + b_3 c_3 + b_4 c_4)\, a_1 i - (b_3 c_3 + b_4 c_4 + b_1 c_1)\, a_2 j - (b_4 c_4 + b_1 c_1 + b_2 c_2)\, a_3 k - (b_1 c_1 + b_2 c_2 + b_3 c_3)\, a_4 u.$$

$$\Sigma ijk = \begin{vmatrix} a_1 & a_2 & a^3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} ijk + \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} jku + \begin{vmatrix} a_3 & a_4 & a_1 \\ b_3 & b_4 & b_1 \\ c_3 & c_4 & c_1 \end{vmatrix} kui + \begin{vmatrix} a_4 & a_1 & a_2 \\ b_4 & b_1 & b_2 \\ c_4 & c_1 & c_2 \end{vmatrix} uij$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ jku & kui & uij & ijk \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ -i & j & -k & u \end{vmatrix}.$$

Thus, in a space of three dimensions $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is a true imaginary; its axis being the fourth axis in a space of four dimensions.

Product of four vectors in space of four dimensions.—By means of the types, given above, the complete product may be formed. In space of three dimensions all the types exist excepting the last. It has commonly been supposed that the product of four lines is impossible. For instance, De Morgan (Double Algebra, p. 107) says that *ABCD* is unintelligible, space not having four dimensions; and Gregory, in his paper on the "Application of Algebraical Symbols to Geometry," says, "If we combine more symbols than three, we find no geometrical interpretation for the result. In fact, it may be looked on as an impossible geometrical operation; just as $\sqrt{-1}$ is an impossible arithmetical one."

QUATERNIONS.

Definition.—By a quaternion proper is meant an arithmetical ratio combined with an amount of turning. It contains three elements: a ratio, an axis and an amount of angle. Let **a** denote a quaternion, *a* its ratio, *a* its axis and *A* the amount of angle; then $\mathbf{a} = a a^{A}$. It is called a quaternion, because *a* requires two numbers to specify it, while *a* and *A* each requires one; in all, four numbers. The ratio of two vectors is a more determinate quantity; it may involve a physical ratio, and the angle is fixed (fig. 15). If **A** and **B** are line-vectors, they define a quaternion, provided they are free to rotate round the axis $\overline{\alpha\beta}$.

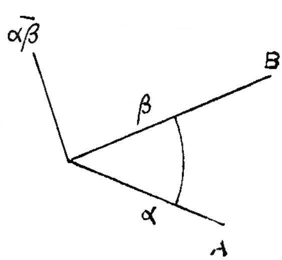

FIG. 15.

Components of a quaternion. — A quaternion may be expressed as the sum of two components, one of which has an indefinite axis, and the other the same axis as the quaternion. Consider the quaternion $a\alpha^A$. If A is less than a quadrant

$$a\alpha^A = a\left(\cos A \cdot \alpha^0 + \sin A \cdot \alpha^{\frac{\pi}{2}}\right)$$

If A is between one and two quadrants

$$a\alpha^A = a\left(\cos A \cdot \alpha^\pi + \sin A \cdot \alpha^{\frac{\pi}{2}}\right)$$

If A is between two and three quadrants

$$a\alpha^A = a\left(\cos A \cdot \alpha^\pi + \sin A \cdot \alpha^{\frac{3\pi}{2}}\right)$$

If A is between three and four quadrants

$$a\alpha^A = a\left(\cos A \cdot \alpha^{2\pi} + \sin A \cdot \alpha^{\frac{3\pi}{2}}\right)$$

and so on, for any amount of angle. Here $\cos A$ and $\sin A$ are looked upon as signless ratios. If the number of half revolutions is thrown into the ratios $\cos A$ and $\sin A$, making them algebraic ratios, then, when A is less than a revolution

$$a\alpha^A = a\left(\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}}\right)$$

and generally $a\alpha^{2r\pi + A} = a\alpha^{2r\pi}\left(\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}}\right)$

When the quaternions are all in one plane, α is constant, and need not be expressed. The quaternion takes the form of the complex ratio

$$a \cdot A = a\left(\cos A + \sin A \cdot \tfrac{\pi}{2}\right)$$

the angle $\frac{\pi}{2}$ being expressed by $\sqrt{-1}$.

If further, the quaternions are restricted to one line, the angle A can only be 0 or π; and $a \cdot 0 = a$, $a \cdot \pi = -a$.

The above equations are homogeneous; a quaternion is equated to the sum of two quaternions, the only peculiarity being that the axis of one of the components may be any axis.

SUM OF TWO QUATERNIONS.

Let $\mathbf{a} = a\,\alpha^A$ and $\mathbf{b} = b\,\beta^B$ be the two quaternions.

Since
$$\mathbf{a} = a\left(\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}}\right),$$
$$\mathbf{b} = b\left(\cos B + \sin B \cdot \beta^{\frac{\pi}{2}}\right),$$

$$\mathbf{a} + \mathbf{b} = (a\cos A + b\cos B) + \left(a\sin A \cdot \alpha^{\frac{\pi}{2}} + b\sin B \cdot \beta^{\frac{\pi}{2}}\right)$$
$$= (a\cos A + b\cos B) + (a\sin A \cdot \alpha + b\sin B \cdot \beta)^{\frac{\pi}{2}}$$
$$= r\,\varphi^\theta$$

where

$$r = \sqrt{a^2 + b^2 + 2ab\,(\cos A\ \cos B + \sin A\ \sin B\ \cos \alpha\beta)}$$

$$\varphi = \frac{a\sin A \cdot \alpha + b\sin B \cdot \beta}{r}$$

and $\theta = \cos^{-1}\dfrac{a\cos A + b\cos B}{r}$

If **a** is given in the form $a_0 + \mathbf{A}^{\frac{\pi}{2}}$ and **b** in the form $b_0 + \mathbf{B}^{\frac{\pi}{2}}$; then

$$\mathbf{a} + \mathbf{b} = a_0 + b_0 + (\mathbf{A} + \mathbf{B})^{\frac{\pi}{2}}$$

Here **A** and **B** denote vectors of zero dimensions.

If
$$\mathbf{a} = a_0 + (a_1 i + a_2 j + a_3 k)^{\frac{\pi}{2}}$$
$$\mathbf{b} = b_0 + (b_1 i + b_2 j + b_3 k)^{\frac{\pi}{2}}$$

then $$\mathbf{a} + \mathbf{b} = a_0 + b_0 + \left\{ (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k \right\}^{\frac{\pi}{2}}$$

This is the addition of complex numbers not confined to one plane.

PRODUCT OF TWO QUATERNIONS.

By the product of two quaternions is meant the product of the tensors combined with the sum of the versors. The product is a quantity of the same kind as either factor; it is the generalization for space of the product of ratios.

Let the two quaternions be

$$\mathbf{a} = a_0 + (a_1 i + a_2 j + a_3 k)^{\frac{\pi}{2}}$$
$$\mathbf{b} = b_0 + (b_1 i + b_2 j + b_3 k)^{\frac{\pi}{2}},$$

then by the rules for versors (p. 75)

$$\mathbf{ab} = a_0 b_0 - (a_1 b_1 + a_2 b_2 + a_3 b_3)$$
$$+ \left\{ \begin{array}{l} a_0 (b_1 i + b_2 j + b_3 k) + b_0 (a_1 i + a_2 j + a_3 k) \\ \quad - (a_2 b_3 - a_3 b_2) i - (a_3 b_1 - a_1 b_3) j - (a_1 b_2 - a_2 b_1) k \end{array} \right\}^{\frac{\pi}{2}}$$

Let *cos* **ab** denote the cosine of the angle of the product multiplied by the tensors of **a** and **b**, and *Sin* **ab** the directed sine of the same angle multiplied in the same manner; then

$$cos\ \mathbf{ab} = a_0 b_0 - (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

and $$Sin\ \mathbf{ab} = a_0 (b_1 i + b_2 j + b_3 k) + b_0 (a_1 i + a_2 j + a_3 k) - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}$$

If the factors are expressed more synthetically by

$$\mathbf{a} = a_0 + \mathbf{A}^{\frac{\pi}{2}}, \qquad \mathbf{b} = b_0 + \mathbf{B}^{\frac{\pi}{2}},$$

then $$\mathbf{ab} = a_0 b_0 - cos\ \mathbf{AB} + (b_0 \mathbf{A} + a_0 \mathbf{B} - Sin\ \mathbf{AB})^{\frac{\pi}{2}}.$$

Trigonometrical form of the product.—Let

$$\mathbf{a} = a\alpha^{A}, \qquad \mathbf{b} = b\beta^{B};$$

then
$$\mathbf{a} = a\left(cos\,A + sin\,A \cdot \alpha^{\frac{\pi}{2}}\right),$$
$$\mathbf{b} = b\left(cos\,B + sin\,B \cdot \beta^{\frac{\pi}{2}}\right);$$

and $$\mathbf{ab} = ab\left\{cos\,A\,cos\,B - sin\,A\,sin\,B\,cos\,\alpha\beta\right\}$$
$$+ ab\left\{cos\,B\,sin\,A \cdot \alpha + cos\,A\,sin\,B \cdot \beta - sin\,A\,sin\,B\,sin\,\alpha\beta \cdot \overline{\alpha\beta}\right\}^{\frac{\pi}{2}}$$

Let $a = b = 1$; then (fig. 16)

$$\cos \alpha^A \beta^B = \cos A \cos B - \sin A \sin B \cos \alpha\beta,$$

which is the fundamental proposition in spherical trigonometry; it is the cosine of the sum of the angles. Also

$$\mathrm{Sin}\, \alpha^A \beta^B = \cos B \sin A \cdot \alpha + \cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}$$

is the expression for the directed sine of the same sum.

Let β coincide with α; we get the fundamental propositions of plane trigonometry, namely,

$$\cos \alpha^{A+B} = \cos A \cos B - \sin A \sin B,$$

and

$$\mathrm{Sin}\, \alpha^{A+B} = (\cos B \sin A + \cos A \sin B) \cdot \alpha.$$

When only one plane is considered, α may be omitted, and the expressions become

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin (A + B) = \cos B \sin A + \cos A \sin B.$$

Here we have evidence that the consistent order of the factors in a quaternion is from left to right; for, when particularized for a plane, we get the established order in plane trigonometry.

Let $\qquad A = B = \frac{\pi}{2}$;

then $\qquad a\alpha^{\frac{\pi}{2}} b\beta^{\frac{\pi}{2}} = - ab \left(\cos \alpha\beta + \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}\right)$

This is the product of two quadrantal quaternions, which in works on quaternions is identified with the product of two vectors, only the sign of the second term is made positive.

Second power of a quaternion.—By the second power of a quaternion is meant the product of the quaternion by itself. From the general product it follows that $a\alpha^A \; a\alpha^A = a^2\alpha^{2A}$. The ratio is raised to the second power, the axis remains the same, the angle is doubled. This is not a square in the proper sense of the word.

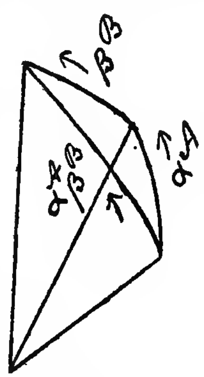

FIG. 16.

Reciprocal of a quaternion.—The quaternion **b** is the reciprocal of **a**, if **ab** $= 1$. Hence its ratio must be the reciprocal of the ratio of a, its axis opposite but its angle equal. Let it be denoted by $\mathbf{a}^{-1}$; then

$$\mathbf{a}^{-1} = \frac{1}{a} \left\{ \cos A + \sin A \; (-\alpha)^{\frac{\pi}{2}} \right\}$$
$$= \frac{1}{a} \left\{ \cos A - \sin A \, \alpha^{\frac{\pi}{2}} \right\}$$

The reciprocal of the versor α^A is the versor $(-\alpha)^A$ or α^{-A}.

Since $\qquad \alpha^A + \alpha^{-A} = 2 \cos A,$

and $\qquad \alpha^A - \alpha^{-A} = 2 \sin A \cdot \alpha^{\frac{\pi}{2}};$

by taking the second power of the former

$$a^{2A} + 2 + a^{-2A} = 4\cos^2 A$$

that is

$$\cos 2A + 1 = 2\cos^2 A;$$

and by taking the second power of the latter

$$a^{2A} - 2 + a^{-2A} = -4\sin^2 A.$$

that is

$$\cos 2A - 1 = -2\sin^2 A.$$

PRODUCT OF THREE QUATERNIONS.

As the product of two quaternions is a quaternion, the product of that product with a third quaternion is found by the same rules as before.

Let $\mathbf{a} = a_0 + \mathbf{A}^{\frac{\pi}{2}}$, $\quad \mathbf{b} = b_0 + \mathbf{B}^{\frac{\pi}{2}}$, $\quad \mathbf{c} = c_0 + \mathbf{C}^{\frac{\pi}{2}}$.

Now, $\quad \mathbf{ab} = a_0 b_0 - \cos \mathbf{AB} + (b_0 \mathbf{A} + a_0 \mathbf{B} - Sin\, \mathbf{AB})^{\frac{\pi}{2}}$;

and by taking the several products of these terms with those of $\mathbf{c}$, we obtain

$$\mathbf{abc} = a_0 b_0 c_0 - a_0 \cos \mathbf{BC} - b_0 \cos \mathbf{AC} - c_0 \cos \mathbf{AB} + \cos\, (Sin\, \mathbf{AB})\, \mathbf{C}$$
$$+ \left\{ \begin{array}{l} b_0 c_0\, \mathbf{A} + c_0 a_0\, \mathbf{B} + a_0 b_0\, \mathbf{C} - \cos \mathbf{AB} \cdot \mathbf{C} - a_0\, Sin\, \mathbf{BC} \\ \quad - b_0\, Sin\, \mathbf{AC} - c_0\, Sin\, \mathbf{AB} + Sin\, (Sin\, \mathbf{AB})\, \mathbf{C} \end{array} \right\}^{\frac{\pi}{2}}$$

As this is itself a quaternion, the former term may be denoted by $\cos \mathbf{abc}$, and the latter by $Sin\, \mathbf{abc}$. The latter may be written in the more symmetrical form

$$b_0 c_0\, \mathbf{A} + c_0 a_0\, \mathbf{B} + a_0 b_0\, \mathbf{C} - \cos \mathbf{BC} \cdot \mathbf{A} + \cos \mathbf{CA} \cdot \mathbf{B} - \cos \mathbf{AB} \cdot \mathbf{C}$$
$$- a_0\, Sin\, \mathbf{BC} + b_0\, Sin\, \mathbf{CA} - c_0\, Sin\, \mathbf{AB}.$$

Let $\mathbf{a} = a\alpha^{A}$, $\quad \mathbf{b} = b\beta^{B}$, $\quad \mathbf{c} = c\gamma^{C}$.

the above expressions become

$$\cos \mathbf{abc} = abc \left\{ \begin{array}{l} \cos A \cos B \cos C - \cos A \sin B \sin C \cos \beta\gamma \\ \quad - \cos B \sin C \sin A \cos \gamma\alpha - \cos C \sin A \sin C \cos \alpha\beta \\ \quad + \sin A \sin B \sin C \sin \alpha\beta \cos \overline{\alpha\beta}\gamma \end{array} \right\}$$

and

$$Sin\, \mathbf{abc} = abc \left\{ \begin{array}{l} \cos B \cos C \sin A \cdot \alpha + \cos C \cos A \sin B \cdot \beta \\ + \cos A \cos B \sin C \cdot \gamma \\ - \sin A \sin B \sin C\ (\cos \beta\gamma\, .\, \alpha - \cos \gamma\alpha \cdot \beta + \cos \alpha\beta \cdot \gamma) \\ - \cos A \sin B \sin C \sin \beta\gamma \cdot \overline{\beta\gamma} \\ + \cos B \sin C \sin A \sin \gamma\alpha \cdot \overline{\gamma\alpha} \\ - \cos C \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta} \end{array} \right\}$$

As all the terms are evidently symmetrical with respect to β, with the exception of the fifth, it follows that $(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc})$ provided

$$\sin \alpha\beta \cos \overline{\alpha\beta}\gamma \text{ is equal to } \sin \beta\gamma \cos \alpha\overline{\beta\gamma};$$

but this is a known truth. Hence in this species of multiplication the mode of association of the factors is indifferent.

When a, b and c are coplanar, $\alpha = \beta = \gamma$; and

$$\cos \alpha^{A+B+C} = \cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin C \sin A$$
$$- \cos C \sin A \sin B,$$

and

$$\sin \alpha^{A+B+C} = \cos B \cos C \sin A + \cos C \cos A \sin B + \cos A \cos B \sin C$$
$$- \sin A \sin B \sin C,$$

which are identical with the formulæ in plane trigonometry.

If further $A = B = C$,

$$\alpha^{3A} = \cos^3 A - 3\cos A \sin^2 A + \left\{ 3\cos^2 A \sin A - \sin^3 A \right\} \alpha^{\frac{\pi}{2}}$$

Let $A = B = C = \frac{\pi}{2}$; then

$$\alpha^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} = \sin \alpha\beta \quad \cos \quad \overline{\alpha\beta}\gamma + \left\{ - \cos \beta\gamma \cdot \alpha + \cos \gamma\alpha \cdot \beta - \cos \alpha\beta \cdot \gamma \right\}^{\frac{\pi}{2}}$$

Finite rotation.—The effect of a finite rotation on a line is in general not an algebraic product. Let α be the axis and θ the amount of the rotation, **R** a line of length r and axis ρ. Then (fig. 17)

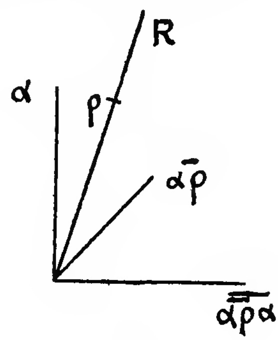

FIG. 17.

$$\alpha^{\theta}\mathbf{R} = r \left\{ \cos \alpha\rho \cdot \alpha + \sin \alpha\rho \sin \theta \cdot \overline{\alpha\rho} + \sin \alpha\rho \cos \theta \cdot \overline{\overline{\alpha\rho\alpha}} \right\}$$

The effect of a subsequent rotation β^{ϕ} is got by applying the same rule to each of the components of **R** in its new position.

In the expression for the quaternion $\alpha^A\beta^B\gamma^C$, let $\alpha^A = \gamma^{-C}$; it will be found on making the reductions that

$$\gamma^{-C}\beta^B\gamma^C = \cos B$$
$$+ \sin B \left\{ \begin{array}{l} \cos^2 C \cdot \beta - \sin^2 C \sin \gamma\beta \;.\; \overline{\overline{\gamma\beta\gamma}} + \sin^2 C \cos \gamma\beta \cdot \gamma \\ \quad 2\sin C \cos C \sin \gamma\beta \cdot \overline{\gamma\beta}\,; \end{array} \right\}$$
$$= \cos B$$
$$+ \sin B \left\{ \begin{array}{l} \cos \gamma\beta \cdot \gamma + 2\sin C \cos C \sin \gamma\beta \cdot \overline{\gamma\beta} \\ \quad + (\cos^2 C - \sin^2 C) \sin \gamma\beta \cdot \overline{\overline{\gamma\beta\gamma}} \end{array} \right\}$$
$$= (\gamma^{2C}\beta)^B.$$

Thus the effect of γ^{-C} ( ) γ^{C} upon the quaternion β^{B} is to rotate its axis by an angle of $2C$ round γ. Hence the effect of a rotation a^{θ} upon any line $r\rho$ is

$$r a^{\theta} \rho = r a^{-\frac{\theta}{2}} \rho^{\frac{\pi}{2}} a^{\frac{\theta}{2}}.$$

The effect of a subsequent rotation β^{ϕ} is

$$\beta^{\phi} a^{\theta} \rho = \beta^{-\frac{\phi}{2}} (a^{-\frac{\theta}{2}} \rho^{\frac{\pi}{2}} a^{\frac{\theta}{2}}) \beta^{\frac{\phi}{2}}$$
$$= (\beta^{-\frac{\phi}{2}} a^{-\frac{\theta}{2}}) \rho^{\frac{\pi}{2}} (a^{\frac{\theta}{2}} \beta^{\frac{\phi}{2}})$$

for the multiplication may be associated in any manner. Now $\beta^{-\frac{\phi}{2}} a^{-\frac{\theta}{2}}$ is the reciprocal of $a^{\frac{\theta}{2}} \beta^{\frac{\phi}{2}}$; hence the axis of the rotation $a^{\theta} \beta^{\phi}$ is the same as that of the versor $a^{\frac{\theta}{2}} \beta^{\frac{\phi}{2}}$ and its angle is double that of the versor.

DE MOIVRE'S THEOREM.

It has been shown that $a^{A} a^{B} = a^{A+B}$; it follows that n being any number whole or fractional,

$$(a^{\theta})^{n} = a^{n\theta}.$$

Hence by decomposing into component quaternions,

$$\cos n\theta + \sin n\theta \cdot a^{\frac{\pi}{2}} = (\cos \theta + \sin \theta \cdot a^{\frac{\pi}{2}})^{n}$$
$$= \cos^{n} \theta - \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} \theta \sin^{2}\theta +$$
$$+ \left\{ n \cos^{n-1} \theta \sin \theta - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cos^{n-3} \theta \sin^{3} \theta + \right\} a^{\frac{\pi}{2}}$$

The component $\cos n\theta$ having an indefinite axis is equal to the sum of the components which have an indefinite axis and $\sin n\theta$ which has a definite axis is equal to the sum of the components having the same definite axis.

When the value of θ is not restricted to be less than a revolution, let

$$a^{\theta} = a^{2r\pi + \phi}$$

where φ is less than a revolution. Then

$$(a^{\theta})^{n} = a^{2rn\pi + n\phi} = a^{2rn\pi} (\cos \varphi + \sin \varphi \cdot a^{\frac{\pi}{2}})^{n}$$

The quaternion $(a^{\theta}\beta^{\phi})^{n}$ may be expanded in a similar manner. For

$$(a^{\theta}\beta^{\phi})^{n} = (a_{0} + \mathbf{A}^{\frac{\pi}{2}})^{n},$$

where

$$a_{0} = \cos \theta \cos \varphi - \sin \theta \sin \varphi \cos a\beta$$

and

$$\mathbf{A} = \cos \varphi \sin \theta \cdot a + \cos \theta \sin \varphi \cdot \beta - \sin \theta \sin \varphi \sin a\beta \cdot \overline{a\beta}.$$

But it is not true that

$$(a^{\theta}\beta^{\phi})^{n} = a^{n\theta}\beta^{n\phi};$$

for such law of indices assumes that the factors are commutative.

Expansion of $\cos^n\theta$ *and* $\sin^n\theta$ *in cosines or sines of multiples of* θ.

Since

$$a^{n\theta} = \cos n\theta + \sin n\theta \cdot a^{\frac{\pi}{2}}$$

and

$$a^{-n\theta} = \cos n\theta - \sin n\theta \cdot a^{\frac{\pi}{2}};$$

$$a^{n\theta} + a^{-n\theta} = 2\cos n\theta \text{ and } a^{n\theta} - a^{-n\theta} = 2\sin n\theta \cdot a^{\frac{\pi}{2}}.$$

Now

$$(a^{\theta} + a^{-\theta})^n = a^{n\theta} + a^{-n\theta} + n\,(a^{(n-2)\theta} + a^{-(n-2)\theta}) + \text{etc.},$$

therefore

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \text{etc.}$$

When n is odd,

$$(a^{\theta} - a^{-\theta})^n = a^{n\theta} - a^{-n\theta} - n\,(a^{(n-2)\theta} - a^{-(n-2)\theta}) + \text{etc.},$$

therefore

$$2^{n-1}\sin^n\theta \cdot a^{\frac{n-1}{2}\pi} = \sin n\theta - n\sin(n-2)\theta +$$

When n is even,

$$(a^{\theta} - a^{-\theta})^n = a^{n\theta} + a^{-n\theta} - n\,(a^{(n-2)\theta} + a^{-(n-2)\theta}) +$$

therefore

$$2^{n-1}\sin^n\theta \cdot a^{\frac{n}{2}\pi} = \cos n\theta - n\cos(n-2)\theta +$$

QUATERNION EXPONENTIALS.

Expression of a versor as an exponential.

The versor $\quad a^{\theta} = \cos\theta + \sin\theta \cdot a^{\frac{\pi}{2}};$

but $\quad \cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} -$

$$= 1 + \frac{(\theta \cdot a^{\frac{\pi}{2}})^2}{\lfloor 2} + \frac{(\theta \cdot a^{\frac{\pi}{2}})^4}{\lfloor 4};$$

and $\quad \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} -$

therefore $\quad \sin\theta \cdot a^{\frac{\pi}{2}} = \theta \cdot a^{\frac{\pi}{2}} + \frac{(\theta \cdot a^{\frac{\pi}{2}})^3}{\lfloor 3} + \frac{(\theta \cdot a^{\frac{\pi}{2}})^5}{\lfloor 5} + ;$

therefore $\quad a^{\theta} = 1 + \theta \cdot a^{\frac{\pi}{2}} + \frac{(\theta \cdot a^{\frac{\pi}{2}})^2}{\lfloor 2} +$

$$= \varepsilon^{\theta \cdot a^{\frac{\pi}{2}}}$$

Similarly $\quad a^{-\theta} = \varepsilon^{-\theta \cdot a^{\frac{\pi}{2}}} = \varepsilon^{\theta \cdot a^{\frac{3\pi}{2}}}$

Let n be any even number; then

$$a^{n\pi} = 1 = \varepsilon^{n\pi \cdot a^{\frac{\pi}{2}}} = 1 + n\pi \cdot a^{\frac{\pi}{2}} - \frac{(n\pi)^2}{\underline{|2}} + \frac{(n\pi)^3}{\underline{|3}} a^{3\frac{\pi}{2}} + \text{etc.};$$

therefore $$1 = 1 - \frac{(n\pi)^2}{\underline{|2}} + \frac{(n\pi)^4}{\underline{|4}} -$$

and $$0 = n\pi - \frac{(n\pi)^3}{\underline{|3}} + \text{etc.}$$

If n is an odd number

$$-1 = 1 - \frac{(n\pi)^2}{\underline{|2}} + \frac{(n\pi)^4}{\underline{|4}} - \text{etc.}$$

$$0 = n\pi - \frac{(n\pi)^3}{\underline{|3}} + \text{etc.}$$

Logarithm of a quaternion.

The general quaternion is $ra^{\theta} = r\varepsilon^{\theta \cdot a^{\frac{\pi}{2}}} = \varepsilon^{\log r + \theta \cdot a^{\frac{\pi}{2}}}$

Hence $$\log (ra^{\theta}) = \log r + \theta \cdot a^{\frac{\pi}{2}}.$$

If the quaternion is given as $\mathbf{a} = a + b \cdot a^{\frac{\pi}{2}}$

then $$\log \mathbf{a} = \tfrac{1}{2} \log (a^2 + b^2) + \tan^{-1} \frac{b}{a} \cdot a^{\frac{\pi}{2}}$$

Hence $$\log 1 = 0 \quad \text{but } \log (-1) = \pi \cdot a^{\frac{\pi}{2}}.$$

The more general form is $\mathbf{a} = a^{2r\pi} (a + b \cdot a^{\frac{\pi}{2}})$,

and $$\log \mathbf{a} = \tfrac{1}{2} \log (a^2 + b^2) + \left(\tan^{-1} \frac{b}{a} + 2r\pi\right) \cdot a^{\frac{\pi}{2}}$$

Quaternion exponential.

Since

$$a^{\theta} = \cos\theta + \sin\theta \cdot a^{\frac{\pi}{2}}$$

$$\varepsilon^{a^{\theta}} = \varepsilon^{\cos\theta + \sin\theta \cdot a^{\frac{\pi}{2}}} = \varepsilon^{\cos\theta}\, \varepsilon^{\sin\theta \cdot a^{\frac{\pi}{2}}},$$

$$= \left(1 + \cos\theta + \frac{\cos^2\theta}{\underline{|2}} + \right) \left(1 + \sin\theta \cdot a^{\frac{\pi}{2}} + \frac{\sin^2\theta \cdot a^{\pi}}{\underline{|2}} + \right)$$

$$= \left(1 + \cos\theta + \frac{\cos^2\theta}{\underline{|2}} + \right) \left(1 - \frac{\sin^2\theta}{\underline{|2}} + \frac{\sin^4\theta}{\underline{|4}} - \right)$$

$$+ \left(1 + \cos\theta + \frac{\cos^2\theta}{\underline{|2}} + \right) \left(\sin\theta - \frac{\sin^3\theta}{\underline{|3}} + \right) . a^{\frac{\pi}{2}}$$

Let $\theta = \frac{\pi}{2}$; then $\varepsilon^{a^{\frac{\pi}{2}}} = a^{1}\cdot$

$$= 1 - \frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} -$$

$$+\left\{1-\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}-\right\}\cdot a^{\frac{\pi}{2}}.$$

Let $\theta = 0$; then $\varepsilon^{a0'} = \varepsilon$

SCALAR DIFFERENTIATION.

By scalar differentiation is meant differentiation with respect to a variable which has no axis, or the only axis considered; for instance, time, or length along a curve, or distance along an axis if one axis only is considered.

Differentiation of a vector.—Consider the radius-vector of a point, $\mathbf{R} = r\rho$, where r denotes the length and ρ the axis.

The velocity-vector $\frac{\mathrm{d}\mathbf{R}}{dt}$ is obtained by differentiating $r\rho$ in the same manner as an ordinary product;

$$\frac{\mathrm{d}\mathbf{R}}{dt} = \frac{dr}{dt}\rho + r\frac{\mathrm{d}\rho}{dt}.$$

Here a small Roman **d** is used to denote a directed differential. The whole velocity may be denoted in accordance with the

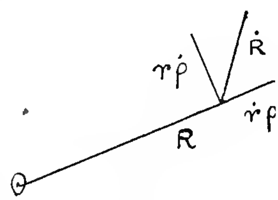

FIG. 18.

fluxional notation by $\dot{\mathbf{R}}$, the component along the radius vector by $\dot{r}\rho$ and the component transverse to the radius-vector by $r\dot{\rho}$ (fig. 18). By differentiating each component of the velocity according to the same rule, we obtain the acceleration-vector

$$\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = \frac{d^2r}{dt^2}\rho + 2\frac{dr}{dt}\frac{\mathrm{d}\rho}{dt} + r\frac{\mathrm{d}^2\rho}{dt^2}$$

or

$$\ddot{\mathbf{R}} = \ddot{r}\rho + 2\dot{r}\dot{\rho} + r\ddot{\rho}$$

The angular velocity $\frac{\mathrm{d}\rho}{dt}$ may be analyzed into $\frac{d\rho}{dt}\bar{\rho}$, where $\frac{d\rho}{dt}$ denotes its ratio magnitude and $\bar{\rho}$ its direction, which is perpendicular to ρ.

Hence $\frac{\mathrm{d}\mathbf{R}}{dt} = \frac{dr}{dt}\cdot\rho + r\frac{d\rho}{dt}\cdot\bar{\rho}$;

and $\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = \frac{d^2r}{dt^2}\cdot\rho + \left(2\frac{dr}{dt}\frac{d\rho}{dt} + r\frac{d^2\rho}{dt^2}\right)\cdot\bar{\rho} + r\frac{d\rho}{dt}\frac{d\bar{\rho}}{dt}\cdot\bar{\bar{\rho}}$

The direction of the third component $\bar{\bar{\rho}}$ is perpendicular to the perpendicular to ρ; in a plane it is $= -\rho$, and then

$$\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\rho}{dt}\right)^2\right\}\cdot\rho + \left\{2\frac{dr}{dt}\frac{d\rho}{dt} + r\frac{d^2\rho}{dt^2}\right\}\cdot\bar{\rho}.$$

The expression for the magnitude of $\frac{\mathrm{d}\mathbf{R}}{dt}$ is $\frac{ds}{dt}$ and for its axis $\frac{\mathrm{d}\mathbf{R}}{ds}$; thus $\frac{\mathrm{d}\mathbf{R}}{dt} = \frac{ds}{dt}\frac{\mathrm{d}\mathbf{R}}{ds}$, and by applying the rule for differentiating a vector,

$$\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = \frac{d^2s}{dt^2}\frac{\mathrm{d}\mathbf{R}}{ds} + \left(\frac{ds}{dt}\right)^2\frac{\mathrm{d}^2\mathbf{R}}{ds^2},$$

the former component expressing the acceleration along the tangent, and the latter that along the radius of curvature.

Let $\mathbf{C} = u \cdot \xi + v \cdot \eta + w \cdot \zeta$ where each of the six elements may vary, then

$$\frac{\mathrm{d}\mathbf{C}}{dt} = \frac{du}{dt} \cdot \xi + \frac{dv}{dt} \cdot \eta + \frac{dw}{dt} \cdot \zeta$$
$$+ u \cdot \frac{\mathrm{d}\xi}{dt} + v \cdot \frac{\mathrm{d}\eta}{dt} + w \cdot \frac{\mathrm{d}\zeta}{dt}$$

If ξ, η and ζ are constant, the second expression vanishes. The simplest case is

$$\mathbf{R} = xi + yj + zk$$

giving
$$\frac{\mathrm{d}\mathbf{R}}{dt} = \frac{dx}{dt}\, i + \frac{dy}{dt}\, j + \frac{dz}{dt}\, k;$$

and for the same reason, it follows that

$$\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = \frac{d^2x}{dt^2}\, i + \frac{d^2y}{dt^2}\, j + \frac{d^2z}{dt^2}\, k.$$

Products.—A velocity-vector, or an acceleration-vector is combined the same as a simple vector. For example,

$$\mathbf{R}\,\frac{\mathrm{d}\mathbf{R}}{dt} = (xi + yj + zk)\,\left(\frac{dx}{dt}\, i + \frac{dy}{dt}\, j + \frac{dz}{dt}\, k\right)$$
$$= x\,\frac{dx}{dt} + y\,\frac{dy}{dt} + z\,\frac{dz}{dt} + \begin{vmatrix} x & y & z \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ i & j & k \end{vmatrix}$$

Also
$$\mathbf{R}\,\frac{\mathrm{d}\mathbf{R}}{dt} = r\rho\,\left(\frac{dr}{dt}\,\rho + r\,\frac{\mathrm{d}\rho}{dt}\right)$$
$$= r\,\frac{dr}{dt} + r^2\,\rho\,\frac{\mathrm{d}\rho}{dt}\,,$$

thus
$$\cos \mathbf{R}\,\frac{\mathrm{d}\mathbf{R}}{dt} = r\,\frac{dr}{dt} \quad \text{and } Sin\, \mathbf{R}\,\frac{\mathrm{d}\mathbf{R}}{dt} = r^2\,\rho\,\frac{\mathrm{d}\rho}{dt}$$

Similarly
$$\mathbf{R}^{-1}\,\frac{\mathrm{d}\mathbf{R}}{dt} = \frac{1}{r}\,\frac{dr}{dt} + \rho\,\frac{\mathrm{d}\rho}{dt}\,.$$

Also
$$\mathbf{R}\,\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = r\rho\,\left(\frac{d^2r}{dt^2}\,\rho + 2\,\frac{dr}{dt}\,\frac{\mathrm{d}\rho}{dt} + r\,\frac{\mathrm{d}^2\rho}{dt^2}\right),$$
$$= r\,\frac{d^2r}{dt^2} + 2r\,\frac{dr}{dt}\,\rho\,\frac{\mathrm{d}\rho}{dt} + r^2\rho\,\frac{\mathrm{d}^2\rho}{dt^2}\;;$$

thus
$$\cos \mathbf{R}\,\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = r\,\frac{d^2r}{dt^2} + r^2 \cos\left(\rho\,\frac{\mathrm{d}^2\rho}{dt^2}\right)$$

and
$$Sin\, \mathbf{R}\,\frac{\mathrm{d}^2\mathbf{R}}{dt^2} = 2r\,\frac{dr}{dt}\,\rho\,\frac{\mathrm{d}\rho}{dt} + r^2\, Sin\left(\rho\,\frac{\mathrm{d}^2\rho}{dt^2}\right)$$

Differentiation of a product of two vectors.—Let $\mathbf{B} = b\,\beta$ and $\mathbf{C} = c\,\gamma$ be any two vectors; it is required to differentiate their product **BC** with respect to time or any scalar variable. The rule is to apply the rule of differentiation (p. 104) to each factor of the product, supposing the other constant, and preserving the order of the factors. This is a generalization of the rule for the ordinary algebraic product. Thus

$$\mathrm{d}\,\frac{(\mathbf{BC})}{dt} = \frac{\mathrm{d}\mathbf{B}}{dt}\,\mathbf{C} + \mathbf{B}\,\frac{\mathrm{d}\mathbf{C}}{dt}$$
$$= \left(\frac{db}{dt}\,\beta + b\,\frac{\mathrm{d}\beta}{dt}\right)\,c\gamma + b\beta\,\left(\frac{dc}{dt}\,\gamma + c\,\frac{\mathrm{d}\gamma}{dt}\right)$$
$$= \left(c\,\frac{db}{dt} + b\,\frac{dc}{dt}\right)\beta\gamma + \,bc\,\left(\frac{\mathrm{d}\beta}{dt}\,\gamma + \beta\,\frac{\mathrm{d}\gamma}{dt}\right)$$

Hence $\frac{d\mathbf{B}^2}{dt} = 2b\,\frac{db}{dt} = 2\cos \mathbf{B}\,\frac{d\mathbf{B}}{dt}$

Let $\mathbf{B} = ai + bj + ck, \quad \mathbf{C} = ui + vj + wk;$

then $$\mathbf{B\,C} = au + bv + cw + \begin{vmatrix} a & b & c \\ u & v & w \\ i & j & k \end{vmatrix},$$

and
$$\frac{d(\mathbf{BC})}{dt} = \frac{da}{dt}u + \frac{db}{dt}v + \frac{dc}{dt}w$$
$$+ a\frac{du}{dt} + b\frac{dv}{dt} + c\frac{dw}{dt}$$
$$+ \begin{vmatrix} \frac{da}{dt} & \frac{db}{dt} & \frac{dc}{dt} \\ u & v & w \\ i & j & k \end{vmatrix} + \begin{vmatrix} a & b & c \\ \frac{du}{dt} & \frac{dv}{dt} & \frac{dw}{dt} \\ i & j & k \end{vmatrix}$$

Hence $$\frac{d\mathbf{B}^2}{dt} = 2\left(a\frac{da}{dt} + b\frac{db}{dt} + c\frac{dc}{dt} \right)$$

Differentiation of a product of three vectors.—Let **B, C, D** be any three vectors, **B** and **C** having elements as before, and $\mathbf{D} = d\delta = fi + gj + hk.$

Then $$\frac{d(\mathbf{B\,CD})}{dt} = \frac{d\mathbf{B}}{dt}\,\mathbf{CD} + \mathbf{B}\,\frac{d\mathbf{C}}{dt}\,\mathbf{D} + \mathbf{B\,C}\,\frac{d\mathbf{D}}{dt}$$

where, not only must the order of the factors, but also their mode of association be preserved.

Let $\mathbf{C} = \mathbf{B}$, then $\frac{d(\mathbf{B}^2\mathbf{D})}{dt} = 2b\,\frac{db}{dt}\,\mathbf{D} + b^2\,\frac{d\mathbf{D}}{dt}$

If further $\mathbf{D} = \mathbf{B}$, then $\frac{d\mathbf{B}^3}{dt} = 2b^2\,\frac{db}{dt}\,\beta + b^2\,\frac{d\mathbf{B}}{dt}$

Differentiation of a power of a vector.—It is evident that
$$\frac{d\mathbf{B}^2}{dt} = 2\cos\left(\mathbf{B}\,\frac{d\mathbf{B}}{dt}\right)$$
and
$$\frac{d\mathbf{B}^3}{dt} = 2\cos\left(\mathbf{B}\,\frac{d\mathbf{B}}{dt}\right)\mathbf{B} + \mathbf{B}^2\,\frac{d\mathbf{B}}{dt}$$
are true generalizations of the differentiation which occurs in ordinary algebra. For if the quantity **B** has a constant axis, as is supposed in that algebra, $\frac{d\mathbf{B}^2}{dt}$ becomes $2\mathbf{B}\,\frac{d\mathbf{B}}{dt}$, and $\frac{d\mathbf{B}^3}{dt}$ becomes $3\mathbf{B}^2\,\frac{d\mathbf{B}}{dt}$. According to the principles of quaternions a minus sign would be introduced.

It may be shown generally that when n is even,
$$\frac{d\mathbf{B}^n}{dt} = nb^{n-1}\,\frac{db}{dt};$$
and when n is odd,
$$\frac{d\mathbf{B}^n}{dt} = (n-1)\,b^{n-2}\,\frac{db}{dt}\,\mathbf{B} + b^{n-1}\,\frac{d\mathbf{B}}{dt}.$$

This holds also, when n is negative; for instance, $n = -1$. For by direct differentiation
$$\frac{d\mathbf{B}^{-1}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{B}}{b^2}\right) = -\frac{2\,\frac{db}{dt}}{b^3}\,\mathbf{B} + \frac{1}{b^2}\,\frac{d\mathbf{B}}{dt},$$

which agrees with the formula. The simplicity of this process may be compared with that given in Tait's *Treatise on Quaternions*, p. 97, where a vector is treated as a quadrantal versor.

Differentiation of a quaternion.—Let $\mathbf{r} = r\varphi^{\theta}$ be any quaternion; then

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\varphi^{\theta} + r\frac{d}{dt}(\varphi^{\theta});$$

we have to find how to differentiate the versor φ^{θ}, supposing φ to vary perpendicular to an initial line.

Since

$$\varphi^{\theta} = \cos\theta + \sin\theta \cdot \varphi^{\frac{\pi}{2}}$$

$$\frac{d}{dt}\varphi^{\theta} = -\sin\theta\frac{d\theta}{dt} + \left\{\cos\theta\frac{d\theta}{dt}\cdot\varphi + \sin\theta\frac{d\phi}{dt}\right\}^{\frac{\pi}{2}}$$

$$= \frac{d\theta}{dt}\varphi^{\theta+\frac{\pi}{2}} + \sin\theta\cdot\frac{d\phi^{\frac{\pi}{2}}}{dt}.$$

Hence

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\varphi^{\theta} + r\frac{d\theta}{dt}\varphi^{\theta+\frac{\pi}{2}} + r\sin\theta\frac{d\phi^{\frac{\pi}{2}}}{dt}.$$

By applying the rule found to each of the components of $\frac{d\mathbf{r}}{dt}$ we obtain

$$\frac{d^2\mathbf{r}}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\varphi^{\theta} + \left\{2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right\}\varphi^{\theta+\frac{\pi}{2}}$$

$$+ 2\left\{\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta\right\}\frac{d\phi^{\frac{\pi}{2}}}{dt} + r\sin\theta\frac{d^2\phi^{\frac{\pi}{2}}}{dt^2}$$

Since $\frac{d\phi}{dt} = \frac{d\phi}{dt}\bar{\varphi}$ where $\bar{\varphi}$ denotes an axis perpendicular to φ;

$$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{dt^2}\bar{\varphi} + \frac{d\phi}{dt}\frac{d\bar{\phi}}{dt}\bar{\bar{\varphi}}.$$

In the case of polar coördinates φ is always perpendicular to a constant axis a; then

$$\bar{\varphi} = \overline{\varphi a} \qquad \text{and } \bar{\bar{\varphi}} = -\varphi \qquad \text{and } \frac{d\bar{\phi}}{dt} = \frac{d\phi}{dt}.$$

Hence the components for the acceleration in terms of polar coördinates

$$\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right\}\varphi^{\theta}$$

$$+ \left\{2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt} - r\sin\theta\cos\theta\,\dot{\varphi}^2\right\}\varphi^{\theta+\frac{\pi}{2}}$$

$$+ \left\{2\frac{dr}{dt}\sin\theta\frac{d\phi}{dt} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\phi}{dt} + r\sin\theta\frac{d^2\phi}{dt^2}\right\}\overline{\varphi a}^{\frac{\pi}{2}}.$$

MATRICES.

Dyad.—In order to specify a homogeneous strain the conception of the dyad is required. It specifies the manner in which all lines originally parallel to a given direction are changed in magnitude and direction. If a line **A** (fig. 19) is changed into **B**, and all lines having the axis α are changed homogeneously, such change is expressed by the dyad $\mathbf{A}^{-1}\mathbf{B}$, that is, $\frac{b}{a}\alpha\beta$. Thus, a dyad expresses an arithmetical ratio combined with a change of axis. If such strain is followed by another specified by

$$\mathbf{B}^{-1}\mathbf{C} = \frac{c}{b}\beta\gamma,$$

the result of the two is found by taking the product of the dyads which means multiplying the ratios and adding the angles. Thus,

$$(\mathbf{A}^{-1}\mathbf{B})\ (\mathbf{B}^{-1}\mathbf{C}) = \frac{b}{a}\frac{c}{b}\ (\alpha\beta)\ (\beta\gamma) = \frac{c}{a}\ \alpha\gamma.$$

In ordinary algebra it is indifferent whether a ratio is written $a^{-1}b$ or ba^{-1}, because no angle is involved. But in specifying a physical ratio,

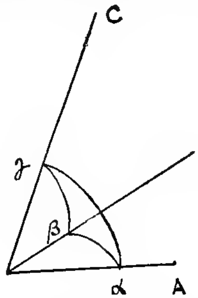

Fig. 19.

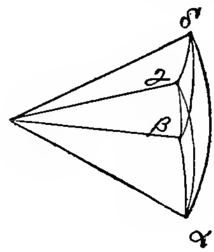

Fig. 20.

where an angle is involved, it is convenient to choose an order; and the proper order appears to be that which specifies the order of the change in the order of writing.

The conjugate dyad is

$$\mathbf{B}\mathbf{A}^{-1} = \frac{b}{a}\ \beta\alpha\,;$$

and the reciprocal dyad is

$$\mathbf{B}^{-1}\mathbf{A} = \frac{a}{b}\ \beta\alpha,$$

for

$$(\mathbf{B}^{-1}\mathbf{A})\ (\mathbf{A}^{-1}\mathbf{B}) = \frac{a}{b}\frac{b}{a}\ (\beta\alpha)\ (\alpha\beta) = 1.$$

If a third change follows specified by

$$\mathbf{C}^{-1}\mathbf{D} = \frac{d}{c}\ \gamma\delta,$$

then the result of the three is

$$(\mathbf{A}^{-1}\mathbf{B})\ (\mathbf{B}^{-1}\mathbf{C})\ (\mathbf{C}^{-1}\mathbf{D}) = \frac{b}{a}\frac{c}{b}\frac{d}{c}\ (\alpha\beta)\ (\beta\gamma)\ \gamma\delta).$$
$$= \frac{d}{a}\ \alpha\delta.$$

The difference between the multiplication of dyads and of quaternions is that in the former the angles are localized and each succeeding one starts from the end of the preceding (fig. 20). The multiplication of quaternions is indifferent with respect to association, it follows *a fortiori* that the multiplication of dyads is also indifferent. This means that we get the same angle $\alpha\delta$ whether we first take the sum of $\alpha\beta$ and $\beta\gamma$ which is $\alpha\gamma$ and then the sum of $\alpha\gamma$ and $\gamma\delta$; or whether we first take the sum of $\beta\gamma$ and $\gamma\delta$ which is $\beta\delta$ and then the sum of $\alpha\beta$ and $\beta\delta$. In a product of three vectors, the two non-associated vectors form a dyad; that is, in (**AB**)**C**, the relation of **B** to **C** is $bc\,\beta\gamma$.

Notation for a matrix.—A homogeneous strain may be considered as a physical quantity, and as such denoted by a single symbol Φ. It is equal to the sum of three dyads, one for each independent axis of space; and is expressed quite generally by the sum of the dyads for each of three mutually rectangular axes. In the figure (fig. 21) the dyad for the axis k is represented. An axis k receives an increment in its own direction, in the direction of j and in the direction of k.

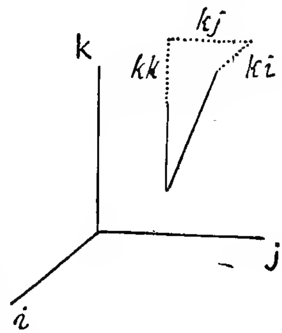

FIG. 21.

Such sum of three dyads is equivalent to the linear-vector operator of Hamilton, or the matrix of algebra. The notation used by Cayley in his *Memoir on Matrices* is

$$(X,\ Y,\ Z) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} (x,\ y,\ z);$$

which represents the coëfficients of a linear transformation separated from the variables, but does not express explicitly the ratios.

As a sum of three mutually rectangular dyads

$$\begin{aligned} \Phi &= (1i)^{-1}\,\mathbf{A} + (1j)^{-1}\,\mathbf{B} + (1k)^{-1}\,\mathbf{C}; \\ &= (1i)^{-1}\,(a_1 i + a_2 j + a_3 k) \\ &\quad + (1j)^{-1}\,(b_1 i + b_2 j + b_3 k) \\ &\quad + (1k)^{-1}\,(c_1 i) + c_2 j + c_3 k); \\ &= a_1 ii + a_2 ij + a_3 ik \\ &\quad + b_1 ji + b_2 jj + b_3 jk \\ &\quad + c_1 ki + c_2 kj + c_3 kk. \end{aligned}$$

In the last expression, where the ratios are expressed, not merely indicated, a_1 means the former a_1 divided by 1, and therefore is a ratio not a length; and ii denotes the angle between the axis i and i, which is nought, while ij denotes that between i and j. Hence, in such an expression as

$$\Phi = i\mathbf{A} + j\mathbf{B} + k\mathbf{C}$$

A, **B** and **C** are ratio-vectors.

Conjugate matrix.—The conjugate of Φ, denoted by Φ' is formed by taking the conjugate of each of the elementary ratios; thus,

$$\begin{aligned}\Phi' &= a_1 ii + b_1 ij + c_1 ik \\ &+ a_2 ji + b_2 jj + c_2 jk \\ &+ a_3 ki + b_3 kj + c_3 kk; \\ &= i\,(a_1 i + b_1 j + c_1 k) \\ &+ j\,(a_2 i + b_2 j + c_2 k) \\ &+ k\,(a_3 i + b_3 j + c_3 k).\end{aligned}$$

Ratio for any axis.—The ratio for any axis ρ may be denoted by $\rho r \rho'$ where r denotes the ratio and ρ' the new axis. It is deduced from the three rectangular ratios in the following manner:

$$\rho r \rho' = \rho \left\{ \cos \rho i\,(a_1 i + a_2 j + a_3 k) + \cos \rho j\,(b_1 i + b_2 j + b_3 k) + \cos \rho k\,(c_1 i + c_2 j + c_3 k) \right\}.$$

Hence any line $R\rho$ becomes

$$rR\rho' = R\,(\cos \rho i \cdot \mathbf{A} + \cos \rho j \cdot \mathbf{B} + \cos \rho k \cdot \mathbf{C}).$$

Product of line and matrix.—Let

$$\mathbf{R} = xi + yj + zk$$

denote any line; then

$$\begin{aligned}\mathbf{R}\Phi &= \mathbf{R}\,(i\mathbf{A} + j\mathbf{B} + k\mathbf{C}) \\ &= (xi + yj + zk)\,(i\mathbf{A} + j\mathbf{B} + k\mathbf{C})\end{aligned}$$

The complete product is the sum of nine partial products of three vectors; the sum of the $\cos \alpha\beta \cdot \gamma$ terms gives the ordinary product, while the sum of the $(Sin\ \alpha\beta)\ \gamma$ terms forms a complementary product. Thus,

$$\mathbf{R}\Phi = x\mathbf{A} + y\mathbf{B} + z\mathbf{C} + \begin{vmatrix} x & y & z \\ i & j & k \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix}.$$

Here we have a product consisting of two parts analogous to the two parts of the product of two vectors, the former may be denoted by $\cos \mathbf{R}\Phi$, the latter by $Sin\ \mathbf{R}\Phi$.

Product of two matrices.

Let
$$\begin{aligned}\Phi &= i\,(a_1 i + a_2 j + a_3 k) \\ &+ j\,(b_1 i + b_2 j + b_3 k) \\ &+ k\,(c_1 i + c_2 j + c_3 k)\end{aligned}$$
and
$$\begin{aligned}\Psi &= i\,(d_1 i + d_2 j + d_3 k) \\ &+ j\,(e_1 i + e_2 j + e_3 k) \\ &+ k\,(f_1 i + f_2 j + f_3 k)\end{aligned}$$

The strain which is the resultant of Φ and Ψ applied in the order named is found by compounding the elementary ratios; for example

$$(a_1 ii)\ (d_1 ii) = a_1 d_1 ii\ ;\quad (a_1 ii)\ (d_2 ij) = a_1 d_2 ij.$$

Hence

$$\Phi\,\Psi = i\left\{(a_1d_1 + a_2e_1 + a_3f_1)i + (a_1d_2 + a_2e_2 + a_3f_2)j + (a_1d_3 + a_2e_3 + a_3f_3)k\right\}$$
$$+ j\left\{(b_1d_1 + b_2e_1 + b_3f_1)i + (b_1d_2 + b_2e_2 + b_3f_2)j + (b_1d_3 + b_2e_3 + b_3f_3)k\right\}$$
$$+ k\left\{(c_1d_1 + c_2e_1 + c_3f_1)i + (c_1d_2 + c_2e_2 + c_3f_2)j + (c_1d_3 + c_2e_3 + c_3f_3)k\right\}$$

Hence if $\Phi = i\mathbf{A} + j\mathbf{B} + k\mathbf{C}$,

and $\Psi = i\mathbf{A}' + j\mathbf{B}' + k\mathbf{C}'$;

$$\Phi\Psi = i\left\{\cos \mathbf{AA}' i + \cos \mathbf{AB}' j + \cos \mathbf{AC}' k\right\}$$
$$+ j\left\{\cos \mathbf{BA}' i + \cos \mathbf{BB}' j + \cos \mathbf{BC}' k\right\}$$
$$+ k\left\{\cos \mathbf{CA}' i + \cos \mathbf{CB}' j + \cos \mathbf{CC}' k\right\}$$

Here the product of the two strains is formed from the nature of a strain apart from the effect upon a given line. As the product of three dyads is associative, this product of three strains is also associative.

Complete product of two matrices.—The ordinary product of $\Phi\,\Psi$ contains only twenty-seven terms, the complete product ought to contain eighty-one. The other fifty-four terms form another term, which is expressed by

$$i\left\{Sin\,\mathbf{AA}' i + Sin\,\mathbf{AB}' j + Sin\,\mathbf{AC}' k\right\}$$
$$+ j\left\{Sin\,\mathbf{BA}' i + Sin\,\mathbf{BB}' j + Sin\,\mathbf{BC}' k\right\}$$
$$+ k\left\{Sin\,\mathbf{CA}' i + Sin\,\mathbf{CB}' j + Sin\,\mathbf{CC}' k\right\}$$

Here we have a product of four axes in which the association begins in the middle.

Product of a matrix and its conjugate.—For the conjugate matrix $\mathbf{A}' = \mathbf{A}$, $\mathbf{B}' = \mathbf{B}$, $\mathbf{C}' = \mathbf{C}$.

Hence $\Phi\,\Phi' = i\left\{\mathbf{A}^2 i + \cos \mathbf{AB} j + \cos \mathbf{AC} k\right\}$

$$+ j\left\{\cos \mathbf{AB} i + \mathbf{B}^2 j + \cos \mathbf{BC} k\right\}$$
$$+ k\left\{\cos \mathbf{AC} i + \cos \mathbf{BC} j + \mathbf{C}^2 k\right\}$$

and the complementary product is

$$i\left\{0 i + Sin\,\mathbf{AB} j + Sin\,\mathbf{AC} k\right\}$$
$$+ j\left\{Sin\,\mathbf{BA} i + 0 j + Sin\,\mathbf{BC} k\right\}$$
$$+ k\left\{Sin\,\mathbf{CA} i + Sin\,\mathbf{CB} j + 0 k\right\}$$

Reciprocal of a matrix.—The reciprocal of Φ is denoted by Φ^{-1} ; it is such that

$$\Phi\,\Phi^{-1} = 1\,ii + 1\,jj + 1\,kk.$$

By solving the equations $cos\,\mathbf{AA'} = 1$, $cos\,\mathbf{BA'} = 0$, $cos\,\mathbf{CA'} = 0$; we find

$$\mathbf{A'} = \frac{Sin\,\mathbf{BC}}{vol\,\mathbf{ABC}}$$

Hence

$$\Phi^{-1} = \frac{i\,Sin\,\mathbf{BC} + j\,Sin\,\mathbf{CA} + k\,Sin\,\mathbf{AB}}{vol\,\mathbf{ABC}}$$

Second power of a matrix.—If $\Psi = \Phi$; then the second power of the ordinary product is

$$\begin{aligned}\Phi^2 = i\Big\{ & (a_1{}^2 + a_2 b_1 + a_3 c_1)i + (a_1 a_2 + a_2 b_2 + a_3 c_2)j \\ & + (a_1 a_3 + a_2 b_3 + a_3 c_3)\,k\Big\} \\ + j\Big\{ & (b_1 a_1 + b_2 b_1 + b_3 c_1)i + (b_1 a_2 + b_2{}^2 + b_3 c_2)\,j \\ & + (b_1 a_3 + b_2 b_3 + b_3 c_3)k\Big\} \\ + k\Big\{ & (c_1 a_1 + c_2 b_1 + c_3 c_1)i + (c_1 a_2 + c_2 b_2 + c_3 c_2)j \\ & + (c_1 a_3 + c_2 b_3 + c_3{}^2)\,k\Big\}\end{aligned}$$

Components of a matrix.— A matrix may be resolved into the sum of three components, as

$$\begin{aligned}\Phi = {} & a_1\,ii + b_2\,jj + c_3\,kk \\ & + \tfrac{1}{2}\Big\{(b_3 - c_2)\,jk + (c_1 - a_3)ki + (a_2 - b_1)\,ij\Big\} \\ & + \tfrac{1}{2}\Big\{(b_3 + c_2)\,jk + (c_1 + a_3)\,ki + (a_2 + b_1)\,ij\Big\};\end{aligned}$$

of which the first component expresses elongation, the second rotation, and the third shear.

Invariant functions of a matrix.—

Let

$$\begin{aligned}\Phi &= i\,(a_1 i + a_2 j + a_3 k) \quad &(1)\\ &+ j\,(b_1 i + b_2 j + b_3 k) \quad &(2)\\ &+ k\,(c_1 i + c_2 j + c_3 k) \quad &(3)\end{aligned}$$

By combining the terms of the three ratios according to the rules of vector multiplication we obtain

$$\begin{aligned} & a_1 + a_2\,k - a_3\,j \\ -\, & b_1 k + b_2 + b_3 i \\ & c_1 j - c_2 i + c_3\end{aligned}$$

and by addition we obtain the scalar and vector invariants

$$a_1 + b_2 + c_3 + (b_3 - c_2)i + (c_1 - a_3)j + (a_2 - b_1)k$$

By combining (1) and (2) we form the ratio for the change of a rectangle having the axes i and j;

$$(ij)\ \left\{a_1b_1 + a_2b_2 + a_3b_3 + (a_1b_2 - a_2b_1)ij + a_2b_3 - a_3b_2)jk \right.$$
$$\left. + (a_3b_1 - a_1b_3)\,ki \right\}$$

and by combining (2) with (3) and (3) with (1) and adding we obtain a scalar and two vectors

$$a_1b_2 - a_2b_1 + b_2c_3 - b_3c_2 + c_3a_1 - c_1a_3;$$
$$(b_1c_1 + b_2c_2 + b_3c_3)i + (c_1a_1 + c_2a_2 + c_3a_3)j + (a_1b_1 + a_2b_2 + a_3b_3)k;$$
$$\left\{(c_1a_2 - c_2a_1) - (a_3b_1 - a_1b_3)\right\}i + \left\{(a_2b_3 - a_3b_2) - (b_1c_2 - b_2c_1)\right\}j$$
$$+ \left\{(b_3c_1 - b_1c_3) - (c_2a_3 - c_3a_3)\right\}k;$$

By combining (1), (2) and (3) together we get the ratio for the change of a rectangular parallelopiped having the axes i, j, k. The scalar which is the same for the three modes of association is the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

In this way the physical meaning is evident of the three scalars which occur in the cubic equation.

VECTOR DIFFERENTIATION.

Of a scalar quantity.—Let u denote any scalar quantity, a function of x, y, z; then $(dxi)^{-1}\,du_x$ denotes its growth per unit of distance in the direction i and $(dyj)^{-1}\,du_y$ the same for the direction j, and $(dzk)^{-1}\,du_z$ the same for the direction k. The reduced expressions for these rates are $i\,\frac{du}{dx}$, $j\,\frac{du}{dy}$, $k\,\frac{du}{dz}$. Their sum

$$i\,\frac{du}{dx} + j\,\frac{du}{dy} + k\,\frac{du}{dz}$$

expresses the rate of growth of u in the direction of the most rapid growth.

Let ν denote that direction and n a distance along it, then

$$i\,\frac{du}{dx} + j\,\frac{du}{dy} + k\,\frac{du}{dz} = \nu\,\frac{du}{dn}.$$

The rate of growth of this quantity per unit of distance in the direction i is expressed by

$$(dxi)^{-1}\,d_x\ \left\{\frac{du}{dx}\,i + \frac{du}{dy}\,j + \frac{du}{dz}\,k\right\}$$

which, when reduced becomes

$$i\,\left(\frac{d^2u}{dx^2}\,i + \frac{d^2u}{dxdy}\,j + \frac{d^2u}{dxdz}\,k\right);$$

and similarly

$$j\,\left(\frac{d^2u}{dydx}\,i + \frac{d^2u}{dy^2}\,j + \frac{d^2u}{dydz}\,k\right)$$

and

$$k\,\left(\frac{d^2u}{dzdx}\,i + \frac{d^2u}{dzdy}\,j + \frac{d^2u}{dz^2}\,k\right).$$

As

$$\frac{d^2u}{dxdy}=\frac{d^2u}{dydx}$$

we obtain on multiplying and adding the scalar

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}.$$

Thus,

$$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}$$

and u combine the same as if the former were a simple vector, the latter being a scalar. The single symbol used to denote it is ∇. By treating ∇ as a quadrantal versor, Hamilton and Tait obtain

$$\nabla(\nabla u)=-\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right).$$

As ∇ cannot be the axis of a quadrantal version, it is not evident where the rotation comes in. By the ordinary rules of multiplication we get

$$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2},$$

precisely equal to Laplace's operator.

The change of u for any other direction ρ is obtained by

$$\frac{d}{d\rho}=\frac{dx}{d\rho}\frac{d}{dx}+\frac{dy}{d\rho}\frac{d}{dy}+\frac{dz}{d\rho}\frac{d}{dz}.$$

Of a vector quantity.—Let

$$\mathbf{C}=ui+vj+wk$$

denote any vector quantity; then

$$\begin{aligned}\nabla\,\mathbf{C}&=(dxi)^{-1}\,d_x\mathbf{C}+(dyj)^{-1}\,d_y\mathbf{C}+(dzk)^{-1}\,d_z\mathbf{C}\\&=i\left(\frac{du}{dx}\,i+\frac{dv}{dx}\,j+\frac{dw}{dx}\,k\right)\\&\quad+j\left(\frac{du}{dy}\,i+\frac{dv}{dy}\,j+\frac{dw}{dy}\,k\right)\\&\quad+k\left(\frac{du}{dz}\,i+\frac{dv}{dz}\,j+\frac{dw}{dz}\,k\right)\end{aligned}$$

from which scalars and vectors may be formed as on p. 112. By combining them simply we obtain

$$\nabla\mathbf{C}=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}+\left(\frac{dw}{dy}-\frac{dv}{dz}\right)i+\left(\frac{du}{dz}-\frac{dw}{dx}\right)j+\left(\frac{dv}{dx}-\frac{du}{dy}\right)k.$$

The scalar

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$$

may be denoted by *cos* $\nabla\mathbf{C}$, and the vector by *Sin* $\nabla\mathbf{C}$. If

$$cos\,\nabla\mathbf{C}=0,$$

then $\mathbf{C}$ is said to be a *solenoidal* vector quantity; and if

$$Sin\,\nabla\mathbf{C}=0,$$

then $\mathbf{C}$ is said to be an irrotational vector quantity.

Successive differentiation.—From the principle that

$$\nabla(\nabla u)=\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2},$$

it follows that $\nabla(\nabla\mathbf{C})$ is not equal to $\nabla^2\mathbf{C}$. For

$$\nabla(\nabla\mathbf{C}) = \left(i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}\right)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$$
$$+\frac{d}{dx}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+\frac{d}{dy}\left(\frac{du}{dz}-\frac{dw}{dx}\right)+\frac{d}{dz}\left(\frac{dv}{dx}-\frac{du}{dy}\right)$$
$$+\left\{\frac{d}{dy}\left(\frac{dv}{dx}-\frac{du}{dy}\right)-\frac{d}{dz}\left(\frac{du}{dz}-\frac{dw}{dx}\right)\right\}i$$
$$+\left\{\frac{d}{dz}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)-\frac{d}{dx}\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}j$$
$$+\left\{\frac{d}{dx}\left(\frac{du}{dz}-\frac{dw}{dx}\right)-\frac{d}{dy}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\right\}k.$$

The scalar term vanishes and the term for i is

$$\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)i+\left\{\frac{d^2v}{dxdy}+\frac{d^2w}{dxdz}-\left(\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)\right\}i$$
$$=2\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)i-\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{d^2z}\right)i.$$

Hence

$$\nabla(\nabla\mathbf{C}) = 2\left(\frac{d}{dx}i+\frac{d}{dy}j+\frac{d}{dz}k\right)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$$
$$-\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(ui+vj+wk)$$

The condition for $\nabla(\nabla\mathbf{C})$ being equal to $\nabla^2\mathbf{C}$ is $\nabla^2\mathbf{C} = \nabla\, cos\, \nabla\mathbf{C}$. It is equal to $-\nabla^2\mathbf{C}$ if $\nabla\, cos\, \nabla\mathbf{C} = 0$.

The following is another investigation. As the rules for ∇ are the same as those for a vector; we form the product

$$\left(\frac{d}{dx}i+\frac{d}{dy}j+\frac{d}{dz}k\right)\left(\frac{d}{dx}i+\frac{d}{dy}j+\frac{d}{dz}k\right)(ui+vj+wk)$$

by finding

$$\Sigma i^3 + \Sigma i\,(ij) + \Sigma i\,(ji) + \Sigma i\,(jj) + \Sigma ijk.$$

Now

$$\Sigma i^3 = \frac{d^2u}{dx^2}i+\frac{d^2v}{dy^2}j+\frac{d^2w}{dz^2}k;$$
$$\Sigma i\,(ij) = -\left\{\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right)j+\left(\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)k+\left(\frac{d^2u}{dz^2}+\frac{d^2u}{dy^2}\right)i\right\}$$
$$\Sigma i\,(ji) = \left(\frac{d^2u}{dxdy}+\frac{d^2w}{dzdy}\right)j+\left(\frac{d^2v}{dydz}+\frac{d^2u}{dxdz}\right)k+\left(\frac{d^2w}{dzdx}+\frac{d^2v}{dydx}\right)i$$
$$\Sigma i\,(jj) = \left(\frac{d^2v}{dxdy}+\frac{d^2w}{dxdz}\right)i+\left(\frac{d^2w}{dydz}+\frac{d^2u}{dydx}\right)j+\left(\frac{d^2u}{dzdx}+\frac{d^2v}{dzdy}\right)k$$

Hence if we combine all the vectors

$$\nabla(\nabla\mathbf{C}) = -\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(ui+vj+wk)$$
$$+2\left(\frac{d}{dx}i+\frac{d}{dy}j+\frac{d}{dz}k\right)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$$

Examples of vector differentiation.

Let $\mathbf{R} = r\rho = xi + yj + zk;$

then

(1) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \rho$

(2) $\nabla\mathbf{R}^2 = \nabla r^2 = 2\mathbf{R}$

(3) $\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\rho$

This is also true when n is negative, the most important case being $n = -1$; then $\nabla \frac{1}{r} = -\frac{1}{r^2}\rho$

(4) $\nabla \mathbf{R} = 3.$

(5) $\nabla \mathbf{R}^3 = \nabla (r^2\mathbf{R}) = (\nabla r^2)\mathbf{R} + r^2\nabla \mathbf{R} = 5r^2$

(6) When n is odd, $\nabla \mathbf{R}^n = \nabla r^{n-1}\mathbf{R}$

$= (n-1)\, r^{n-2} \rho \mathbf{R} + 3r^{n-1} = (n+2)\, r^{n-1}$

(7) $\nabla (u\mathbf{C})$ is not in general $= \nabla (\mathbf{C}u)$

For $\nabla (u\mathbf{C}) = (\nabla u)\, \mathbf{C} + u\, (\nabla \mathbf{C})$

and $\nabla (\mathbf{C}u) = (\nabla \mathbf{C})\, u + \mathbf{C}\, (\nabla u)$;

but $(\nabla u)\,\mathbf{C}$ is not equal to $\mathbf{C}\,(\nabla u)$, unless $\mathbf{C}$ and ∇u have the same axis

(8) $\nabla \rho = \nabla\left(\frac{\mathbf{R}}{r}\right) = (\nabla \mathbf{R})\,\frac{1}{r} + \mathbf{R}\left(\nabla \frac{1}{r}\right) = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$

(9) $\nabla (Sin\, \mathbf{AR}) = \nabla (\mathbf{AR}) - \nabla cos\, \mathbf{AR} = 2\mathbf{A}.$

(10) To prove that $\nabla\, (\nabla \frac{1}{r}) = 0.$ Since $\nabla \frac{1}{r} = -\frac{1}{r^2}\rho$

$\nabla\, (\nabla \frac{1}{r}) = 2\frac{1}{r^3}\rho^2 - \frac{1}{r^2}\nabla \rho = 0.$

(11) $\nabla\, (\nabla \mathbf{R}^2) = \nabla\, 2\mathbf{R} = 3 \cdot 2 = 6$

(12) $\nabla\, (\nabla\, (\nabla \mathbf{R}^3)) = 5 \cdot 2 \cdot 3 = 30$

(13) $\nabla^4\, \mathbf{R}^4 = 4 \cdot 5 \cdot 2 \cdot 3 = 120.$

GENERALIZED ADDITION.

Signless quantities at different points.— Given a mass m, at $\mathbf{A}_1$ and m_2 at $\mathbf{A}_2$; by adding them is meant adding the masses, and finding such a position that the mass-vector of the sum of the masses will be equal to the sum of the mass-vectors. Let m times the vector $\mathbf{A}_1$, be $\mathbf{P}$, and m_2 times the vector $\mathbf{A}_2$ be $\mathbf{Q}$; the resultant $\mathbf{R}$ is the sum of the mass-vectors; take $\mathbf{S}$ equal to $\mathbf{R}$ divided by $m_1 + m_2$ (fig. 22).

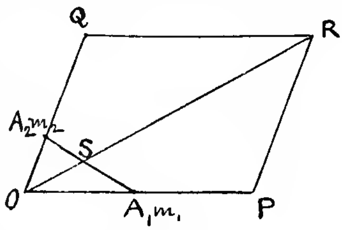

FIG. 22.

Hence $\mathbf{A}_1 \cdot m_1 + \mathbf{A}_2 \cdot m_2 = \frac{m_1\mathbf{A}_1 + m_2\mathbf{A}_2}{m_1 + m_2} \cdot (m_1 + m_2).$

This is generalized addition; for if we put $\mathbf{A}_2 = \mathbf{A}_1$, we get ordinary addition.

Scalar quantities at different points. — The same principle applies to a quantity which may be positive or negative; but there is a special case when the quantities are equal and of opposite sign. Then

$$\mathbf{A}_1 \cdot m - \mathbf{A}_2 \cdot m = \frac{m\mathbf{A}_1 - m\mathbf{A}_2}{m - m} \cdot (m - m)$$
$$= m\,(\mathbf{A}_1 - \mathbf{A}_2)$$

Their sum is then a moment, as in the case of a magnet.

Parallel vector quantities at different points.—If the vector quantities have the same axis, they are added in the same manner as signless quantities; hence (fig. 23).

$$\mathbf{A}_1 \cdot \mathbf{B}_1 + \mathbf{A}_2 \cdot \mathbf{B}_2 = \frac{b_1\mathbf{A}_1 + b_2\mathbf{A}_2}{b_1 + b_2} \cdot (b_1 + b_2)\,\beta$$

If they have opposite axes, they are added like scalar quantities. Suppose $\mathbf{B}_1 = b_1\beta$ and $\mathbf{B}_2 = b_2\,(-\beta)$; then

$$\mathbf{A}_1 \cdot \mathbf{B}_1 + \mathbf{A}_2 \cdot \mathbf{B}_2 = \frac{b_1\mathbf{A}_1 - b_2\mathbf{A}_2}{b_1 - b_2} \cdot (b_1 - b_2)\,\beta\cdot$$

If further $b_1 = b_2$, then their sum is

$$= \mathbf{A}_1\mathbf{B} - \mathbf{A}_2\mathbf{B} = (\mathbf{A}_1 - \mathbf{A}_2)\,\mathbf{B} = \mathit{cos}\,(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{B} + \mathit{Sin}\,(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{B}.$$

The latter term is the moment of a couple.

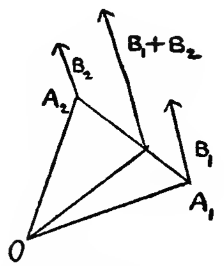

FIG. 23.

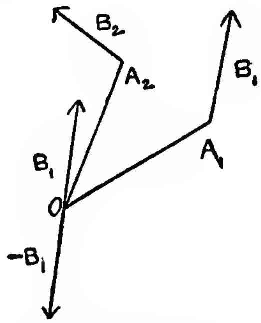

FIG. 24.

Vector quantities at different points.—The following is the most general form of the principle that a quantity is not changed by the simultaneous addition and subtraction of the same quantity (fig. 24).

$$\begin{aligned}\mathbf{A}_1 \cdot \mathbf{B}_1 &= 0 \cdot \mathbf{B}_1 - 0 \cdot \mathbf{B}_1 + \mathbf{A}_1 \cdot \mathbf{B}_1 \\ &= 0 \cdot \mathbf{B}_1 + \mathbf{A}_1\mathbf{B}_1\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{A}_1 \cdot \mathbf{B}_1 + \mathbf{A}_2 \cdot \mathbf{B}_2 &= 0 \cdot (\mathbf{B}_1 + \mathbf{B}_2) + \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 \\ &= 0 \cdot (\mathbf{B}_1 + \mathbf{B}_2) + \mathit{cos}\,\mathbf{A}_1\mathbf{B}_1 + \mathit{cos}\,\mathbf{A}_2\mathbf{B}_2 \\ &\quad + \mathit{Sin}\,\mathbf{A}_1\mathbf{B}_1 + \mathit{Sin}\,\mathbf{A}_2\mathbf{B}_2\end{aligned}$$

And generally $\Sigma\,\mathbf{A} \cdot \mathbf{B} = 0 \cdot \Sigma\mathbf{B} + \Sigma\,\mathit{Sin}\,\mathbf{AB} + \Sigma\,\mathit{cos}\,\mathbf{AB}.$

THE

IMAGINARY OF ALGEBRA

BEING A CONTINUATION OF THE PAPER

"PRINCIPLES OF THE ALGEBRA OF PHYSICS."

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PAPERS READ.

ON THE IMAGINARY OF ALGEBRA. By Prof. A. MACFARLANE, University of Texas, Austin, Texas.

"The student, if he should hereafter inquire into the assertions of different writers, who contend for what each of them considers as *the* explanation of $\sqrt{-1}$, will do well to substitute the indefinite article."— DE MORGAN, *Double Algebra*, p. 94.

WITH respect to the theory and use of $\sqrt{-1}$ analysts may be divided into three classes: *first*, those who have considered it as *undefined* and *uninterpreted*, and consequently make use of it only in a tentative manner; *second*, those who have considered it as *undefinable* and *uninterpretable*, and build upon this supposed fact a special theory of reasoning; *third*, those who, viewing it as capable of definition, have sought for the definition in the ideas of geometry.

Of the first class we have an example in the view laid down by the astronomer Airy (*Cambridge Philosophical Transactions*, vol. x, p. 327). "I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols. I am very glad to use them as conveniently indicating a conclusion which it may afterwards be possible to obtain by strictly logical methods; but until these logical methods shall have been discovered, I regard the result as requiring further demonstration." This view admits that conclusions are indicated by methods which are not strictly logical; that a method which is not strictly logical can indicate and always can indicate a conclusion is a paradox which it is very desirable to explain.

Of the second class we have an example in the mathematician and logician, Boole. Instead of conforming analysis to ordinary reasoning, he endeavors to conform reasoning to analysis by introducing a transcendental species of logic. In his *Laws of Thought*, p. 68, he lays down the following as an axiomatic principle in reasoning: The process of solution or demonstration may be conducted throughout in obedience to certain formal laws of combination of the symbols, without regard to the question of the interpretability of the intermediate results, provided the final result be interpretable. Our knowledge of the foregoing principle is based upon the actual occurrence of an instance, that instance being the imaginary of algebra. In support of this view he says: "A single example of reasoning in which symbols are employed in obedience to laws founded upon their interpretation, but without any sustained reference to that interpretation, the chain of demonstration conducting us through intermediate steps which are not interpretable to a final result which is interpretable, seems not only to establish the validity of the particular application, but to make known to us the general law manifested therein. No accumulation of instances can properly add weight to such evidence. The employment of the uninterpretable symbol $\sqrt{-1}$, in the intermediate processes of trigonometry, furnishes an illustration of what has been said. I apprehend that there is no mode of explaining that application which does not covertly assume the very principle in question. But that principle, though not, as I conceive, warranted by formal reasoning based upon other grounds, seems to deserve a place among those axiomatic truths, which constitute, in some sense, the foundation of the possibility of general knowledge, and which may properly be regarded as expressions of the mind's own laws and constitution."

Inasmuch as the successful use of the undefined symbol $\sqrt{-1}$ by analysts is thus made the basis of a sort of transcendental logic, it is a matter of interest to investigate whether the intermediate steps in such demonstrations are not uninterpretable but merely uninterpreted. If it can be shown that some at least of the expressions in which $\sqrt{-1}$ occurs have a real geometrical meaning, the argument for a transcendental logic will fail.

The "principle of the permanence of equivalent forms," which was by Peacock made the foundation of the operations and results of algebra, is scarcely so transcendental, but is certainly a very vague and unsound principle of generalization. He states it as follows (*Symbolical Algebra*, p. 631): "*Whatever algebraical forms are equivalent, when the symbols are general in form but specific in value, will be equivalent likewise when the symbols are general in value as well as in form.* It will follow from this principle that all the results of arithmetical algebra will be results likewise of symbolical algebra, and the discovery of equivalent forms in the former science possessing the requisite conditions will be not only their discovery in the latter, but *the only* authority for their existence; for there are no definitions of the operations in symbolical algebra by which such equivalent forms can be detected."

The principle is applied to indices in the following manner: "Observing that the indices m and n in the expressions which constitute the equation $a^m \times a^n = a^{m+n}$, though *specific* in value, are *general* in form we are authorized to conclude by the principle of the permanence of equivalent forms that in symbolical algebra the same expressions continue to be equivalent to each other for *all values* of those indices; or, in other words, that $a^m \times a^n = a^{m+n}$ whatever be the values of m and n."

The question is: How general may the symbols be made, yet the equation still retain the same form? This is not a question of nominal definition and merely symbolical truth, but of real definition and of real truth; as may be shown by considering the above principle of indices. For a certain generalized meaning of m and n, Hamilton (Elements of Quaternions, p. 388) investigates whether or not $a^m \times a^n = a^{m+n}$, and concludes that it is not true. With him the question is one of material truth, not of symbolical definition.

The above principle of generalization may be tested in another way. If r denote the ordinary algebraic quantity which may be positive or negative, $r \cdot \theta$ may represent that quantity when generalized so as to have any angle θ with an initial line in a given plane. For this generalized magnitude

$$r \cdot \theta \times r' \cdot \theta' = rr' \cdot \theta + \theta';$$

in words, the length of the product is the product of the lengths, and the angle of the product is the sum of the angles. Now the principle of the permanence of equivalent forms does not help us to generalize this proposition for space. A plausible hypothesis likely to present itself at first is: Let φ denote the angle between the given plane and a fixed plane, is

$$(r \cdot \theta \cdot \varphi) \times (r' \cdot \theta' \cdot \varphi') = rr' \cdot \theta + \theta' \cdot \varphi + \varphi'?$$

This is a question not of symbolism, but of truth.

At the time of De Morgan there was no adequate theory of $\sqrt{-1}$, as is evident from the quotation prefixed; nor is there at the present time. The view at present held about $i = \sqrt{-1}$ by analysts is thus stated by Cayley in a paper "On Multiple Algebra," printed in the *Quarterly Journal of Mathematics*, vol. XXII.

"We have come to regard $a + bi$ as an ordinary analytical magnitude, viz.: in every case an ordinary symbol represents or may represent such a magnitude, and the magnitude (and as a particular case thereof the symbol i) is commutable with the extraordinaries of any system of multiple algebra; and similarly in analytical geometry without seeking for any real representation we deal with imaginary points, lines, etc., that is, with points, lines, etc., depending on parameters of the form $a + bi$."

I propose to review critically the different explanations or elements of explanation which have been contributed, with the hope of finding a theory which will tend to unify them, and to diminish still further that region of analysis where we have mere symbolism without real definition.

The investigation of this subject arose with the celebrated controversy about the nature of the logarithms of negative numbers; whether they are real or impossible. Leibnitz maintained that the logarithm of a negative number is impossible, because if log (—2) is real, so is $\frac{1}{2}$ log (—2), that is log $\sqrt{-2}$, which would lead to the supposed absurdity of the logarithm of an impossible quantity being real. John Bernoulli held that the logarithm of a negative number is as real as the logarithm of a positive number; for the ratio $-m : -n$ does not differ from that of $+m : +n$. The former view was afterwards maintained by Euler, the latter by D'Alembert. Euler claimed to demonstrate that every positive number has an infinite number of logarithms, of which only one is possible; further, that every negative as well as every impossible number has an infinite number of logarithms, which are all impossible. He reasoned from the values of the n^{th} root of $+1$ and of -1, viewing $+$ as denoting an even number, and $-$ as denoting an odd number, of half revolutions. D'Alembert pointed out that the logarithm of a negative number may be real. Thus $e^{\frac{1}{2}} = +\sqrt{e}$ or $-\sqrt{e}$; but the logarithm of $e^{\frac{1}{2}}$ is $\frac{1}{2}$; therefore the logarithm of $-\sqrt{e}$ as well as of $+\sqrt{e}$ is $\frac{1}{2}$.

These opposing views arise from different conceptions of the negative symbol and of the magnitude treated by algebra. The magnitudes considered in elementary algebra are, first, a mere number or ratio; second, a magnitude which may have a given direction, or the opposite, and third, a geometric ratio which combines a number with a certain amount of change of direction. The logarithm of a ratio is itself a ratio, and is unique. If a directed magnitude has a logarithm, it is difficult to see how the direction of the logarithm, if it has any direction, can be different from that of the magnitude. It is of number in the sense of a geometric ratio that Euler's proposition is true. This conception of number immediately transcends representation by a single straight line; consequently a part of the ratio generally appears as impossible.

In his *Geometrie de Position*, Carnot asks the following among other questions: "If two quantities, of which the one is positive and the other negative, are both real, and do not differ excepting in position, why should the root of the one be an imaginary quantity, while that of the other is real? Why should $\sqrt{-a}$ not be as real as $\sqrt{+a}$?" In this question it is assumed that $-a$ and $+a$ denote directed magnitudes, the one being opposite to the other; and if such a quantity has a square root, it is difficult to understand why the one direction should differ from the other. But the $-a$ which has the imaginary square roots, while $+a$ has real, do not differ in direction; they differ in the amount of change of direction.

In 1806, M. Bueé published in the *Philosophical Transactions* a memoir on Imaginary Quantities, and in it he endeavors to answer some of the questions raised by Carnot. His main idea is that $+$, $-$, and $\sqrt{-1}$ are purely descriptive signs; that is, signs which indicate direction. Suppose three equal lines AB, AC, AD, drawn from a point A (fig. 1), of which AC is opposite to AB, and AD perpendicular to BAC; then if the line AB is designated by $+1$, the line AC will be -1, and the line AD will be $\sqrt{-1}$. Thus $\sqrt{-1}$ is the sign of perpendicularity. It follows from this view of $\sqrt{-1}$ that it does not indicate a unique direction, the opposite line AD', or any line in the plane as AD'' is also indicated by $\sqrt{-1}$. Bueé admits the consequence. But it may be asked: If every perpendicular is represented by $\sqrt{-1}$, what meaning is left for $-\sqrt{-1}$?

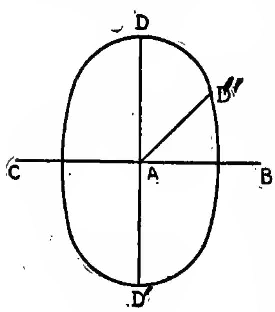

FIG. 1.

Bueé applies his theory to the interpretation of the solution of a quadratic equation which had been considered by Carnot, namely: To divide a line AB into two parts such that the product of the segments shall be equal to half the square of the line.

Let AB (fig. 2) be the given line, and suppose K to be the required point; let AB be denoted by a, and AK by x; then by the given condition

$$x\,(a-x) = \frac{a^2}{2}$$

and by the ordinary process of solution

$$x = \frac{a}{2} \pm \sqrt{-\frac{a^2}{4}} = \frac{a}{2} \pm \sqrt{-1}\,\frac{a}{2}.$$

According to Carnot, the appearance of the imaginary indicates that there is no such point as is required between A and B, but that it is outside AB

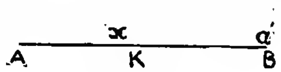

FIG. 2.

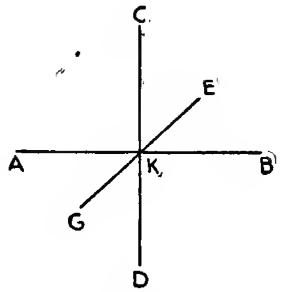

FIG. 3.

on the line prolonged. If it is supposed to be beyond B on the line produced, the equation takes the modified form $x\,(x-a) = \frac{1}{2}\,a^2$, giving

$$x = \tfrac{1}{2}a \pm \sqrt{\frac{3a^2}{4}}$$

Of these two roots he considers

$$x = \tfrac{1}{2}\,a + \sqrt{\frac{3a^2}{4}}$$

only to be a true solution of the question; while

$$x = \frac{a}{2} - \sqrt{\frac{3a^2}{4}}$$

is the solution on the hypothesis that the point is on the line produced, but on the side of A. Bueé views these answers as the solutions of connected equations, not of the given equation. His solution is represented (fig. 3) by drawing two mutual perpendiculars KC and KE to represent $\sqrt{-1}\,\frac{a}{2}$ and their opposites KD and KG to represent $-\sqrt{-1}\,\frac{a}{2}$; C and D or E and G are the points required. But Bueé does not show how the square of $\frac{a}{2} + \sqrt{-1}\,\frac{a}{2}$ is to be represented? If the one component of the line is perpendicular to the other, ought not the square of the sum to be equal to the sum of the squares? But this does not agree with the principles of algebra, for

$$(x + \sqrt{-1}\,y)^2 = x^2 - y^2 + 2\sqrt{-1}\,xy.$$

This is a difficulty which a theory of mere direction cannot get over. Led by his theory of perpendicularity, Bueé considers the question: What does a conic section become, when its ordinates become imaginary? Consider a circle; when x has any value between $-a$ and $+a$, then

$$y = \pm\sqrt{a^2 - x^2}$$

But when x is greater than a, or less than $-a$, let it be denoted by x', and the analogue of y by y', then

$$y' = \pm\sqrt{-1}\sqrt{x'^2 - a^2}.$$

Bueé advances the view that the circle in the plane of the paper changes into an equilateral hyperbola in the plane perpendicular to the plane of the paper; but he does not prove the suggestion, or test it by application to calculation. A similar view has been developed by Phillips and Bêebe in their "Graphic Algebra." It appears to me that here we have a fundamental question in the theory of $\sqrt{-1}$. The expression $\sqrt{a^2-x^2}$ denotes the ordinate of the circle, what is represented by $\sqrt{-1}\sqrt{x'^2-a^2}$, x' being greater than a? The former is constructed by drawing from the extremity of x a straight line at right angles to it in the given plane, and describing with centre O a circle of radius a the point of intersection P determining the length of the ordinate, and $-\sqrt{a^2-x^2}$ is equal and opposite. Now (fig. 4) $\sqrt{x'^2-a^2}$ is equal in length to the tangent from the extremity of x' to the circle, and $\sqrt{-1}$ appears to indicate the direction of the tangent, which varies in inclination to the axis of x, but is determined by always being perpendicular to the radius at the point of contact. Hence if x' be considered a directed magnitude, the expression

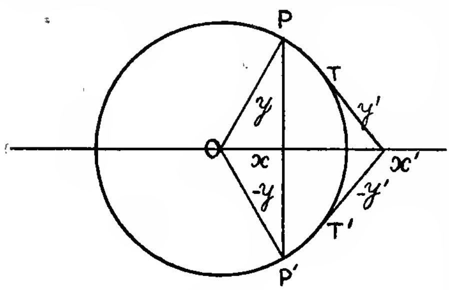

FIG. 4.

$$x' + \sqrt{-1}\sqrt{x'^2-a^2}$$

denotes the radius from O to the one point of contact T, while

$$x' - \sqrt{-1}\sqrt{x^2-a^2}$$

denotes the radius to the other point of contact T'. This construction does not necessitate going out of the given plane; and if space be considered we have a whole complex of ordinates to the sphere, as well as a complex of tangents to the sphere. The ordinary theory of miuus gives no explanation of the double sign in the case of the tangent. It is true in the case of the two ordinates, that the one is opposite to the other in direction, but it is not true of the two tangents. In the case of the sphere the ordinate may have any direction in a plane perpendicular to x, while the tangent may have any direction in a cone of which x is the axis. This other and hitherto unnoticed meaning of $\sqrt{-1}$ will be developed more fully in the investigation which follows (p. 52).

The same year, Argand published his "*Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.*" His method is restricted to a plane (fig. 5). According to his view $+$ is a sign of direction, $-$ of the opposite direction, $\sqrt{-1}$ of the upward perpendicular direction and $-\sqrt{-1}$ of the downward perpendicular direction. The general quantity $a + b\sqrt{-1}$ is represented by a line OP (fig. 5) having a and $b\sqrt{-1}$ for rectangular components. The product of two lines $a + b\sqrt{-1}$ and $a' + b'\sqrt{-1}$ is

$$(a + b\sqrt{-1})\,(a' + b'\sqrt{-1}) = aa' - bb' + \sqrt{-1}(ab' + a'b)$$

and it too is represented by a line, namely, the line which has $aa' - bb'$ and $\sqrt{-1}(ab' + ba')$ for rectangular components.

A very important advance was made by Français, who perceived that $+$, $-$, $\sqrt{-1}$ and $-\sqrt{-1}$ did not denote directions, but rather amounts of angle. He introduced the notation a_α to denote the general line where a denotes its magnitude and α the angle between it and a fixed initial line. Thus $+a$ is a_0, $-a$ is a_π, $\sqrt{-1}a$ is $a_{\frac{\pi}{2}}$, and $-\sqrt{-1}a$ is $a_{-\frac{\pi}{2}}$. So long as α is supposed to denote the angle specifying the position of a line, it is difficult to perceive what is the meaning of the multiplication or division of two lines. It was customary to look upon the product line as forming a fourth proportional to the initial line and the two given lines. But when it is perceived[1] that the angle does not refer to a fixed initial line, but to any line in the plane, it becomes evident that the product of two quantities r_θ and $r'_{\theta'}$ is $rr'_{\theta+\theta'}$, the ratio of the product being the product of the ratios, and the angle of the product being the sum, or what appears to be the sum, of the angles. In the investigation of Français, the symbol $\sqrt{-1}$, though replaced by $\frac{\pi}{2}$ in the primary quantity, reappears again in the exponential expression for a line; he writes

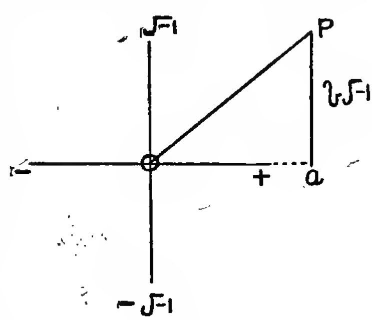

FIG. 5.

$$ae^{\alpha\sqrt{-1}} = a_\alpha.$$

He does not appear to have considered the question: Can the $\sqrt{-1}$ in this index be replaced by $\frac{\pi}{2}$? It is evident that $\frac{\pi}{2}$ cannot be substituted for it as a simple multiplier; does the index really mean $\alpha_{\frac{\pi}{2}}$, a quantity similar to a_α? This question is, I believe, correctly answered by an affirmative. The view which has been commonly taken by analysts is that everything is explained provided $a + b\sqrt{-1}$ is explained, and provided every

[1]*Note on Plane Algebra*, by the author. Proc. R. S. E., 1883, p. 184.

other function involving $\sqrt{-1}$ can be reduced to the form $P + Q\sqrt{-1}$. But it cannot be proved that this reduction is always possible, unless on the assumption that all the imaginaries refer to one plane. For example, De Morgan, in his *Double Algebra*, does not interpret directly $e^{a\sqrt{-1}}$ or the more general expression $(a + b\sqrt{-1})^{p + q\sqrt{-1}}$, but the expression is *reduced to significance* by being reduced to the form $P + Q\sqrt{-1}$. And this is the current mode in modern analysis of explaining functions of the imaginary.

In a subsequent paper Argand adopted the notation of Français for a line in a plane; but used $\frac{1}{4}$ instead of $\frac{\pi}{2}$ to denote the quadrant, which, as Français pointed out, is not an improvement. So imbued was he with the *direction* theory of $\sqrt{-1}$ that he sought to express any direction in space by means of an imaginary function. He arrived at the view that the third mutual perpendicular KP (fig. 6) is expressed by $\sqrt{-1}^{\sqrt{-1}}$, the opposite line KQ by $-\sqrt{-1}^{\sqrt{-1}}$, and any line KM in the perpendicular plane by $\sqrt{-1}^{\cos\mu + \sqrt{-1}\sin\mu}$ where μ denotes the angle between KB and KM. He remarks that if the above be the correct meaning of $\sqrt{-1}^{\sqrt{-1}}$, then it is not true that every function can be reduced to the form $p + q\sqrt{-1}$ and he doubts the validity of the current demonstration which aims at proving that the function $(a + b\sqrt{-1})^{m + n\sqrt{-1}}$ can always be reduced to the form $p + q\sqrt{-1}$. According to that reduction, as was shown by Euler, $\sqrt{-1}^{\sqrt{-1}} = e^{-\frac{\pi}{2}}$, and this meaning of the expression was maintained by Français and Servois. The latter, following the analogy of $a + b\sqrt{-1}$ for a line in one plane, suggested that the expression for a line in space had the form

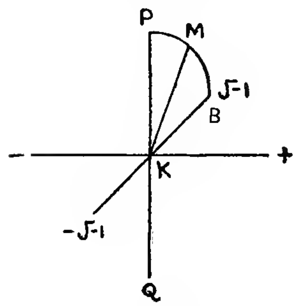

FIG. 6.

$$p\cos\alpha + q\cos\beta + r\cos\gamma,$$

where p, q, r are imaginaries of some sort, but he questioned whether they are each reducible to the form $A + B\sqrt{-1}$. In reply to the criticisms of Français and Servois, Argand maintained that Euler had not demonstrated that

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x$$

but had defined the meaning of $e^{x\sqrt{-1}}$ by extending the theorem

$$e^x = 1 + x + \frac{x^2}{2!} + \text{etc.}$$

It will be shown afterwards that in the equation of Euler, namely

$$\sqrt{-1}^{\sqrt{-1}} = e^{-\frac{\pi}{2}}$$

there is an assumption that the axes of the two angles are coincident; and that Argand's meaning is incorrect.

The ideas of Warren in his *Treatise on the geometrical representation of the square roots of negotive quantities*, 1828, are essentially the same as those of Français, but they receive a more complete development.

It is curious to find, considering the intensely geometrical character of quaternions, that Hamilton was led by the Kantian ideas of space and time to start out with the theory that algebra is the science of time, as geometry is the science of space, and that he strove hard to find on that basis a meaning for the square root of minus one. But having observed the success, so far as the plane is concerned, of the geometrical theory of Argand, Français and Warren, he adopted a geometrical basis and took up the problem of extending their method to space. What he sought for was the product of two directed lines in space, in the sense of a fourth proportional to two given lines and an initial line. He perceived that one root of the difficulty which had been experienced lay in regarding the initial line as real, and the two perpendiculars as expressed by imaginaries; and, looking at the symmetry of space, adopted the view that each of the three axes should be treated as an imaginary. He was thus led to the principle that if i, j, k denote three mutually rectangular axes, then

$$i^2 = -1, j^2 = -1, k^2 = -1,$$

and if Ua denote any vector of unit length $(Ua)^2 = -1$. Hence follows the paradoxical conclusion that the square of a directed magnitude is negative, which is contrary to the principles of analysis. An after development of Hamilton's was to give to i, j, k a double meaning, namely: to signify not only unit vectors, but to signify the axes of quadrantal versors. But in the quaternion we have for the first time the clear distinction between a line and a geometric ratio. In a paper read before this Association last year I have given reasons for believing that the identification of a directed line with a quadrantal quaternion is the principal cause of the obscurity in the method, and of its want of perfect harmony with the other methods of analysis.

The imaginary symbol, notwithstanding its apparent banishment from space, reappears in Hamilton's works as the coëfficient of an unreal quaternion. He appears to hold that there is a scalar $\sqrt{-1}$ distinct from that vector $\sqrt{-1}$ which can be replaced by i, j, k. In the recent edition of Tait's *Treatise on Quaternions*, Prof. Cayley contributes an analytical theory of quaternions, in which the components w, x, y, z of a quaternion are considered in the most general case to have the form $a + b\sqrt{-1}$ where $\sqrt{-1}$ is the imaginary of ordinary algebra. Thus it appears as if we were landed in an analytic theory of quaternions instead of a quaternionic theory of analysis.

In a work recently published on quaternions (*Theorie der Quaternionen*, by Dr. Molenbroek), the principal novelty is the introduction of the symbol $\sqrt{-1}$ with the meaning attached to it by Bueé, namely: to denote perpendicularity. Thus (fig. 7) $\sqrt{-1}\ \alpha$ denotes any vector such as OP or OQ, which is equal in length to α, and perpendicular to α, and $\sqrt{-1}$ is thus made to mean a quadrantal versor with an indefinite axis; but the axis is not entirely indefinite, for it must be perpendicular to α. Doubtless it is convenient to have a notation for any direction from O which is perpendicular to α; but it does not follow that $\sqrt{-1}$ denotes it properly.

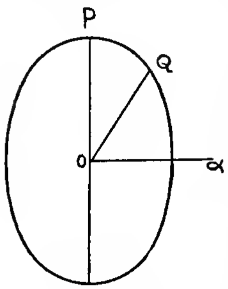

FIG. 7.

I have found the following notation convenient: Let α, β denote two independent axes, then the axis perpendicular to both may be denoted by $\overline{\alpha\beta}$. In harmony with this notation $\overline{\alpha}$ denotes any of the perpendiculars to α; but $\overline{\alpha}$ may also be used to denote a definite perpendicular, when the conditions make the perpendicular definite.

In a paper read before this Association last year[1] I showed that the products of directed magnitudes may be considered in complete independence of the idea of rotation; consequently that the method of dealing with such quantities forms a special branch of the algebra of space, of great importance to the physicist. The method of dealing with versors forms another distinct branch; and in the idea of a versor, or more generally of a geometric ratio or quaternion we find a true explanation of $\sqrt{-1}$, and I believe that the following development will show that it has at least one other geometric meaning.

SPHERICAL TRIGONOMETRY.

Notation for a quaternion.

A quaternion, or geometric ratio, will be denoted synthetically by **a**, and analytically by $a\alpha^A$ where a denotes the arithmetical ratio, α the axis, and A the angle in circular measure. The factor α^A forms the versor or circular sector. Let A become $\frac{\pi}{2}$, then $\alpha^{\frac{\pi}{2}}$ is an imaginary made definite; $\beta^{\frac{\pi}{2}}$ is another differing from the former as regards its axis.

According to the notation of Hamilton, α^1 denotes a quadrantal versor, whereas, according to the above definition, it denotes a circular sector of which the arc is unity the radius also being unity. Viewed merely as a matter of convenience in writing and printing, the notation α^A is preferable to $\alpha^{\frac{2A}{\pi}}$. For the sake of the extension to hyperbolic sectors, it is found necessary to consider A as denoting not the circular arc but double the

[1]Proc. A. A. A. S., Vol. XL, p. 65.

area of the sector included by the arc. This notation is capable of generalization, while the other is not.

Meaning of the equation $a^A = \cos A + \sin A \cdot a^{\frac{\pi}{2}}$

Let OP (fig. 8) be any line of unit length in the plane of a, and let OQ be the line from O to the extremity of the circular sector of area $\frac{A}{2}$ enclosed between OP and the circular arc: then

$$\begin{aligned} OQ &= OM + MQ \\ &= \cos A \cdot OP + \sin A\, a^{\frac{\pi}{2}} \cdot OP \\ &= \left(\cos A + \sin A \cdot a^{\frac{\pi}{2}}\right) OP \\ &= a^A\ OP \end{aligned}$$

therefore $$a^A = \cos A + \sin A \cdot a^{\frac{\pi}{2}}.$$

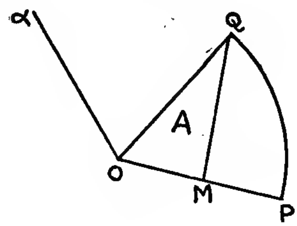

FIG 8.

This equation is true so far as the amount of angle is concerned but not it may be as regards the whole amount of turning. In this sense $\cos A$ and $\sin A \cdot a^{\frac{\pi}{2}}$ are the components of a^A.

To prove that $a^A = e^{Aa^{\frac{\pi}{2}}}$.

We have $$a^A = \cos A + \sin A \cdot a^{\frac{\pi}{2}},$$

and $$\cos A = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - ,$$

and $$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - .$$

By restoring the powers of $a^{\frac{\pi}{2}}$ in the expression for $\cos A$ we obtain

$$\cos A = 1 + \frac{A^2 a^{2\frac{\pi}{2}}}{2!} + \frac{A^4 a^{4\frac{\pi}{2}}}{4!} + ;$$

and by a similar restoration in the series for *Sin A*

$$\sin A \cdot a^{\frac{\pi}{2}} = A a^{\frac{\pi}{2}} + \frac{A^3 a^{3\frac{\pi}{2}}}{3!} + ;$$

and by adding the two series together we get

$$a^A = 1 + Aa^{\frac{\pi}{2}} + \frac{A^2 a^{2\frac{\pi}{2}}}{2!} + \frac{A^3 a^{3\frac{\pi}{2}}}{3!} + ;$$

$$= e\ Aa^{\frac{\pi}{2}}$$

Also $$(-a)^A = a^{-A} = e^{Aa^{-\frac{\pi}{2}}} = e^{-A \cdot a^{\frac{\pi}{2}}}$$

and $$a^{2\pi - A} = e^{Aa^{3\frac{\pi}{2}}}.$$

So far as angle is concerned, irrespective of the whole amount of turning, we have

$$a^{-A} = a^{2\pi - A}.$$

It follows that $Aa^{\frac{\pi}{2}}$ is the logarithm of a^A; and $a^{\frac{\pi}{2}}$ the logarithm of a^I.

As the most general expression for minus is $a^{(2n+1)\pi}$,

$$log\ (-1) = (2n+1)\pi \cdot a^{\frac{\pi}{2}}.$$

The general expression for $\sqrt{-1}$ is $a^{\frac{\pi}{2}+2n\pi}$, therefore

$log\sqrt{-1} = (2n\pi + \frac{\pi}{2}) \cdot a^{\frac{\pi}{2}}$; and for $+$ it is $a^{2n\pi}$, therefore $log + = 2n\pi \cdot a^{\frac{\pi}{2}}$.

Hence generally $\quad log\ (aa^A) = log\, a + A \cdot a^{\frac{\pi}{2}}$.

In his *Geometrie de Position* Carnot says, in reference to the celebrated discussion about the logarithms of negative quantities "Quoique cette discussion soit aujourd'hui terminé, il reste ce paradoxe savoir que quoiqu' on ait $log\ (-z)^2 = log\ (z)^2$, on n'a cependant pas $2\ log\ (-z) = 2\ log\ z$."

The paradox may be explained as follows: Suppose the complete expression for z to be $za^{2n\pi}$, then that for $-z$ is $za^{(2n+1)\pi}$; then

$$log\ z^2 = 2\ log\ z + 4n\pi \cdot a^{\frac{\pi}{2}} \text{ and } log\ (-z)^2 = 2\ log\, z + (4n+2)\pi \cdot a^{\frac{\pi}{2}}.$$

As the latter is twice the logarithm of $za^{(2n+1)\pi}$, the supposed paradox vanishes.

To prove that

$$a^A \beta^B = cos\ A\, cos\ B - sin\ A\, sin\ B\, cos\ a\beta$$
$$+ cos\ A\ sin\ B \cdot \beta^{\frac{\pi}{2}} + cos\ B\, sin\ A \cdot a^{\frac{\pi}{2}} - sin\ A\ sin\ B\ sin\ a\beta \cdot \overline{a\beta}^{\frac{\pi}{2}}.$$

Since $$a^A = cos\ A + sin\ A \cdot a^{\frac{\pi}{2}},$$

and $$\beta^B = cos\ B + sin\ B \cdot \beta^{\frac{\pi}{2}},$$

by multiplying the two equations together we obtain

$$a^A \beta^B = cos\, A\, cos\, B + cos\, A\, sin\, B \cdot \beta^{\frac{\pi}{2}} + cos\, B\, sin\, A \cdot a^{\frac{\pi}{2}} + sin\, A\, sin\, B \cdot a^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}}.$$

Now, as was shown in the previous paper (p. 98)

$$a^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}} = -\ cos\ a\beta - sin\ a\beta \cdot \overline{a\beta}^{\frac{\pi}{2}};$$

hence

$$\cos \alpha^A \beta^B = \cos A \cos B - \sin A \sin B \cos \alpha\beta \qquad (1),$$

and

$$\mathrm{Sin}\, \alpha^A\beta^B = \left\{\cos A \sin B \cdot \beta + \cos B \sin A \cdot \alpha - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}\right\}^{\frac{\pi}{2}} \quad (2).$$

Equation (1) expresses what is held to be the fundamental theorem of spherical trigonometry; but the complementary theorem expressed by (2) is never considered. So far as magnitude is concerned, it may be derived from (1) by the relation $\cos^2 \theta + \sin^2 \theta = 1$; but it is not so as regards the axis. Equation (1) is the generalization of the theorem of plane trigonometry

$$\cos (A + B) = \cos A \cos B - \sin A \sin B;$$

while equation (2) is the true generalization of the complementary theorem

$$\sin (A + B) = \cos A \sin B + \cos B \sin A.$$

The one theorem may perhaps be derived logically from the other, when restricted to the plane, but it is not so in space. The two equations form together what is called the addition theorem in plane trigonometry. Why do we have addition on the one side of the equation, while we have multiplication on the other? Because $A + B$ is the sum of two indices of an axis which is not expressed, the complete expression being

$$\cos \alpha^{A+B} = \cos A \cos B - \sin A \sin B$$

$$\mathrm{Sin}\, \alpha^{A+B} = (\cos A \sin B + \cos B \sin A) \cdot \alpha^{\frac{\pi}{2}}.$$

Prosthaphaeresis in spherical trigonometry.

The formula for $\alpha^A \beta^{-B}$ is obtained from that for $\alpha^A\beta^B$ by putting a minus before the $\sin B$ factor. Hence

$$\cos \alpha^A \beta^{-B} = \cos A \cos B + \sin A \sin B \cos \alpha\beta, \text{ and}$$

$$\mathrm{Sin}\, \alpha^A \beta^{-B} = -\cos A \sin B \cdot \beta^{\frac{\pi}{2}} + \cos B \sin A \cdot \alpha^{\frac{\pi}{2}} + \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}.$$

Hence the generalizations for space of

$$\cos (A-B) + \cos (A+B) = 2 \cos A \cos B,$$
$$\cos (A-B) - \cos (A+B) = 2 \sin A \sin B,$$
$$\sin (A+B) + \sin (A-B) = 2 \cos B \sin A,$$
$$\sin (A+B) - \sin (A-B) = 2 \cos A \sin B,$$

are respectively

$$\cos \alpha^A \beta^{-B} + \cos \alpha^A \beta^B = 2 \cos A \cos B,$$

$$\cos \alpha^A \beta^{-B} - \cos \alpha^A \beta^B = 2 \sin A \sin B \cos \alpha\beta,$$

$$\mathrm{Sin}\, \alpha^A \beta^B + \mathrm{Sin}\, \alpha^A \beta^{-B} = 2 \cos B \sin A \cdot \alpha^{\frac{\pi}{2}},$$

$$\mathrm{Sin}\, \alpha^A \beta^B - \mathrm{Sin}\, \alpha^A \beta^{-B} = 2\left\{\cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}\right\}^{\frac{\pi}{2}}$$

Let

$$\alpha^A \beta^B = \gamma^C \text{ and } \alpha^A \beta^{-B} = \delta^D \text{ (fig. 9)}$$

then $$\beta^B \alpha^{-A} \alpha^A \beta^B = \delta^{-D}\gamma^C = \beta^{2B};$$

therefore $$\frac{\delta^{-D}\gamma^C}{2} = \beta^B$$

Also $$\delta^D \frac{\delta^{-D}\gamma^C}{2} = \alpha \quad ;$$

but this does not reduce to $$\frac{\delta^D\gamma^C}{2} = \alpha^A.$$

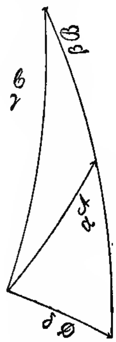

FIG. 9.

Hence

$$\cos \delta^D + \cos \gamma^C = 2\cos \left\{ \delta^D \frac{\delta^{-D}\gamma^C}{2} \right\} \cos \frac{\delta^{-D}\gamma^C}{2};$$

$$\cos \delta^D - \cos \gamma^C = 2\sin \left\{ \delta^D \frac{\delta^{-D}\gamma^C}{2} \right\} \sin \frac{\delta^{-D}\gamma^C}{2} \cos \alpha\beta;$$

etc.

To prove that $\alpha^A\beta^B = e^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}}$.

Since $$\alpha^A = 1 + A\alpha^{\frac{\pi}{2}} + \frac{A^2\alpha^{2\frac{\pi}{2}}}{2!} + \frac{A^3\alpha^{3\frac{\pi}{2}}}{3!} +,$$

and $$\beta^B = 1 + B\beta^{\frac{\pi}{2}} + \frac{B^2\beta^{2\frac{\pi}{2}}}{2!} + \frac{B^3\beta^{3\frac{\pi}{2}}}{3!} +,$$

$$\alpha^A\beta^B = 1 + A\alpha^{\frac{\pi}{2}} + \frac{A^2\alpha^{\pi}}{2!} + \frac{A^3\alpha^{3\frac{\pi}{2}}}{3!} +$$

$$+ B\beta^{\frac{\pi}{2}} + AB\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} + \frac{A^2B}{2!}\alpha^{\pi}\beta^{\frac{\pi}{2}} +$$

$$+ \frac{B^2\beta^{\pi}}{2!} + \frac{AB^2}{2!}\alpha^{\frac{\pi}{2}}\beta^{\pi} +$$

$$+ \frac{B^3}{3!}\beta^{3\frac{\pi}{2}} +$$

$$= 1 + (A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}) + \frac{(A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}})^2}{2!} + \ldots + \frac{(A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}})^n}{n!} +$$

$$= e^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}}.$$

The general term is

$$\frac{1}{n!}\left\{A^n \alpha^{n\frac{\pi}{2}} + nA^{n-1} B\alpha^{(n-1)\frac{\pi}{2}}\beta^{\frac{\pi}{2}} + \frac{n(n-1)}{2!}A^{n-2}B^2\alpha^{(n-2)\frac{\pi}{2}}\beta^{\pi} + \right\}$$

which is formed according to the binomial theorem, only the order of α, β must be preserved in each term.

The binomial here is the sum of two logarithms, not a sum of two quaternions. It is not true that

$$e^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}} = e^{(A\alpha + B\beta)^{\frac{\pi}{2}}},$$

for

$$e^{(A\alpha + B\beta)^{\frac{\pi}{2}}} = 1 + (A\alpha + B\beta)^{\frac{\pi}{2}} + \frac{\{(A\alpha + B\beta)^2\}^{\pi}}{2!} +$$

$$= 1 - \frac{A^2 + B^2 + 2\,A\,B\cos\alpha\beta}{2!} + \frac{(A^2 + B^2 + 2\,A\,B\cos\alpha\beta)^2}{4!} -$$

$$+ \left\{1 - \frac{A^2 + B^2 + 2\,A\,B\cos\alpha\beta}{3!} + \right\}(A\alpha + B\beta)^{\frac{\pi}{2}}$$

In a similar manner it may be shown that

$$\alpha^A\beta^B\gamma^C = 1 + A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}} + C\gamma^{\frac{\pi}{2}}$$

$$+ \frac{1}{2!}\left\{A^2\alpha^{\pi} + B^2\beta^{\pi} + C^2\gamma^{\pi} + 2AB\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} + 2AC\alpha^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}} + 2BC\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}\right\}$$

$$+ \frac{1}{3!}\left\{A^3\alpha^{3\frac{\pi}{2}} + B^3\beta^{3\frac{\pi}{2}} + C^3\gamma^{3\frac{\pi}{2}} + 3A^2B\alpha^{\pi}\beta^{\frac{\pi}{2}} + 3A^2C\alpha^{\pi}\gamma^{\frac{\pi}{2}} + 3B^2C\beta^{\pi}\gamma^{\frac{\pi}{2}}\right.$$

$$\left. + 3\,AB^2\alpha^{\frac{\pi}{2}}\beta^{\pi} + 3AC^2\alpha^{\frac{\pi}{2}}\gamma^{\pi} + 3BC^2\beta^{\frac{\pi}{2}}\gamma^{\pi} + 6ABC\,\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}\right\}$$

$+$ etc.

where the terms are formed according to the rule of the trinomial theorem, but the order α, β, γ, must be preserved in each term. And the multinomial theorem is true, provided the above condition is observed.

Circular Spirals.

Meaning of α^A_w.

The series $e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} +$ may be viewed as having a logarithmic angle or period 0 or more generally $2n\pi$, so that it is expressed more fully by $e^{A\alpha^0}$ or $e^{A\alpha^{2n\pi}}$. Similarly the logarithmic angle or period of α^A, that is of

$$e^{A\alpha^{\frac{\pi}{2}}} = 1 + A\alpha^{\frac{\pi}{2}} + \frac{A^2\alpha^{\pi}}{2!} +$$

is $\frac{\pi}{2}$ or more generally $2n\pi + \frac{\pi}{2}$.

By α^A_w is meant $e^{A\alpha^w}$ where the logarithmic angle is w, so that

$$\alpha_w^A = e^{A\alpha^w} = 1 + A\alpha^w + \frac{A^2 \alpha^{2w}}{2!} + \frac{A^3 \alpha^{3w}}{3!} + .$$

What is the geometrical meaning of α_w^A? It is a sector of the logarithmic spiral which has α for axis, w for the angle between the tangent and the radius vector and $A \sin w$ for the angle at the apex.

On account of the new element w the quantity may be named a *quinternion*, for when a multiplier is prefixed we have five elements.

To prove that $\alpha_w^A = e^{A \cos w + A \sin w \cdot \alpha^{\frac{\pi}{2}}}$

For $\alpha_w^A = e^{A\alpha^w} = 1 + A\alpha^w + \frac{A^2\alpha^{2w}}{2!} + \frac{A^3\alpha^{3w}}{3!} + ,$

$$= 1 + A \cos w + \frac{A^2 \cos 2w}{2!} + \frac{A^3 \cos 3w}{3!} +$$

$$+ \left\{ A \sin w + \frac{A^2 \sin 2w}{2!} + \frac{A^3 \sin 3w}{3!} + \right\} \cdot \alpha^{\frac{\pi}{2}}$$

But $e^{A \cos w + A \sin w \cdot \alpha^{\frac{\pi}{2}}} = e^{A \cos w} e^{A \sin w \cdot \alpha^{\frac{\pi}{2}}}$

$$= \left\{1 + A \cos w + \frac{A^2 \cos^2 w}{2!} + \right\} \left\{1 + A \sin w \cdot \alpha^{\frac{\pi}{2}} - \frac{A^2 \sin^2 w}{2!} + \right\},$$

$$= 1 + A \cos w + \frac{A^2}{2!} (\cos^2 w - \sin^2 w) +$$

$$+ \left\{A \sin w + \frac{A^2}{2!} 2 \sin w \cos w + \right\} \cdot \alpha^{\frac{\pi}{2}};$$

$$= 1 + A \cos w + \frac{A^2}{2!} \cos 2w +$$

$$+ A \sin w + \frac{A^2}{2!} \sin 2w +$$

therefore $e^{A\alpha^w} = e^{A \cos w + A \sin w \cdot \alpha^{\frac{\pi}{2}}}$.

To prove that $\alpha_w^A \beta_w^B = e^{A\alpha^w + B\beta^w}$.

Since $\alpha_w^A = e^{A \cos w + A \sin w \cdot \alpha^{\frac{\pi}{2}}}$

and $\beta_w^B = e^{B \cos w + B \sin w \cdot \beta^{\frac{\pi}{2}}}$

$$\alpha_w^A \beta_w^B = e^{A \cos w} e^{A \sin w \cdot \alpha^{\frac{\pi}{2}}} e^{B \cos w} e^{B \sin w \cdot \beta^{\frac{\pi}{2}}}$$

$$= e^{A \cos w + B \cos w} e^{A \sin w \cdot \alpha^{\frac{\pi}{2}} + B \sin w \cdot \beta^{\frac{\pi}{2}}}$$

$$= e^{(A + B) \cos w} e^{\sin w \left\{ A \cdot \alpha^{\frac{\pi}{2}} + B \cdot \beta^{\frac{\pi}{2}} \right\}}$$

But $e^{A\alpha^w + B\beta^w} = e^{(A \cos w + A \sin w \cdot \alpha^{\frac{\pi}{2}}) + (B \cos w + B \sin w \cdot \beta^{\frac{\pi}{2}})}$

$$= e^{A\cos w + B\cos w}\, e^{A\sin w \cdot \alpha^{\frac{\pi}{2}} + B\sin w \cdot \beta^{\frac{\pi}{2}}}.$$

Because $e^{A\cos w}$ and $e^{B\cos w}$ are independent of axis, they can be changed from the order in which they occur in the sum of indices.

The meaning of $\alpha_w^A\, \beta_w^B$ is the sector of the spiral which joins the beginning of the former with the end of the latter.

Hence when $\beta = \alpha$,

$$\alpha_w^A\, \alpha_w^B = e^{(A+B)\cos w}\, e^{(A+B)\sin w \cdot \alpha^{\frac{\pi}{2}}}$$

$$= e^{(A+B)\alpha^w}$$

$$= \alpha_w^{A+B}$$

which is the addition theorem for the logarithmic spiral, the two component sectors being in the same plane.

Exponent of a compound angle.

We have

$$e^{x\alpha^A\beta^B} = 1 + x\,\alpha^A\beta^B + \frac{x^2}{2!}(\alpha^A\beta^B)^2 + \frac{x^3}{3!}(\alpha^A\beta^B)^3 +;$$

where $\alpha^A\beta^B$ is expanded as shown above, and $(\alpha^A\beta^B)^2$ is double of the compound angle, $(\alpha^A\beta^B)^3$ is three times the compound angle and so on. It is to be observed that $(\alpha^A\beta^B)^2$ is not in general equal to $\alpha^{2A}\beta^{2B}$.

Let $x = A = B = \frac{\pi}{2}$ and let β be identical with α, then we have

$$e^{\frac{\pi}{2}\alpha^{\pi}} = 1 - \frac{\pi}{2} + \left(\frac{\pi}{2}\right)^2 \frac{1}{2!} -.$$

But $e^{\frac{\pi}{2}\alpha^{\pi}} = e^{-\frac{\pi}{2}}$ and it is also $= \alpha^{\frac{\pi}{2}\alpha^{\frac{\pi}{2}}}$;

and thus $e^{-\frac{\pi}{2}} = \alpha^{\frac{\pi}{2}\alpha^{\frac{\pi}{2}}}$,

which is a rational expression for the celebrated equation of Euler

$$\sqrt{-1}^{\sqrt{-1}} = e^{-\frac{\pi}{2}}.$$

By taking logs we obtain

$$\alpha^{\frac{\pi}{2}} \log\left(\alpha^{\frac{\pi}{2}}\right) = -\frac{\pi}{2},$$

that is

$$\frac{\log\left(\alpha^{\frac{\pi}{2}}\right)}{\alpha^{\frac{\pi}{2}}} = \frac{\pi}{2}.$$

To differentiate α^A.

Since $$\alpha^A = e^{A\alpha^{\frac{\pi}{2}}} = \cos A + \sin A \cdot \alpha^{\frac{\pi}{2}};$$

therefore

$$d(\alpha^A) = e^{A\alpha^{\frac{\pi}{2}}} d\,(A\alpha^{\frac{\pi}{2}}) = (-\sin A + \cos A \cdot \alpha^{\frac{\pi}{2}})\, dA + \sin A\, d\alpha \cdot \bar{\alpha}^{\frac{\pi}{2}}$$

therefore

$$\alpha^A d(A\alpha^{\frac{\pi}{2}}) = (-\sin A + \cos A \cdot \alpha^{\frac{\pi}{2}}) dA + \sin A\, d\alpha \cdot \bar{\alpha}^{\frac{\pi}{2}}$$

But since

$$\alpha^A \alpha^{-A} = 1,$$

$$d(\alpha^A)\, \alpha^{-A} + \alpha^A d\,(\alpha^{-A}) = 0;$$

therefore

$$\alpha^A d(A\alpha^{\frac{\pi}{2}}) \alpha^{-A} + \alpha^A \alpha^{-A}\, d(-A\alpha^{\frac{\pi}{2}}) = 0;$$

therefore $$\alpha^A d(A\alpha^{\frac{\pi}{2}})\, \alpha^{-A} = d(A\alpha^{\frac{\pi}{2}}).$$

Hence $$d(A\alpha^{\frac{\pi}{2}}) = \alpha^{A+\frac{\pi}{2}} dA\, \alpha^{-A} + \sin A\, d\alpha \cdot \bar{\alpha}^{\frac{\pi}{2}} \alpha^{-A}$$

$$= dA \cdot \alpha^{\frac{\pi}{2}} + d\alpha\, (\sin A \cos A \cdot \bar{\alpha}^{\frac{\pi}{2}} - \sin^2 A \cdot \bar{\alpha}^{\frac{\pi}{2}} \alpha^{\frac{\pi}{2}})$$

$$= dA \cdot \alpha^{\frac{\pi}{2}} + d\alpha\, (\sin A \cos A \cdot \bar{\alpha}^{\frac{\pi}{2}} + \sin^2 A \cdot \overline{\bar{\alpha}\alpha}^{\frac{\pi}{2}}$$

$$= \left\{ dA \cdot \alpha + d\alpha\, (\sin A \cos A \cdot \bar{\alpha} + \sin^2 A \cdot \overline{\bar{\alpha}\alpha} \right\}^{\frac{\pi}{2}}$$

To differentiate $\alpha^A \beta^B$.

$$d(\alpha^A \beta^B) = (d\alpha^A)\, \beta^B + \alpha^A d\,(\beta^B),$$

$$= \alpha^A d\,(A\alpha^{\frac{\pi}{2}})\, \beta^B + \alpha^A \beta^B d\,(B\beta^{\frac{\pi}{2}}),$$

which is not $$= \alpha^A \beta^B \left\{ d(A\alpha^{\frac{\pi}{2}}) + d(B\beta^{\frac{\pi}{2}}) \right\} \text{unless } \beta = \alpha.$$

But $$\alpha^A\, \beta^B = e^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}}$$

and $$d(\alpha^A \beta^B) = e^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}}\, d\,(A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}),$$

provided it be understood that in the final terms the order of α, β be observed.

To differentiate $e^{(A\alpha + B\beta)^{\frac{\pi}{2}}}$ is more simple, because then we have but one index, not a binomial, and

$$d\left\{ e^{(A\alpha + B\beta)^{\frac{\pi}{2}}} \right\} = e^{(A\alpha + B\beta)^{\frac{\pi}{2}}} d\left\{ (A\alpha + B\beta)^{\frac{\pi}{2}} \right\}.$$

Hyperbolic Trigonometry.

Meaning of the equation

$$h\alpha^A = \cosh\, A + \sinh\, A \cdot \alpha^{\frac{\pi}{2}}.$$

The expression α^A, when no period is expressed, is understood to have the period $\frac{\pi}{2}$; in other words the area $\frac{A}{2}$ is bounded by a circular arc. Let $h\alpha^A$ denote the same when the bounding arc is the equilateral hyperbola (fig. 10). Then the rectangular components OM and MQ of the hyperbolic versor which has the axis α and the area $\frac{A}{2}$ are commonly denoted by *cosh* A and *sinh* A, so that

$$h\alpha^A = \cosh A + \sinh A \cdot \alpha^{\frac{\pi}{2}}$$

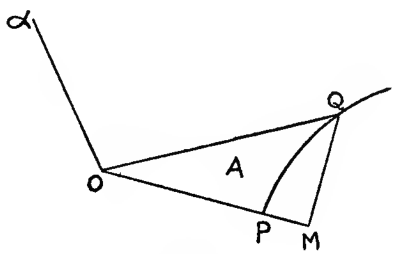

FIG. 10.

The hyperbolic versor $h\alpha^A$ is equivalent to the multiplier *cosh* A together with the circular versor $\sinh A \cdot \alpha^{\frac{\pi}{2}}$.

To prove that $h\alpha^A = he^{A\alpha^{\frac{\pi}{2}}}$.

We have
$$h\alpha^A = \cosh A + \sinh A \cdot \alpha^{\frac{\pi}{2}},$$
$$= 1 + \frac{A^2}{2!} + \frac{A^4}{4!} +$$
$$+ \left(A + \frac{A^3}{3!} + \right) \cdot \alpha^{\frac{\pi}{2}}$$

This is an essentially different expansion from the circular. It may be denoted by $h\, e^{A\alpha^{\frac{\pi}{2}}}$, and it differs from that for $e^{A\alpha^{\frac{\pi}{2}}}$ in having $\alpha^{\frac{\pi}{2}}\, \alpha^{\frac{\pi}{2}} = 1$.

Similarly
$$h\alpha^{-A} = \cosh A - \sinh A \cdot \alpha^{\frac{\pi}{2}},$$
$$= h\, e^{-A\alpha^{\frac{\pi}{2}}}.$$

To compare $h\alpha^A$ *with* $e^{A\alpha^\pi}$.

$$e^{A\alpha^\pi} = \cosh A + \sinh A \cdot \alpha^\pi,$$
$$= \cosh A + \alpha^{\frac{\pi}{2}} \sinh A \cdot \alpha^{\frac{\pi}{2}};$$

that is
$$e^{(A\alpha^{\frac{\pi}{2}})\alpha^{\frac{\pi}{2}}} = \cosh A + \alpha^{\frac{\pi}{2}} \sinh A \cdot \alpha^{\frac{\pi}{2}};$$

therefore $$\cosh A = \cos\left(Aa^{\frac{\pi}{2}}\right),$$

and $$a^{\frac{\pi}{2}} \sinh A = \sin\left(Aa^{\frac{\pi}{2}}\right).$$

Also $$ha^{-A} = \cosh A - \sinh A \cdot a \ ,$$
$$= \cos\left(Aa^{\frac{\pi}{2}}\right) - a^{\frac{\pi}{2}} \sin\left(Aa^{\frac{\pi}{2}}\right) \cdot a^{\frac{\pi}{2}}.$$

To find the value of $ha^A \, h\beta^B$, the analogue of $a^A\beta^B$.

We have $$ha^A = \cosh A + \sinh A \cdot a^{\frac{\pi}{2}},$$

and $$h\beta^B = \cosh B + \sinh B \cdot \beta^{\frac{\pi}{2}};$$

therefore $$ha^A h\beta^B = \cosh A \cosh B + \cosh A \sinh B \cdot \beta^{\frac{\pi}{2}}$$
$$+ \cosh B \sinh A \cdot a^{\frac{\pi}{2}} + \sinh A \sinh B \cdot a^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}.$$

The problem is reduced to finding the value of $a^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}}$. Now for a plane, in which case $a = \beta$, we have

$$ha^A \, ha^B = \cosh A \cosh B + \sinh A \sinh B$$
$$+ \left\{\cosh A \sinh B \cdot a + \cosh B \sinh A \cdot a\right\}^{\frac{\pi}{2}}$$

from which it appears that the second term of the *cosh* for space is $\sinh A \sinh B \cos a\beta$. The term in *Sinh* must be of the form

$$x \sinh A \sinh B \sin a\beta \cdot \overline{a\beta},$$

the value of x to be determined by the condition that $\cosh^2 - \sinh^2 = 1$.

Now

$$\cosh^2 = \cosh^2 A \cosh^2 B + \sinh^2 A \sinh^2 B \cos^2 a\beta$$
$$+ 2 \cosh A \cosh B \sinh A \sinh B \cos a\beta.$$

and $$\sinh^2 = \cosh^2 A \sinh^2 B + \cosh^2 B \sinh^2 A$$
$$+ 2 \cosh A \cosh B \sinh A \sinh B \cos a\beta$$
$$+ x^2 \sinh^2 A \sinh^2 B \sin^2 a\beta.$$

and $$\cosh^2 - \sinh^2 = \cosh^2 A \left(\cosh^2 B - \sinh^2 B\right)$$
$$- \sinh^2 A \left\{\cosh^2 B - \sinh^2 B \left(\cos^2 a\beta - x^2 \sin^2 a\beta\right)\right\},$$

which is equal to 1, if $x^2 = -1$, or $x = \sqrt{-1}$.

Hence $$\cosh a^A\beta^B = \cosh A \cosh B + \sinh A \sinh B \cos a\beta \qquad (1)$$

and $$\operatorname{Sinh} a^A\beta^B = \left\{\cosh A \sinh B \cdot \beta + \cosh B \sinh A \cdot a + \sqrt{-1} \sinh A \sinh B \sin a\beta \cdot \overline{a\beta}\right\}. \qquad (2)$$

Equation (1) is the fundamental theorem in hyperbolic non-Euclidian geometry. Equation (2) gives the complementary theorem, and we propose to investigate its geometrical meaning. Guided by the analogy to the circular sectors we conclude that equation (1) suffices to determine the amount of hyperbolic sector of the product, while equation (2) serves to determine the plane of the sector. How can the expression in (2) determine a plane? Compound (fig. 11) $\cosh A \sinh B \cdot \beta$ with $\cosh B \sinh A \cdot \alpha$ and from the extremity P describe a circle with radius $\sinh A \sinh B \sin \alpha\beta$ in the plane of OP and the perpendicular $\overline{\alpha\beta}$. The positive tangent OT, drawn from O to the circle has the direction of the perpendicular to the plane.

This may be readily verified in the case of the product of equal sectors.

Let $$\alpha^A = x + y \cdot \alpha^{\frac{\pi}{2}}$$

$$\beta^A = x + y \cdot \beta^{\frac{\pi}{2}}$$

then according to the rule for the product in space

$$\alpha^A \beta^A = x^2 + y^2 \cos \alpha\beta$$

$$+ \left\{ xy(\alpha + \beta) + \sqrt{-1}\, y^2 \sin \alpha\beta \cdot \overline{\alpha\beta} \right\}^{\frac{\pi}{2}}$$

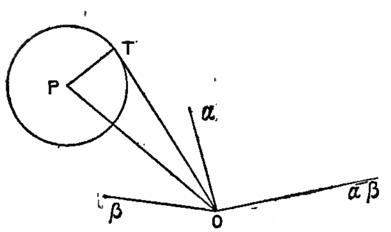

FIG. 11.

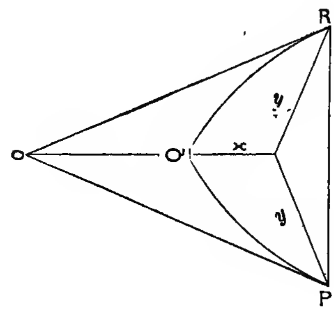

FIG. 12.

Suppose that the straight line PR (fig. 12) joining the extremities of the arcs is the chord of the product; it is symmetrical with respect to the axis $\overline{\alpha\beta}$. Then

$$\sinh \frac{\alpha^A \beta^A}{2} = \tfrac{1}{2}\sqrt{2y^2 + 2y^2 \cos \alpha\beta} = \frac{y}{\sqrt{2}} \sqrt{1 + \cos \alpha\beta};$$

therefore $$\cosh \frac{\alpha^A \beta^A}{2} = \sqrt{1 + \frac{y^2}{2}(1 + \cos \alpha\beta)};$$

therefore by the rule for the plane, which is known to be true,

$$\cosh \alpha^A \beta^A = \frac{y^2}{2}(1 + \cos \alpha\beta) + 1 + \frac{y^2}{2}(1 + \cos \alpha\beta),$$

$$= y^2 (1 + \cos \alpha\beta) + 1,$$

$$= y^2 + 1 + y^2 \cos \alpha\beta,$$

$$= x^2 + y^2 \cos \alpha\beta.$$

But this last is the value given above by the rule found for space.

Prosthaphaeresis in hyperbolic trigonometry.

We have $cosh\ \alpha^A\beta^B = cosh\ A\ cosh\ B + sinh\ A\ sinh\ B\ cos\ \alpha\beta$;

and $Sinh\ \alpha^A\beta^B = \left\{cosh\ A\ sinh\ B \cdot \beta + cosh\ B\ sinh\ A \cdot \alpha + \sqrt{-1}\ sinh\ A\ sinh\ B\ sin\ \alpha\beta \cdot \overline{\alpha\beta}\right\}^{\frac{\pi}{2}}$

By putting in $-\ sinh\ B$ instead of $sinh\ B$ we get

$$cosh\ \alpha^A\beta^{-B} = cosh\ A\ cosh\ B - sinh\ A\ sinh\ B\ cos\ \alpha\beta;$$

and $Sinh\ \alpha^A\beta^{-B} = -\ cosh\ A\ sinh\ B \cdot \beta + cosh\ B\ sinh\ A \cdot \alpha - \sqrt{-1}\, sinh\ A\ sinh\ B\ sin\ \alpha\beta \cdot \overline{\alpha\beta}.$

Therefore $cosh\ \alpha^A\beta^B + cosh\ \alpha^A\beta^{-B} = 2\ cosh\ A\ cosh\ B$;

$$cosh\ \alpha^A\beta^B - cosh\ \alpha^A\beta^{-B} = 2\ sinh\ A\ sinh\ B\ cos\ \alpha\beta;$$

$$Sinh\ \alpha^A\beta^B + Sinh\ \alpha^A\beta^{-B} = 2\ cosh\ B\ sinh\ A \cdot \alpha;$$

$$Sinh\ \alpha^A\beta^B - Sinh\ \alpha^A\beta^{-B} = 2\ cosh\ A\ sinh\ B \cdot \beta + 2\sqrt{-1}\ sinh\ A\ sinh\ B\ sin\ \alpha\beta \cdot \overline{\alpha\beta}.$$

To prove that $h\alpha^A h\beta^B = h\, e^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}}$.

Since $h\alpha^A = 1 + A\alpha^{\frac{\pi}{2}} + \frac{A^2\alpha^{2\frac{\pi}{2}}}{2!} + \frac{A^3\alpha^{3\frac{\pi}{2}}}{3!} +,$

and $h\beta^B = 1 + B\beta^{\frac{\pi}{2}} + \frac{B^2\beta^{2\frac{\pi}{2}}}{2!} + \frac{B^3\beta^{3\frac{\pi}{2}}}{3!} +;$

$$h\alpha^A h\beta^B = 1 + A\alpha^{\frac{\pi}{2}} + \frac{A^2\alpha^{\pi}}{2!} + \frac{A^3\alpha^{3\frac{\pi}{2}}}{3!} + + B\beta^{\frac{\pi}{2}} + AB\,\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} + \frac{A^2B}{2!}\alpha^{\pi}\beta^{\frac{\pi}{2}} + + \frac{B^2\beta^{\pi}}{2!} + \frac{AB^2}{2!}\alpha^{\frac{\pi}{2}}\beta^{\pi} + + \frac{B^3}{3!}\beta^{3\frac{\pi}{2}} +$$

$$= 1 + (A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}) + \frac{(A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}})^2}{2!} +$$

$$= he^{A\alpha^{\frac{\pi}{2}} + B\beta^{\frac{\pi}{2}}}.$$

The expansion is the same as for the product of circular sectors, excepting that we have

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = cos\ \alpha\beta + \sqrt{-1}\ sin\ \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}$$

and (as a special case) $\alpha^{\pi} = \beta^{\pi} = 1.$

Hyperbolic Spirals.

To investigate the meaning of ha_w^A *the analogue of* a_w^A.

We must have $h\,a_w^A = he^{A\cosh w}\, he^{A\sinh w \cdot a^{\frac{\pi}{2}}},$

$$= \left(1 + A\cosh w + \frac{A^2}{2!}\cosh^2 w + \frac{A^3}{3!}\cosh^3 w + \right)$$

$$\times \left(1 + A\sinh w \cdot a^{\frac{\pi}{2}} + \frac{A^2}{2!}\sinh^2 w + \frac{A^3}{3!}\sinh^3 w \cdot a^{\frac{\pi}{2}} + \right)$$

$$= 1 + A\cosh w + \frac{A^2}{2!}(\cosh^2 w + \sinh^2 w) + \frac{A^3}{3!}\left\{\cosh^3 w + 3\cosh w \sinh^2 w\right\} +$$

$$+ \left\{A\sinh w + \frac{A^2}{2!}2\cosh w \sinh w + \frac{A^3}{3!}\left\{3\cosh^2 w \sinh w + \sinh^3 w\right\} + \right\} \cdot a^{\frac{\pi}{2}}$$

$$= 1 + A(\cosh w + \sinh w \cdot a^{\frac{\pi}{2}}) + \frac{A^2}{2!}(\cosh w + \sinh w \cdot a^{\frac{\pi}{2}})^2 + \frac{A^3}{3!}(\cosh w + \sinh w \cdot a^{\frac{\pi}{2}})^3 +$$

$$= 1 + A\cosh w + \frac{A^2}{2!}\cosh 2w + \frac{A^3}{3!}\cosh 3w +$$

$$+ \left\{A\sinh w + \frac{A^2}{2!}\sinh 2w + \frac{A^3}{3!}\sinh 3w + \right\} \cdot a^{\frac{\pi}{2}}$$

$$= 1 + Aa^w + \frac{A^2}{2!}a^{2w} + \frac{A^3}{3!}a^{3w} + .$$

It follows as in the case of the circular spirals, that

$$ha_w^A\, h\beta_w^B = he^{Aa^w + B\beta^w}$$

$$= e^{A\cosh w + B\cosh w}\, ha^{A\sinh w}\, h\beta^{B\sinh w}$$

THE

FUNDAMENTAL THEOREMS OF ANALYSIS

GENERALIZED FOR SPACE.

BY

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THE FUNDAMENTAL THEOREMS OF ANALYSIS GENERALIZED FOR SPACE.

By Alexander Macfarlane, D.Sc., LL.D., *University of Texas.*

[Read before the New York Mathematical Society, May 7, 1892.]

The fundamental theorem of plane trigonometry expresses the cosine and the sine of the sum of two angles in terms of the cosines and sines of the component angles; namely,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B, \tag{1}$$

and

$$\sin(A+B) = \sin A \cos B + \cos A \sin B. \tag{2}$$

The complementary theorem gives the cosine and the sine of the difference of two angles; namely,

$$\cos(A-B) = \cos A \cos B + \sin A \sin B, \tag{3}$$

and

$$\sin(A-B) = \sin A \cos B - \cos A \sin B. \tag{4}$$

Now the fundamental theorem of spherical trigonometry is, c denoting the angle between the arcs A and B, and C denoting the opposite side.

$$\cos C = \cos A \cos B + \sin A \sin B \cos c.$$

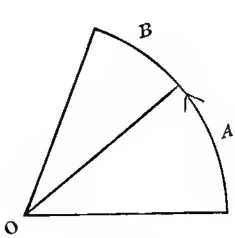

Fig. 1.

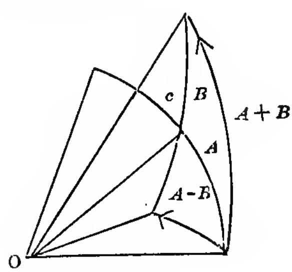

Fig. 2.

But suppose that the angle B of Fig. 1 is tilted up, and let c denote the angle by which it has been tilted (Fig. 2), then in a certain sense the arc of the great circle from the beginning of A

to the end of B is the sum of the arcs A and B. We obtain for this more general sum the formula

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \cos c,$$

which is the generalization of (1); and

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \cos c,$$

which is the generalization of (3). But in treatises on spherical trigonometry there is no formula corresponding to (2); the only place where I have observed such a formula is Hamilton's *Lectures on Quaternions*, p. 537. The supposition appears to be that (2) is not essentially different from (1), and therefore that no generalization of it is necessary. No doubt the magnitude of the sine may be deduced from the cosine by the relation

$$\sin^2(A+B) = 1 - \cos^2(A+B);$$

but this is not the generalization of (2).

In order to investigate this question we require a notation for an angle in space.

Such an angle is fully specified by the axis and the amount of arc at unit radius; the axis will be denoted by a Greek letter, such as α, and the amount of arc at unit radius, that is, the circular measure, by an italic capital, such as A. The arc (Fig. 3) may be rotated round α to any position in the circle; it does not suppose a fixed initial line; it is symmetrical with respect to α. The angle itself is properly denoted by α^A; for let α^B be another angle, then $\alpha^A\alpha^B = \alpha^{A+B}$, so that α and A are truly related as base to index. According to this view the above theorem in plane trigonometry relates to the addition of arcs, but to the product of angles. Let α denote the axis of the constant plane, then (1) takes the form

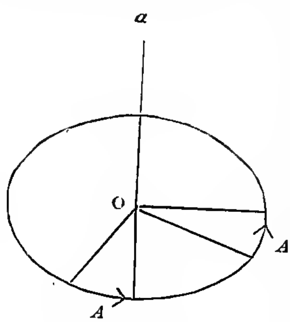

FIG. 3.

$$\cos \alpha^A\alpha^B = \cos \alpha^{A+B} = \cos A \cos B - \sin A \sin B,$$

and (2) takes the form

$$\sin \alpha^A \alpha^B = \sin \alpha^{A+B} = \cos A \sin B + \cos B \sin A.$$

We may also view A as denoting twice the area of the circular sector, the radius being unity; and this view of the notation is important, for it applies to the equilateral hyperbola, while the former view does not.

An angle which is the negative of a given angle has an equal arc, but the opposite axis; $(-\alpha)^A$ is the negative of α^A. The minus may be removed from the base and attached to the index; thus $(-\alpha)^A = \alpha^{-A}$, and $\alpha^A(-\alpha)^B = \alpha^{A-B}$. So long as the axis remains the same or the opposite, the arcs are combined like ordinary indices. But suppose that a different axis β is introduced, it is evident that then the rule for indices must be generalized. The $\sqrt{-1}$ in the ordinary complex quantity denotes an angle whose arc is a quadrant, but it leaves the axis of the plane unspecified.

The angle α^A is a quaternion with unity for ratio; that is, a versor. The general quaternion may be denoted by a single symbol such as $\mathbf{a}$; and if a denote the ratio, α the axis, and A the arc at unit distance, then

$$\mathbf{a} = a\alpha^A.$$

Any versor can be expressed as the sum of two quaternions which have arcs differing by a quadrant.

Let the arc A be less than a quadrant. Then

$$\alpha^A = \cos A \cdot \alpha^0 + \sin A \cdot \alpha^{\frac{\pi}{2}}$$

is a complete equivalence. The versor α^A applied to any line in its plane leaves the magnitude of the line unchanged, but turns it round α by an amount A. This is equivalent, both as regards final position and the whole amount of turning, to multiplying the line by $\cos A$ and turning it round α by no amount, together with the effect of multiplying the line by $\sin A$, and turning it round α by a quadrant.

But the above form of equation provides a complete equivalence for an angle however large, and also distinguishes between a positive and negative angle. Thus we have for the quadrants indicated:

Quadrant.	Angle.	Components.
first	α^A	$\cos A \cdot \alpha^0 \quad + \sin A \cdot \alpha^{\frac{\pi}{2}}$
second	α^A	$\cos A \cdot \alpha^{\pi} \quad + \sin A \cdot \alpha^{\frac{\pi}{2}}$
third	α^A	$\cos A \cdot \alpha \quad + \sin A \cdot \alpha^{3\frac{\pi}{2}}$
fourth	α^A	$\cos A \cdot \alpha^{2\pi} \quad + \sin A \cdot \alpha^{3\frac{\pi}{2}}$
fifth	α^A	$\cos A \cdot \alpha^{2\pi} \quad + \sin A \cdot \alpha^{5\frac{\pi}{2}}$
sixth	α^A	$\cos A \cdot \alpha^{3\pi} \quad + \sin A \cdot \alpha^{5\frac{\pi}{2}}$
first negative	$(-\alpha)^A$	$\cos A \cdot (-\alpha)^0 + \sin A \cdot (-\alpha)^{\frac{\pi}{2}}$
second negative	$(-\alpha)^A$	$\cos A \cdot (-\alpha)^{\pi} + \sin A \cdot (-\alpha)^{\frac{\pi}{2}}$
etc.	etc.	etc.

In the above expressions $\cos A$ and $\sin A$ are supposed to be signless ratios. For an arc less than 2π the different quadrants can be distinguished by making $\cos A$ and $\sin A$ algebraic quantities, that is, either positive or negative; so that a complete equivalence for any positive angle less than a whole turn is

$$\alpha^A = \cos A + \sin A \cdot \alpha^{\frac{\pi}{2}},$$

while the complete equivalence for any negative angle less than a whole turn is

$$(-\alpha)^A = \cos A + \sin A \cdot (-\alpha)^{\frac{\pi}{2}}.$$

But if the angle exceeds a whole turn, then the complete equivalence requires a factor to express the number of whole turns. Suppose that r is the number of times which A contains 2π, then the complete equivalence is

$$\alpha^A = \alpha^{r2\pi}(\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}}).$$

Similarly, the complete equivalence for any negative angle is

$$(-\alpha)^A = (-\alpha)^{r2\pi}\{\cos A + \sin A \cdot (-\alpha)^{\frac{\pi}{2}}\}.$$

Suppose A to be less than 2π, and m to be an integer, then

$$\alpha^{mA} = \cos mA + \sin mA \cdot \alpha^{\frac{\pi}{2}}$$

may be an equivalence only so far as the final position is concerned, not as regards the whole amount of turning.

Suppose that p is the number of times which mA contains 2π, then

$$\alpha^{mA} = \alpha^{p2\pi}\{\cos mA + \sin mA \cdot \alpha^{\frac{\pi}{2}}\}.$$

The root of an incomplete equivalence is ambiguous, while that of a complete equivalence is unique. Thus, as either p or $p-1$ or $p-2$ is exactly divisible by 3, the cube root of α^A is *some* one of the three following:

$$\alpha^{\frac{p2\pi}{3}}\left\{\cos\frac{A-p2\pi}{3} + \sin\frac{A-p2\pi}{3}\cdot\alpha^{\frac{\pi}{2}}\right\},$$

$$\alpha^{\frac{(p-1)2\pi}{3}}\left\{\cos\frac{A-(p-1)2\pi}{3} + \sin\frac{A-(p-1)2\pi}{3}\cdot\alpha^{\frac{\pi}{2}}\right\},$$

$$\alpha^{\frac{(p-2)2\pi}{3}}\left\{\cos\frac{A-(p-2)2\pi}{3} + \sin\frac{A-(p-2)2\pi}{3}\cdot\alpha^{\frac{\pi}{2}}\right\}.$$

But the cube root of the incomplete equivalence

$$\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}}$$

is *any* one of the three following:

$$\cos\frac{A-p2\pi}{3} + \sin\frac{A-p2\pi}{3}\cdot\alpha^{\frac{\pi}{2}},$$

$$\cos\frac{A-(p-1)2\pi}{3} + \sin\frac{A-(p-1)2\pi}{3}\cdot\alpha^{\frac{\pi}{2}},$$

$$\cos\frac{A-(p-2)2\pi}{3} + \sin\frac{A-(p-2)2\pi}{3}\cdot\alpha^{\frac{\pi}{2}}.$$

In the treatment of angles in space, we commonly take only the incomplete equivalence, as in most questions a whole turn counts for nothing.

GENERALIZATION OF THE TRIGONOMETRIC THEOREM.

Product of two angles in space.

Let α^A and β^B denote any two angles in space, having a common apex O (Fig. 4).

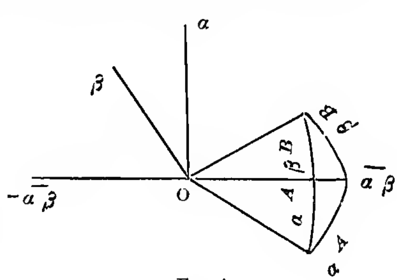

FIG. 4.

Now $\alpha^A = \cos A + \sin A \cdot \alpha^{\frac{\pi}{2}}$,

and $\beta^B = \cos B + \sin B \cdot \beta^{\frac{\pi}{2}}$, therefore

$$\alpha^A\beta^B = (\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}})(\cos B + \sin B \cdot \beta^{\frac{\pi}{2}})$$
$$= \cos A \cos B + \cos A \sin B \cdot \beta^{\frac{\pi}{2}} + \cos B \sin A \cdot \alpha^{\frac{\pi}{2}} + \sin A \sin B \cdot \alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}},$$

if the distributive rule holds. We propose to investigate the meaning of these terms on the supposition that the product $\alpha^A\beta^B$ means the angle from the beginning of α^A to the end of β^B when these two angles are brought to a common intersection, or any angle in the same plane having an equal arc.

The meaning of the first three terms is evident, but not that of the fourth. To investigate and express the value of $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$, we require a notation for the axis which is perpendicular to α and β.

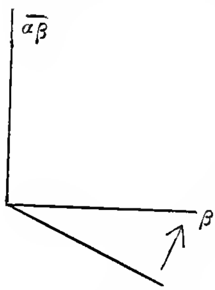

FIG. 5.

Suppose (Fig. 5) α and β to be in a horizontal plane, and that we look down from above; then the arrow indicates the direction of positive turning, and the corresponding axis is the perpendicular to α and β drawn upwards. Let this axis be denoted by $\overline{\alpha\beta}$, then $\overline{\beta\alpha}$ denotes the axis of negative rotation; and as it is opposite to $\overline{\alpha\beta}$, we have $\overline{\beta\alpha} = -\overline{\alpha\beta}$. This is the right-handed system. Place the thumb of the right hand perpendicular to the outstretched palm, and consider the base of the thumb as the centre of rotation; then the axis of the rotation from the forefinger to the small finger is given by the thumb however the hand be placed.

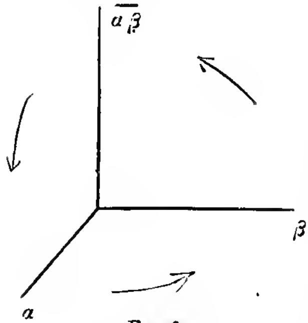

FIG. 6.

The axis of $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$ is evidently $\overline{\alpha\beta}$; let then

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = a \cos \alpha\beta + b \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}},$$

where a and b are coefficients to be determined. First, let α and β coincide; then $\alpha^{\frac{\pi}{2}}\alpha^{\frac{\pi}{2}} = \alpha^{\pi} = -1$; therefore a is -1.

Next let α and β be at right angles. The three axes α, β, $\overline{\alpha\beta}$ are now mutually rectangular, and the diagram (Fig. 6) shows the directions of positive rotation round the three axes. For if the thumb be successively held along the directions of α, β, and $\overline{\alpha\beta}$, the successive directions of rotation from the forefinger to the small finger will be given by the respective arrows. But $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$ means a quadrant round α followed by a quadrant round β, and in the particular case considered (where α and β are at right angles) it is evident that the result is a quadrant round the opposite of $\overline{\alpha\beta}$; therefore b is -1.

Hence

$$\alpha^A\beta^B = \cos A \cos B - \sin A \sin B \cos \alpha\beta$$
$$+\cos A \sin B \cdot \beta^{\frac{\pi}{2}} + \cos B \sin A \cdot \alpha^{\frac{\pi}{2}} - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}$$
$$= \cos A \cos B - \sin A \sin B \cos \alpha\beta$$
$$+ \{\cos A \sin B \cdot \beta + \cos B \sin A \cdot \alpha - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}\}^{\frac{\pi}{2}}$$

Now $\alpha^A\beta^B$ denoting the angle of the great circle between the extreme points $\cos(\alpha^A\beta^B) = \cos A \cos B - \sin A \sin B \cos \alpha\beta$ expresses the fundamental theorem of spherical trigonometry (p. 1); while

$$\text{Sin}\, \alpha^A\beta^B = \cos A \sin B \cdot \beta + \cos B \sin A \cdot \alpha - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}$$

expresses the generalization for the sine. For the square of the above quantity is

$$\cos^2 A \sin^2 B + \cos^2 B \sin^2 A + \sin^2 A \sin^2 B \sin^2 \alpha\beta$$
$$+ 2 \cos A \cos B \sin A \sin B \cos \alpha\beta,$$

and the square of the cosine is

$$\cos^2 A \cos^2 B + \sin^2 A \sin^2 B \cos^2 \alpha\beta - 2 \cos A \cos B \sin A \sin B \cos \alpha\beta,$$

and the sum of these is 1. Also that the direction of this directed sine is that of the axis to the great circle passing through the extreme points may be tested by actual construction, or by trial of special cases.

By supposing β identical with α we get the theorem for the plane, namely,

$$\alpha^{A+B} = \cos A \cos B - \sin A \sin B$$
$$+ \{\cos A \sin B + \cos B \sin A\} \cdot \alpha^{\frac{\pi}{2}}.$$

The generalization of the theorem for the difference of two angles is

$$\alpha^A\beta^{-B} = \cos A \cos B + \sin A \sin B \cos \alpha\beta$$
$$+\{-\cos A \sin B \cdot \beta + \cos B \sin A \cdot \alpha + \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}\}^{\frac{\pi}{2}},$$

which is obtained from the former by changing the sign of each term in which sin B occurs.

GENERALIZATION OF DE MOIVRE'S THEOREM.

Product of three angles in space.

Let α^A, β^B, γ^C be any three angles in space, having a common apex O (Fig. 7); it is required to find their product when taken in the order of enumeration. We first find the product of α^A and β^B, which is represented by the arc PQ; and as PQ and RT will not in general intersect in Q, PQ must be shifted along to SR; the ST, which is the product of SR and RT, represents the product of the three angles in the specified order. By assuming the distributive law, we get

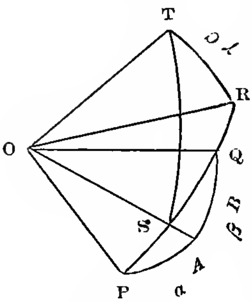

FIG. 7.

$$\alpha^A\beta^B\gamma^C = (\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}})(\cos B + \sin B \cdot \beta^{\frac{\pi}{2}})(\cos C + \sin C \cdot \gamma^{\frac{\pi}{2}})$$
$$= \cos A \cos B \cos C$$
$$+ \cos A \cos B \sin C \cdot \gamma^{\frac{\pi}{2}} + \cos A \cos C \sin A \cdot \alpha^{\frac{\pi}{2}}$$
$$+ \cos B \cos C \sin B \cdot \beta^{\frac{\pi}{2}} + \cos A \sin B \sin C \cdot \beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}$$
$$+ \cos B \sin A \sin C \cdot \alpha^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}} + \cos C \sin A \sin B \cdot \alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$$
$$+ \sin A \sin B \sin C \cdot \alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}.$$

The sixth and seventh space coefficients are not formed from the fifth by cyclical permutation; the order of the factors in the product must be retained in each of the terms; thus it is $\alpha^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}$, not $\gamma^{\frac{\pi}{2}}\alpha^{\frac{\pi}{2}}$. These double coefficients are expanded by the rule already obtained; namely,

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = -\cos \alpha\beta - \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}.$$

The last coefficient is of a new kind, and is expanded as follows :

Since $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = -\cos\alpha\beta - \sin\alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}$,

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}} = -(\cos\alpha\beta + \sin\alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}})\gamma^{\frac{\pi}{2}}$$

$$= -\cos\alpha\beta \cdot \gamma^{\frac{\pi}{2}} + \sin\alpha\beta\cos\overline{\alpha\beta}\gamma + \sin\alpha\beta\sin\overline{\alpha\beta}\gamma \cdot \overline{\overline{\alpha\beta}\gamma}^{\frac{\pi}{2}},$$

where $\cos\overline{\alpha\beta}\gamma$ denotes the cosine between the axes $\overline{\alpha\beta}$ and γ, and $\overline{\overline{\alpha\beta}\gamma}$ denotes the axis which is perpendicular to $\overline{\alpha\beta}$ and γ.

Now it may be shown* that

$$\sin\alpha\beta\sin\overline{\alpha\beta}\gamma \cdot \overline{\overline{\alpha\beta}\gamma} = \cos\alpha\gamma \cdot \beta - \cos\beta\gamma \cdot \alpha\,;$$

hence the last term of the product when expanded is

$$\sin A\sin B\sin C\{-\cos\alpha\beta \cdot \gamma^{\frac{\pi}{2}} + \cos\alpha\gamma \cdot \beta^{\frac{\pi}{2}} - \cos\beta\gamma \cdot \alpha^{\frac{\pi}{2}} + \cos\overline{\alpha\beta}\gamma\}.$$

Hence we obtain for the cosine

$$\begin{aligned}\cos\alpha^A\beta^B\gamma^C = {} & \cos A\cos B\cos C - \cos A\sin B\sin C\cos\beta\gamma \\ & - \cos B\sin A\sin C\cos\alpha\gamma - \cos C\sin A\sin B\cos\alpha\beta \\ & + \sin A\sin B\sin C\sin\alpha\beta\cos\overline{\alpha\beta}\gamma\,;\end{aligned}$$

and for the directed sine

$$\begin{aligned}\mathrm{Sin}\,\alpha^A\beta^B\gamma^C = {} & \cos A\cos B\sin C \cdot \gamma + \cos A\cos C\sin B \cdot \beta \\ & + \cos B\cos C\sin A \cdot \alpha - \cos A\sin B\sin C\sin\beta\gamma \cdot \overline{\beta\gamma} \\ & - \cos B\sin A\sin C\sin\alpha\gamma \cdot \overline{\alpha\gamma} - \cos C\sin A\sin B\sin\alpha\beta \cdot \overline{\alpha\beta} \\ & - \sin A\sin B\sin C\{\cos\alpha\beta \cdot \gamma - \cos\alpha\gamma \cdot \beta + \cos\beta\gamma \cdot \alpha\}.\end{aligned}$$

By Sin with a capital S is meant the directed sine.

Let $\alpha = \beta = \gamma$; the above formulæ then become identical with the formulæ in plane trigonometry for the cosine and sine of the sum of three arcs.

As the above theorem is true for any three angles in space, it is also true in the special case when the arcs form the sides of a spherical polygon. It has its most general meaning in the composition of the finite rotations of a rigid body.

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* Principles of the Algebra of Physics, Proceedings A. A. A. S., Vol. XL., p. 89.

Product of any number of angles in space.—Let a denote the cosine component, and $\mathbf{a}$ the sine component of an angle in space, and let a_r denote the product formed from any r cosine components, $\mathbf{a}_s$ the product formed from any s sine components; then by the distributive rule,

$$\alpha^A\beta^B\gamma^C\cdots\nu^N = a_n + \Sigma a_{n-1}\mathbf{a} + \Sigma a_{n-2}\mathbf{a}_2 + \cdots + \Sigma a_1\mathbf{a}_{n-1} + \mathbf{a}_n.$$

We have already found the value of $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$ the kind of space-coefficient which occurs in the third term, and by the rule obtained we have deduced the value of $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}$ the kind of space-coefficient which occurs in the fourth term. The value of the kind of coefficient which occurs in the fifth term is deduced from that of the fourth by another application of the same rule. Thus

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}\delta^{\frac{\pi}{2}} = \{-\cos\alpha\beta\cdot\gamma^{\frac{\pi}{2}} + \cos\alpha\gamma\cdot\beta^{\frac{\pi}{2}} - \cos\beta\gamma\cdot\alpha^{\frac{\pi}{2}} + \cos\overline{\alpha\beta\gamma}\}\delta^{\frac{\pi}{2}}$$

$$= \cos\alpha\beta\cos\gamma\delta - \cos\alpha\gamma\cos\beta\delta + \cos\beta\gamma\cos\alpha\delta$$

$$+ \cos\alpha\beta\sin\gamma\delta\cdot\overline{\gamma\delta}^{\frac{\pi}{2}} - \cos\alpha\gamma\sin\beta\delta\cdot\overline{\beta\delta}^{\frac{\pi}{2}} + \cos\beta\gamma\sin\alpha\delta\cdot\overline{\alpha\delta}^{\frac{\pi}{2}}$$

$$+ \cos\overline{\alpha\beta}\gamma\cdot\delta^{\frac{\pi}{2}}.$$

In a similar manner the space-coefficients for any subsequent terms may be developed. De Moivre's theorem is obtained from the above, by making the n axes coincident, and the n arcs equal. Then it becomes

$$\alpha^{nA} = \cos nA + \sin nA\cdot\alpha^{\frac{\pi}{2}}$$

$$= a^n + na^{n-1}\mathbf{a} + \frac{n(n-1)}{2!}a^{n-2}\mathbf{a}^2 + \cdots + na\mathbf{a}^{n-1} + \mathbf{a}^n,$$

where $a = \cos A$ and $\mathbf{a} = \sin A\cdot\alpha^{\frac{\pi}{2}}$.

PRODUCT OF TWO ANGLES IN SPACE, WHEN EXPRESSED IN TERMS OF OBLIQUE COMPONENTS.

We may equate the angle α^A to the sum of two components, the arcs of which differ by any amount greater than 0 and less than π. Let A contain r whole turns, and let A' denote the remainder; then the complete equivalence is expressed by

$$\alpha^A = \alpha^{r2\pi}\{\cos A'\cdot\alpha^0 + \sin A'\cdot\alpha^w\},$$

where the components differ by an arc w, and $\cos A'$ and $\sin A'$ are the oblique cosine and sine for the difference of arc w (Fig. 8). In the figure these are denoted for shortness by x and y; and they are connected by the relation

$$x^2 + y^2 + 2\,xy \cos w = 1.$$

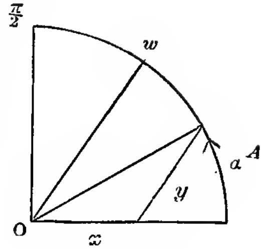

FIG. 8.

The incomplete equivalence is

$$\alpha^A = x + y \cdot \alpha^w.$$

To prove that the distributive rule still applies, namely that

$$(x + y \cdot \alpha^w)(x' + y' \cdot \beta^w) = xx + xy' \cdot \beta^w + x'y \cdot \alpha^w + yy' \cdot \alpha^w\beta^w.$$

Since $\qquad \alpha^A = x + y \cdot \alpha^w = x + y \cos w + y \sin w \cdot \alpha^{\frac{\pi}{2}}$

and $\qquad \beta^B = x' + y' \cdot \beta^w = x' + y' \cos w + y' \sin w \cdot \beta^{\frac{\pi}{2}},$

$$\alpha^A\beta^B = \{(x + y \cos w) + y \sin w \cdot \alpha^{\frac{\pi}{2}}\}\{(x' + y' \cos w) + y' \sin w \cdot \beta^{\frac{\pi}{2}}\};$$

therefore, by applying the rule for rectangular components,

$$\alpha^A\beta^B = (x + y \cos w)(x' + y' \cos w) - yy' \sin^2 w \cos \alpha\beta$$

$$+ \{(x + y \cos w) y' \sin w \cdot \beta + (x' + y' \cos w) y \sin w \cdot \alpha - yy' \sin^2 w \sin \alpha\beta \cdot \overline{\alpha\beta}\}^{\frac{\pi}{2}}$$

$$= xx' + xy' \cos w + x'y \cos w + yy'(\cos^2 w - \sin^2 w \cos \alpha\beta)$$

$$+ [xy' \sin w \cdot \beta + x'y \sin w \cdot \alpha + yy'\{\cos w \sin w(\alpha + \beta) - \sin^2 w \sin \alpha\beta \cdot \overline{\alpha\beta}\}]^{\frac{\pi}{2}}$$

$$= xx' + xy' \cdot \beta^w + x'y \cdot \alpha^w + yy' \cdot \alpha^w\beta^w.$$

To express the product angle in terms of oblique components of the same kind with that of the factor-angles.

From the above we see that

$$\alpha^A\beta^B = xx' + (xy' + yx') \cos w + yy'(\cos^2 w - \sin^2 w \cos \alpha\beta)$$

$$+ \sin w\,[xy' \cdot \beta + x'y \cdot \alpha + yy'\{\cos w(\alpha + \beta) - \sin w \sin \alpha\beta \cdot \overline{\alpha\beta}\}]^{\frac{\pi}{2}}.$$

The axis is the same whether the components are rectangular or oblique; the magnitude of the w sine is obtained by dividing the rectangular sine by $\sin w$; and the w cosine is obtained from the rectangular cosine by subtracting the magnitude of the w sine multiplied by cos w. Hence

$$\alpha^A\beta^B = xx' + (xy' + yx')\cos w + yy'(\cos^2 w - \sin^2 w \cos\alpha\beta) - Y\cos w$$
$$+ \{xy' \cdot \beta + x'y \cdot \alpha + yy'\{\cos w(\alpha + \beta) - \sin w \sin\alpha\beta \cdot \overline{\alpha\beta}\}^w,$$

where Y denotes the square root of the square of the vector

$$(xy' + yy' \cos w) \cdot \beta + (x'y + yy' \cos w) \cdot \alpha - yy' \sin w \sin\alpha\beta \cdot \overline{\alpha\beta}.$$

Suppose that β is identical with α. Then

$$\alpha^{A+B} = xx' + (xy' + yx')\cos w + yy'(\cos^2 w - \sin^2 w)$$
$$+ \sin w\{xy' + x'y + 2\,yy' \cos w\} \cdot \alpha^{\frac{\pi}{2}}$$
$$= xx' + xy' \cdot \alpha^w + x'y \cdot \alpha^w + yy' \cdot \alpha^{2w}$$
$$= xx' - yy' + \{xy' + x'y + 2\,yy' \cos w\} \cdot \alpha^w.$$

This last result for the plane agrees with the oblique trigonometry of Biehringer and Unverzagt.*

To find the product when the obliquity is different for the two factor-angles.

Let $\alpha^A = x + y \cdot \alpha^w$ and $\beta^B = x' + y' \cdot \beta^{w'}$;

then it may be shown in the same way as before that

$$\alpha^A\beta^B = xx' + xy'\cos w' + x'y\cos w + yy'(\cos w \cos w' - \sin w \sin w' \cos\alpha\beta)$$
$$+ \{xy' \sin w' \cdot \beta + x'y \sin w \cdot \alpha$$
$$+ yy'(\cos w \sin w' \cdot \beta + \cos w' \sin w \cdot \alpha - \sin w \sin w' \sin\alpha\beta \cdot \overline{\alpha\beta}\}^{\frac{\pi}{2}}$$

from which the components for either kind of oblique axes may be deduced as before.

We have also

$$\alpha^A\beta^B = xx' + xy' \cdot \beta^{w'} + x'y \cdot \alpha^w + yy' \cdot \alpha^w\beta^{w'}.$$

For the plane this becomes

$$\alpha^A\alpha^B = \alpha^{A+B} = xx' + xy' \cdot \alpha^{w'} + x'y \cdot \alpha^w + yy' \cdot \alpha^{w+w'}.$$

Let $\mathbf{a} = a\alpha^A$, $\mathbf{b} = b\alpha^B$;

then $\mathbf{ab} = ab\alpha^A\alpha^B = ab\alpha^{A+B}$.

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* Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunctionen; von Dr. Siegm. Günther, p. 359.

The product of **ab** is obtained by taking the product of the ratios, leaving the axis the same, and taking the sum of the arcs. This is the product of Plane Algebra,* and the above result shows that the distributive rule holds for such product.

GENERALIZATION OF THE EXPONENTIAL THEOREM.

We have seen that

$$\cos \alpha^A\beta^B = \cos A \cos B - \sin A \sin B \cos \alpha\beta$$

and $$(\operatorname{Sin} \alpha^A\beta^B)^{\frac{\pi}{2}} = \cos B \sin A \cdot \alpha^{\frac{\pi}{2}} + \cos A \sin B \cdot \beta^{\frac{\pi}{2}} - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}.$$

Now $$\cos A = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} +,$$

and $$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - .$$

Substitute these series for $\cos A$, $\sin A$, $\cos B$, and $\sin B$ in the above expressions, multiply out, and group the homogeneous terms together. It will be found that

$$\begin{aligned}\cos \alpha^A\beta^B = 1 &- \frac{1}{2!}\{A^2 + 2\,AB \cos \alpha\beta + B^2\}\\ &+ \frac{1}{4!}\{A^4 + 4\,A^3B \cos \alpha\beta + 6\,A^2B^2 + 4\,AB^3 \cos \alpha\beta + B^4\}\\ &- \frac{1}{6!}\{A^6 + 6\,A^5B \cos \alpha\beta + 15\,A^4B^2 + 20\,A^3B^3 \cos \alpha\beta + 15\,A^2B^4\\ &\qquad + 6\,AB^5 \cos \alpha\beta + \beta^6\}\\ &+ \text{etc.,}\end{aligned}$$

where the coefficients are those of the binomial theorem, the only difference being that $\cos \alpha\beta$ occurs in all the odd terms as a factor.

Similarly, by expanding the terms of the sine, we obtain

$$\begin{aligned}(\operatorname{Sin} \alpha^A\beta^B)^{\frac{\pi}{2}} &= A \cdot \alpha^{\frac{\pi}{2}} + B \cdot \beta^{\frac{\pi}{2}} - AB \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}\\ &- \frac{1}{3!}\{A^3 \cdot \alpha^{\frac{\pi}{2}} + 3\,A^2B \cdot \beta^{\frac{\pi}{2}} + 3\,AB^2 \cdot \alpha^{\frac{\pi}{2}} + B^3 \cdot \beta^{\frac{\pi}{2}}\}\end{aligned}$$

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* Note on Plane Algebra by the author. See Appendix, p. 28.

$$+\frac{1}{3\,!}\{AB^3 + A^3B\}\sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}$$

$$+\frac{1}{5\,!}\{A^5\cdot\alpha^{\frac{\pi}{2}} + 5\,A^4B\cdot\beta^{\frac{\pi}{2}} + 10\,A^3B^2\cdot\alpha^{\frac{\pi}{2}} + 10\,A^2B^3\cdot\beta^{\frac{\pi}{2}}$$
$$+ 5\,AB^4\cdot\alpha^{\frac{\pi}{2}} + B^5\cdot\beta^{\frac{\pi}{2}}\}$$

$$-\frac{1}{5\,!}\left\{AB^5 + \frac{5\cdot 4}{2\cdot 3}A^2B^3 + A^5B\right\}\sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}$$

— etc.

By adding the two together we get the expansion for $\alpha^A\beta^B$; namely,

$$\alpha^A\beta^B = 1 + A\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}$$
$$-\frac{1}{2\,!}\{A^2 + 2\,AB(\cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}) + B^2\}$$
$$-\frac{1}{3\,!}\{A^3\cdot\alpha^{\frac{\pi}{2}} + 3\,A^2B\cdot\beta^{\frac{\pi}{2}} + 3\,AB^2\cdot\alpha^{\frac{\pi}{2}} + B^3\cdot\beta^{\frac{\pi}{2}}\}$$
$$+\frac{1}{4\,!}\{A^4 + 4\,A^3B(\cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}) + 6\,A^2B^2$$
$$+ 4\,AB^3(\cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}) + B^4\}$$
$$+ \text{etc.}$$

Now by restoring the minus, we find that the terms on the second line can be thrown into the form

$$\frac{1}{2\,!}\{A^2\cdot\alpha^{\pi} + 2\,AB\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} + B^2\cdot\beta^{\pi}\},$$

and this is equal to

$$\frac{1}{2\,!}\{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}\}^2,$$

provided that in forming the two cross-terms the order of the terms in the binomial is followed, not any supposed order of a first and second factor.

In a similar manner the terms on the third line can be restored to

$$\frac{1}{3\,!}\{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}\}^3,$$

on the understanding that the cube is formed by preserving in each term the order of the axes as given in the binomial; that is,

$$A^3\cdot\alpha^{3\frac{\pi}{2}} + 3\,A^2B\cdot\alpha^{\pi}\beta^{\frac{\pi}{2}} + 3\,AB^2\cdot\alpha^{\frac{\pi}{2}}\beta^{\pi} + B^3\cdot\beta^{3\frac{\pi}{2}}.$$

Hence

$$\alpha^A\beta^B = 1 + (A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}) + \frac{1}{2!}\{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}\}^2$$

$$+\frac{1}{3!}\{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}\}^3 + \frac{1}{4!}\{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}\}^4 +$$

$$= e^{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}}.$$

Hence, also, $\log\alpha^A\beta^B = A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}.$

Let $B = 0;$

then $$\alpha^A = 1 + A\cdot\alpha^{\frac{\pi}{2}} + \frac{1}{2!}(A\cdot\alpha^{\frac{\pi}{2}})^2 +$$

$$= e^{A\cdot\alpha^{\frac{\pi}{2}}},$$

and $\log\alpha^A = A\cdot\alpha^{\frac{\pi}{2}}.$

The quaternion is the complex quantity in space, and is expressed by $a\alpha^A$. Hence

$$\log(a\alpha^A) = \log a + A\cdot\alpha^{\frac{\pi}{2}},$$

which is the generalization for space of a well-known result for the plane.

We also see that $\alpha^A\beta^B$ is a true generalization of the product of algebra, for the logarithm of $\alpha^A\beta^B$ is the sum of the logarithms of α^A and of β^B.

This result is different from that which is taught in Quaternions. At page 386 of his *Elements of Quaternions* Hamilton says: "In the present theory of diplanar quaternions we cannot expect to find that the sum of the logarithms of any two proposed factors shall be generally equal to the logarithm of the product; but for the simpler and earlier case of complanar quaternions, that algebraic property may be considered to exist, with due modification for multiplicity of value."

Hamilton was led to the above view by erroneously identifying a vector with a quadrantal quaternion, and both with a quadrantal index, or logarithm. We have three essentially different binomials to consider. Let $a\alpha$ and $b\beta$ be any two vectors having a common point of application; their sum is $a\alpha + b\beta$, and it means the geometrical or physical resultant, a vector of the same kind as either component. Then

$$(a\alpha + b\beta)^2 = a^2 + b^2 + 2\,ab\cos\alpha\beta,$$

for the square of any vector is the square of its magnitude. The sum of two quadrantal quaternions $a \cdot \alpha^{\frac{\pi}{2}}$ and $b \cdot \beta^{\frac{\pi}{2}}$ is

$$a \cdot \alpha^{\frac{\pi}{2}} + b \cdot \beta^{\frac{\pi}{2}} = (a\alpha + b\beta)^{\frac{\pi}{2}};$$

the square of which is

$$-(a^2 + b^2 + 2\,ab \cos \alpha\beta).$$

But the sum of two quadrantal indices or logarithms $a \cdot \alpha^{\frac{\pi}{2}}$ and $b \cdot \beta^{\frac{\pi}{2}}$ is not $(a\alpha + b\beta)^{\frac{\pi}{2}}$; and $(a\alpha^{\frac{\pi}{2}} + b\beta^{\frac{\pi}{2}})^2$ is not

$$-(a^2 + b^2 + 2\,ab \cos \alpha\beta),$$

but
$$-(a^2 + b^2 + 2\,ab \cos \alpha\beta) - 2\,ab \sin \alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}.$$

The sum of two simultaneous vectors is independent of order; hence the square does not involve the sine term, for it supposes an order. The sum of two quadrantal indices is a successive sum; hence the square involves the sine term.

FURTHER GENERALIZATION OF THE EXPONENTIAL THEOREM.

We have found that for an angle in space

$$\alpha^A = e^{A \cdot \alpha^{\frac{\pi}{2}}}$$

The occurrence of the constant $\frac{\pi}{2}$ suggests that by generalizing it we shall get a more general idea of which α^A is the $\frac{\pi}{2}$ case. Let the more general idea be denoted by α_w^A, which means that

$$\begin{aligned}
\alpha_w^A &= e^{A \cdot \alpha^w} \\
&= 1 + A \cdot \alpha^w + \frac{A^2}{2!} \cdot \alpha^{2w} + \frac{A^3}{3!} \cdot \alpha^{3w} + \\
&= 1 + A \cos w + \frac{A^2}{2!} \cos 2\,w + \frac{A^3}{3!} \cos 3\,w + \\
&\qquad + A \sin w \cdot \alpha^{\frac{\pi}{2}} + \frac{A^2}{2!} \sin 2\,w \cdot \alpha^{\frac{\pi}{2}} + \frac{A^3}{3!} \sin 3\,w \cdot \alpha^{\frac{\pi}{2}} + .
\end{aligned}$$

To prove that $\alpha_w^A = e^{A\cos w + A\sin w\,.\,\alpha^{\frac{\pi}{2}}}$.

For

$$e^{A\cos w + A\sin w\,.\,\alpha^{\frac{\pi}{2}}} = e^{A\cos w}\, e^{A\sin w\,.\,\alpha^{\frac{\pi}{2}}}$$

$$=\{1 + A\cos w + \frac{A^2}{2!}\cos^2 w + \frac{A^3}{3!}\cos^3 w + \}$$

$$\times\{1 + A\sin w\cdot\alpha^{\frac{\pi}{2}} - \frac{A^2}{2!}\sin^2 w - \frac{A^3}{3!}\sin^3 w\cdot\alpha^{\frac{\pi}{2}} + \}$$

$$= 1 + A\cos w + \frac{A^2}{2!}(\cos^2 w - \sin^2 w) + \frac{A^3}{3!}(\cos^3 w - 3\cos w\sin^2 w) +$$

$$+ A\sin w\cdot\alpha^{\frac{\pi}{2}} + \frac{A^2}{2!}2\sin w\cos w\cdot\alpha^{\frac{\pi}{2}} + \frac{A^3}{3!}(3\cos^2 w\sin w - \sin^3 w)\cdot\alpha^{\frac{\pi}{2}} +$$

$$= 1 + A\cdot\alpha^w + \frac{A^2}{2!}\cdot\alpha^{2w} + \frac{A^3}{3!}\cdot\alpha^{3w} +$$

$$= e^{A\cdot\alpha^w}.$$

Meaning of α_w^A.

Since $\alpha_w^A = e^{A\cos w} e^{A\sin w\,.\,\alpha^{\frac{\pi}{2}}}$,

therefore it is $= e^{A\cos w}\alpha^{A\sin w}$.

It involves a versor of axis α and arc $A\sin w$, and an exponential multiplier $e^{A\cos w}$. Let the arc $A\sin w$ be denoted by θ, then

$$\alpha_w^A = e^{\frac{\theta}{\tan w}}\alpha^\theta.$$

Now this is the equation to a logarithmic spiral OMP (Fig. 9) in the plane of α, OM being of unit length, and w being the constant angle between the radius-vector and the tangent.

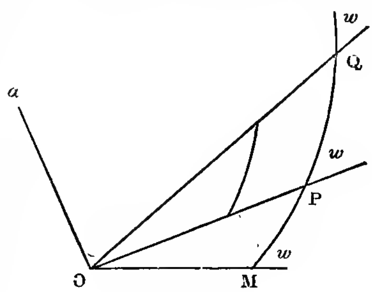

FIG. 9.

In the case of the circle $w = \frac{\pi}{2}$

and $$\alpha_{\frac{\pi}{2}}^A = e^{\frac{\theta}{\tan\frac{\pi}{2}}}\alpha^\theta = \alpha^A.$$

As $\alpha\alpha_w^A$ involves one element more than the quaternion $\alpha\alpha^A$, it may be called a *quinternion*.

To find the product of two spiral versors α_w^A and β_w^B.

Since $$\alpha_w^A = e^{A\cos w} e^{A \sin w \,.\, \alpha^{\frac{\pi}{2}}}$$

and $$\beta_w^B = e^{B\cos w} e^{B \sin w \,.\, \beta^{\frac{\pi}{2}}},$$

therefore $$\alpha_w^A \beta_w^B = e^{(A+B)\cos w} e^{A \sin w \,.\, \alpha^{\frac{\pi}{2}} + B \sin w \,.\, \beta^{\frac{\pi}{2}}}$$

$$= e^{(A+B)\cos w} e^{\sin w (A \,.\, \alpha^{\frac{\pi}{2}} + B \,.\, \beta^{\frac{\pi}{2}})}$$

$$= e^{(A+B)\cos w} \{1 + \sin w (A \cdot \alpha^{\frac{\pi}{2}} + B \cdot \beta^{\frac{\pi}{2}}) + \frac{\sin^2 w}{2\,!}(A \cdot \alpha^{\frac{\pi}{2}} + B \cdot \beta^{\frac{\pi}{2}})^2 + \}.$$

Thus the ratio of the product is the product of the ratios, and the angle of the product is the product of the angles.

Suppose β to be identical with α, then (Fig. 9)

$$\alpha_w^A \alpha_w^B = e^{(A+B)\cos w} e^{(A+B)\sin w \,.\, \alpha^{\frac{\pi}{2}}} = \alpha_w^{A+B}.$$

This is the addition theorem for the logarithmic spiral.

To find the product of two quinternions of the most general kind.

Let $\mathbf{a} = a\alpha_{w_1}^A$ and $\mathbf{b} = b\beta_{w_2}^B$ be any two quinternions. Then

$$\mathbf{ab} = a e^{A \cos w_1} \alpha^{A \sin w_1} b e^{B \cos w_2} \beta^{B \sin w_2}$$

$$= ab e^{A \cos w_1 + B \cos w_2} e^{A \sin w_1 \,.\, \alpha^{\frac{\pi}{2}} + B \sin w_2 \,.\, \beta^{\frac{\pi}{2}}}.$$

The ratio of the product is

$$ab e^{A \cos w_1 + B \cos w_2},$$

and the angle of the product is

$$\alpha^{A \sin w_1} \beta^{B \sin w_2}.$$

Also $$\mathbf{ab} = ab e^{A \,.\, \alpha^{w_1} + B \,.\, \beta^{w_2}}$$

$$= ab \left\{ 1 + A \cdot \alpha^{w_1} + B \cdot \beta^{w_2} + \frac{1}{2\,!}(A \cdot \alpha^{w_1} + B \cdot \beta^{w_2})^2 + \frac{1}{3\,!}(A \cdot \alpha^{w_1} + B \cdot \beta^{w_2})^3 + \right\}.$$

The square term is expanded as follows

$$(A \cdot \alpha^{w_1} + B \cdot \beta^{w_2})^2 = A^2 \cdot \alpha^{2w_1} + 2\,AB \cdot \alpha^{w_1}\beta^{w_2} + B^2 \cdot \beta^{2w_2},$$

and the cube term as follows

$$(A \cdot \alpha^{w_1} + B \cdot \beta^{w_2})^3 = A^3 \cdot \alpha^{3w_1} + 3A^2B \cdot \alpha^{2w_1}\beta^{w_2} + 3AB^2 \cdot \alpha^{w_1}\beta^{2w_2} + B^3 \cdot \beta^{3w_2},$$

and so on.

GENERALIZATION OF THE BINOMIAL THEOREM.

By the preceding investigation (p 14) we arrived at the conclusion that for the sum of any two quadrantal logarithms the nth power is given by the formula

$$\{A\cdot\alpha^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}\}^n=A^n\cdot\alpha^{n\frac{\pi}{2}}+nA^{n-1}B\cdot\alpha^{(n-1)\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$$
$$+\frac{n(n-1)}{1\cdot 2}A^{n-2}B^2\cdot\alpha^{(n-2)\frac{\pi}{2}}\beta^{\pi}+\text{etc.}$$

Doubtless this theorem is true also when n is negative or fractional.

But we obtain a still more general form, by taking the sum of two logarithms of the most general kind $A\alpha^{w_1}$ and $B\beta^{w_2}$. Let a denote $A\alpha^{w_1}$ and b denote $B\beta^{w_2}$, then

$$(\mathsf{a}+\mathsf{b})^n=\mathsf{a}^n+n\mathsf{a}^{n-1}\mathsf{b}+\frac{n(n-1)}{1\cdot 2}\mathsf{a}^{n-2}\mathsf{b}^2+,$$

the general term being

$$\frac{n\,!}{r\,!\,(n-r)\,!}\mathsf{a}^{n-r}\mathsf{b}^r;$$

that is,
$$\frac{n\,!}{r\,!\,(n-r)\,!}a^{n-r}b^r\alpha^{(n-r)w_1}\beta^{rw_2}.$$

The binomial theorem of algebra applies to the sum of two algebraic terms, that is, terms of the nature of a cosine component; the binomial theorem of trigonometry applies to the case where one term is a cosine, the other a sine component; the former of the two theorems above applies to the case where both terms are of the nature of the sine; while the latter theorem includes all the others as particular cases.

GENERALIZATION OF THE MULTINOMIAL THEOREM.

In the expressions obtained (p. 9) for $\cos\alpha^A\beta^B\gamma^C$ and $(\sin\alpha^A\beta^B\gamma^C)^{\frac{\pi}{2}}$ insert the series for $\cos A$, $\sin A$, etc., and multiply out, collecting the homogeneous terms. The sum of the terms of the first order is

$$A\cdot\alpha^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}.$$

The sum of the terms of the second order becomes, when the minus is restored,

$$\frac{1}{2!}\{A^2\cdot\alpha^\pi+B^2\cdot\beta^\pi+C^2\cdot\gamma^\pi+2(AB\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}+AC\cdot\alpha^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}+BC\cdot\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}})\}$$

$$=\frac{1}{2!}\{\Sigma A^2\cdot\alpha^\pi+2\Sigma AB\cdot\alpha^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}\}.$$

The order of the axes in the products is the order of the axes in the trinomial; that is, α is before β and before γ, and β is before γ. Hence the terms form

$$\frac{1}{2!}(A\cdot\alpha^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}})^2.$$

The sum of the terms of the third order is $\frac{1}{3!}$ of

$$\begin{aligned}&A^3\cdot\alpha^{3\frac{\pi}{2}}+B^3\cdot\beta^{3\frac{\pi}{2}}+C^3\cdot\gamma^{3\frac{\pi}{2}}\\&+3\{A^2B\cdot\alpha^\pi\beta^{\frac{\pi}{2}}+A^2C\cdot\alpha^\pi\gamma^{\frac{\pi}{2}}+B^2C\cdot\beta^\pi\gamma^{\frac{\pi}{2}}\}\\&+3\{AB^2\cdot\alpha^{\frac{\pi}{2}}\beta^\pi+AC^2\cdot\alpha^{\frac{\pi}{2}}\gamma^\pi+BC^2\cdot\beta^{\frac{\pi}{2}}\gamma^\pi\}\\&+6\,ABC\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}}.\end{aligned}$$

$$=\Sigma A^3\cdot\alpha^{3\frac{\pi}{2}}+3\,\Sigma A^2B\cdot\alpha^\pi\beta^{\frac{\pi}{2}}+3\,\Sigma AB^2\cdot\alpha^{\frac{\pi}{2}}\beta^\pi+6\,ABC\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}};$$

therefore the sum of these terms is

$$\frac{1}{3!}\{A\cdot\alpha^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}\}^3.$$

As the same is true for the nth term, we have

$$\alpha^A\beta^B\gamma^C=e^{A\cdot\alpha^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}}.$$

Thus the multinomial theorem of algebra may be applied to the sum of a number of quadrantal indices, provided that in all the terms the order of the axes is preserved; that is, is made to follow the order of the indices in the multinomial.

The most general form is where we have a multinomial in which the indices may have any angle. Let a, b, c be three such indices, then

$$(\mathsf{a}+\mathsf{b}+\mathsf{c})^n=n!\,\Sigma\frac{\mathsf{a}^r\mathsf{b}^s\mathsf{c}^t}{r!\,s!\,t!},\text{ where } r+s+t=n.$$

An application of the multinomial theorem.

We may apply the multinomial theorem to develop the product $\gamma^{-C}\beta^{B}\gamma^{C}$.

By the exponential theorem

$$\begin{aligned}\gamma^{-C}\beta^{B}\gamma^{C} &= e^{-C\cdot\gamma^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}}\\ &=1+\{-C\cdot\gamma^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}\}\\ &\quad+\frac{1}{2!}\{-C\cdot\gamma^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}\}^{2}\\ &\quad+\frac{1}{3!}\{-C\cdot\gamma^{\frac{\pi}{2}}+B\cdot\beta^{\frac{\pi}{2}}+C\cdot\gamma^{\frac{\pi}{2}}\}^{3}\\ &\quad+\text{etc.}\end{aligned}$$

Now the first power of the trinomial reduces to $B\cdot\beta$,

the square of the trinomial to $\frac{1}{2!}\{-B^{2}+4BC\sin\gamma\beta\cdot\overline{\gamma\beta}\}$,

the cube to $\frac{1}{3!}\{-B^{3}\cdot\beta-12BC^{2}\cdot\beta+12C^{2}B\cos\beta\gamma\cdot\gamma\}$,

etc.

Hence $\gamma^{-C}\beta^{B}\gamma^{C}=1-\frac{1}{2!}B^{2}+\frac{1}{4!}B^{4}-$

$$+\left\{\begin{array}{l}B\cdot\beta-\frac{1}{3!}B^{3}\cdot\beta+\frac{1}{5!}B^{5}\cdot\beta-\\ +2BC\sin\gamma\beta\cdot\overline{\gamma\beta}-2BC^{2}\cdot\beta+2BC^{2}\cos\beta\gamma\cdot\gamma+\end{array}\right\}^{\frac{\pi}{2}}.$$

It is shown in Professor Tait's *Treatise on Quaternions* that $\gamma^{-C}\beta^{B}\gamma^{C}$ turns the axis β round γ by an amount $2\,C$. The above development shows that the amount of the angle is unchanged, for the cosine is unchanged; while the sine term gives the development for the new axis in terms of B, C, β, and γ.

GENERALIZATION OF THE LOGARITHMIC THEOREM.

It follows from the above principles that the logarithmic theorem

$$\log(1+x)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+,\text{ etc.},$$

x being less than 1, is true when instead of x we insert the general quaternion $\mathbf{x}=x\cdot\xi^{X}$. Thus,

$$\log(1+\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x}^2}{2} + \frac{\mathbf{x}^3}{3} - \frac{\mathbf{x}^4}{4} +$$

$$= x \cdot \xi^X - \frac{x^2}{2} \cdot \xi^{2X} + \frac{x^3}{3} \cdot \xi^{3x} -, \text{ etc.},$$

$$= x \cos X - \frac{x^2}{2} \cos 2X + \frac{x^3}{3} \cos 3X -$$

$$+ \{x \sin X - \frac{x^2}{2} \sin 2X + \frac{x^3}{3} \sin 3X - \} \cdot \xi^{\frac{\pi}{2}}$$

$$= x(\cos X + \sin X \cdot \xi^{\frac{\pi}{2}}) - \frac{x^2}{2} (\cos X + \sin X \cdot \xi^{\frac{\pi}{2}})^2 +, \text{ etc.}$$

It is true even more generally, namely, when we insert the quinternion $\mathbf{x} = x \cdot \xi_{w}^{X}$, provided $xe^{X \cos w}$ is less than unity.

Application to prove Gregory's series.

We have $\quad \log(\alpha^A) = \log(\cos A + \sin A \cdot \alpha^{\frac{\pi}{2}})$.

Suppose that $\sin A$ is not greater than $\cos A$, then

$$\log \alpha^A = \log \cos A + \log(1 + \tan A \cdot \alpha^{\frac{\pi}{2}})$$

$$= \log \cos A + \tan A \cdot \alpha^{\frac{\pi}{2}} - \frac{\tan^2 A}{2} \cdot \alpha^{\pi} + \frac{\tan^3 A}{3} \cdot \alpha^{3\frac{\pi}{2}} -$$

$$= \log \cos A + \frac{\tan^2 A}{2} - \frac{\tan^2 A}{4} +$$

$$+ \{\tan A - \frac{\tan^3 A}{3} + \frac{\tan^5 A}{5} - \} \cdot \alpha^{\frac{\pi}{2}}.$$

But $\quad \log(\alpha^A) = A \cdot \alpha^{\frac{\pi}{2}}$,

therefore $\quad -\log \cos A = \dfrac{\tan^2 A}{2} - \dfrac{\tan^4 A}{4} +,$

and $\quad A = \tan A - \dfrac{\tan^3 A}{3} + \dfrac{\tan^5 A}{5} -.$

Thus we obtain not only Gregory's series for the arc in terms of the tangent of the arc, but also a complementary series for the logarithm of the cosine of the arc.

Application to find log $(\log(\alpha^A\beta^B))$.

Suppose that B is not greater than A.

Since $$\log(\alpha^A\beta^B) = A \cdot \alpha^{\frac{\pi}{2}} + B \cdot \beta^{\frac{\pi}{2}}$$
$$= A \cdot \alpha^{\frac{\pi}{2}}\left\{1 + \frac{B}{A} \cdot \alpha^{-\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\right\},$$

therefore $$\log\log(\alpha^A\beta^B) = \log\left(A \cdot \alpha^{\frac{\pi}{2}}\right) + \log\left\{1 + \frac{B}{A} \cdot \alpha^{-\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\right\}.$$

Now $$\log(A \cdot \alpha^{\frac{\pi}{2}}) = \log A + \frac{\pi}{2} \cdot \alpha^{\frac{\pi}{2}}$$

and $$\log\left(1 + \frac{B}{A} \cdot \alpha^{-\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\right) = \frac{B}{A} \cdot \alpha^{-\frac{\pi}{2}}\beta^{\frac{\pi}{2}} - \frac{1}{2}\frac{B^2}{A^2} \cdot (\alpha^{-\frac{\pi}{2}}\beta^{\frac{\pi}{2}})^2 + \text{etc.},$$

where $$\alpha^{-\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = -\cos\alpha\beta + \sin\alpha\beta \cdot \overline{\alpha\beta}^{\frac{\pi}{2}}.$$

Let this angle be denoted by γ^C,

then $$\log\log(\alpha^A\beta^B) = \log A + \frac{\pi}{2} \cdot \alpha^{\frac{\pi}{2}}$$
$$+ \frac{B}{A} \cdot \gamma^C - \frac{1}{2}\frac{B^2}{A^2} \cdot \gamma^{2C} + \frac{1}{3}\frac{B^3}{A^3} \cdot \gamma^{3C} - .$$

It is to be observed that $(\alpha^A\beta^B)^n$ is not equal to $\alpha^{nA}\beta^{nB}$ unless β is identical with α. Twice the angle $\alpha^A\beta^B$ is not equal to the angle $\alpha^{2A}\beta^{2B}$.

GENERALIZATION OF HYPERBOLIC TRIGONOMETRY.

The fundamental theorem of hyperbolic trigonometry is

$$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$$

and $$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B,$$

where A now denotes twice the area of the hyperbolic sector, not the length of the bounding arc.

Let OM (Fig. 10) be of unit length, and OX and XP the projections of OP on the principal diameter OM and perpendicular to that diameter. Then OX represents $\cosh A$ and XP represents $\sinh A$. But $\cosh A$ is a ratio, namely, the ratio of the line OX to the line OM; and $\sinh A$ is a ratio, namely, that of the line XP to the line OM. In the case of the sector B starting from the diameter OP, draw QV parallel to the tangent at P; then OV/OP and VQ/OP have the same magnitude as the rectangular projections of the radius-vector, obtained when the sector is shifted without change of area to start from the principal diameter.

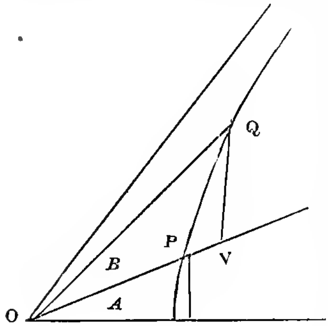

FIG. 10.

Let hyp α^A denote the hyperbolic sector or versor determined by α, the axis of the plane, and A twice the area enclosed. Then as in the case of the circular versor we have the equivalence, which in this case is complete,

$$\text{hyp}\, \alpha^A = \cosh A + \sinh A \cdot \alpha^{\frac{\pi}{2}}.$$

Here we equate the hyperbolic versor to the sum of two quaternions differing by a right angle.

To find the product of two hyperbolic versors.

Let one hyperbolic versor be

$$\text{hyp}\, \alpha^A = \cosh A + \sinh A \cdot \alpha^{\frac{\pi}{2}},$$

and the other

$$\text{hyp}\, \beta^B = \cosh B + \sinh B \cdot \beta^{\frac{\pi}{2}};$$

then since the distributive rule holds good,

$$\text{hyp}\, \alpha^A \,\text{hyp}\, \beta^B = \cosh A \cosh B + \cosh A \sinh B \cdot \beta^{\frac{\pi}{2}}$$
$$+ \cosh B \sinh A \cdot \alpha^{\frac{\pi}{2}} + \sinh A \sinh B \cdot \alpha^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}}.$$

The meaning of the first three terms is known; it remains to find the meaning of $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}$. As the fundamental theorem in plane hyperbolic trigonometry differs from that for plane circular trigonometry in the sign of the plane component of the fourth term, we form the hypothesis that for the equilateral hyperbola

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = \cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}.$$

This would give

$$\cosh\alpha^A\beta^B = \cosh A\cosh B + \sinh A\sinh B\cos\alpha\beta,$$

and $$\sinh\alpha^A\beta^B = \cosh A\sinh B\cdot\beta + \cosh B\sinh A\cdot\alpha$$
$$+ \sinh A\sinh B\sin\alpha\beta\cdot\overline{\alpha\beta}.$$

If we test this expression for $\sinh\alpha^A\beta^B$ by the relation

$$\sinh^2\alpha^A\beta^B = 1 + \cosh^2\alpha^A\beta^B,$$

we find that the relation is not satisfied. But when $\sqrt{-1}$ is introduced as a coefficient of $\sin\alpha\beta$, the relation is satisfied. Hence the fundamental principle in extending hyperbolic trigonometry to space is

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = \cos\alpha\beta + \sqrt{-1}\sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}.$$

As a special case we see $\alpha^\pi = 1$.

Hyperbolic exponentials.

$$\begin{aligned}\operatorname{hyp}\alpha^A &= \operatorname{hyp} e^{A\alpha^{\frac{\pi}{2}}}\\ &= 1 + A\cdot\alpha^{\frac{\pi}{2}} + \frac{A^2}{2!}\cdot\alpha^\pi + \frac{A^3}{3!}\cdot\alpha^{3\frac{\pi}{2}} + \\ &= 1 + \frac{A^2}{2!} + \frac{A^4}{4!} + \\ &+ \left\{A + \frac{A^3}{3!} + \frac{A^5}{5!} + \right\}\cdot\alpha^{\frac{\pi}{2}}\end{aligned}$$

since $\alpha^\pi = 1$.

Also, $$\operatorname{hyp}\alpha^A\operatorname{hyp}\beta^B = e^{A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}}$$
$$= 1 + (A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}) + \frac{1}{2!}(A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}})^2 +,$$

where the terms are expanded as before, only instead of

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = -(\cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}})$$

we have $$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = \cos\alpha\beta + \sqrt{-1}\sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}.$$

We deduce that for hyperbolic versors

$$\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\gamma^{\frac{\pi}{2}} = (\cos\alpha\beta + \sqrt{-1}\sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}})\gamma^{\frac{\pi}{2}}$$
$$= \cos\,\alpha\beta\cdot\gamma^{\frac{\pi}{2}} + \sqrt{-1}\sin\,\alpha\beta\cos\overline{\alpha\beta}\gamma - \sin\,\alpha\beta\sin\overline{\alpha\beta}\gamma\cdot\overline{\overline{\alpha\beta}\gamma}^{\frac{\pi}{2}}$$
$$= \sqrt{-1}\sin\alpha\beta\cos\overline{\alpha\beta}\gamma + \{\cos\alpha\beta\cdot\gamma + \cos\beta\gamma\cdot\alpha - \cos\gamma\alpha\cdot\beta\}^{\frac{\pi}{2}}.$$

Hence we have the three fundamental principles :

first, for vectors, $\alpha\beta = \cos\alpha\beta + \sin\alpha\beta\cdot\overline{\alpha\beta}$;

second, for circular versors, $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = -\cos\alpha\beta - \sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}$;

third, for hyperbolic versors, $\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} = \cos\alpha\beta + \sqrt{-1}\sin\alpha\beta\cdot\overline{\alpha\beta}^{\frac{\pi}{2}}$.

GENERALIZATION OF DIFFERENTIATION.

To differentiate a circular versor with respect to a scalar variable such as time.

If we take the incomplete equivalence

$$\alpha^A = \cos A + \sin A\cdot\alpha^{\frac{\pi}{2}},$$

then $$\mathbf{d}(\alpha^A) = dA(-\sin A + \cos A\cdot\alpha^{\frac{\pi}{2}}) + \sin A\cdot\mathbf{d}\alpha^{\frac{\pi}{2}}$$
$$= dA\alpha^{A+\frac{\pi}{2}} + \sin A d\alpha\cdot\bar{\alpha}^{\frac{\pi}{2}},$$

where $\bar{\alpha}$ denotes an axis perpendicular to α.

It is worthy of remark that the cosine term is differentiated with respect to A only; and is treated as independent of α.

When α^A denotes an angular velocity, A is infinitely small, and from the above we get the angular acceleration

$$\frac{\mathbf{d}\alpha^A}{dt} = \left\{\frac{dA}{dt}\cdot\alpha + A\frac{d\alpha}{dt}\cdot\bar{\alpha}\right\}^{\frac{\pi}{2}};$$

that is, an angle whose cosine is 1, and whose directed sine is the iufinitely small quantity

$$\frac{dA}{dt}\cdot\alpha + A\frac{d\alpha}{dt}\cdot\bar{\alpha}.$$

The former term expresses the change of speed, the latter the change of axis.

The differential of a quaternion involves the additional term

$$da\cdot\alpha^A.$$

To find the differential of a product of angles in space.

Since

$$\alpha^A\beta^B = \cos A\cos B + \cos A\sin B\cdot\beta^{\frac{\pi}{2}} + \cos B\sin A\cdot\alpha^{\frac{\pi}{2}} + \sin A\sin B\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}},$$

$$\begin{aligned}
\mathrm{d}(\alpha^A\beta^B) &= dA\{-\sin A\cos B - \sin A\sin B\cdot\beta^{\frac{\pi}{2}} + \cos B\cos A\cdot\alpha^{\frac{\pi}{2}} + \cos A\sin B\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\},\\
&+ dB\{-\cos A\sin B + \cos A\cos B\cdot\beta^{\frac{\pi}{2}} - \sin B\sin A\cdot\alpha^{\frac{\pi}{2}} + \sin A\cos \mathrm{B}\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\}\\
&+ d\alpha\{\cos B\sin A\cdot\overline{\alpha}^{\frac{\pi}{2}} + \sin A\sin B\cdot\overline{\alpha}^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\},\\
&+ d\beta\{\cos A\sin B\cdot\overline{\beta}^{\frac{\pi}{2}} + \sin A\sin B\cdot\alpha^{\frac{\pi}{2}}\overline{\beta}^{\frac{\pi}{2}}\},\\
&= dA\alpha^{A+\frac{\pi}{2}}\beta^B + dB\alpha^A\beta^{B+\frac{\pi}{2}},\\
&+ d\alpha\{\cos \mathrm{B}\sin A\cdot\overline{\alpha}^{\frac{\pi}{2}} + \sin A\sin B\cdot\overline{\alpha}^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}\},\\
&+ d\beta\{\cos A\sin B\cdot\overline{\beta}^{\frac{\pi}{2}} + \sin A\sin B\cdot\alpha^{\frac{\pi}{2}}\overline{\beta}^{\frac{\pi}{2}}\},\\
&= \left.\begin{matrix} -(\sin A\cos B + \cos A\sin B\cos\alpha\beta)dA \\ -(\cos A\sin B + \sin A\cos B\cos\alpha\beta)dB \\ -\sin A\sin B\{\cos(\mathrm{d}\alpha)\beta + \cos\alpha(\mathrm{d}\beta)\}\end{matrix}\right\}\\
&\left.\begin{matrix} +(-\sin A\sin B\,dA + \cos A\cos B\,dB)\cdot\beta \\ +(-\sin B\sin A\,dB + \cos B\cos A\,dA)\cdot\alpha \\ -(\cos A\sin B\,dA + \sin A\cos B\,dB)\sin\alpha\beta\cdot\overline{\alpha\beta} \\ +\cos A\sin B\cdot\mathrm{d}\beta + \cos B\sin A\cdot\mathrm{d}\alpha \\ -\sin A\sin B\{\sin(\mathrm{d}\alpha)\beta + \sin\alpha(\mathrm{d}\beta)\}\end{matrix}\right\}^{\frac{\pi}{2}}
\end{aligned}$$

We obtain successive approximations by differentiating the terms of the series

$$1 + (A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}}) + \frac{1}{2\,!}(A\cdot\alpha^{\frac{\pi}{2}} + B\cdot\beta^{\frac{\pi}{2}})^2 + .$$

Thus the first approximation is:

$$\mathrm{d}(\alpha^A\beta^B) = \{dA\cdot\alpha + dB\cdot\beta + A\cdot\mathrm{d}\alpha + B\cdot\mathrm{d}\beta\}^{\frac{\pi}{2}}.$$

The second approximation adds to the above

$$-AdA - BdB + (AdB + BdA)\cdot\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}} + AB\,\mathrm{d}(\alpha^{\frac{\pi}{2}}\beta^{\frac{\pi}{2}}).$$

To find the differential of a power of a quaternion.

Let
$$\mathbf{a}^n = a^n \alpha^{nA},$$
then
$$\mathbf{d}(\mathbf{a}^n) = na^{n-1}\alpha^{nA} + a^n n dA \cdot \alpha^{nA+\frac{\pi}{2}} + a^n \sin nA \, (\mathbf{d}\alpha)^{\frac{\pi}{2}}.$$

Let A be infinitely small, then
$$\mathbf{d}(\mathbf{a}^n) = na^n \left\{ \frac{1}{a}\alpha^{nA} + dA \cdot \alpha^{nA+\frac{\pi}{2}} + A d\alpha \cdot \overline{\alpha}^{\frac{\pi}{2}} \right\}.$$

To find the differential of a spiral versor.

$$\begin{aligned}
\mathbf{d}(\alpha_w^A) &= \mathbf{d}(e^{A\cos w}\alpha^{A\sin w}) \\
&= e^{A\cos w}\alpha^{A\sin w}(dA\cos w - A\sin w\, dw) \\
&\quad + e^{A\cos w}\alpha^{A\sin w+\frac{\pi}{2}}(dA\sin w + A\cos w\, dw) \\
&\quad + e^{A\cos w}\sin(A\sin w)\, d\alpha \cdot \overline{\alpha}^{\frac{\pi}{2}}. \\
&= e^{A\cos w}\alpha^{A\sin w}(\cos w + \sin w \cdot \alpha^{\frac{\pi}{2}})\, dA \\
&\quad + e^{A\cos w}\alpha^{A\sin w}(-\sin w + \cos w \cdot \alpha^{\frac{\pi}{2}})\, A dw \\
&\quad + e^{A\cos w}\sin(A\sin w)\, d\alpha \cdot \overline{\alpha}^{\frac{\pi}{2}} \\
&= e^{A\cos w}\alpha^{A\sin w+w}\, dA \\
&\quad + e^{A\cos w}\alpha^{A\sin w+w+\frac{\pi}{2}}\, A dw \\
&\quad + e^{A\cos w}\sin(A\sin w)\, d\alpha \cdot \overline{\alpha}^{\frac{\pi}{2}}.
\end{aligned}$$

APPENDIX.

NOTE ON PLANE ALGEBRA.

From the Proceedings of the Royal Society of Edinburgh, 1883, *p.* 184.

By Plane Algebra I mean what De Morgan called Double Algebra. While ordinary algebra deals with quantities which are represented on a straight line, and Quaternions with quantities which are represented in space, Double Algebra deals with those which are represented on a plane. The object of this paper is to show some applications of this intermediate method.

The quantities considered are conveniently denoted by small Roman letters, leaving their Tensor component to be denoted by the corresponding Italic letter, and the Versor component by the corresponding Greek letter. Thus a denotes a line of length a and angle α; b a line of length b, and angle β. Quantities of this kind are related to those of ordinary algebra as genus and species, and the laws of operation for the former are very easily generalized from those for the latter.

Expansions can be obtained by altering the order of the operations performed; for example, first by applying the Binomial Theorem, and then resolving; and second, by resolving and then applying the Binomial Theorem.

For example —

$$\frac{1}{\mathsf{a}-\mathsf{b}}=\frac{1}{\mathsf{a}}\left(1-\frac{\mathsf{b}}{\mathsf{a}}\right)^{-1}=\frac{1}{\mathsf{a}}+\frac{\mathsf{b}}{\mathsf{a}^2}+\frac{\mathsf{b}^2}{\mathsf{a}^3}+$$

$$=\frac{1}{a}\cos(-\alpha)+\frac{b}{a^2}\cos(\beta-2\alpha)+\frac{b^2}{a^3}\cos(2\beta-3\alpha)+$$

$$+i\left\{\frac{1}{a}\sin(-\alpha)+\frac{b}{a^2}\sin(\beta-2\alpha)+\frac{b^2}{a^3}\sin(2\beta-3\alpha)+\right\}.$$

Again,

$$\frac{1}{\mathsf{a}-\mathsf{b}}=\frac{1}{a\cos\alpha-b\cos\beta+i(a\sin\alpha-b\sin\beta)}$$

$$=\frac{1}{a\cos\alpha-b\cos\beta}\left\{1-i\frac{a\sin\alpha-b\sin\beta}{a\cos\alpha-b\cos\beta}\right.$$

$$\left.+i^2\left(\frac{a\sin\alpha-b\sin\beta}{a\cos\alpha-b\cos\beta}\right)^2-i^3\left(\frac{a\sin\alpha-b\sin\beta}{a\cos\alpha-b\cos\beta}\right)^3+\right\}.$$

Hence, by equating the components along the initial axis,

$$\frac{1}{a\cos\alpha-b\cos\beta}\left\{1-\left(\frac{a\sin\alpha-b\sin\beta}{a\cos\alpha-b\cos\beta}\right)^2+\left(\frac{a\sin\alpha-b\cos\beta}{a\cos\alpha-b\cos\beta}\right)^4-\right\}$$

$$=\frac{1}{a}\cos\alpha+\frac{b}{a^2}\cos(2\alpha-\beta)+\frac{b^2}{a^3}\cos(3\alpha-2\beta)+.$$

Another identity is obtained by eqnating the components along the perpendicular axis.

By treating $(1+\mathbf{a})^{\frac{1}{2}}$ in a similar manner we get

$$1+\frac{1}{2}a\cos\alpha-\frac{1}{2\cdot 4}a^2\cos 2\alpha+\frac{1\cdot 3}{2\cdot 4\cdot 6}a^3\cos 3\alpha-$$

$$=(1+a\cos\alpha)^{\frac{1}{2}}\left\{1+\frac{1}{2\cdot 4}\left(\frac{a\sin\alpha}{1+\cos\alpha}\right)^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\left(\frac{a\sin\alpha}{1+\cos\alpha}\right)^4+\right\},$$

and

$$\frac{1}{2}a\sin\alpha-\frac{1}{2\cdot 4}a^2\sin 2\alpha+\frac{1\cdot 3}{2\cdot 4\cdot 6}a^3\sin 3\alpha-$$

$$=(1+a\cos\alpha)^{\frac{1}{2}}\left\{\frac{1}{2}\frac{a\sin\alpha}{1+a\cos\alpha}-\frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{a\sin\alpha}{1+a\cos\alpha}\right)^3+\right\}.$$

An expansion for $\log\{a^2+b^2+2ab\cos\theta\}^{\frac{1}{2}}$ is derived as follows:

$$\log(\mathbf{a}+\mathbf{b})=\log\mathbf{a}+\log\left(1+\frac{\mathbf{b}}{\mathbf{a}}\right).$$

Now $$\log\mathbf{a}=\log a+i\log\alpha,$$

and $$\log\left(1+\frac{\mathbf{b}}{\mathbf{a}}\right)=\frac{\mathbf{b}}{\mathbf{a}}-\frac{1}{2}\left(\frac{\mathbf{b}}{\mathbf{a}}\right)^2+\frac{1}{3}\left(\frac{\mathbf{b}}{\mathbf{a}}\right)^3-$$

$$=\frac{b}{a}\cos(\beta-\alpha)-\frac{1}{2}\left(\frac{b}{a}\right)^2\cos 2(\beta-\alpha)+$$

$$+i\left\{\frac{b}{a}\sin(\beta-\alpha)-\frac{1}{2}\left(\frac{b}{a}\right)^2\sin 2(\beta-\alpha)+\right\}.$$

Also,

$$\log\{\mathbf{a}+\mathbf{b}\}=\log\left[(a^2+b^2+2ab\cos(\beta-\alpha))^{\frac{1}{2}}\cdot\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right]$$

$$=\frac{1}{2}\log(a^2+b^2+2ab\cos(\beta-\alpha))+i\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}.$$

Equate the components along the initial axis, and put $\beta-\alpha=\theta$.

The direct logical power of the method is illustrated by the mode in which it deduces the expressions for the accleration along and perpendicular to the radius vector for a point moving in any plane curve from the expression for the velocity.

Given $$\mathbf{r} = r \cdot \theta$$

then $$\frac{d\mathbf{r}}{dt} = dr \cdot \theta + ird\theta \cdot \theta.$$

Apply that principle again;

$$\frac{d^2\mathbf{r}}{dt^2} = d^2r \cdot \theta + idrd\theta \cdot \theta + idrd\theta \cdot \theta + ird^2\theta \cdot \theta + i^2r(d\theta)^2 \cdot \theta$$

$$= (d^2r - r(d\theta)^2) \cdot \theta + i(2\, drd\theta + rd^2\theta) \cdot \theta.$$

ON THE DEFINITIONS

OF

THE TRIGONOMETRIC FUNCTIONS

BY

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ON THE DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS.

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In a paper on "*The Principles of the Algebra of Physics*" I introduced a trigonometric notation for the partial products of two vectors, writing

$$\mathsf{AB} = \cos \mathsf{AB} + \mathrm{Sin}\, \mathsf{AB},$$

where cos AB denotes the positive scalar product, and Sin AB the directed vector product. To denote the magnitude of the vector product I used the notation sin AB without a capital: it is not the exact equivalent of the *tensor*, because the magnitude may be positive or negative. With the additional device of using the Greek letters α, β, γ, etc., to denote axes, it is possible to dispense with the peculiar symbols introduced into analysis by Hamilton, namely, *S*, *V*, *T*, *U*, *K*, *I*; and the space-analysis then assumes to a large extent the more familiar features of the ordinary analysis. The notation raises the question of the relation of space-analysis to trigonometry. If cos and sin are correct appellations of the products mentioned, are there products of two vectors which are correctly designated by tan, sec, cotan, cosec? At p. 87 of the *Principles* I give a brief answer to this question; but a complete answer called for a more thorough investigation than I had then time to make.

This trigonometrical notation has been briefly discussed by Mr. Heaviside (*The Electrician*, Dec. 9, 1892). He takes the position that vector algebra is far more simple and fundamental than trigonometry, and that it is a mistake to base vectorial notation upon that of a special application thereof of a more complicated nature. I believe that this paper will show that trigonometry is not an application of space-analysis, but an element of it; and that the ideas of this element are of the greatest importance in developing the higher elements of the analysis.

The notation has also been discussed by Professor Alfred Lodge (*Nature*, Nov. 3, 1892). He takes the following view: "The particular symbol used to denote a scalar or a vector product is a matter of secondary importance, but is a matter which must sooner or later be settled if vector algebra is to come into general use. Lord Kelvin is of opinion that a function-symbol should be written with not less than three letters, and Professor Macfarlane's notation obeys that law, and is, moreover, easy to work with; but is incomplete, being applicable to products of two vectors only."

I consider that the notation is a matter not of secondary, but of paramount importance. If the notation is arbitrary, it gives us no help in the further development of analysis; if on the other hand it is systematic and logically connected with the existing notation of analysis, it points the way to more general principles and results. I believe that this paper will show that my notation is systematic and logical.

It is not true that the notation is applicable to products of only two vectors. In the *Principles* I have shown that the complete product of three vectors consists of three partial products, and that of four consists of five partial products: these several products are specified by means of the *cos* and *Sin* notation. The additional principle introduced is that in space of three dimensions the aspect of an area can be specified by the axis which it wants; hence that the complete product of an area-vector and a line-vector consists of two partial products which may be denominated the *cos* of the area and line, and the *Sin* of the area and line.

In this paper I propose first to review critically the historical definitions of the trigonometric terms, and the definitions, triangular, circular, or hyperbolic, given in the best modern treatises at my command; then to devise a logical system of definitions which will apply to space-analysis and that modern trigonometry which, as Professor Greenhill * shows, includes the properties both of circular and hyperbolic functions, and will be able to bring within the same domain the properties of the elliptic, general hyperbolic, and other functions. In this paper attention is mostly given to trigonometry in a plane; in a paper on *The Prin-*

* Differential and Integral Calculus, p. 61.

ciples of Elliptic and Hyperbolic Analysis I consider trigonometry in space.

The ancient method of defining the trigonometric terms is described by De Morgan at p. 18 of his "*Trigonometry and Double Algebra.*" A straight line *OP* of constant length (Fig. 1) revolves round *O* from a starting position *OA*; the arc *AP* traced out by the extremity of the revolving radius represents the angle *AOP*. From *P* draw a line *PM* perpendicular to *OA*; from *A* draw a line *AT* at right angles to *OA*, and terminating in *OP* produced; draw *OB* at right angles to *OA* and equal to *OA*, and from *B* draw *BV* at right angles to *OB* and terminating in the line of

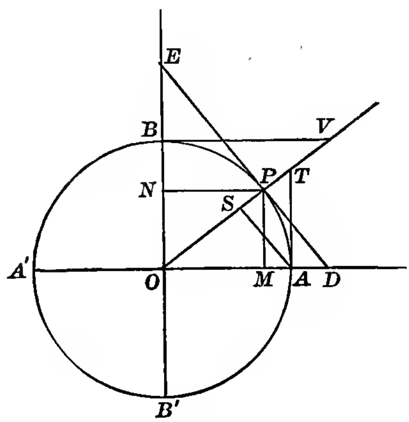

FIG. 1.

OP. The line *PM* is called the *sine* of the arc *AP*, the line *OM* is called the *cosine*, the line *AM* the *versed-sine*, the line *AT* the *tangent*, the line *OT* the *secant*, the line *BV* the *cotangent*, and the line *OV* the *cosecant*.

Here the terms sine, cosine, versed-sine, etc., are applied to certain lines drawn in and about a sector of a circle. These lines are commonly called the *trigonometric* lines; but inasmuch as they have reference to a circular sector and not to a triangle in general, they are more properly denominated *circular* lines. The trigonometric lines proper may be defined independently of the circle or any other curve.

We also remark that for the purposes of the higher analysis the circular lines must be defined with the utmost exactness;

difference of sense is not immaterial, still less is difference of direction. The sine-line is MP not PM, still less AS drawn from A perpendicular to OP. According to the account given by Dr. Hobson* of the ancient method, the tangent-line is not AT, but PD drawn a tangent to the circle at P, and terminated in the line of OA. Thus there are four logically distinct ways of defining the tangent line: *first*, it may mean the line drawn from A at right angles to OA; *second*, the line drawn from A a tangent to the circle at A; *third*, the line drawn from P at right angles to OP; *fourth*, the line drawn from P so as to touch the circle at P. The first definition agrees with the most ancient conceptions of the tangent; namely, the *umbra versa* of Abû'l Wafâ,† and the καθετος of Copernicus; ‡ the fourth view is taken by Professor Greenhill. § These four lines may be all unequal and differently directed when another curve such as the logarithmic spiral is substituted for the circle. It is necessary then to devise a separate notation for each.

In the same way there are four logically distinct definitions of the secant-line. It may mean, *first*, OT cut off by the perpendicular from A; *second*, OT cut off by the tangent at A; *third*, OD cut off by the perpendicular from P; *fourth*, OD cut off by the tangent at P. The first conception agrees with the ὑποτεινουσα of Copernicus,‖ while the fourth answers to the etymological conception of the tangent.

It is instructive to remember that the primary conception of the *sine* was the half of the chord of the double arc, and that it was long before the conception of the *cosine* was developed beyond that of the sine of the complementary arc.

The *circular ratios* are thus defined by De Morgan.¶ Let θ denote the angle AOP (Fig. 1); then

$$\sin\theta = \frac{MP}{OP}, \quad \cos\theta = \frac{OM}{OP}, \quad \text{vers}\,\theta = \frac{AM}{OP}, \quad \tan\theta = \frac{AT}{OA},$$

$$\sec\theta = \frac{OT}{OA}, \quad \text{cotant}\,\theta = \frac{BV}{OB}, \quad \text{cosec}\,\theta = \frac{OV}{OB}.$$

---

* Treatise on Plane Trigonometry, p. 16.

† Cantor's Vorlesungen über Geschichte der Mathematik, Vol. I., p. 642.

‡ *Ibid.*, Vol. II., p. 433.

§ Differential and Integral Calculus, p. 29.

‖ Cantor's Geschichte, Vol. II., p. 433.

¶ Double Algebra, p. 19.

Here three different radii OA, OP, OB are introduced, but no reason is given why in a particular case one should be preferred to either of the others. Why should the secant be defined with respect to OA while the cosine is defined with respect to OP? Is it a matter of indifference which radius is taken? It may be as regards mere numerical ratios, but it is not so as regards geometric ratios. Accuracy of definition is essential to the higher development of trigonometry.

In consequence of defining some of the ratios with respect to the revolving line OP (Fig. 1) instead of the initial line OA, a difficulty in the signs is introduced; to wit, OP is always positive, even when coincident with OA' or OB', which are held to be negative. This view in my judgment partakes of the nature of a paradox. De Morgan attempts to dissolve it by the following explanation (*Double Algebra*, p. 8):—

"When the revolving line comes into the position OA', is it negative? I answer, no: OA' as a projection is considered as part of a line which makes an angle 0° with the starting-line; and on a line *so described* is negative. But OA' as a position of the line of revolution is part of a line which makes 180° with the starting-line; and thus considered it is positive. The same considerations apply to the other axis. A line may be considered as making *with itself* an angle of 0° or an angle of 180°; whatever signs its parts have in the first case they have the opposite ones in the second."

Now the terms *positive* and *negative*, symbolized by + and − respectively, are essentially relative; they in their simplest application compare one line with another. If the line compared has the same direction as the line of reference, it is positive with respect to that line; if it has the opposite direction, it is negative with respect to that line. The line OA' is negative with respect to OA, and it is equally true that OP when coincident with OA' is negative with respect to OA. The line OB' is negative with respect to OB, and OP when coincident with OB' is negative with respect to OB. There is no meaning in saying that OP is always positive. The fact is that we cannot dispense with the idea of an initial line as a basis of reference, and I propose to show in the development which follows that the ratios are properly defined with respect to this initial line. The radius which should appear in each of the definitions is the radius OA.

The modern method seeks to define the trigonometric ratios independently of the circle merely by means of two intersecting lines. In elementary works this is first done under the limitation that the lines intersect at an acute angle. For instance, Todhunter proceeds thus (*Plane Trigonometry*, p. 14):—

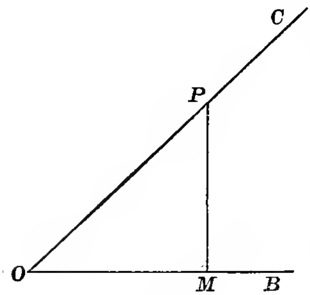

Fig. 2.

"Let BOC (Fig. 2) be any angle; take any point in either of the containing straight lines, and from it draw a perpendicular to the other straight line; let P be the point in the straight line OC, and PM perpendicular to OB. We shall use the letter A to denote the angle BOC. Then

$\frac{PM}{OP}$, that is $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is called the *sine* of the angle A;

$\frac{OM}{OP}$, that is $\frac{\text{base}}{\text{hypotenuse}}$, is called the *cosine* of the angle A;

$\frac{PM}{OM}$, that is $\frac{\text{perpendicular}}{\text{base}}$, is called the *tangent* of the angle A;

$\frac{OM}{PM}$, that is $\frac{\text{base}}{\text{perpendicular}}$, is called the *cotangent* of the angle A;

$\frac{OP}{OM}$, that is $\frac{\text{hypotenuse}}{\text{base}}$, is called the *secant* of the angle A;

$\frac{OP}{PM}$, that is $\frac{\text{hypotenuse}}{\text{perpendicular}}$, is called the *cosecant* of the angle A.

If the cosine of A be subtracted from unity, the remainder is called the *versed-sine* of A. If the sine of A be subtracted from unity, the remainder is called the coversed-sine of A." Equivalent definitions are given by Levett and Davison * and by Hobson.†

The definitions quoted are accurate only so far as arithmetical magnitude is concerned; they take no account of sense or direction. For exact purposes it is not indifferent whether the per-

* Plane Trigonometry, p. 4. † Plane Trigonometry, p. 16.

pendicular be drawn from OB or from OC, and whether the sine be defined as $\frac{PM}{OP}$ or $\frac{MP}{OP}$. In consequence of dropping out the idea of an initial line it is necessary to compare OM and MP with OP, which does not coincide with the axis on which the projection is made. The cotangent so defined answers to the old conception of the *umbra*, the tangent to that of the *umbra versa*, and the secant to that of the *hypotenuse* of Copernicus. A difficulty is encountered with the *versed-sine;* for it is not defined geometrically like the others, as the ratio of two lines; it is defined analytically. Why this breakdown in the scheme of definitions? But the above definitions are not comprehensive enough even for the simple case where the lines meet at an obtuse angle, because then the triangle POM encloses not the angle BOC, but its supplement.

The definitions are extended by dropping the idea of a right-angled triangle, and substituting the idea of projection. Thus Levett and Davison, following De Morgan, say (p. 93): —

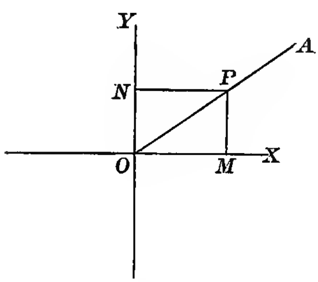

FIG. 3.

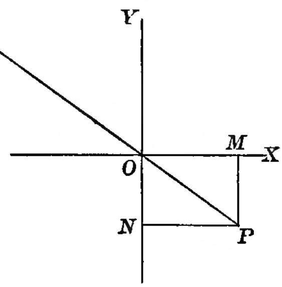

FIG. 4.

"Let a line rotate about O (Figs. 3 and 4) from OX through any positive or negative angle α to the position OA; let OY be a line making an angle $\frac{\pi}{2}$ in the positive sense with OX; and let OA, OX, OY be the positive senses of the lines OA, OX, OY. Let a length OP, of any magnitude and of either sense, be measured along OA; and let OM, ON be the projections of OP on OX, OY respectively. The ratio $OM:OP$ is called the *cosine* of the angle α, $ON:OP$ the *sine* of α, $ON:OM$ the *tangent* of α, $OP:OM$ the *secant* of α, $OP:ON$ the *cosecant* of α, and $OM:ON$ the *cotangent* of α. These ratios are called the Circular Functions of the angle α." The following is added in small print: "Two other ratios are occasionally used, and are defined as follows: If the length OP be equal in magnitude to OX, and positive in sense, and if $OY = OX$, the ratio $MX : OP$ is called the *versine* of α, and $NY : OP$ the *coversine* of α."

The above mode of defining assumes that a line may be positive in itself, whereas there are reasons for believing that positive and negative have their primary meaning in the comparison of two lines. Again, in order to define the versine, the two intersecting lines are given up, and conditions are imposed equivalent to introducing the circle; for OP is made of constant length, and is supposed to be always positive.

Mr. Carr in his *Synopsis of Pure Mathematics* defines the sine, cosine, and tangent geometrically; but the secant, cosecant, and cotangent as the respective reciprocals of these. It is surely more logical to define each function geometrically and independently, and afterwards prove what relations exist between them.

From the definitions examined we may conclude that under the one name of trigonometric ratios are comprised two species: the geometric, or rather *triangular*, and the *circular* proper. The triangular ratios are defined independently of the circle, and they include some of the circular ratios as special cases.

Further light on this subject may be obtained by considering those functions analogous to the circular which depend on the equilateral hyperbola, or ex-circle. The convenient terms "ex-circle" and "ex-circular" have been introduced by Mr. Hayward for the phrases "equilateral hyperbola" and "equilateral hyperbolic," commonly called "hyperbolic" (*Vector Algebra and Trigonometry*, p. 128). The following method of defining these ratios is adopted by Messrs. Levett and Davison (*Plane Trigonometry*, p. 258):—

"Let a point move along the curve (Fig. 5) from the vertex A of one branch of a rectangular hyperbola, whose centre is O and semi-axis equal to a, to the position P; let A be the area of the hyperbolic sector AOP, and let $u = \frac{2A}{a^2}$; that is, let u be the measure of the sector AOP, the unit of measurement being the square whose diagonal is the semi-axis.

"Take OY, a line making an angle of 90° in the positive sense with the transverse axis OAX, and let OM, ON be the projections of OP on OX, OY respectively; then the ratio

$OM : OA$ is called the *hyperbolic cosine* of u,
$ON : OA$ the *hyperbolic sine* of u,
$ON : OM$ the *hyperbolic tangent* of u,
$OA : OM$ the *hyperbolic secant* of u,
$OA : ON$ the *hyperbolic cosecant* of u,
$OM : ON$ the *hyperbolic cotangent* of u."

We observe that here the ratios are not defined with respect to the radius-vector OP, but with respect to OA the initial line; to

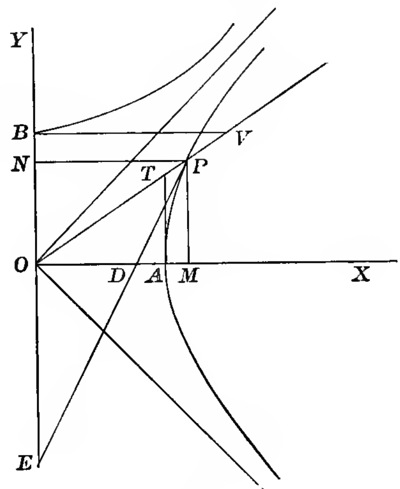

FIG 5.

define them with respect to OP would be an error. Wherefore, we conclude that it is the analogue of OA, not the analogue of OP, which should be introduced into the definitions of the circular ratios. We also observe that the hyperbolic argument is not the ratio of the arc to the initial radius, but the ratio of twice the area of the sector to the square on the initial radius; hence the true analytical argument for the circular ratios is not the ratio of the arc to the radius, but the ratios of twice the area of the sector to the square on the radius. This leads us to the idea that the trigonometric ratios may be ratios of areas as well as ratios of lines.

Dr. Günther,* following M. Laisant,† gives the definitions of the circular lines which appear to furnish most readily the definitions of the analogous ex-circular lines. Let *APB* (Fig. 1) be a circle of unit radius, and let u denote double the area of the sector *AOP*; draw *PM* perpendicular to *OA*, and *PN* to *OB*; draw *AT* a tangent to the circle from *A* terminating in *OP* produced, and *BV* a tangent to the circle at *B* also terminating in *OP* produced; draw a tangent to the circle at *P* cutting the axis of *OA* in *D*, and that of *OB* in *E*. Then the line *PM* or *ON* represents $\sin u$, the line *OM* or *NP* $\cos u$; *AT* represents $\tan u$, and *BV* $\operatorname{cotan} u$; while *OD*, not *OT*, represents $\sec u$, and *OE*, not *OV*, $\operatorname{cosec} u$. The six ratios are represented by lines along the axes of projection,—three along the axis of abscissæ, and three along the axis of ordinates; none have the direction of the radius-vector. The definition of the tangent takes the second view, while that of the secant takes the fourth view of it mentioned at page 4 above.

The analogous lines are defined in the following manner: Let *APB* (Fig. 5) be an equilateral hyperbola of unit semi-diameter, and let u denote double the area of the sector *AOP*; draw *PM* perpendicular to *OA*, and *PN* to *OB*; draw *AT* a tangent to the hyperbola at *A* terminating in *OP*, and *BV* a tangent to the conjugate hyperbola at *B* also terminating in *OP*; draw a tangent to the hyperbola at *P* cutting the axis of *OA* in *D*, and that of *OB* in *E*. Then the line *MP* represents $\sinh u$, *OM* $\cosh u$, *AT* $\tanh u$, *BV* $\coth u$, *OD* $\operatorname{sech} u$, and *OE* $\operatorname{cosech} u$. The analogous ratios are represented by the analogous lines. We observe that *AT* and *BV* might have been defined as drawn at right angles to *OA* and *OB* respectively, that is, according to the first view of the tangent; but that *OD* corresponds to the fourth view of the secant, and to it only. Why is it that analysts find it easier to deal with lines which have the directions of the axes than with lines having any other direction such as that of the radius-vector, or of the true tangent? Because the former involve scalar products only, while the latter involve vector products.

M. Laisant, in his admirable *Essai*, extends his definitions of the trigonometric lines to the ellipse and general hyperbola.‡

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* Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunctionen, p. 92. † Essai sur les fonctions hyperboliques.

‡ Essai sur les fonctions hyperboliques, p. 269.

Let APB' (Fig. 13) be an ellipse of such size that the product of its two semi-axes OA and OB' is unity. By u is meant twice the area of the sector AOP; elliptic $\cos u$ is represented by OM, elliptic $\sin u$ by MP, elliptic $\sec u$ by OD, elliptic $\tan u$ by AT, elliptic $\operatorname{cotan} u$ by $B'V$, and elliptic $\operatorname{cosec} u$ by OE. Here the denominator of the ratio u is the product of the two semi-axes.

Many analysts hold that the circular functions might be defined by purely algebraic ideas. For instance, De Morgan (*Double Algebra*, p. 34): "I said that we should soon make it very evident that a purely algebraical basis *might* have been made for trigonometry. If we had chosen to call the preceding functions of z, namely,

$$1-\frac{z^2}{2!}+, \qquad z-\frac{z^3}{3!}+, \qquad z+\frac{z^3}{3}+,$$

by the names of *cosine, sine,* and *tangent* of z (and their reciprocals *secant, cosecant,* and *cotangent*), we might have investigated the properties of these series, and we should at last have arrived at all our preceding formulæ of connection; but with much more difficulty."

Again, Dr. Hobson (*Plane Trigonometry*, p. 279): "It is possible to give purely analytical definitions of the circular functions, and to deduce from these definitions their fundamental analytical properties, so that the calculus of circular functions can be placed upon a basis independent of all geometrical considerations; these definitions will include the circular functions of a complex quantity. We can define the cosine and sine of z by means of the equations

$$\cos z = \tfrac{1}{2}\{e^{iz}+e^{-iz}\},$$

$$\sin z = \frac{1}{2i}\{e^{iz}-e^{-iz}\},$$

where e^z denotes the series $1+z+\dfrac{z^2}{2!}+$, etc. In other words, we define $\cos z$ as the sum of the series $1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-$, and $\sin z$ as the sum of the series $z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-$. We may regard this then as the generalized definition of the cosine and sine functions, and it includes the case of a complex argument, which was not included in the earlier geometric definitions."

A definition which has only an algebraic basis is, in my opinion, of the species which logicians call *nominal;* while one which has a geometrical basis is of the species called *real.* It may be doubted whether nominal definitions are of much scientific value. The primary geometric idea which is the basis of the primary trigonometric function can also be generalized, and in more ways than one; how can the analyst secure a correspondence between his arbitrarily generalized definition and the more general ideas which develop from the primary geometrical idea? In the present paper and in a paper on "*The Principles of Elliptic and Hyperbolic Analysis*" I show that there are several geometrically real generalizations of the circular functions, and that the algebraic series for the simple functions generalize in ways that would never be deduced by taking the elementary series as the general definitions.

I now proceed to consider how the several species of trigonometric functions — the triangular, the circular, and the ex-circular, — may be defined in harmony with one another. The method adopted is afterwards shown to be applicable to the logarithmic spiral, ellipse and general hyperbola, and to a mixed curve composed partly of a circle, partly of an ex-circle; further, in the paper on "*The Principles of Elliptic and Hyperbolic Analysis,*" it is applied to ellipsoidal and hyperboloidal trigonometry.

THE TRIANGULAR FUNCTIONS.

Let OA and OP represent (Fig. 6) any two finite straight lines, or vectors, meeting at the point O. A triangle is formed

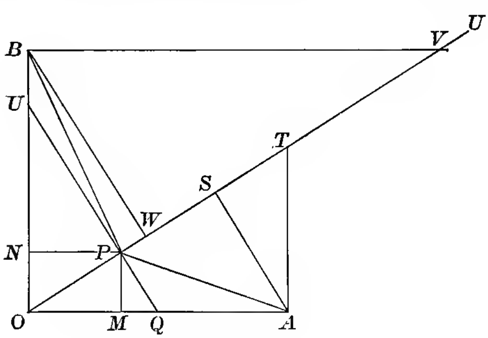

FIG. 6.

by joining A and P. From P draw PM at right angles to OA, and PQ at right angles to OP; from A draw AT at right angles to OA, and AS at right angles to OP.

First: we consider OM and MP the orthogonal projections of OP on OA. In a certain sense

$$OP = OM + MP;$$

to wit, in the ordinary sense of a vector equation. By prefixing OA to each term, we derive an area equation

$$(OA)(OP) = (OA)(OM) + (OA)(MP).$$

What is the meaning of this area equation? It is that the parallelogram $(OA)(OP)$ is equivalent to the product $(OA)(OM)$ together with the rectangle formed by OA and MP. This, in my opinion, is the fundamental principle of vector analysis (*Principles of the Algebra of Physics*, p. 72).

Let the vector OA be denoted by the black letter A, and the vector OP by the black letter R; let the rectangular co-ordinates of A be a,* b, c, and those of R be x, y, z, so that

$$\mathsf{A} = ai + bj + ck \text{ and } \mathsf{R} = xi + yj + zk.$$

Then the analytical product of the two vectors is

$$\begin{aligned}\mathsf{AR} &= (ai + bj + ck)(xi + yj + zk)\\ &= ax + by + cz + (bz - cy)jk + (cx - az)ki + (ay - bx)ij,\end{aligned}$$

and of the two partial products into which the complete product breaks up, the former, $ax + by + cz$, expresses $(OA)(OM)$, while the latter,

$$(bz - cy)jk + (cx - az)ki + (ay - bx)ij,$$

expresses $(OA)(MP)$.

It appears to me that the former partial product is correctly denoted by the expression cos AR, and the latter by the complementary expression Sin AR. The latter function is written with

* The letter a is in some places used to denote the magnitude of OA according to the usage of analysis; the context shows clearly whether it is the whole magnitude or the magnitude of the i-component which is meant.

a capital because it has an aspect or axis; it is not a simple area, but a directed area. The equation

$$(OA)(OP) = (OA)(OM) + (OA)(MP)$$

is then written

$$\mathsf{AR} = \cos \mathsf{AR} + \text{Sin } \mathsf{AR}.$$

The notation sin AR serves for the magnitude of the sine product apart from its aspect or axis; it is the equivalent of the unwieldy Cartesian expression

$$\sqrt{(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2}.$$

While $(OA)(MP)$ will be used to denote Sin AR; the notation $OA \times MP$ will be used to denote sin AR.

The function Sin AR cannot be expressed in rectangular coordinates without introducing symbols for the axes; hence it cannot be treated by the Cartesian analysis except indirectly.

Corresponding to the line equation

$$OP = OM + MP$$

there is the scalar equation

$$(OP)^2 = (OM)^2 + (MP)^2;$$

and corresponding to the area equation

$$\mathsf{AR} = \cos \mathsf{AR} + \text{Sin } \mathsf{AR}$$

there is the scalar equation

$$\mathsf{A}^2\mathsf{R}^2 = (\cos \mathsf{AR})^2 + (\text{Sin } \mathsf{AR})^2,$$

which, expanded in Cartesians, becomes

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2 + (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2.$$

If we take the vector which is the reciprocal of A, we get

$$\frac{1}{\mathsf{A}}\mathsf{R} = \frac{1}{OA} OM + \frac{1}{OA} MP$$

$$= \frac{OM}{OA} + \frac{1}{OA} MP.$$

When the order of the factors in a quotient is immaterial as in the cosine term, the quotient may be written in the ordinary way; when the order of the factors is essential as in the Sine term, the order will be indicated by introducing the reciprocal before or after according to the manner in which it enters. Hence by introducing OA in both numerator and denominator,

$$\frac{1}{\mathsf{A}}\mathsf{R} = \frac{(OA)(OM)}{(OA)^2} + \frac{(OA)(MP)}{(OA)^2}$$

$$= \frac{\mathsf{AR}}{\mathsf{A}^2}.$$

Hence $$\cos\frac{1}{\mathsf{A}}\mathsf{R} = \frac{OM}{OA} = \frac{(OA)(OM)}{(OA)^2} = \frac{ax+by+cz}{a^2+b^2+c^2} = \frac{\cos \mathsf{AR}}{\mathsf{A}^2},$$

and $$\text{Sin}\frac{1}{\mathsf{A}}\mathsf{R} = \frac{1}{OA}MP = \frac{(OA)(MP)}{(OA)^2}$$

$$= \frac{(bz-cy)jk + (cx-az)ki + (ay-bx)ij}{a^2+b^2+c^2}$$

$$= \frac{\text{Sin } \mathsf{AR}}{\mathsf{A}^2}.$$

Here no relation is imposed connecting A and R; their extremities are not restricted to lying on a circle or any other curve. Thus the functions are triangular or trigonometric in the primary sense of the word. We are introduced to the consideration of trigonometric areas as well as trigonometric lines and trigonometric ratios.

Second: we consider the lines OT and TA obtained by drawing AT at right angles to OA. As a line-vector equation we have

$$OA = OT + TA,$$

and from it we derive the area-vector equation

$$(OA)(OA) = (OA)(OT) + (OA)(TA),$$

or $$(OA)^2 = (OA)(OT) - (OA)(AT).$$

The latter equation means that the square of OA is in a certain sense equal to the difference between the parallelogram formed by OA and OT and the rectangle formed by OA and AT. In form it is merely a transformation of the area equation considered above (p. 13).

Let $(OA)(OT)$ be denoted by Sec AR and $(OA)(AT)$ by Tan AR, then the above equation is written

$$\mathsf{A}^2 = \operatorname{Sec} \mathsf{AR} - \operatorname{Tan} \mathsf{AR}.$$

Both functions are written with a capital, because each involves an aspect or axis.

After dividing by A^2 we obtain

$$1 = \frac{\operatorname{Sec} \mathsf{AR}}{\mathsf{A}^2} - \frac{\operatorname{Tan} \mathsf{AR}}{\mathsf{A}^2}$$

$$= \operatorname{Sec} \frac{1}{\mathsf{A}} \mathsf{R} - \operatorname{Tan} \frac{1}{\mathsf{A}} \mathsf{R}.$$

Corresponding to the line equation we have the scalar equation

$$(OA)^2 = (OT)^2 - (AT)^2,$$

and corresponding to the area equation we have the scalar equations

$$\mathsf{A}^4 = (\operatorname{Sec} \mathsf{AR})^2 - (\operatorname{Tan} \mathsf{AR})^2,$$

and $$1 = \left(\operatorname{Sec} \frac{1}{\mathsf{A}} \mathsf{R}\right)^2 - \left(\operatorname{Tan} \frac{1}{\mathsf{A}} \mathsf{R}\right)^2.$$

To find the expressions for these trigonometrical functions in terms of rectangular co-ordinates, we proceed as follows. Since

$$OT = \frac{OA}{OM} OP$$

and $$AT = \frac{OA}{OM} MP;$$

therefore

$$(OA)^2 = \frac{OA}{OM}(OA)(OP) - \frac{OA}{OM}(OA)(MP)$$

$$= \frac{(OA)^2}{(OM)(OA)}(OA)(OP) - \frac{(OA)^2}{(OM)(OA)}(OA)(MP);$$

that is, $$\mathsf{A}^2 = \frac{\mathsf{A}^2}{\cos \mathsf{AR}} \mathsf{AR} - \frac{\mathsf{A}^2}{\cos \mathsf{AR}} \operatorname{Sin} \mathsf{AR}.$$

Hence $\text{Sec AR} = \dfrac{\mathsf{A}^2}{\cos \mathsf{AR}} \mathsf{AR}$

$$= \frac{a^2+b^2+c^2}{ax+by+cz}(ai+bj+ck)(xi+yj+zk),$$

and $\text{Tan AR} = \dfrac{\mathsf{A}^2}{\cos \mathsf{AR}} \text{Sin AR}$

$$= \frac{a^2+b^2+c^2}{ax+by+cz}\{(bz-cy)jk+(cx-az)ki+(ay-bx)ij\}.$$

Hence $\text{Sec} \dfrac{1}{\mathsf{A}} \mathsf{R} = \dfrac{\mathsf{AR}}{\cos \mathsf{AR}} = \dfrac{(ai+bj+ck)(xi+yj+ck)}{ax+by+cz},$

and $\text{Tan} \dfrac{1}{\mathsf{A}} \mathsf{R} = \dfrac{\text{Sin AR}}{\cos \mathsf{AR}} = \dfrac{(bz-cy)jk+(cx-az)ki+(ay-bx)ij}{ax+by+cz}.$

The function sec AR is obtained from Sec AR by substituting the appropriate square roots of $(ai+bj+ck)^2$ and $(xi+yj+zk)^2$ Similarly, the function tan AR is obtained from Tan AR by substituting the appropriate square root of $(\text{Sin AR})^2$. By sec AR is meant the magnitude of Sec AR, and by tan AR the magnitude of Tan AR.

Third: we consider the lines OQ and QP obtained by drawing PQ at right angles to OP. We have the line-vector equation

$$OP = OQ + QP$$

with the corresponding scalar equation

$$(OP)^2 = (OQ)^2 - (QP)^2.$$

From the former we derive the area-vector equation

$$(OA)(OP) = (OA)(OQ) + (OA)(QP),$$

which means that the parallelogram OA, OP is in a certain sense equivalent to the product of the two codirectional lines OA and OQ together with the parallelogram OA, QP. The two parallelograms are on the same base and between the same parallels, but the angle of the latter exceeds the angle of the former by a quadrant. For the sake of clearness it is absolutely necessary to devise a distinctive notation for the products in question. As the line PQ is drawn from OP in the same manner as AT from

OA, the line OQ partakes of the nature of the Sec line OT, and the line QP partakes of the nature of the Tan line AT. By changing the initial consonants from light to heavy, we obtain a notation which is suggestive and easily remembered, and will serve at least for the purpose of this investigation.

Let, then, $(OA)(OQ)$ be denoted by zec AR, and $(OA)(QP)$ by Dan AR; the above equation is then written

$$\mathsf{AR} = \text{zec}\,\mathsf{AR} + \text{Dan}\,\mathsf{AR}.$$

As
$$\frac{\mathsf{AR}}{\mathsf{A}^2} = \frac{\text{zec}\,\mathsf{AR}}{\mathsf{A}^2} + \frac{\text{Dan}\,\mathsf{AR}}{\mathsf{A}^2},$$

therefore
$$\frac{1}{\mathsf{A}}\mathsf{R} = \text{zec}\frac{1}{\mathsf{A}}\mathsf{R} + \text{Dan}\frac{1}{\mathsf{A}}\mathsf{R}.$$

The corresponding scalar equations are

$$\mathsf{A}^2\mathsf{R}^2 = (\text{zec}\,\mathsf{AR})^2 - (\text{Dan}\,\mathsf{AR})^2,$$

and
$$\frac{\mathsf{R}^2}{\mathsf{A}^2} = \left(\text{zec}\frac{1}{\mathsf{A}}\mathsf{R}\right)^2 - \left(\text{Dan}\frac{1}{\mathsf{A}}\mathsf{R}\right)^2.$$

To find the expressions for these functions in terms of rectangular co-ordinates, we proceed as follows:

Since $OQ = \dfrac{(OP)^2}{OM}$, and $QP = \dfrac{MP}{OM}\sqrt{-1}\,OP$, where $\sqrt{-1}\,OP$ denotes that the line OP is turned through a positive quadrant in the given plane; we deduce that

$$(OA)(OP) = \frac{(OA)(OP)^2}{OM} + \frac{MP}{OM}(OA)(\sqrt{-1}\,OP)$$

$$= \frac{(OA)^2(OP)^2}{(OA)(OM)} + \frac{(OA)(MP)}{(OA)(OM)}(OA)(\sqrt{-1}\,OP),$$

therefore
$$\mathsf{AR} = \frac{\mathsf{A}^2\mathsf{R}^2}{\cos\mathsf{AR}} + \frac{\sin\mathsf{AR}}{\cos\mathsf{AR}}\mathsf{A}\sqrt{-1}\,\mathsf{R}.$$

Hence
$$\text{zec}\,\mathsf{AR} = \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax+by+cz},$$

and Dan $\mathsf{AR} =$

$$\frac{\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}}{ax+by+cz}(ai+bj+ck)\sqrt{-1}(xi+yj+zk).$$

Similarly $$\text{zec}\frac{1}{\mathsf{A}}\mathsf{R}=\frac{x^2+y^2+z^2}{ax+by+cz},$$

while $\text{Dan}\frac{1}{\mathsf{A}}\mathsf{R}$ is obtained by dividing Dan AR by $a^2+b^2+c^2$,

and $$\text{dan}\,\mathsf{AR}=\frac{\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}}{ax+by+cz}.$$

Thus zec AR is the reciprocal of cos AR, not with respect to unity, but with respect to $\mathsf{A}^2\mathsf{R}^2$; while $\text{zec}\frac{1}{\mathsf{A}}\mathsf{R}$ is the reciprocal of $\cos\frac{1}{\mathsf{A}}\mathsf{R}$ with respect to $\frac{\mathsf{R}^2}{\mathsf{A}^2}$.

Fourth: we consider the lines OS and SA obtained by drawing AS perpendicular to OP.

We have the line-vector equation

$$OA=OS+SA,$$

and from it we derive the area-vector equation

$$\begin{aligned}(OA)^2&=(OA)(OS)+(OA)(SA)\\&=(OA)(OS)-(OA)(AS).\end{aligned}$$

This equation means that the square of OA is equal to the difference between the parallelograms OA, OS and OA, AS. As the lines OS and AS have a certain analogy to the lines OM and MP, let the products be denominated by Gos and Zin, the initial consonants of the functions being changed from light to heavy. The above equation is then written

$$A^2=\text{Gos}\,\mathsf{AR}-\text{Zin}\,\mathsf{AR}.$$

Since $$OS=\frac{OA}{OQ}OP=\frac{(OM)(OA)}{OP^2}OP=\frac{\cos\mathsf{AR}}{\mathsf{R}^2}OP,$$

and $$AS=\frac{OA}{OQ}QP=\frac{(OA)(MP)}{(OP)^2}\sqrt{-1}\,OP=\frac{\sin\mathsf{AR}}{\mathsf{R}^2}\sqrt{-1}\,OP,$$

the above equation becomes

$$\mathsf{A}^2=\frac{\cos\mathsf{AR}}{\mathsf{R}^2}\mathsf{AR}-\frac{\sin\mathsf{AR}}{\mathsf{R}^2}\mathsf{A}\sqrt{-1}\,\mathsf{R}.$$

$$\text{Hence } \mathrm{Gos}\,\mathsf{AR} = \frac{ax+by+cz}{x^2+y^2+z^2}(ai+bj+ck)(xi+yj+zk),$$

and $$\mathrm{gos}\,\mathsf{AR} = (ax+by+cz)\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{x^2+y^2+z^2}}.$$

The *versed-sine* product is obtained by considering AP the third side of the triangle. Because

$$AP = AO + OP,$$

therefore $$(OA)(AP) = (OA)(AO) + (OA)(OP)$$
$$= -(OA)^2 + (OA)(OP).$$

Hence $$\cos(OA)(AP) = -(OA)^2 + \cos(OA)(OP),$$

and $$\mathrm{Sin}(OA)(AP) = \mathrm{Sin}\,(OA)(OP).$$

It is the new product $\cos(OA)(AP)$ which is properly called vers AR; so that

$$\text{vers}\,\mathsf{AR} = -\mathsf{A}^2 + \cos\mathsf{AR}$$
$$= (OA)(AM).$$

Similarly $$\text{vers}\,\frac{1}{\mathsf{A}}\mathsf{R} = -1 + \frac{\cos\mathsf{AR}}{\mathsf{A}^2}$$
$$= \frac{AM}{OA}.$$

According to this definition the versine is negative when the point M falls to the left of A; for OA and AM then have opposite directions. In circular trigonometry it is commonly stated that the versine is always positive; it is more correct to say that in the case of the circular functions the versine is always negative.

Finally, we have to consider the definitions of the complementary functions. By the *complementary-vector* of A with respect to R is meant the vector OB (Fig. 6), which is equal and perpendicular to A in the plane of A and R, and drawn to the side of A on which R is (*Principles of the Algebra of Physics*, p. 87). Let it be denoted by $\overline{\mathsf{A}}$, the horizontal bar denoting "perpendicular to." When all the lines lie in a common plane, this notation is definite. Grassmann uses a vertical bar prefixed to the vector it refers to, as |A. The horizontal bar is preferable, because in space it must be attached to a pair of vectors, and the horizontal form allows this to be done conveniently. The complementary vector is expressed in terms of A and R by the equation

$$\overline{\mathsf{A}} = \frac{\text{Sin}(\text{Sin}\,\mathsf{AR})\,\mathsf{A}}{\sin \mathsf{AR}}$$

where
$$\begin{aligned}\text{Sin}(\text{Sin}\,\mathsf{AR})\,\mathsf{A} = &\{(cx - az)c - (ay - bx)b\}i \\ &+ \{(ay - bx)a - (bz - cy)c\}j \\ &+ \{(bz - cy)b - (cx - az)a\}k.\end{aligned}$$

By the complementary function is meant the function which is obtained when $\overline{\mathsf{A}}$ is substituted for A in the original function. Draw *PN* perpendicular to *OB*, and *PU* to *OP*; *BV* perpendicular to *OB*, and *BW* to *OP*. The prefix *co-* may be used to denote the complementary function. The geometrical definitions then are

co-cos AR $= (OB)(ON)$,	co-Sin AR $= (OB)(NP)$,
co-Sec AR $= (OB)(OV)$,	co-Tan AR $= (OB)(BV)$,
co-zec AR $= (OB)(OU)$,	co-Dan AR $= (OB)(UP)$,
co-Gos AR $= (OB)(OW)$,	co-Zin AR $= (OB)(BW)$.

It may be shown that co-cos AR = sin AR. Also co-Sin AR may be denoted by Cos AR; it is equal to $-\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}}$ Sin AR.

The several trigonometric areas are exhibited synoptically in the following table. It is evident that Hamilton's *S* and *V* are entirely inadequate to express the various scalar and vector functions of the product of two vectors.

TRIGONOMETRIC AREAS.

Function.	Geometric Definition.	Analytical Definition.
AR	$(OA)(OP)$	$(ai+bj+ck)(xi+yj+zk)$
cos AR	$(OA)(OM)$	$ax+by+cz$
Sin AR	$(OA)(MP)$	$(bz-cy)jk+(cx-az)ki+(ay-bx)ij$
sin AR	$OA \times MP$	$\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}$
Sec AR	$(OA)(OT)$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}}\mathrm{AR}$
sec AR	$OA \times OT$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}}\sqrt{\mathrm{A}^2\mathrm{R}^2}$
Tan AR	$(OA)(AT)$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}}\,\mathrm{Sin\ AR}$
tan AR	$OA \times AT$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}}\sin \mathrm{AR}$
zec AR	$(OA)(OQ)$	$\frac{\mathrm{A}^2\mathrm{R}^2}{\cos \mathrm{AR}}$
Dan AR	$(OA)(QP)$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}}\mathrm{A}\sqrt{-1}\,\mathrm{R}$
dan AR	$OA \times QP$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}}\sqrt{\mathrm{A}^2\mathrm{R}^2}$
Gos AR	$(OA)(OS)$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2}\mathrm{AR}$
gos AR	$OA \times OS$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2}\sqrt{\mathrm{A}^2\mathrm{R}^2}$
Zin AR	$(OA)(AS)$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2}\mathrm{A}\sqrt{-1}\,\mathrm{R}$
zin AR	$OA \times AS$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2}\sqrt{\mathrm{A}^2\mathrm{R}^2}$
vers AR	$(OA)(AM)$	$-\mathrm{A}^2+\cos \mathrm{AR}$
co-cos AR	$(OB)(ON)$	$\sin \mathrm{AR}$

TRIGONOMETRIC AREAS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
co-Sin AR	$(OB)(NP)$	$-\frac{\cos AR}{\sin AR}\text{Sin } AR = \text{Cos } AR$
co-sin AR	$OB \times NP$	$\cos AR$
co-Sec AR	$(OB)(OV)$	$\frac{A^2}{\sin AR}\overline{A}R$
co-sec AR	$OB \times OV$	$\frac{A^2}{\sin AR}\sqrt{A^2R^2}$
co-Tan AR	$(OB)(BV)$	$\frac{A^2}{\sin AR}\text{Cos } AR$
co-tan AR	$OB \times BV$	$A^2\frac{\cos AR}{\sin AR}$
co-zec AR	$(OB)(OU)$	$\frac{A^2R^2}{\sin AR}$
co-Dan AR	$(OB)(UP)$	$\frac{\cos AR}{\sin AR}\overline{A}\sqrt{-1}\,R$
co-dan AR	$OB \times UP$	$\frac{\cos AR}{\sin AR}\sqrt{A^2R^2}$
co-Gos AR	$(OB)(OW)$	$\frac{\sin AR}{R^2}\overline{A}R$
co-gos AR	$OB \times OW$	$\frac{\sin AR}{R^2}\sqrt{A^2R^2}$
co-Zin AR	$(OB)(BW)$	$\frac{\cos AR}{R^2}\overline{A}\sqrt{-1}\,R$
co-zin AR	$OB \times BW$	$\frac{\cos AR}{R^2}\sqrt{A^2R^2}$
co-vers AR	$(OB)(BN)$	$-A^2 + \sin AR$

THE CIRCULAR FUNCTIONS.

In the case of the circular functions the variable vector R is always of the same length as the initial vector A; in other words, *OP* is limited by the condition that its extremity must lie on a circle of radius *OA* (Fig. 7). There is a definite area enclosed

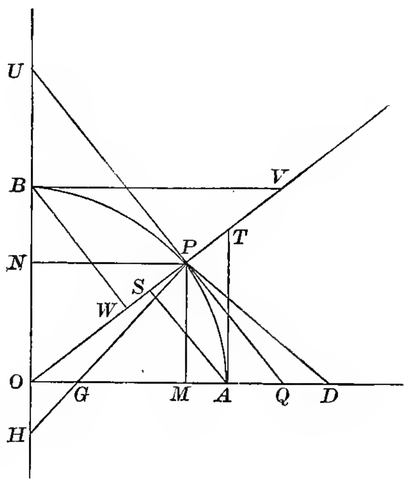

FIG. 7.

between *OA*, *OP* and the arc *AP*; and the triangular functions can be expressed as functions of this area. Let *A* denote the area of the sector *AOP*, *s* the length of the arc *AP*, and *a* the magnitude of *OA*; then $\frac{2A}{a^2} = \frac{s}{a}$. Let this quantity be denoted by *u*; it is the circular measure of the angle *AOP*, and is more properly regarded as the ratio of twice the area of the sector *AOP* to the square on *OA* than as the ratio of the arc *AP* to the line *OA*.

The following table shows that the circular ratio is deduced from the corresponding trigonometric area by dividing by A^2, and introducing the special relation that

$$(\cos AR)^2 + (\text{Sin } AR)^2 = A^4,$$

or

$$R^2 = A^2.$$

In addition to the triangular lines there are the curve lines or circular lines proper; namely, the tangent, the secant, the normal, etc. By the *tangent* is meant the line DP drawn from a point in OA so as to touch the curve at P, and by the *secant* is meant the line OD cut off. By the *normal* is meant the line GP which starts from the line OA, and is at right angles to the tangent at P, while OG is the complementary line. Let these functions be denoted by Tnt, Sct, Nor, respectively.

Since $$DP = DM + MP,$$

$$(OA)(DP) = (OA)(DM) + (OA)(MP)$$
$$= (OA)(DM) + \text{Sin AR}.$$

But, generally, $$DM = \sin\frac{d(\cos)}{d(\sin)}OA,$$

which, for the special case of the circle, becomes

$$DM = -\frac{\sin^2 u}{\cos u}OA,$$

therefore $$(OA)(DP) = -\frac{(\text{Sin AR})^2}{\cos \text{AR}} + \text{Sin AR}.$$

Again, $$(OA)(OD) = (OA)(OM) + (OA)(MD)$$
$$= \cos \text{AR} + (OA)(MD),$$

which, for the case of the circle, becomes

$$(OA)(OD) = \cos \text{AR} + \frac{(\text{Sin AR})^2}{\cos \text{AR}}$$
$$= \frac{\text{A}^2\text{R}^2}{\cos \text{AR}}$$
$$= \frac{\text{A}^4}{\cos \text{AR}}.$$

For the normal we have the general relation

$$GP = GM + MP,$$

therefore $$(OA)(GP) = (OA)(GM) + (OA)(MP)$$
$$= (OA)(GM) + \text{Sin AR}$$
$$= -\sin\frac{d(\sin)}{d(\cos)}(OA)^2 + \text{Sin AR}.$$

Hence for the special case of the circle

$$(OA)(GP) = \cos \mathsf{AR} + \operatorname{Sin} \mathsf{AR}$$
$$= \mathsf{AR};$$

hence GP is identical with OP.

Finally, $$OG = OM + MG,$$

$$(OA)(OG) = (OA)(OM) + (OA)(MG)$$
$$= \cos \mathsf{AR} + \sin \frac{d(\sin)}{d(\cos)} (OA)^2,$$

therefore for the special case of the circle

$$(OA)(OG) = \cos \mathsf{AR} - \cos \mathsf{AR}$$
$$= 0.$$

The ratios are defined by taking the ratio of the corresponding area to A^2; thus

$$\operatorname{sct} u = \frac{OD}{OA} = \frac{\mathsf{A}^2}{\cos \mathsf{AR}} = \operatorname{zec} u,$$

$$\operatorname{Tnt} u = \frac{1}{OA} DP = -\frac{(\operatorname{Sin} \mathsf{AR})^2}{\mathsf{A}^2 \cos \mathsf{AR}} + \frac{\operatorname{Sin} \mathsf{AR}}{\mathsf{A}^2} = \operatorname{Dan} u,$$

$$\operatorname{tnt} u = \frac{DP}{OA} = \frac{\sin \mathsf{AR}}{\cos \mathsf{AR}} = \tan u = \operatorname{dan} u,$$

$$\operatorname{Nor} u = \frac{1}{OA} GP = \frac{\mathsf{AR}}{\mathsf{A}^2},$$

$$\operatorname{nor} u = \frac{GP}{OA} = 1,$$

$$\operatorname{anon} u = \frac{OG}{OA} = 0.$$

Answering to each curve-ratio there is a complementary curve-ratio. In Fig. 7 EP is the co-tangent line, and HP represents the co-normal line. For the circle, E coincides with U. Then

$$\text{co-sct}\, u = \frac{OE}{OB}, \quad \text{co-Tnt}\, u = \frac{1}{OB} EP, \quad \text{co-tnt}\, u = \frac{EP}{OB},$$

$$\text{co-Nor}\, u = \frac{1}{OB} HP, \quad \text{co-nor}\, u = \frac{HP}{OB}.$$

CIRCULAR RATIOS.

Function.	Geometric Definition.	Analytical Definition.
u^{u}	$\frac{1}{OA}OP$	$\frac{\mathsf{AR}}{\mathsf{A}^2}$
$\cos u$	$\frac{OM}{OA}$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
$\mathrm{Sin}\, u$	$\frac{1}{OA}MP$	$\frac{\mathrm{Sin}\, \mathsf{AR}}{\mathsf{A}^2}$
$\sin u$	$\frac{MP}{OA}$	$\frac{\sin \mathsf{AR}}{\mathsf{A}^2}$
$\mathrm{Sec}\, u$	$\frac{1}{OA}OT$	$\frac{\mathsf{AR}}{\cos \mathsf{AR}}$
$\sec u$	$\frac{OT}{OA}$	$\frac{\mathsf{A}^2}{\cos \mathsf{AR}}$
$\mathrm{Tan}\, u$	$\frac{1}{OA}AT$	$\frac{\mathrm{Sin}\, \mathsf{AR}}{\cos \mathsf{AR}}$
$\tan u$	$\frac{AT}{OA}$	$\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}}$
$\mathrm{zec}\, u$	$\frac{OQ}{OA}$	$\frac{\mathsf{A}^2}{\cos \mathsf{AR}}$
$\mathrm{Dan}\, u$	$\frac{1}{OA}QP$	$\frac{1}{\mathsf{A}^2}\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}}\mathsf{A}\sqrt{-1}\,\mathsf{R}$
$\mathrm{dan}\, u$	$\frac{QP}{OA}$	$\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}}$
$\mathrm{Gos}\, u$	$\frac{1}{OA}OS$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^4}\mathsf{AR}$
$\mathrm{gos}\, u$	$\frac{OS}{OA}$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
$\mathrm{Zin}\, u$	$\frac{1}{OA}AS$	$\frac{\sin \mathsf{AR}}{\mathsf{A}^4}\mathsf{A}\sqrt{-1}\,\mathsf{R}$
$\mathrm{zin}\, u$	$\frac{AS}{OA}$	$\frac{\sin \mathsf{AR}}{\mathsf{A}^2}$
$\mathrm{vers}\, u$	$\frac{AM}{OA}$	$-1+\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
$\mathrm{sct}\, u$	$\frac{OD}{OA}$	$\frac{\mathsf{A}^2}{\cos \mathsf{AR}}$

CIRCULAR RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
Tnt u	$\frac{1}{OA}DP$	$-\frac{(\text{Sin AR})^2}{\text{A}^2\cos\text{AR}}+\frac{\text{Sin AR}}{\text{A}^2}$
tnt u	$\frac{DP}{OA}$	$\frac{\sin\text{AR}}{\cos\text{AR}}$
Nor u	$\frac{1}{OA}GP$	$\frac{\text{AR}}{\text{A}^2}$
nor u	$\frac{GP}{OA}$	1
co-cos u	$\frac{ON}{OB}$	$\frac{\sin\text{AR}}{\text{A}^2}$
co-Sin u	$\frac{1}{OB}NP$	$\frac{\text{Cos AR}}{\text{A}^2}$
co-sin u	$\frac{NP}{OB}$	$\frac{\cos\text{AR}}{\text{A}^2}$
co-Sec u	$\frac{1}{OB}OV$	$\frac{\overline{\text{A}}\text{R}}{\sin\text{AR}}$
co-sec u	$\frac{OV}{OB}$	$\frac{\text{A}^2}{\sin\text{AR}}$
co-Tan u	$\frac{1}{OB}BV$	$\frac{\text{Cos AR}}{\sin\text{AR}}$
co-tan u	$\frac{BV}{OB}$	$\frac{\cos\text{AR}}{\sin\text{AR}}$
co-zec u	$\frac{OU}{OB}$	$\frac{\text{A}^2}{\sin\text{AR}}$
co-Dan u	$\frac{1}{OB}UP$	$\frac{\cos\text{AR}}{\sin\text{AR}}\ \frac{\overline{\text{A}}\sqrt{-1}\,\text{R}}{\text{A}^2}$
co-dan u	$\frac{UP}{OB}$	$\frac{\cos\text{AR}}{\sin\text{AR}}$
co-Gos u	$\frac{1}{OB}OW$	$\frac{\sin\text{AR}}{\text{A}^2}\ \frac{\overline{\text{A}}\text{R}}{\text{A}^2}$
co-gos u	$\frac{OW}{OB}$	$\frac{\sin\text{AR}}{\text{A}^2}$
co-Zin u	$\frac{1}{OB}BW$	$\frac{\cos\text{AR}}{\text{A}^2}\ \frac{\overline{\text{A}}\sqrt{-1}\,\text{R}}{\text{A}^2}$

CIRCULAR RATIOS (Continued).

Function.	Geometric Definition.	Analytical Definition.
co-zin u	$\frac{BW}{OB}$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
co-sct u	$\frac{OE}{OB}$	$\frac{\mathsf{A}^2}{\sin \mathsf{AR}}$
co-Tnt u	$\frac{1}{OB} EP$	$-\frac{(\mathrm{Cos}\, \mathsf{AR})^2}{\mathsf{A}^2 \sin \mathsf{AR}} + \frac{\mathrm{Cos}\, \mathsf{AR}}{\mathsf{A}^2}$
co-tnt u	$\frac{EP}{OB}$	$\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}}$
co-Nor u	$\frac{1}{OB} HP$	$\frac{\overline{\mathsf{A}}\mathsf{R}}{\mathsf{A}^2}$
co-nor u	$\frac{HP}{OB}$	1

As a test of the accuracy of these definitions, let us consider how they apply to the proof of the addition theorem for two circular sectors having a common plane. Let AOP and POQ be the successive coplanar sectors (Fig. 8); PM and QK are drawn perpendicular to OA, QN is drawn perpendicular to OP, and from the point N so determined NL is drawn perpendicular to OA, and NR perpendicular to QK. By definition,

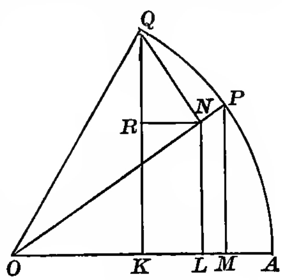

Fig. 8.

$$\cos u = \frac{OM}{OA}, \qquad \sin u = \frac{MP}{OA},$$

$$\cos v = \frac{ON}{OP}, \qquad \sin v = \frac{NQ}{OP},$$

and $$\cos(u+v) = \frac{OK}{OA}, \qquad \sin(u+v) = \frac{KQ}{OA}.$$

Now $$\cos(u+v) = \frac{OK}{OA}$$

$$= \frac{OL}{OA} + \frac{LK}{OA},$$

and $OL = \frac{ON}{OP} OM$ on account of the similarity of the triangles LON and MOP,

and $$LK = NR = \frac{MP}{OP} QN,$$

on account of the similarity of the triangles MOP and RQN, and the negative nature of NR with respect to OA;

therefore $$\cos(u+v) = \frac{ON}{OP}\frac{OM}{OA} + \frac{MP}{OP}\frac{QN}{OA}$$

$$= \frac{OM}{OA}\frac{ON}{OP} - \frac{MP}{OA}\frac{NQ}{OP}$$

$$= \cos u \cos v - \sin u \sin v.$$

In a similar manner

$$\sin(u+v) = \frac{KQ}{OA}$$

$$= \frac{LN}{OA} + \frac{RQ}{OA}$$

$$= \frac{ON}{OA}\frac{MP}{OP} + \frac{NQ}{OP}\frac{OM}{OA}$$

$$= \frac{MP}{OA}\frac{ON}{OP} + \frac{OM}{OA}\frac{NQ}{OP}$$

$$= \sin u \cos v + \cos u \sin v.$$

THE EXCIRCULAR FUNCTIONS.

In this case the bounding line AP (Fig. 9) is part of a rectangular hyperbola or excircle, having OA for principal axis. Let s denote the length of the arc AP, a the length of OA, and A the area of AOP; the analogue of the circular u is no longer

$\frac{s}{a}$, but it still is $\frac{2A}{a^2}$. All the triangular ideas and all the curve ideas which apply to the circle apply also to the excircle, and they are expressed by analogous functions of u. These functions are appropriately denominated by the same names, while for dis-

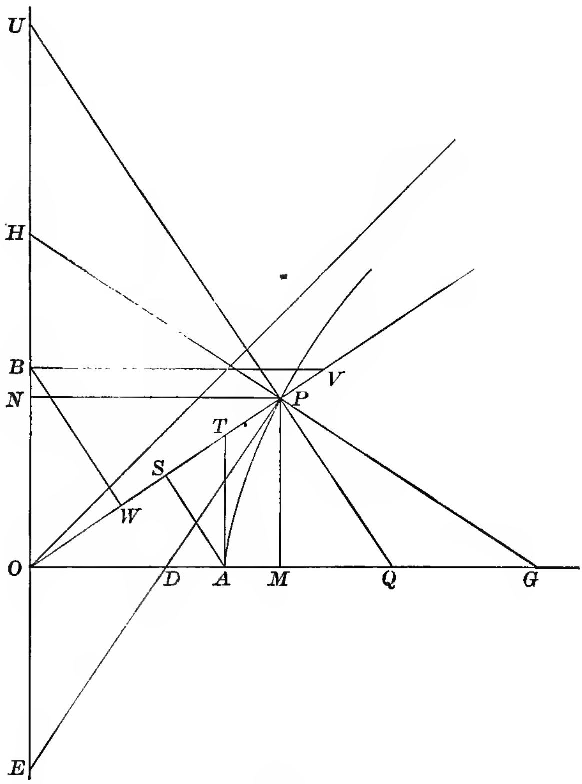

Fig. 9.

tinction the qualification "hyperbolic" is introduced. The abbreviations for the functions are distinguished by an appended h.

The analytical definition is obtained by dividing the corresponding area function by A^2, and adding the condition that

$$(\cos \mathsf{AR})^2 - (\text{Sin } \mathsf{AR})^2 = \mathsf{A}^4.$$

In the case of the excircle

$$DM = \sinh \frac{d(\cosh)}{d(\sinh)} OA$$

$$= \frac{(\sinh u)^2}{\cosh u} OA$$

$$= \frac{(\text{Sin AR})^2}{a \cos \text{AR}}.$$

Consequently, $(OA)(DP) = \frac{(\text{Sin AR})^2}{\cos \text{AR}} + \text{Sin AR},$

and $(OA)(OD) = \cos \text{AR} - \frac{(\text{Sin AR})^2}{\cos \text{AR}}$

$$= \frac{\text{A}^4}{\cos \text{AR}}.$$

Again, for the excircle

$$GM = -\sinh \frac{d(\sinh)}{d(\cosh)} OA$$

$$= -\cosh u \, OA$$

$$= -\frac{\cos \text{AR}}{a};$$

consequently, $(OA)(GP) = -\cos \text{AR} + \text{Sin AR}.$

Hence GP is the reflection of OP with respect to MP,

and $(OA)(OG) = 2 \cos \text{AR}.$

When the radius-vector is subject to the hyperbolic condition, the several lines drawn according to their definitions are all different from one another; from which we see the necessity for these exact definitions.

EXCIRCULAR RATIOS.

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
α^{iu}	$\frac{1}{OA} OP$	$\frac{AR}{A^2}$
cosh u	$\frac{OM}{OA}$	$\frac{\cos AR}{A^2}$
Sinh u	$\frac{1}{OA} MP$	$\frac{\text{Sin } AR}{A^2}$
sinh u	$\frac{MP}{OA}$	$\frac{\sin AR}{A^2}$
Sech u	$\frac{1}{OA} OT$	$\frac{AR}{\cos AR}$
sech u	$\frac{OT}{OA}$	$\frac{\sqrt{A^2R^2}}{\cos AR}$
Tanh u	$\frac{1}{OA} AT$	$\frac{\text{Sin } AR}{\cos AR}$
tanh u	$\frac{AT}{OA}$	$\frac{\sin AR}{\cos AR}$
zech u	$\frac{OQ}{OA}$	$\frac{R^2}{\cos AR}$
Danh u	$\frac{1}{OA} QP$	$\frac{\sin AR}{\cos AR} \frac{A\sqrt{-1}\,R}{A^2}$
danh u	$\frac{QP}{OA}$	$\frac{\sin AR}{\cos AR} \frac{\sqrt{A^2R^2}}{A^2}$
Gosh u	$\frac{1}{OA} OS$	$\frac{\cos AR}{A^2R^2} AR$
gosh u	$\frac{OS}{OA}$	$\frac{\cos AR}{\sqrt{A^2R^2}}$
Zinh u	$\frac{1}{OA} AS$	$\frac{\sin AR}{A^2R^2} A\sqrt{-1}\,R$
zinh u	$\frac{AS}{OA}$	$\frac{\sin AR}{\sqrt{A^2R^2}}$
versh u	$\frac{AM}{OA}$	$-1+\frac{\cos AR}{A^2}$
scth u	$\frac{OD}{OA}$	$\frac{A^2}{\cos AR}$

EXCIRCULAR RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
Tnth u	$\frac{1}{OA}DP$	$\frac{(\text{Sin AR})^2}{A^2 \cos AR} + \frac{\text{Sin AR}}{A^2}$
tnth u	$\frac{DP}{OA}$	$\frac{\sin AR}{\cos AR}\ \frac{\sqrt{A^2R^2}}{A^2}$
Norh u	$\frac{1}{OA}GP$	$-\frac{\cos AR}{A^2} + \frac{\text{Sin AR}}{A^2}$
norh u	$\frac{GP}{OA}$	$\frac{\sqrt{A^2R^2}}{A^2}$
co-cosh u	$\frac{ON}{OB}$	$\frac{\sin AR}{A^2}$
co-Sinh u	$\frac{1}{OB}NP$	$\frac{\text{Cos AR}}{A^2}$
co-sinh u	$\frac{NP}{OB}$	$\frac{\cos AR}{A^2}$
co-Sech u	$\frac{1}{OB}OV$	$\frac{\bar{A}R}{\sin AR}$
co-sech u	$\frac{OV}{OB}$	$\frac{\sqrt{A^2R^2}}{\sin AR}$
co-Tanh u	$\frac{1}{OB}BV$	$\frac{\text{Cos AR}}{\sin AR}$
co-tanh u	$\frac{BV}{OB}$	$\frac{\cos AR}{\sin AR}$
co-zech u	$\frac{OU}{OB}$	$\frac{R^2}{\sin AR}$
co-Danh u	$\frac{1}{OB}UP$	$\frac{\cos AR}{\sin AR}\ \frac{\bar{A}\sqrt{-1}\,R}{A^2}$
co-danh u	$\frac{UP}{OB}$	$\frac{\cos AR}{\sin AR}\ \frac{\sqrt{A^2R^2}}{A^2}$
co-Gosh u	$\frac{1}{OB}OW$	$\frac{\sin AR}{A^2R^2}\bar{A}R$
co-gosh u	$\frac{OW}{OB}$	$\frac{\sin AR}{\sqrt{A^2R^2}}$
co-Zinh u	$\frac{1}{OB}BW$	$\frac{\cos AR}{A^2R^2}\bar{A}\sqrt{-1}\,R$

EXCIRCULAR RATIOS (Continued).

Function.	Geometric Definition.	Analytical Definition.
co-zinh u	$\frac{BW}{OB}$	$\frac{\cos AR}{\sqrt{A^2R^2}}$
co-versh u	$\frac{BN}{OB}$	$-1+\frac{\sin AR}{A^2}$
co-scth u	$\frac{OE}{OB}$	$-\frac{A^2}{\sin AR}$
co-Tnth u	$\frac{1}{OB}EP$	$\frac{(\text{Cos } AR)^2}{A^2 \sin AR}+\frac{\text{Cos } AR}{A^2}$
co-tnth u	$\frac{EP}{OB}$	$\frac{\cos AR}{\sin AR}\,\frac{\sqrt{A^2R^2}}{A^2}$
co-Norh u	$\frac{1}{OB}HP$	$-\frac{\sin AR}{A^2}+\frac{\text{Cos } AR}{A^2}$
co-norh u	$\frac{HP}{OB}$	$\frac{\sqrt{A^2R^2}}{A^2}$

Consider now the proof of the addition theorem for two successive excircular sectors, of which the former starts from the principal axis. Let AOP and POQ be two such sectors (Fig. 10);

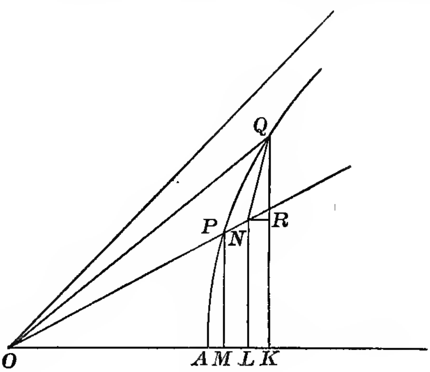

Fig. 10.

the lines PM and QK are drawn perpendicular to OA as before, but QN must now be drawn parallel to the tangent at P; NR is drawn perpendicular to QK as before. Let u denote the ratio to a^2 of twice the area of AOP, and v that of POQ. By definition,

$$\cosh u = \frac{OM}{OA} \quad \text{and} \quad \sinh u = \frac{MP}{OA}.$$

By $\cosh v$ is meant the ratio $\frac{ON}{OP}$, when the sector v is moved back so as to start from OA, the area being retained constant; and by $\sinh v$ is meant the ratio $\frac{NQ}{OP}$ under the same conditions. Now it may be shown that whatever the position of P, these ratios are constant, provided the area of the sector is constant in magnitude; hence,

$$\cosh v = \frac{ON}{OP}, \quad \sinh v = \frac{NQ}{OP}.$$

By the property of the tangent to the curve, the triangles MOP and RQN are similar as before, but now NR is positive with respect to OA. With that modification, the same proof applies as before, giving

$$\frac{OK}{OA} = \frac{OM}{OA}\frac{ON}{OP} + \frac{MP}{OA}\frac{NQ}{OP};$$

that is, $\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v$,

and $$\frac{KQ}{OA} = \frac{MP}{OA}\frac{ON}{OP} + \frac{OM}{OA}\frac{NQ}{OP};$$

that is, $\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v$.

THE LOGARITHMIC FUNCTIONS.

The circle is a special case of the logarithmic spiral, and consequently each circular ratio is a special case of what may be called the *logarithmic* ratio. To understand this generalization it is necessary to observe (*Fundamental Theorems of Analysis generalized for Space*, p. 16), that in the case of the circle, u is not a simple scalar, but the index of an exponential expression α^u, in which α denotes the axis of the plane of the circle. In plane analysis, the α is apt to drop out of sight; but in space analysis it must be introduced explicitly, in order to distinguish one plane from another. The exponential expression α^u is equal to $e^{u\alpha^{\frac{\pi}{2}}}$, and the generalization is obtained by making the angle $\frac{\pi}{2}$ any angle w. Then

$$e^{u\alpha^w} = e^{u\cos w + u\sin w\,.\,\alpha^{\frac{\pi}{2}}}.$$

Now $u \sin w$ expresses the ratio to the square of OA of twice the area of the circular sector AOP', corresponding to the logarithmic sector AOP (Fig. 11); while $e^{u\cos w}$ denotes the manner in which the radius is lengthened.

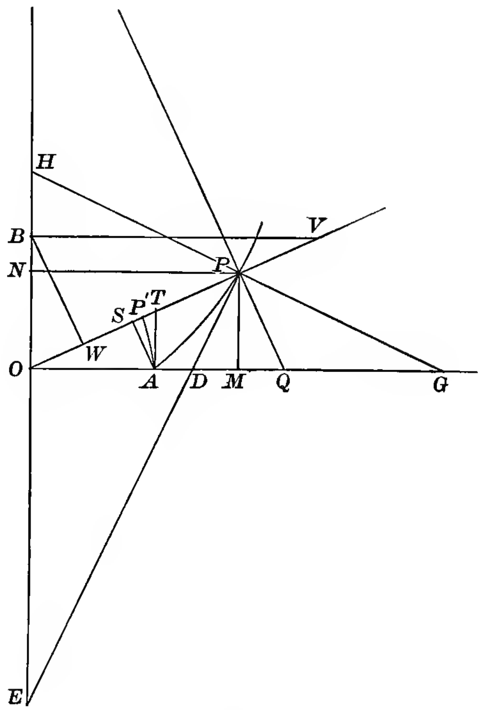

FIG. 11.

The lines PM, PQ, AT, AS, PD, PG, which refer to the axis of OA, are drawn as before; so also the complementary lines which refer to the axis of OB. The geometric definitions of the ratios are the same as before; the analytical definitions are obtained by taking the ratios of the trigonometric areas to A^2, and introducing the special condition,

$$(\cos \mathsf{AR})^2 + (\mathrm{Sin}\, \mathsf{AR})^2 = \mathsf{A}^4 e^{2u\cos w};$$

or, $$\mathsf{R}^2 = \mathsf{A}^2 e^{2u\cos w}.$$

Thus,

$$\cos u, w = \frac{OM}{OA} = \frac{\cos \mathsf{AR}}{\mathsf{A}^2},$$

$$\mathrm{Sin}\, u, w = \frac{1}{OA} MP = \frac{\mathrm{Sin}\, \mathsf{AR}}{\mathsf{A}^2},$$

$$\sin u, w = \frac{MP}{OA} = \frac{\sin \mathsf{AR}}{\mathsf{A}^2},$$

etc., etc., etc.

The series for $\cos u, w$ is

$$1 + u \cos w + \frac{u^2}{2!}\cos 2w + \frac{u^3}{3!}\cos 3w + \text{etc.},$$

and that for $\sin u, w$ is

$$u \sin w + \frac{u^2}{2!}\sin 2w + \frac{u^3}{3!}\sin 3w + \text{etc.}$$

The values of the secant and tangent areas are deduced as before, by finding the value of DM. Now

$$DM = (\sin u, w)\frac{d(\cos u, w)}{d(\sin u, w)} OA,$$

the differentiation being with respect to u; but the ratio of the differentials does not simplify as it does in the special case of the circle.

Similarly, $$GM = -(\sin u, w)\frac{d(\sin u, w)}{d(\cos u, w)} OA.$$

From the areas the ratios are deduced by dividing by A^2.

When the logarithmic ratios are defined in the manner described, the addition theorem remains true. Let u, w denote the initial ratio $\frac{1}{OA}OP$ (Fig. 12), and v, w the subsequent ratio $\frac{1}{OP}OQ$. As in the case of the circle, draw QN perpendicular to

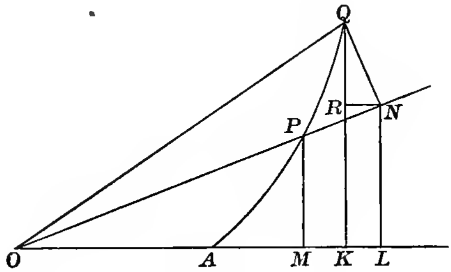

FIG. 12.

OP; PM, QK, NL perpendicular to OA; and NR perpendicular to QK. By definition,

$$\cos u, w = \frac{OM}{OA}, \quad \sin u, w = \frac{MP}{OA},$$

and
$$\cos v, w = \frac{ON}{OP}, \quad \sin v, w = \frac{NQ}{OP}.$$

Now, just as in the special case of the circle, the triangles LON and MOP are similar, and the triangles NQR and POM are similar. Hence, as before,

$$\cos \frac{1}{OA} OQ = \frac{OK}{OA} = \frac{OM}{OA}\frac{ON}{OP} - \frac{MP}{OA}\frac{NQ}{OP},$$

and
$$\sin \frac{1}{OA} OQ = \frac{KQ}{OA} = \frac{MP}{OA}\frac{ON}{OP} + \frac{OM}{OA}\frac{NQ}{OP}.$$

But the versor of $\frac{1}{OA}OQ$ is $\alpha^{u \sin w}\alpha^{v \sin w}$, that is, $\alpha^{(u+v)\sin w}$, and its ratio is $\frac{OP}{OA}\frac{OQ}{OP}$, that is, $e^{(u+v)\cos w}$. Hence $\frac{1}{OA}OQ = u + v, w$. Therefore,

$$\cos u + v, w = \cos u, w \cos v, w - \sin u, w \sin v, w$$

and
$$\sin u + v, w = \sin u, w \cos v, w + \cos u, w \sin v, w.$$

THE ELLIPTIC RATIOS.

Let the bounding line be an ellipse of which OA is the semi-major axis. The ellipse may be regarded as the orthogonal projection of a circle of radius OA upon a plane which passes through OA and makes an angle λ with the plane of the circle. Let $\cos\lambda$ be denoted by k. All lines in the circle parallel to OA remain unaltered in the projection, while all lines perpendicular to OA are diminished by the ratio $\cos\lambda$. Let A denote the area of a sector AOP of the ellipse, and as before let $u = \frac{2A}{a^2}$.

The trigonometric and the curve lines (Fig. 13) are drawn according to the same definitions as before; the geometric defi-

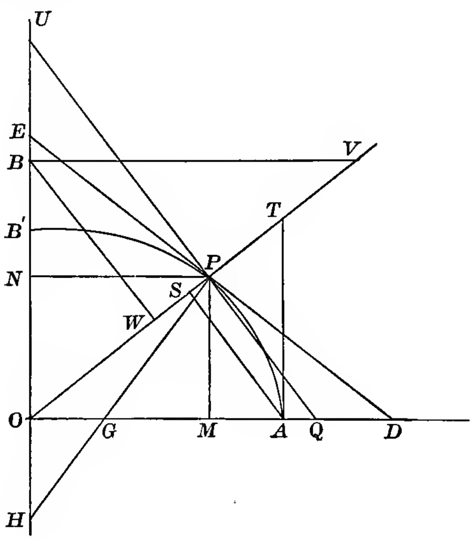

FIG. 13.

nitions of the ratios are the same as before. The analytical definitions of the ratios are obtained by taking the ratio of the corresponding area to A^2, and introducing the special condition that

$$(\cos \mathsf{AR})^2 + \frac{(\operatorname{Sin} \mathsf{AR})^2}{k^2} = \mathsf{A}^4.$$

Thus

$$\cos u, k = \frac{OM}{OA} = \frac{\cos \mathrm{AR}}{\mathrm{A}^2},$$

$$\operatorname{Sin} u, k = \frac{1}{OA} MP = \frac{\operatorname{Sin} \mathrm{AR}}{\mathrm{A}^2},$$

$$\sin u, k = \frac{MP}{OA} = \frac{\sin \mathrm{AR}}{\mathrm{A}^2},$$

etc., etc., etc.

The series for the elliptic cosine is obtained by the principle that $\cos u, k = \cos\frac{u}{k}$, and the series for the elliptic sine by the principle that $\sin u, k = k \sin\frac{u}{k}$.

It is found, by application of the principle stated at p. 25, that

$$DM = -\frac{\sin^2\frac{u}{k}}{\cos\frac{u}{k}} OA,$$

and

$$GM = k^2 \cos\frac{u}{k} OA.$$

Hence

$$(OA)(OD) = \frac{\mathrm{A}^4}{\cos \mathrm{AR}},$$

$$(OA)(DP) = -\frac{1}{k^2}\frac{(\sin \mathrm{AR})^2}{\cos \mathrm{AR}} + \operatorname{Sin} \mathrm{AR},$$

$$(OA)(GP) = k^2 \cos \mathrm{AR} + \operatorname{Sin} \mathrm{AR},$$

$$(OA)(OG) = (1 - k^2) \cos \mathrm{AR},$$

and from these the secant, tangent, normal, and the anonymous ratio are derived by dividing by A^2.

A question arises whether the complementary ratios should be defined with respect to *OB*, Fig. 13, which is equal to *OA*, or with respect to *OB'*, the semi-minor axis. I consider that they ought to be defined with respect to *OB*; the corresponding functions for *OB'* can be deduced from them by dividing by *k*.

In order to obtain the complementary curve ratios it is necessary to find NE and HN.

Now
$$NE = -\cos \frac{d \sin}{d \cos} OB$$

$$= -\cos\frac{u}{k}\,\frac{d\left(k\sin\frac{u}{k}\right)}{d\left(\cos\frac{u}{k}\right)} OB$$

$$= \frac{k\cos^2\frac{u}{k}}{\sin\frac{u}{k}} OB$$

therefore
$$(OB)(NE) = \frac{k^2(\cos \mathsf{AR})^2}{\sin \mathsf{AR}}$$

therefore
$$(OB)(OE) = \sin \mathsf{AR} + \frac{k^2(\cos \mathsf{AR})^2}{\sin \mathsf{AR}}$$

$$= \frac{k^2 \mathsf{A}^4}{\sin \mathsf{AR}},$$

and
$$(OB)(EP) = -\frac{k^2(\cos \mathsf{AR})^2}{\sin \mathsf{AR}} + \mathrm{Cos}\, \mathsf{AR}.$$

Again,
$$HN = -\cos\frac{d\cos}{d\sin} OB$$

$$= \frac{1}{k}\sin\frac{u}{k} OB$$

therefore
$$(OB)(HN) = \frac{\sin \mathsf{AR}}{k^2}$$

therefore
$$(OB)(HP) = \frac{\sin \mathsf{AR}}{k^2} + \mathrm{Cos}\, \mathsf{AR},$$

and
$$(OB)(OH) = -\sin \mathsf{AR}\,\frac{1-k^2}{k^2}.$$

ELLIPTIC RATIOS.

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
$\alpha^{u,k}$	$\frac{1}{OA} OP$	$\frac{\mathsf{AR}}{\mathsf{A}^2}$
$\cos u, k$	$\frac{OM}{OA}$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
$\text{Sin}\, u, k$	$\frac{1}{OA} MP$	$\frac{\text{Sin}\, \mathsf{AR}}{\mathsf{A}^2}$
$\sin u, k$	$\frac{MP}{OA}$	$\frac{\sin \mathsf{AR}}{\mathsf{A}^2}$
$\text{Sec}\, u, k$	$\frac{1}{OA} OT$	$\frac{\mathsf{AR}}{\cos \mathsf{AR}}$
$\sec u, k$	$\frac{OT}{OA}$	$\frac{\sqrt{\mathsf{A}^2\mathsf{R}^2}}{\cos \mathsf{AR}}$
$\text{Tan}\, u, k$	$\frac{1}{OA} AT$	$\frac{\text{Sin}\, \mathsf{AR}}{\cos \mathsf{AR}}$
$\tan u, k$	$\frac{AT}{OA}$	$\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}}$
$\text{zec}\, u, k$	$\frac{OQ}{OA}$	$\frac{\mathsf{R}^2}{\cos \mathsf{AR}}$
$\text{Dan}\, u, k$	$\frac{1}{OA} QP$	$\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}} \frac{\mathsf{A}\sqrt{-1}\,\mathsf{R}}{\mathsf{A}^2}$
$\text{dan}\, u, k$	$\frac{QP}{OA}$	$\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}} \frac{\sqrt{\mathsf{A}^2\mathsf{R}^2}}{\mathsf{A}^2}$
$\text{Gos}\, u, k$	$\frac{1}{OA} OS$	$\frac{\cos \mathsf{AR}}{\mathsf{R}^2} \frac{\mathsf{AR}}{\mathsf{A}^2}$
$\text{gos}\, u, k$	$\frac{OS}{OA}$	$\frac{\cos \mathsf{AR}}{\sqrt{\mathsf{A}^2\mathsf{R}^2}}$
$\text{Zin}\, u, k$	$\frac{1}{OA} AS$	$\frac{\sin \mathsf{AR}}{\mathsf{R}^2} \frac{\mathsf{A}\sqrt{-1}\,\mathsf{R}}{\mathsf{A}^2}$
$\text{zin}\, u, k$	$\frac{AS}{OA}$	$\frac{\sin \mathsf{AR}}{\sqrt{\mathsf{A}^2\mathsf{R}^2}}$
$\text{vers}\, u, k$	$\frac{AM}{OA}$	$-1 + \frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
$\text{sct}\, u, k$	$\frac{OD}{OA}$	$\frac{\mathsf{A}^2}{\cos \mathsf{AR}}$

ELLIPTIC RATIOS (Continued).

Function.	Geometric Definition.	Analytical Definition.
Tnt u, k	$\frac{1}{OA}DP$	$-\frac{(\sin \mathrm{AR})^2}{k^2\mathrm{A}^2\cos \mathrm{AR}}+\frac{\mathrm{Sin\,AR}}{\mathrm{A}^2}$
tnt u, k	$\frac{DP}{OA}$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}}\frac{\sqrt{\sin^2 \mathrm{AR}+k^4\cos^2 \mathrm{AR}}}{k^2\mathrm{A}^2}$
Nor u, k	$\frac{1}{OA}GP$	$\frac{k^2\cos \mathrm{AR}+\mathrm{Sin\,AR}}{\mathrm{A}^2}$
nor u, k	$\frac{GP}{OA}$	$\frac{\sqrt{k^4\cos^2 \mathrm{AR}+\mathrm{Sin}^2\,\mathrm{AR}}}{\mathrm{A}^2}$
co-cos u, k	$\frac{ON}{OB}$	$\frac{\sin \mathrm{AR}}{\mathrm{A}^2}$
co-Sin u, k	$\frac{1}{OB}NP$	$\frac{\mathrm{Cos\,AR}}{\mathrm{A}^2}$
co-sin u, k	$\frac{NP}{OB}$	$\frac{\cos \mathrm{AR}}{\mathrm{A}^2}$
co-Sec u, k	$\frac{1}{OB}OV$	$\frac{\bar{\mathrm{A}}\mathrm{R}}{\sin \mathrm{AR}}$
co-sec u, k	$\frac{OV}{OB}$	$\frac{\sqrt{\mathrm{A}^2\mathrm{R}^2}}{\sin \mathrm{AR}}$
co-Tan u, k	$\frac{1}{OB}BV$	$\frac{\mathrm{Cos\,AR}}{\sin \mathrm{AR}}$
co-tan u, k	$\frac{BV}{OB}$	$\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}}$
co-zec u, k	$\frac{OU}{OB}$	$\frac{\mathrm{R}^2}{\sin \mathrm{AR}}$
co-Dan u, k	$\frac{1}{OB}UP$	$\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}}\frac{\bar{\mathrm{A}}\sqrt{-1}\,\mathrm{R}}{\mathrm{A}^2}$
co-dan u, k	$\frac{UP}{OB}$	$\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}}\frac{\sqrt{\mathrm{A}^2\mathrm{R}^2}}{\mathrm{A}^2}$
co-Gos u, k	$\frac{1}{OB}OW$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2}\frac{\bar{\mathrm{A}}\mathrm{R}}{\mathrm{A}^2}$
co-gos u, k	$\frac{OW}{OB}$	$\frac{\sin \mathrm{AR}}{\sqrt{\mathrm{A}^2\mathrm{R}^2}}$
co-Zin u, k	$\frac{1}{OB}BW$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2}\frac{\bar{\mathrm{A}}\sqrt{-1}\,\mathrm{R}}{\mathrm{A}^2}$

ELLIPTIC RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
co-zin u, k	$\frac{BW}{OB}$	$\frac{\cos \mathrm{AR}}{\sqrt{\mathrm{R}^2\mathrm{A}^2}}$
co-sct u, k	$\frac{OE}{OB}$	$\frac{k^2\mathrm{A}^2}{\sin \mathrm{AR}}$
co-Tnt u, k	$\frac{1}{OB} EP$	$-\frac{k^2(\cos \mathrm{AR})^2}{\mathrm{A}^2 \sin \mathrm{AR}} + \frac{\mathrm{Cos\ AR}}{\mathrm{A}^2}$
co-tnt u, k	$\frac{EP}{OB}$	$\frac{\cos \mathrm{AR}}{\mathrm{A}^2 \sin \mathrm{AR}} \sqrt{k^4 \cos^2 \mathrm{AR} + \sin^2 \mathrm{AR}}$
co-Nor u, k	$\frac{1}{OB} HP$	$\frac{\sin \mathrm{AR}}{k^2\mathrm{A}^2} + \frac{\mathrm{Cos\ AR}}{\mathrm{A}^2}$
co-nor u, k	$\frac{HP}{OB}$	$\frac{\sqrt{k^4 \cos^2 \mathrm{AR} + \sin^2 \mathrm{AR}}}{k^2\mathrm{A}^2}$
anon u, k	$\frac{OH}{OB}$	$-\frac{\sin \mathrm{AR}}{\mathrm{A}^2} \frac{1-k^2}{k^2}$

When the elliptic ratios are so defined it is not difficult to obtain the generalized addition theorem. Let AOP and POQ (Fig. 14) be two successive elliptic sectors of which the former starts from the principal axis. Draw QN parallel to the tangent at P; and PM, QK, NL perpendicular to QA, and NR perpendicular to QK. Let u denote the ratio of twice the area of the sector AOP to the square on OA, and v that of twice the area of the sector POQ to the square on OA; it follows that $u+v$ is the ratio of twice the area of the sector AOQ to the square on OA. By definition

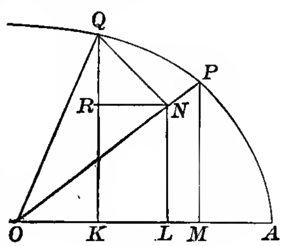

FIG. 14.

$$\cos u, k = \frac{OM}{OA} \qquad \sin u, k = \frac{MP}{OA},$$

and
$$\cos u+v, k = \frac{OK}{OA} \qquad \sin u+v, k = \frac{KQ}{OA}.$$

Now $$\cos v,\, k = \cos\frac{v}{k} = \frac{ON}{OP},$$

because the lines ON and OP have the same direction and therefore the same ratio as the corresponding lines in the circle. But as NQ and OP have different directions, and are in general lines which do not coincide with the principal axes, the relation of their ratio to $\sin\frac{v}{k}$ is more complex. It will be found by examination of the projection that

$$\frac{NQ}{OP}\sqrt{\frac{\cos^2\frac{u}{k} + k^2\sin^2\frac{u}{k}}{\sin^2\frac{u}{k} + k^2\cos^2\frac{u}{k}}} = \sin\frac{v}{k}.$$

For the sake of brevity let the radical be denoted by q. The triangle NQR is no longer similar to the triangle POM; instead of the relation

$$\frac{NR}{NQ} = -\frac{MP}{OP}$$

we have the relation

$$\frac{NR}{NQ} = -\frac{MP}{OP}\frac{q}{k}.$$

Now $$\cos u + v,\, k = \frac{OK}{OL}$$

$$= \frac{OL}{OA} + \frac{LK}{OA}$$

$$= \frac{OM}{OP}\frac{ON}{OA} - \frac{MP}{OP}\frac{NQ}{OA}\frac{q}{k}$$

$$= \frac{OM}{OA}\frac{ON}{OP} - \frac{MP}{OA}\frac{NQ}{OP}\frac{q}{k}$$

$$= \cos u,\, k \cos v,\, k - \frac{\sin u,\, k \sin v,\, k}{k^2}.$$

Again, $\sin u+v,\, k=\frac{KQ}{OA}$

$$=\frac{LN}{OA}+\frac{RQ}{OA}$$

$$=\frac{ON}{OA}\,\frac{MP}{OP}+\frac{OM}{OA}\,\frac{NQ}{OP}kq$$

$$=\frac{MP}{OA}\,\frac{ON}{OP}+\frac{OM}{OA}\,\frac{NQ}{OP}kq$$

$$=\sin u,\, k\cos v,\, k+\cos u,\, k\sin v,\, k.$$

By $\sin v,\, k$ is meant the ratio of NQ to OP when the sector is shifted back without change of area so as to start from the principal axis.

THE HYPERBOLIC RATIOS.

Let the bounding line be an hyperbola of which OA is the semi-major axis. The hyperbola may be regarded as the orthogonal projection of an excircle of radius OA upon a plane which passes through OA and makes an angle λ with the plane of the circle. As before, let $\cos\lambda$ be denoted by k. Let A denote the area of a sector of the hyperbola, and let $u=\frac{2A}{a^2}$.

The triangular and the curve lines are drawn according to the same definitions as before; the geometric definitions of the several functions of u and k are the same as before. The analytical definitions of the ratios are obtained by taking the ratio of the corresponding area to A^2, and introducing the special condition that

$$(\cos \mathsf{AR})^2-\frac{(\mathrm{Sin}\ \mathsf{AR})^2}{k^2}=\mathsf{A}^4.$$

Thus $$\cosh u,\, k=\frac{OM}{OA}=\frac{\cos \mathsf{AR}}{\mathsf{A}^2},$$

etc., etc., etc.

THE COMPLEX RATIOS.

Our method of definition applies also to the complex ratios. Let AOQ (Fig. 15) be a complex sector made up of a circular

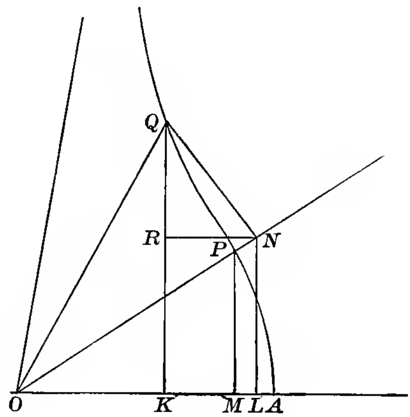

FIG. 15.

sector AOP and an excircular sector POQ. Draw QN perpendicular to OP, and PM, QK, NL perpendicular to OA, also NR perpendicular to QK. Let u denote the ratio of twice the area of AOP to the square on OA, and v that of twice the area of POQ to the square of OP. To distinguish the form of the area let i be prefixed to v; then $u+iv$ denotes the ratio of twice the area of the complex sector AOQ to the square of OA. By definition

$$\cos u = \frac{OM}{OA}, \qquad \sin u = \frac{MP}{OA},$$

$$\cos iv = \frac{ON}{OP}, \qquad \sin iv = \frac{NQ}{OP},$$

$$\cos u+iv = \frac{OK}{OA}, \qquad \sin u+iv = \frac{KQ}{OA}.$$

Now as in the case of the circle

$$\frac{OK}{OA} = \frac{OM}{OA}\,\frac{ON}{OP} - \frac{MP}{OA}\,\frac{NQ}{OP},$$

therefore $$\cos u + iv = \cos u \cos iv - \sin u \sin iv$$

$$= \cos u \cosh v - \sin u \sinh v.$$

Similarly $$\frac{KQ}{OA} = \frac{MP}{OA}\frac{ON}{OP} + \frac{OM}{OA}\frac{NQ}{OP}$$

$$= \sin u \cos iv + \cos u \sin iv$$

$$= \sin u \cosh v + \cos u \sinh v.$$

The function $\cos iv$ is obtained from $\cos v$ by supposing $i = \sqrt{-1}$; and $\sin iv$ from $\sin v$ by the same process, only the $\sqrt{-1}$ common to all the terms must be removed.

From the symmetry of the formulæ it is evident that the order of circular-excircular or excircular-circular is indifferent.

THE PRINCIPLES

OF

ELLIPTIC AND HYPERBOLIC ANALYSIS

BY

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THE PRINCIPLES OF ELLIPTIC AND HYPERBOLIC ANALYSIS.

[ABSTRACT READ BEFORE THE MATHEMATICAL CONGRESS AT CHICAGO, AUGUST 24, 1893.*]

IN several papers recently published, entitled "Principles of the Algebra of Physics," "The Imaginary of Algebra," and "The Fundamental Theorems of Analysis generalized for Space," I have considered the principles of vector analysis; and also the principles of versor analysis, the versor being circular, logarithmic, or equilateral-hyperbolic. In the present paper, I propose to consider the versor part of space analysis more fully, and to extend the investigation to elliptic and hyperbolic versors. The order of the investigation is as follows: The fundamental theorem of trigonometry is investigated for the sphere, the ellipsoid of revolution, and the general ellipsoid; then for the equilateral hyperboloid of two sheets, the equilateral hyperboloid of one sheet, and the general hyperboloid. Subsequently, the principles arrived at are applied to find the complete form of other theorems in spherical trigonometry, and to deduce the generalized theorems for the ellipsoid and the hyperboloid. At the end, the analogues of the rotation theorem are deduced.

FUNDAMENTAL THEOREM FOR THE SPHERE.

Let α^A and β^B denote any two spherical versors; their planes will intersect in the axis which is perpendicular to α and β, and

* Jan. 8, 1894. I have rewritten and extended the original paper so as to include the trigonometry of the general ellipsoid and hyperboloid. At the time of reading the paper, I had discovered how to make this extension, but had not had time to work it out.

which we denote by $\overline{\alpha\beta}$. Let OPA (Fig. 1) represent α^A, and OAQ represent β^B; then OPQ, the third side of the spherical triangle, represents the product $\alpha^A\beta^B$.

To prove that

$$\alpha^A\beta^B = \cos A \cos B - \sin A \sin B \cos \alpha\beta$$
$$+ \{\cos B \sin A \cdot \alpha + \cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta}\}^{\frac{\pi}{2}}.$$

The first part of this proposition, namely, that

$$\cos \alpha^A\beta^B = \cos A \cos B - \sin A \sin B \cos \alpha\beta,$$

is equivalent to the well-known fundamental theorem of Spherical Trigonometry; the only difference is, that $\alpha\beta$ denotes, not the angle included by the sides, but the angle between the planes; or, to speak more accurately, the angle between the axes α and β. It is more difficult to prove the complementary proposition, namely, that

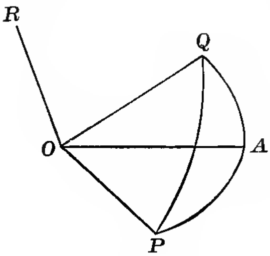

FIG. 1.

$$\text{Sin}\, \alpha^A\beta^B = \cos B \sin A \cdot \alpha + \cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta},$$

for it is necessary to prove, not only that the magnitude of the right-hand member is equal to $\sqrt{1 - \cos^2 \alpha^A\beta^B}$, but also that its direction coincides with the axis normal to the plane of OPQ. At page 7 of "*Fundamental Theorems*," I have proved the above statement as regards the magnitude, but I was then unable to give a general proof as regards the axis. Now, however, I am able to supply a general proof, and it will be found of the highest importance in the further development of the analysis.

In Fig. 1, OP is the initial line of α^A, and OQ the terminal line of β^B; let OR be drawn equal to

$$\cos B \sin A \cdot \alpha + \cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta};$$

it is required to prove that OR is perpendicular to OP and to OQ.

Now, $$OP = \alpha^{-A}\,\overline{\alpha\beta} = (\cos A - \sin A \cdot \alpha^{\frac{\pi}{2}}) \cdot \overline{\alpha\beta}$$
$$= \cos A \cdot \overline{\alpha\beta} - \sin A \cdot \alpha^{\frac{\pi}{2}}\, \overline{\alpha\beta}.$$

Similarly, $OQ = \beta^B \overline{\alpha\beta} \;= (\cos B + \sin B \cdot \beta^{\frac{\pi}{2}}) \cdot \overline{\alpha\beta}$

$$= \cos B \cdot \overline{\alpha\beta} + \sin B \cdot \beta^{\frac{\pi}{2}} \overline{\alpha\beta}.$$

By $\alpha^{\frac{\pi}{2}} \overline{\alpha\beta}$ is meant the axis which is perpendicular to α and β, after it is rotated by a quadrant round α. In Fig. 2, let OA and OB represent α and β, any two axes drawn from O, then $\overline{\alpha\beta}$ is drawn from O upwards, normal to the plane of the paper. Hence $\alpha^{\frac{\pi}{2}} \overline{\alpha\beta}$ is OL, which is of unit length, and drawn in the plane of the paper, perpendicular to α. It is required to find the components of OL along α and β. Draw LN parallel to β, and LM parallel to α. Now OM or NL is $-\frac{1}{\sin \alpha\beta} \cdot \beta$, and ON is $\frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \alpha$; hence,

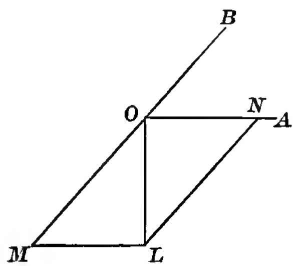

FIG. 2.

$$\alpha^{\frac{\pi}{2}} \overline{\alpha\beta} = \frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \alpha - \frac{1}{\sin \alpha\beta} \cdot \beta.$$

Similarly, $\beta^{\frac{\pi}{2}} \overline{\alpha\beta} = -\beta^{\frac{\pi}{2}} \overline{\beta\alpha} = -\frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \beta + \frac{1}{\sin \alpha\beta} \cdot \alpha.$

Consequently, the three lines expressed in terms of the axes α, β, and $\overline{\alpha\beta}$, are

$$OR = \quad \cos B \sin A \;\cdot \alpha + \cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta};$$

$$OP = -\sin A \frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \alpha + \sin A \frac{1}{\sin \alpha\beta} \cdot \beta + \cos A \cdot \overline{\alpha\beta};$$

$$OQ = \quad \sin B \frac{1}{\sin \alpha\beta} \;\cdot \alpha - \sin B \frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \beta + \cos B \cdot \overline{\alpha\beta}.$$

Hence $\cos(OR)(OP)$

$$= -\cos B \sin^2 A \left(\frac{\cos \alpha\beta}{\sin \alpha\beta} - \frac{\cos \alpha\beta}{\sin \alpha\beta}\right)$$

$$- \cos A \sin A \sin B \left(\frac{\cos^2 \alpha\beta}{\sin \alpha\beta} - \frac{1}{\sin \alpha\beta} + \sin \alpha\beta\right)$$

$$= 0.$$

Similarly, it may be shown that $\cos(OR)(OQ) = 0$; hence OR has the direction of the normal to the plane of OPQ.

To find the general expression for a spherical versor, when reference is made to a principal axis.

Let OA represent the principal axis (Fig. 3), and let it be denoted by α. Any versor OPA, which passes through the principal axis, may be denoted by β^u, where β denotes a unit axis perpendicular to α. Similarly, OAQ, another versor passing through the principal axis, may be denoted by γ^v, where γ denotes a unit axis perpendicular to α. The product versor OPQ is circular, but it will not, in general, pass through OA; let it be denoted by ξ^θ. Now

$$\begin{aligned}\xi^\theta &= \beta^u \gamma^v \\ &= \cos u \cos v - \sin u \sin v \cos \beta\gamma \\ &\quad + \{\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma - \sin u \sin v \sin \beta\gamma \cdot \overline{\beta\gamma}\}^{\frac{\pi}{2}}.\end{aligned}$$

We observe that the directed sine may be broken up into two components, namely, $\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma$, which is perpendicular to the principal axis, and $-\sin u \sin v \sin \beta\gamma \cdot \overline{\beta\gamma}$, which has the direction of the negative of the principal axis, for $\overline{\beta\gamma} = \alpha$.

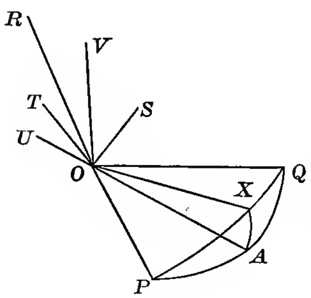

FIG. 3.

Draw OS to represent the first component $\cos v \sin u \cdot \beta$, OT to represent the second component $\cos u \sin v \cdot \gamma$, and OU to represent the third component $-\cos u \cos v \sin \beta\gamma \cdot \alpha$. Draw OV, the resultant of the first two, and OR, the resultant of all three. The plane of OA and OV passes through OR, which is normal to the plane OPQ; hence these planes cut at right angles in a line OX; and the angle between OA and OX is equal to that between OV and OR, for OV is perpendicular to OA, and OR to OX. Let ϕ denote the angle AOX, then

$$\cos \phi = \frac{\sqrt{\cos^2 v \sin^2 u + \cos^2 u \sin^2 v + 2 \cos u \cos v \sin u \sin v \cos \beta\gamma}}{\sqrt{1 - (\cos u \cos v - \sin u \sin v \cos \beta\gamma)^2}},$$

and

$$\sin \phi = \frac{\sin u \sin v \sin \beta\gamma}{\sqrt{1 - (\cos u \cos v - \sin u \sin v \cos \beta\gamma)^2}}.$$

Figure 4 represents a section through the plane of OA and OV. Let XM be drawn from X perpendicular to OA; it is equal in magnitude to $\sin\phi$; and OM is equal in magnitude to $\cos\phi$.

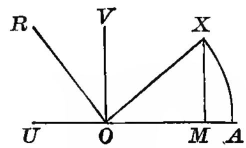

FIG. 4.

Hence the axis ξ has the form

$$\cos\phi\cdot\epsilon - \sin\phi\cdot\alpha,$$

where ϵ denotes a unit axis perpendicular to α. And

$$\xi^\theta = \cos\theta + \sin\theta(\cos\phi\cdot\epsilon - \sin\phi\cdot\alpha)^{\frac{\pi}{2}}$$

is determined by the equations,

$$\cos\theta = \cos u\cos v - \sin u\sin v\cos\beta\gamma, \quad (1)$$

$$\sin\theta\sin\phi = \sin u\sin v\sin\beta\gamma, \quad (2)$$

$$\sin\theta\cos\phi\cdot\epsilon = \cos v\sin u\cdot\beta + \cos u\sin v\cdot\gamma. \quad (3)$$

The unit axis ϵ may be expressed in terms of two axes β and γ, which are at right angles to one another and to α, and the angle which ϵ makes with β. Hence the more general expression for any spherical versor is

$$\xi^\theta = \cos\theta + \sin\theta\{\cos\phi(\cos\psi\cdot\beta + \sin\psi\cdot\gamma) - \sin\phi\cdot\alpha\}^{\frac{\pi}{2}}.$$

We observe that the line OX is the principal axis of the product versor POQ.

To find the product of two spherical versors of the general form given above.

The two factor versors may be expressed by

$$\xi^u = \cos u + \sin u(\cos\phi\cdot\beta - \sin\phi\cdot\alpha)^{\frac{\pi}{2}},$$

and $$\eta^v = \cos v + \sin v\,(\cos\phi'\cdot\gamma - \sin\phi'\cdot\alpha)^{\frac{\pi}{2}},$$

where β and γ denote any unit axes perpendicular to α. The product has the form

$$\zeta^w = \cos w + \sin w(\cos\phi''\cdot\gamma - \sin\phi''\cdot\alpha)^{\frac{\pi}{2}}.$$

Since $$\xi^u\eta^v = \cos u\cos v - \sin u\sin v\cos\xi\eta$$
$$+\{\cos v\sin u\cdot\xi + \cos u\sin v\cdot\eta - \sin u\sin v\sin\xi\eta\cdot\overline{\xi\eta}\}^{\frac{\pi}{2}},$$

and $$\cos \xi\eta = \cos\phi \cos\phi' \cos\beta\gamma + \sin\phi \sin\phi',$$

and $$\operatorname{Sin} \xi\eta = \cos\phi \cos\phi' \sin\beta\gamma \cdot \overline{\beta\gamma} - (\cos\phi \sin\phi' \cdot \overline{\beta\alpha} + \cos\phi' \sin\phi \cdot \overline{\alpha\gamma}),$$

therefore $$\cos w = \cos u \cos v - \sin u \sin v (\cos\phi \cos\phi' \cos\beta\gamma + \sin\phi \sin\phi'), \quad (1)$$

$$\sin w \sin\phi'' = \cos u \sin v \sin\phi' + \cos v \sin u \sin\phi + \sin u \sin v \cos\phi \cos\phi' \sin\beta\gamma, \quad (2)$$

$$\sin w \cos\phi'' \cdot \epsilon = \cos u \sin v \cos\phi' \cdot \gamma + \cos v \sin u \cos\phi \cdot \beta + \sin u \sin v (\cos\phi \sin\phi' \cdot \overline{\beta\alpha} + \cos\phi' \sin\phi \cdot \overline{\alpha\gamma}). \quad (3)$$

From equation (1) we obtain w, then from (2) we obtain ϕ'', and finally from (3) we obtain ϵ.

When the factor versors are restricted to one plane, the axes coincide; that is, $\eta = \xi$. The above formula then becomes

$$\xi^{\theta+\theta'} = \cos\theta \cos\theta' - \sin\theta \sin\theta' + (\cos\theta \sin\theta' + \cos\theta' \sin\theta) \{\cos\phi \cdot \beta - \sin\phi \cdot \alpha\}^{\frac{\pi}{2}},$$

which is the fundamental theorem for trigonometry in any plane.

When the axes are coplanar with the initial line, we have γ identical with β, but ϕ', in general, different from ϕ. The theorem then becomes

$$\begin{aligned}\xi^{\theta}\eta^{\theta'} = {} & \cos\theta \cos\theta' - \sin\theta \sin\theta' \cos(\phi' - \phi) \\ & + \{(\cos\theta \sin\theta' \cos\phi' + \cos\theta' \sin\theta \cos\phi) \cdot \beta \\ & + \sin\theta \sin\theta' \sin(\phi' - \phi) \cdot \overline{\beta\alpha} \\ & - (\cos\theta \sin\theta' \sin\phi' + \cos\theta' \sin\theta \cos\phi) \cdot \alpha\}^{\frac{\pi}{2}}.\end{aligned}$$

If, in addition, the middle term of the sine vanishes, the axis of the product will also be in the same plane with the other axes and the initial line.

To prove that the sum of the squares of the three components of the product of two general spherical versors is unity.

For shortness, let $x = \cos\theta$, $y = \sin\theta \cos\phi$, $z = \sin\theta \sin\phi$; $x' = \cos\theta'$, $y' = \sin\theta' \cos\phi'$, $z' = \sin\theta' \sin\phi'$. Then

$$\cos^2\theta'' = (xx' - yy'\cos\beta\gamma - zz')^2$$
$$= x^2x'^2 + y^2y'^2\cos^2\beta\gamma + z^2z'^2 - 2xx'yy'\cos\beta\gamma - 2xx'zz'$$
$$+ 2yy'zz'\cos\beta\gamma,$$

$$(\sin\theta''\cos\phi''\cdot\epsilon)^2 = \{xy'\cdot\gamma + x'y\cdot\beta + yz'\cdot\overline{\beta\alpha} - zy'\cdot\overline{\gamma\alpha}\}^2$$
$$= x^2y'^2 + x'^2y^2 + y^2z'^2 + z^2y'^2 + 2xx'yy'\cos\beta\gamma + 2xyy'z'\cos\gamma\overline{\beta\alpha}$$
$$- 2yz'x'y'\cos\beta\overline{\gamma\alpha} - 2yzy'z'\cos\overline{\beta\alpha}\cdot\overline{\gamma\alpha},$$

$$(\sin\theta''\sin\phi'')^2 = \{xz' + x'z + yy'\sin\beta\gamma\}^2$$
$$= x^2z'^2 + z^2x'^2 + y^2y'^2\sin^2\beta\gamma + 2xx'zz' + 2xyy'z'\sin\beta\gamma$$
$$+ 2x'yy'z\sin\beta\gamma.$$

The sum of the square terms is $(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2)$, that is, 1; and the sum of the product terms reduces to

$$2yy'zz'(\cos\beta\gamma - \cos\overline{\beta\alpha}\cdot\overline{\gamma\alpha}) + 2xyy'z'(\cos\gamma\overline{\beta\alpha} + \sin\beta\gamma)$$
$$- 2yz'x'y'(\cos\beta\overline{\gamma\alpha} - \sin\beta\gamma).$$

Now, β and γ both being perpendicular to α, $\cos\beta\gamma = \cos\overline{\beta\alpha}\cdot\overline{\gamma\alpha}$, and $\sin\beta\gamma = -\cos\gamma\overline{\beta\alpha} = \cos\beta\overline{\gamma\alpha}$. Hence the sum of the product terms vanishes.

FUNDAMENTAL THEOREM FOR THE ELLIPSOID OF REVOLUTION.

Imagine a circle APB (Fig. 5) to be projected on the plane of AQB, by means of lines drawn from the points of the circle, perpendicular to the plane, as PQ from P; the projection of the circle is an ellipse, having the initial line for semi-major axis. Let λ denote the axis of the circle, and β that of the plane; all lines perpendicular to the initial line are in the projected figure, diminished by the ratio $\cos\lambda\beta$, while all lines parallel to the initial line remain unaltered. Any area A in the circle will be changed into $A\cos\lambda\beta$ in the ellipse; and this is true whatever the form of the area. For shortness, $\cos\lambda\beta$ will be denoted by k.

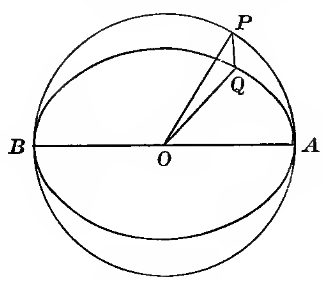

FIG. 5.

The projecting lines, instead of being drawn perpendicular to the plane of projection, may be drawn perpendicular to the plane of the circle; the ratio of projection then becomes $\sec\lambda\beta$, which may likewise be denoted by k, but k is then always greater than unity. The figure obtained is an ellipse, having the initial line for semi-minor axis. By the revolution of the former ellipse round the initial line we obtain a prolate ellipsoid; by the revolution of the latter, an oblate ellipsoid.

The Fundamental Equation of Elliptic Trigonometry.

The elliptic versor is expressed by $\frac{1}{OA}OP$ (Fig. 6), and

$$\frac{1}{OA}OP = \frac{OM}{OA} + \frac{1}{OA}MP.$$

The problem is, to find the correct analytical expressions for these three terms. If by u we denote the ratio of twice the area of the sector AOP to the square on OA, then,

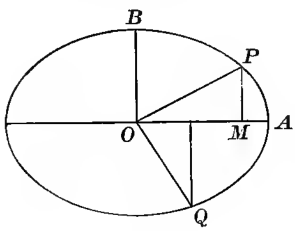

Fig. 6.

$$\frac{OM}{OA} = \cos\frac{u}{k} \text{ and } \frac{MP}{OA} = k\sin\frac{u}{k}.$$

Hence, if β denote a unit axis normal to the plane of the ellipse, the equation may be written

$$(k\beta)^u = \cos\frac{u}{k} + \sin\frac{u}{k}\cdot(k\beta)^{\frac{\pi}{2}}.$$

But we observe that it is much simpler to define u as the ratio of twice the area of AOP to the rectangle formed by OA and OB, the semi-axes; for then we have

$$(k\beta)^u = \cos u + \sin u\cdot(k\beta)^{\frac{\pi}{2}}.$$

We attach the k to the axis rather than to the ratio, because in forming a product of versors it does not enter as an ordinary multiplier. When the elliptic sector does not start from the principal axis, the element u must still be taken as the ratio of twice the area of the sector to the rectangle formed by the axes. The index $\frac{\pi}{2}$ is due to the rectangular nature of the components; it expresses the circular versor between OA and MP. When oblique components are used, the index is then w, the angle of the obliquity. This is proved in *Fundamental Theorems*, page 10.

To find the product of two elliptic versors which are in one plane passing through the principal axis.

Let the two versors be represented by OQA and OAP (Fig. 6); then their product is represented by OQP. Let β denote a unit axis normal to the plane; the former versor may be denoted by $(k\beta)^u$, and the latter by $(k\beta)^v$. Then

$$\begin{aligned}(k\beta)^u(k\beta)^v &= \{\cos u + \sin u \cdot (k\beta)^{\frac{\pi}{2}}\}\{\cos v + \sin v \cdot (k\beta)^{\frac{\pi}{2}}\} \\ &= \cos u \cos v + \cos u \sin v \cdot (k\beta)^{\frac{\pi}{2}} + \cos v \sin u \cdot (k\beta)^{\frac{\pi}{2}} \\ &\quad + \sin u \sin v \cdot (k\beta)^{\frac{\pi}{2}}(k\beta)^{\frac{\pi}{2}}.\end{aligned}$$

Now $(k\beta)^u(k\beta)^v = (k\beta)^{u+v}$

$$\begin{aligned}&= \cos(u+v) + \sin(u+v) \cdot (k\beta)^{\frac{\pi}{2}} \\ &= \cos u \cos v - \sin u \sin v \\ &\quad + (\cos u \sin v + \cos v \sin u) \cdot (k\beta)^{\frac{\pi}{2}}.\end{aligned}$$

Hence $(k\beta)^{\frac{\pi}{2}}(k\beta)^{\frac{\pi}{2}} = \beta^{\pi} = -1$. From this we infer that k is such a multiplier that it does not affect the terms of the cosine.

To find the product of two elliptic versors which intersect in the principal axis of the ellipsoid of revolution.

Let $\frac{1}{OP}OA$ and $\frac{1}{OA}OQ$ (Fig. 7) represent the two versors; their axes are β and γ, respectively, each being perpendicular to α, the direction of the principal axis OA. Let u denote the ratio of twice the area of OPA to the rectangle formed by the semi-axes of its ellipse, and v the ratio of twice the area of OAQ to the rectangle formed by the semi-axes of its ellipse. The versors are denoted by $(k\beta)^u$ and $(k\gamma)^v$. Now

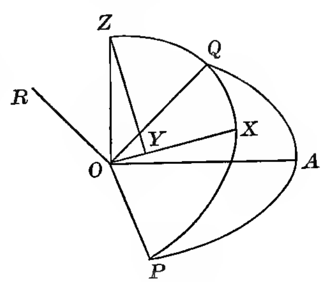

FIG. 7.

$$(k\beta)^u = \cos u + \sin u \cdot (k\beta)^{\frac{\pi}{2}},$$

and $(k\gamma)^v = \cos v + \sin v \cdot (k\gamma)^{\frac{\pi}{2}}$,

therefore $(k\beta)^u(k\gamma)^v = \cos u \cos v + \cos v \sin u \cdot (k\beta)^{\frac{\pi}{2}}$

$$+ \cos u \sin v \cdot (k\gamma)^{\frac{\pi}{2}} + \sin u \sin v \cdot (k\beta)^{\frac{\pi}{2}}(k\gamma)^{\frac{\pi}{2}}.$$

By means of the principle that the first power of k is k, we see that the second and third terms contribute

$$k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma)$$

to the Sine component. It remains to determine the meaning of the fourth term, that is, the values of the coefficients x and y in the equation

$$(k\beta)^{\frac{\pi}{2}}(k\gamma)^{\frac{\pi}{2}} = x \cos \beta\gamma + y \sin \beta\gamma \cdot \overline{\beta\gamma}^{\frac{\pi}{2}}.$$

From the form of the product of two coplanar versors (page 9), it appears that x is -1; the value of y appears to be either $-k^2$ or -1.

On the former hypothesis the directed sine OR would be

$$k \cos v \sin u \cdot \beta + k \cos u \sin v \cdot \gamma - k^2 \sin u \sin v \sin \beta\gamma \cdot \alpha.$$

Now $\qquad OP = \cos u \cdot \alpha - k \sin u \cdot \beta^{\frac{\pi}{2}} \overline{\beta\gamma},$

and $\qquad OQ = \cos v \cdot \alpha + k \sin v \cdot \gamma^{\frac{\pi}{2}} \overline{\beta\gamma};$

consequently $\qquad \cos(OR)(OP) = -k^2 \cos v \sin^2 u \left(\frac{\cos \beta\gamma}{\sin \beta\gamma} - \frac{\cos \beta\gamma}{\sin \beta\gamma}\right)$

$$- k^2 \cos u \sin u \sin v \left(\frac{\cos^2 \beta\gamma}{\sin^2 \beta\gamma} - \frac{1}{\sin \beta\gamma} + \sin \beta\gamma\right),$$

which vanishes, as before (page 3). Similarly $\cos(OR)(OQ) = 0$. Hence the above expression gives the direction of the normal to the plane of the product versor. But suppose that $\frac{1}{OP} OA$ and $\frac{1}{OA} OQ$ are quadrantal elliptic versors, then $\cos u = \cos v = 0$, and $\sin u = \sin v = 1$; consequently the cosine of the product would then be $-\cos \beta\gamma$ and the sine of the product $-k^2 \sin \beta\gamma \cdot \alpha^{\frac{\pi}{2}}$. But it is evident that in this case the product versor is circular, namely, $-(\cos \beta\gamma + \sin \beta\gamma \cdot \alpha^{\frac{\pi}{2}})$. Hence it appears that k^2 cannot enter as a factor of the third term of the Sine.

On the other hypothesis the directed sine is

$$k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma) - \sin u \sin v \sin \beta\gamma \cdot \alpha.$$

This expression satisfies the test of becoming circular under the conditions mentioned; but its direction is not normal to the plane of the product versor. How then, is its direction related to that plane? It will be found that it has the direction of the conjugate axis to the plane. Draw OV (Fig. 8), to represent $k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma)$, the component perpendicular to the principal axis OA, and OU' in the direction opposite to the

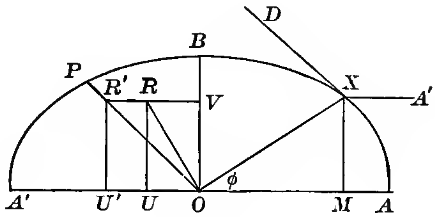

FIG. 8.

principal axis to represent $-\sin u \sin v \sin \beta\gamma$, also OU to represent the same quantity multiplied by k^2; and draw OR' and OR, the two resultants. The plane through OA and OV will cut the ellipsoid in a principal ellipse AXB, and as it passes through the normal OR it will cut the plane of the product ellipse at right angles; let OX denote the line of intersection. Draw XA' parallel to OA and XD the tangent at X, and let θ denote the circular versor between AO and OX. Now

$$\tan\theta = \frac{MX}{OM} = \frac{OU}{OV}$$

$$= \frac{k \sin u \sin v \sin \beta\gamma}{\sqrt{\cos^2 v \sin^2 u + \cos^2 u \sin^2 v + 2 \cos u \cos v \sin u \sin v \cos \beta\gamma}};$$

but $\tan A'XD = -k^2 \operatorname{cotan}\theta$

$$= -\frac{k\sqrt{\cos^2 v \sin^2 u + \cos^2 u \sin^2 v + 2 \cos u \cos v \sin u \sin v \cos \beta\gamma}}{\sin u \sin v \sin \beta\gamma}$$

$$= \operatorname{cotan} VOR' = \tan AOR'.$$

Thus the direction of OR' is that of the conjugate axis of the plane of the product versor.

Let ϕ denote the ratio of twice the area of AOX to the square of OA; it is equal to the angle which OX made with OA before the contraction. The direction of the axis was then $\cos\phi$ along OB, and $\sin\phi$ along OA'; by the contraction, $\cos\phi$ has been changed into $k \cos\phi$; hence the axis of the ellipsoid, along the direction of OR', is $k \cos\phi \cdot \epsilon - \sin\phi \cdot \alpha$, where ϵ denotes a unit axis in the direction of OB.

The magnitude of the product versor is determined by the cosine function,

$$\cos u \cos v - \sin u \sin v \cos \beta\gamma.$$

Suppose that an elliptic sector OXZ (Fig. 7), having the area of the third side of the ellipsoidal triangle, starts from the semi-major axis OX, and let OY and OZ be the rectangular projections of the bounding radius vector OZ. As the small ellipse OPQ is derived from a principal ellipse by diminishing all lines parallel to OX in the ratio of OX to OA, that is, in the ratio of $\sqrt{\cos^2\phi + k^2 \sin^2\phi}$ to 1, while the transverse lines remain unaltered; the ratio of OY to OX is equal to the corresponding ratio in the principal ellipse; hence the ratio of OY to OX is equal to $\cos u \cos v - \sin u \sin v \cos \beta\gamma$.

Let w denote the ratio of twice the sector OPQ to the rectangle formed by OX and the minor semi-axis of the ellipse OPQ; this ratio is equal to the ratio of twice the corresponding circular sector to the square of OA. By the corresponding circular sector is meant that circular sector from which the elliptic sector was formed by contraction along the two axes. Also, let ξ denote the elliptic axis, $\cos\phi \cdot k\epsilon - \sin\phi \cdot \alpha$. The product versor then takes the form

$$\xi^w = \cos w + \sin w (\cos\phi \cdot k\epsilon - \sin\phi \cdot \alpha)^{\frac{\pi}{2}},$$

the quantities w, ϕ, and ϵ being determined by

$$\cos w = \cos u \cos v - \sin u \sin v \cos \beta\gamma, \qquad (1)$$

$$\sin\phi = \frac{\sin u \sin v \sin \beta\gamma}{\sqrt{1 - \cos^2 w}}, \qquad (2)$$

$$\epsilon = \frac{\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma}{\sin w \cos\phi}. \qquad (3)$$

Consequently we have for the elliptic axis OP,

$$\xi = \frac{k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma) - \sin u \sin v \sin \beta\gamma \cdot \alpha}{\sqrt{1 - \cos^2 w}}.$$

The locus of the poles of the several elliptic areas is the original ellipsoid.

To find the product of two ellipsoidal versors of the above general form.

The two factor versors are expressed by

$$\xi^u = \cos u + \sin u(\cos\phi\cdot k\beta - \sin\phi\cdot\alpha)^{\frac{\pi}{2}},$$

and $$\eta^v = \cos v + \sin v(\cos\phi'\cdot k\gamma - \sin\phi'\cdot\alpha)^{\frac{\pi}{2}};$$

it is required to show that their product has the form

$$\zeta^w = \cos w + \sin w(\cos\phi''\cdot k\epsilon - \sin\phi''\cdot\alpha)^{\frac{\pi}{2}}.$$

We have

$$\begin{aligned}\xi^u\eta^v &= (\cos u + \sin u\cdot\xi^{\frac{\pi}{2}})(\cos v + \sin v\cdot\eta^{\frac{\pi}{2}})\\ &= \cos u\cos v - \sin u\sin v\cos\xi\eta\\ &\quad + \{\cos u\sin v\cdot\eta + \cos v\sin u\cdot\xi - \sin u\sin v\,\mathrm{Sin}\,\xi\eta\}^{\frac{\pi}{2}}.\end{aligned}$$

The problem is reduced to finding the value of $\cos\xi\eta$ and $\mathrm{Sin}\,\xi\eta$. Now $\xi\eta$ means the elliptic versor between the elliptic axes

$$\cos\phi\cdot k\beta - \sin\phi\cdot\alpha \quad\text{and}\quad \cos\phi'\cdot k\gamma - \sin\phi'\cdot\alpha.$$

To find them, we apply the following principle:

Restore the elliptic axes to their spherical originals, find the versor between these unit axes according to the ordinary rule, and reduce its axes back to the ellipsoidal form. Applied to the above, the rule means: suppose $k=1$, form the cosine and the directed Sine, and introduce k as a multiplier of those components of the directed Sine which are perpendicular to α. Hence

$$\cos\xi\eta = \cos\phi\cos\phi'\cos\beta\gamma + \sin\phi\sin\phi',$$

and $$\mathrm{Sin}\,\xi\eta = \cos\phi\cos\phi'\sin\beta\gamma\cdot\alpha - k(\cos\phi\sin\phi'\cdot\overline{\beta\alpha} + \sin\phi\cos\phi'\cdot\overline{\alpha\gamma}).$$

If we express $\mathrm{Sin}\,\xi\eta$ as $\sin\xi\eta\cdot\overline{\xi\eta}$, what must $\overline{\xi\eta}$ now mean? Its length is not unity, nor is it normal to the plane of ξ and η. It means

$$\frac{\cos\phi\cos\phi'\sin\beta\gamma\cdot\alpha - k(\cos\phi\sin\phi'\cdot\overline{\beta\alpha} + \sin\phi\cos\phi'\cdot\overline{\alpha\gamma})}{\sqrt{1-\cos^2\xi\eta}};$$

that is, the elliptic axis conjugate to the plane of ξ and η.

Hence

$$\cos w = \cos u \cos v - \sin u \sin v (\cos\phi \cos\phi' \cos\beta\gamma + \sin\phi \sin\phi'), \quad (1)$$

$$\sin w \sin\phi'' = \cos u \sin v \sin\phi' + \cos v \sin u \sin\phi + \sin u \sin v \cos\phi \cos\phi' \sin\beta\gamma, \quad (2)$$

$$\sin w \sin\phi'' \cdot \epsilon = \cos u \sin v \cos\phi' \cdot \gamma + \cos v \sin u \cos\phi \cdot \beta + \sin u \sin v (\cos\phi \sin\phi' \cdot \overline{\beta\alpha} + \cos\phi' \sin\phi \cdot \overline{\alpha\gamma}). \quad (3)$$

FUNDAMENTAL THEOREM FOR THE GENERAL ELLIPSOID.

To find the product of two ellipsoidal versors whose axes have the same directions as the minor axes of the ellipsoid.

In the general ellipsoid there are three principal axes mutually rectangular; in Fig. 9 they are represented by OA, OB, OC. We shall suppose the greatest semi-axis to be taken as the initial line, but either of the others might be chosen. Let unit axes along OA, OB, and OC be denoted by α, β, γ, respectively; let k' denote the ratio of OB to OA, and k that of OC to OA. A versor POA in the plane COA is expressed by $(k\beta)^u$, while a versor AOQ in the plane of AOB is expressed by $(k'\gamma)^v$; u denoting the ratio of twice POA to the rectangle COA, and v that of twice AOQ to the rectangle AOB.

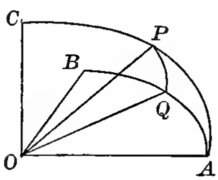

FIG. 9.

Now $(k\beta)^u (k'\gamma)^v = \{\cos u + \sin u \cdot (k\beta)^{\frac{\pi}{2}}\}\{\cos v + \sin v \cdot (k'\gamma)\}^{\frac{\pi}{2}}$

$$= \cos u \cos v + \cos v \sin u \cdot (k\beta)^{\frac{\pi}{2}} + \cos u \sin v \cdot (k'\gamma)^{\frac{\pi}{2}} + \sin u \sin v \cdot (k\beta)^{\frac{\pi}{2}} (k'\gamma)^{\frac{\pi}{2}}.$$

The fourth term, as it involves two axes which are at right angles, can contribute nothing to the cosine; the cosine is $\cos u \cos v$. The second and third terms contribute $k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma$ to the directed Sine; while the fourth contributes either $-kk' \sin u \sin v \cdot \alpha$ or $-\sin u \sin v \cdot \alpha$.

It may be shown, in the same manner as before (page 2), that

$$k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma - kk' \sin u \sin v \cdot \alpha$$

is perpendicular to both OP and OQ, hence has the direction of the normal to their plane; and, by the principle stated at page 13, it is seen that

$$k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma - \operatorname{Sin} u \sin v \cdot \alpha$$

is the axis conjugate to the plane of POQ.

Let a plane pass through the principal axis and the perpendicular component $k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma$; as it passes through the normal to the plane POQ it must cut that plane at right angles, and OX, the line of intersection, is the principal axis of the ellipse PQ. Let ϕ denote the elliptic ratio of AOX, and ψ the angle between β and $\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma$, and w the ratio of twice the elliptic versor POQ to the rectangle of the semi-axes of its ellipse; then the product versor takes the form

$$\xi^w = \cos w + \sin w \{\cos \phi (k \cos \psi \cdot \beta + k' \sin \psi \cdot \gamma) - \sin \phi \cdot \alpha\}^{\frac{\pi}{2}}.$$

For
$$\cos w = \cos u \cos v, \tag{1}$$
$$\sin w \sin \phi = \sin u \sin v, \tag{2}$$
$$\sin w \cos \phi \cos \psi = \cos v \sin u, \tag{3}$$
$$\sin w \cos \phi \sin \psi = \cos u \sin v. \tag{4}$$

To find the product of two ellipsoidal versors of the above form.

Let the one versor be ξ^u, where

$$\xi = \cos \phi (k \cos \psi \cdot \beta + k' \sin \psi \cdot \gamma) - \sin \phi \cdot \alpha,$$

and let the other be η^v, where

$$\eta = \cos \phi' (k \cos \psi' \cdot \beta + k' \sin \psi' \cdot \gamma) - \sin \phi' \cdot \alpha;$$

it is required to show that $\xi^u \eta^v$ has the form ζ^w, where

$$\zeta = \cos \phi'' (k \cos \psi'' \cdot \beta + k' \sin \psi \cdot \gamma) - \sin \phi'' \cdot \alpha.$$

Since
$$\xi^u \eta^v = \cos u \cos v - \sin u \sin v \cos \xi\eta$$
$$+ \{\cos v \sin u \cdot \xi + \cos u \sin v \cdot \eta - \sin u \sin v \operatorname{Sin} \xi\eta\}^{\frac{\pi}{2}},$$

the problem reduces to finding $\cos \xi\eta$ and $\operatorname{Sin} \xi\eta$. By $\xi\eta$ is meant the elliptic angle between the elliptic axes ξ and η; the ratio of the sector $\xi\eta$ to the rectangle of its ellipse is the same as the ratio of the sector of the primitives of ξ and η to 1. Hence the cosine is obtained by supposing k and k' to be one, and the Sine is obtained by the same method, and then reducing by k the component having the axis β, and by k' the component having the axis γ. We obtain

$$\cos \xi\eta = \cos\phi \cos\phi' \cos(\psi - \psi') + \sin\phi \sin\phi',$$

and $$\mathrm{Sin}\,\xi\eta = \cos\phi \cos\phi' \sin(\psi - \psi') \cdot \alpha$$
$$+ k'(\cos\phi \cos\psi \sin\phi' - \cos\phi' \cos\psi' \sin\phi) \cdot \gamma$$
$$- k(\cos\phi \sin\psi \sin\phi' - \cos\phi' \sin\psi' \sin\phi) \cdot \beta.$$

Hence $\cos w$

$$= \cos u \cos v - \sin u \sin v \{\cos\phi \cos\phi' \cos(\psi - \psi') + \sin\phi \sin\phi'\}, \quad (1)$$

$$\sin w \cos\phi'' \cos\psi''$$
$$= \cos u \sin v \cos\phi' \cos\psi' + \cos v \sin u \cos\phi \cos\psi$$
$$+ \sin u \sin v (\cos\phi \sin\psi \sin\phi' - \cos\phi' \sin\psi' \sin\phi), \quad (2)$$

$$\sin w \cos\phi'' \sin\psi''$$
$$= \cos u \sin v \cos\phi' \sin\psi' + \cos v \sin u \cos\phi \sin\psi$$
$$- \sin u \sin v (\cos\phi \cos\psi \sin\phi' - \cos\phi' \cos\psi' \sin\phi), \quad (3)$$

$$\sin w \sin\phi'' = \cos u \sin v \sin\phi' + \cos v \sin u \sin\phi$$
$$- \sin u \sin v \cos\phi \cos\phi' \sin(\psi - \psi'). \quad (4)$$

The elliptic axis is given in magnitude and direction by $\dfrac{\mathrm{Sin}\,\xi\eta}{\sqrt{1 - \cos^2 \xi\eta}}$. The locus of these axes is an ellipsoid derived from the original ellipsoid by interchanging the ratios k and k'.

FUNDAMENTAL THEOREM FOR THE EQUILATERAL HYPERBOLOID OF TWO SHEETS.

In order to distinguish readily the equilateral from the general hyperbola, it is desirable to have a single term for the equilateral hyperbola. The term *excircle,* with the corresponding adjective *excircular,* have been introduced by Mr. Hayward, in his "Algebra of Coplanar Vectors." These terms are brief and suggestive, for the equilateral hyperbola is the analogue of the circle. If we consider the sphere, we find that its hyperbolic analogue consists of three sheets. Two of these are similar, the one being merely the negative of the other with respect to the centre, and are classed together as the equilateral hyperboloid of two sheets; the third is called the equilateral hyperboloid of one sheet. For brevity we propose to call these the *exsphere of two sheets*, and the *exsphere of one sheet*, the two together being called the *exsphere*. In treating of the exsphere of two sheets, we shall generally consider the positive sheet.

To find the expression for an exspherical versor, the plane of which passes through the principal axis.

Let OA (Fig. 10) be the principal axis of an equilateral hyperboloid of two sheets, QAP a section through OA, AOP the sector of a versor in that plane, and PM perpendicular to OA. The versor is denoted by $\frac{1}{OA}OP$, or $(OA)(OP)$, if OA is of unit length. Now

$$\frac{1}{OA}OP = \frac{1}{OA}(OM + MP)$$

$$= \frac{OM}{OA} + \frac{1}{OA}MP.$$

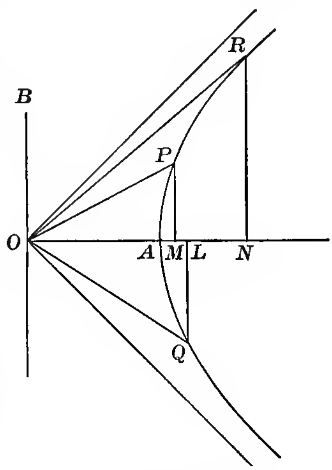

FIG. 10.

The problem is to find the proper analytical expression for this equation. Let β denote a unit axis normal to the plane of QAP, and u the ratio of twice the area of the sector AOP to the square of OA, or rather to the area of the rectangle AOB, and let i denote $\sqrt{-1}$. The above equation, if the starting line is indifferent, is expressed by

$$\beta^{iu} = \cos iu + \sin iu \cdot \beta^{\frac{\pi}{2}}$$

$$= \cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}}.$$

We observe that $\cosh u = \frac{OM}{OA}$, and $\sinh u = \frac{MP}{OA}$, and that $\beta^{\frac{\pi}{2}}$ expresses the circular versor between OA and MP. What is the geometrical meaning of the i? It expresses the fact that $\cosh u$ and $\sinh u$ are related, not by the condition

$$\cosh^2 u + \sinh^2 u = 1,$$

but by the condition $\cosh^2 u - \sinh^2 u = 1$.

With this notation, we can deduce readily from any spherical theorem the corresponding exspherical theorem.

A plausible hypothesis is that the i before $\sinh u$ may be considered as an index $\frac{\pi}{2}$ to be given to the axis β, making

$$\beta^{iu} = \cosh u + \sinh u \cdot \beta^{\pi};$$

but this would leave out entirely the axis of the plane, for the equation would reduce to

$$\beta^{iu} = \cosh u - \sinh u.$$

The quantity here denoted by i is the scalar $\sqrt{-1}$, while the index $\frac{\pi}{2}$ expresses the vector $\sqrt{-1}$.

The series for e^{iu} is wholly scalar; but the series for $e^{iu \cdot \beta^{\frac{\pi}{2}}}$ breaks up into a scalar and a vector part.

In specifying an exspherical versor, it is necessary to give not only the ratio and the perpendicular axis of the plane, but also the principal axis of the versor. This is the reason why the spherical versor has to be treated with reference to a principal axis, in order to obtain theorems which can be translated into theorems for the exspherical versor.

To find the product of two coplanar exspherical versors, when the common plane passes through the principal axis.

Suppose the versors shifted without change of area until the line of meeting coincides with the principal axis. Let QOA (Fig. 10) be denoted by β^{iu}, and AOP by β^{iv}, expressions which are independent of the shifting. Then

$$\beta^{iu} = \cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}},$$

$$\beta^{iv} = \cosh v + i \sinh v \cdot \beta^{\frac{\pi}{2}};$$

therefore $$\beta^{iu}\beta^{iv} = (\cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}})(\cosh v + \sinh v \cdot \beta^{\frac{\pi}{2}})$$

$$= \cosh u \cosh v + i(\cosh u \sinh v + \cosh v \sinh u) \cdot \beta^{\frac{\pi}{2}} + i^2 \sinh u \sinh v \cdot \beta^{\pi};$$

but $i^2 = -$, and $\beta^{\pi} = -$; hence

$$\beta^{iu}\beta^{iv} = \cosh u \cosh v + \sinh u \sinh v + i(\cosh u \sinh v + \cosh v \sinh u) \cdot \beta^{\frac{\pi}{2}}.$$

Hence $$\beta^{iu}\beta^{iv} = \beta^{i(u+v)}.$$

Suppose that the sector QOP is shifted without change of area till it starts from OA, and becomes AOR. Then

$$\frac{ON}{OA} = \cosh u \cosh v + \sinh u \sinh v,$$

and

$$\frac{NR}{OA} = \cosh u \sinh v + \cosh v \sinh u.$$

To find the product of two diplanar exspherical versors when the plane of each passes through the principal axis.

Let the two versors POA and AOQ (Fig. 11) be denoted by β^{iu} and γ^{iv}, the axes β and γ being each perpendicular to the principal axis α. Then

$$\begin{aligned}\beta^{iu}\gamma^{iv} &= (\cos iu + \sin iu \cdot \beta^{\frac{\pi}{2}})(\cos iv + \sin iv \cdot \gamma^{\frac{\pi}{2}}) \\ &= \cos iu \cos iv - \sin iu \sin iv \cos \beta\gamma \\ &\quad + \{\cos iv \sin iu \cdot \beta + \cos iu \sin iv \cdot \gamma - \sin iu \sin iv \sin \beta\gamma \cdot \alpha\}^{\frac{\pi}{2}}.\end{aligned}$$

But $\cos iu = \cosh u$, and $\sin iu = i \sinh u$, therefore,

$$\begin{aligned}\beta^{iu}\gamma^{iv} &= \cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma, \\ &+ i\{\cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma - i \sinh u \sinh v \sin \beta\gamma \cdot \alpha\}^{\frac{\pi}{2}}.\end{aligned}$$

Hence $\cosh \beta^{iu}\gamma^{iv} = \cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma$

and

$$\begin{aligned}\operatorname{Sinh} \beta^{iu}\gamma^{iv} &= \cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma \\ &\quad - i \sinh u \sinh v \sin \beta\gamma \cdot \alpha.\end{aligned}$$

By expanding, it may be shown that

$$(\cosh \beta^{iu}\gamma^{iv})^2 - (\operatorname{Sinh} \beta^{iu}\gamma^{iv})^2 = 1,$$

or

$$(\cos \beta^{iu}\gamma^{iu})^2 + (\operatorname{Sin} \beta^{iu}\gamma^{iv})^2 = 1.$$

The function Sinh is the same as Sin, only an i has been dropped from all the terms of the latter. The product versor is also represented by a sector of an excircle of unit semi-axis.

The first and second components of the excircular Sine are perpendicular to the principal axis; hence their resultant,

$$\cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma,$$

is also perpendicular to the principal axis. Let it be represented by OV (Fig. 11). The difficulty consists in finding the true direction of the third component, $- i \sinh u \sinh v \sin \beta\gamma \cdot \alpha$. At page 53 of *The Imaginary of Algebra,* I suggested the following construction:

With V as centre, and radius equal to $\sinh u \sinh v \sin \beta\gamma$, describe a circle in the plane of OA and OV, and draw OS or OS' a tangent to this circle.

But another hypothesis presents itself; namely, to make the same construction as in the case of the sphere.

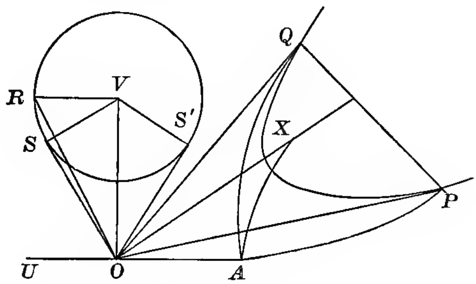

FIG. 11.

Draw OU opposite to OA, and equal to $\sinh u \sinh v \sin \beta\gamma$; and find OR, the resultant of OV and OU. We shall show that OR satisfies the condition of being normal to the plane POQ, while OS or OS' does not.

The reasoning at page 2 applies to give the expression for the vectors OP and OQ. Hence the expressions for the three vectors OR, OP, OQ, are

$$OR = \cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma - \sinh u \sinh v \sin \beta\gamma \cdot \overline{\beta\gamma},$$

$$OP = -\sinh u \frac{\cos \beta\gamma}{\sin \beta\gamma} \cdot \beta + \sinh u \frac{1}{\sin \beta\gamma} \cdot \gamma + \cosh u \cdot \overline{\beta\gamma},$$

$$OQ = -\sinh v \frac{1}{\sin \beta\gamma} \cdot \beta - \sinh v \frac{\cos \beta\gamma}{\sin \beta\gamma} \cdot \gamma + \cosh v \cdot \overline{\beta\gamma}.$$

It follows, as there, that

$$\cos(OR)(OP) = 0, \quad \text{and} \quad \cos(OR)(OQ) = 0.$$

Hence OR is normal to the plane POQ, and OS is not.

The function of the i before the third component of the Sine is to indicate that the magnitude of the Sine is not $\sqrt{OV^2 + VR^2}$ but $\sqrt{OV^2 - VR^2}$. This gives

$\sinh \beta^{iu}\gamma^{iv}$

$$= \sqrt{\{\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v + 2\cosh u \cosh v \sinh u \sinh v \cos \beta\gamma - \sinh^2 u \sinh^2 v \sin^2 \beta\gamma\}}$$

$$= \sqrt{(\cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma)^2 - 1}.$$

The expression $\dfrac{OR}{\sqrt{OV^2 - VR^2}}$ gives the excircular axis both in magnitude and direction. The plane of OA and OV cuts the exsphere in an excircle, and as it passes through the normal OR, it must cut the plane POQ at right angles. Let OX be the line of intersection (Fig. 12). Draw XM perpendicular to OA;

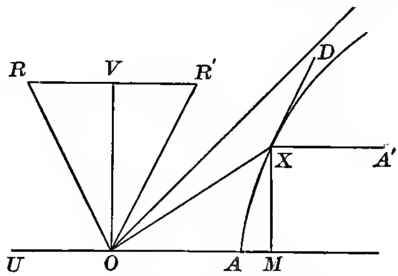

FIG. 12.

draw XD a tangent to the excircle at X, and XA' parallel to OA, and OR' the reflection of OR with respect to OV. Let ϕ denote the excircular angle of AOX; that is, the ratio of twice the area of AOX to the square of OA.

As OR is normal to the plane POQ, it is perpendicular to OX; but OV is perpendicular to OA; therefore the angle AOX is equal to the angle VOR. Also as the angle AOR' is the complement of $R'OV$ and $A'XD$ the complement of AOX, the line OR' is parallel to the tangent XD.

Hence $\cosh \phi = \dfrac{OM}{OA} = \dfrac{OV}{\sqrt{OV^2 - VR^2}} =$

$$\sqrt{\frac{\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v + 2\cosh u \cosh v \sinh u \sinh v \cos \beta\gamma}{(\cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma)^2 - 1}}$$

and $\qquad \sinh \phi = \dfrac{MX}{OA} = \dfrac{VR}{\sqrt{OV^2 - VR^2}}$

$$= \frac{\sinh u \sinh v \sin \beta\gamma}{\sqrt{(\cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma)^2 - 1}}.$$

The above analysis shows that the product versor of POQ may be specified by three elements: *first,* ϵ a unit axis drawn perpendicular to OA in the plane of OA and the normal to the plane of POQ; *second,* ϕ the excircular angle of AOX determined by OA and OX drawn at right angles to the normal in the plane of OA and the normal; *third,* w the versor of a unit excircle determined by the conditions of passing through the points P and Q and having its vertex on the line OX.

When u and v are equal, half of the line joining PQ is the sinh of half of the versor of the product. Let y denote the sinh of each of the factor versors, then it is easy to see from geometrical considerations (v. *The Imaginary of Algebra,* page 53), that

$$\sinh\frac{w}{2}=\frac{1}{\sqrt{2}}y\sqrt{1+\cos\beta\gamma},$$

therefore
$$\cosh\frac{w}{2}=\frac{1}{\sqrt{2}}\sqrt{2+y^2(1+\cos\beta\gamma)}.$$

But it is also evident that the distance from O to the mid-point of PQ is

$$\sqrt{\frac{y^2(1-\cos\beta\gamma)+2(y^2+1)}{y^2(1+\cos\beta\gamma)+2}}.$$

The excess of this distance over $\cosh\frac{w}{2}$ gives the distance by which the axis has been displaced along OX.

Hence the product versor may be expressed by an excircular axis and an excircular versor as ξ^{iw}, where

$$\xi=\cosh\phi\cdot\epsilon-i\sinh\phi\cdot\alpha.$$

To determine these quantities, we have, as in the case of the sphere, the three equations

$$\cosh w=\cosh u\cosh v+\sinh u\sinh v\cos\beta\gamma, \quad (1)$$

$$\sinh w\cosh\phi=\sinh u\sinh v\sin\beta\gamma, \quad (2)$$

$$\sinh w\sinh\phi\cdot\epsilon=\cosh v\sinh u\cdot\beta+\cosh u\sinh v\cdot\gamma. \quad (3)$$

The axis ϵ may be expressed in terms of two axes β and γ forming with α a set of mutually rectangular axes, and the angle ψ which it makes with β; so that for the excircular axis we have

$$\xi=\cosh\phi(\cos\psi\cdot\beta+\sin\psi\cdot\gamma)-i\sinh\phi\cdot\alpha.$$

In the above investigation it is assumed that the magnitude of the perpendicular component of the Sine is necessarily greater than the component parallel to the principal axis. This means that

$$\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v + 2 \cosh u \cosh v \sinh u \sinh v \cos \beta\gamma$$

is necessarily greater than $\sinh^2 u \sinh^2 v \sin^2 \beta\gamma$.

Let $\sin \beta\gamma = 1$; then $\cos \beta\gamma = 0$; and we have to compare

$$\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v \quad \text{with} \quad \sinh^2 u \sinh^2 v.$$

Now each term on the left is greater than the term on the right; therefore their sum must be greater, for each term is the square of a real quantity. Next let $\sin \beta\gamma = 0$; then $\cos \beta\gamma = 1$; the former term becomes a complete square while the latter is 0; hence the former must always be greater than the latter.

To find the product of two exspherical versors of the general kind.

The two versors are expressed by

$$\xi^{iu} = \cosh u + i \sinh u (\cosh \phi \cdot \beta - i \sinh \phi \cdot \alpha)^{\frac{\pi}{2}},$$

and $$\eta^{iv} = \cosh v + i \sinh v (\cosh \phi' \cdot \gamma - i \sinh \phi' \cdot \alpha)^{\frac{\pi}{2}};$$

it is required to show that their product has the form

$$\zeta^{iw} = \cosh w + i \sinh w (\cosh \phi'' \cdot \epsilon - i \sinh \phi'' \cdot \alpha)^{\frac{\pi}{2}}.$$

We have $\xi^{iu} = \cosh u + i \sinh u \cdot \xi^{\frac{\pi}{2}}$

and $\eta^{iv} = \cosh v + i \sinh v \cdot \eta^{\frac{\pi}{2}}$,

therefore

$$\xi^{iu}\eta^{iv} = \cosh u \cosh v + \sinh u \sinh v \cos \xi\eta$$
$$+ i \{\cosh u \sinh v \cdot \eta + \cosh v \sinh u \cdot \xi - i \sinh u \sinh v \sin \xi\eta \cdot \overline{\xi\eta}\}^{\frac{\pi}{2}}.$$

It remains to determine $\cos \xi\eta$ and $\operatorname{Sin} \xi\eta$.

Since $\xi = \cosh \phi \cdot \beta - i \sinh \phi \cdot \alpha$,

and $\eta = \cosh \phi' \cdot \gamma - i \sinh \phi' \cdot \alpha$,

and as we have seen that the i is merely scalar, and does not affect the direction, we conclude that

$$\cos \xi\eta = \cosh\phi \cosh\phi' \cos\beta\gamma - \sinh\phi \sinh\phi',$$

$$\begin{aligned}\operatorname{Sin}\xi\eta = &\cosh\phi\cosh\phi'\sin\beta\gamma\cdot\alpha \\ &- i(\cosh\phi\sinh\phi'\cdot\overline{\beta\alpha} + \cosh\phi'\sinh\phi\cdot\overline{\alpha\gamma}).\end{aligned}$$

Substituting these values of $\cos\xi\eta$ and $\operatorname{Sin}\xi\eta$, we obtain

$$\begin{aligned}\cosh w = &\cosh u \cosh v \\ &+ \sinh u \sinh v (\cosh\phi\cosh\phi'\cos\beta\gamma - \sinh\phi\sinh\phi'), \quad (1)\end{aligned}$$

$$\begin{aligned}\sinh w \sinh\phi'' = &\cosh u \sinh v \sinh\phi' + \cosh v \sinh u \sinh\phi \\ &+ \sinh u \sinh v \cosh\phi \cosh\phi' \sin\beta\gamma, \quad (2)\end{aligned}$$

$$\begin{aligned}\sinh w \cosh\phi''\cdot\epsilon = &\cosh u \sinh v \cosh\phi'\cdot\gamma + \cosh v \sinh u \cosh\phi\cdot\beta \\ &- \sinh u \sinh v (\cosh\phi\sinh\phi'\cdot\overline{\beta\alpha} + \cosh\phi'\sinh\phi\cdot\overline{\alpha\gamma}). (3)\end{aligned}$$

Let us consider, more minutely, the above equations

$$\cos\xi\eta = \cosh\phi\cosh\phi'\cos\beta\gamma - \sinh\phi\sinh\phi',$$

and
$$\begin{aligned}\operatorname{Sin}\xi\eta = &\cosh\phi\cosh\phi'\sin\beta\gamma\cdot\alpha \\ &- i(\cosh\phi\sinh\phi'\cdot\overline{\beta\alpha} + \cosh\phi'\sinh\phi\cdot\overline{\alpha\gamma}).\end{aligned}$$

If we square these functions, we find

$$\begin{aligned}(\cos\xi\eta)^2 = &\cosh^2\phi\cosh^2\phi'\cos^2\beta\gamma + \sinh^2\phi\sinh^2\phi' \\ &- 2\cosh\phi\cosh\phi'\sinh\phi\sinh\phi'\cos\beta\gamma,\end{aligned}$$

$$\begin{aligned}(\operatorname{Sin}\xi\eta)^2 = &\cosh^2\phi\cosh^2\phi'\sin^2\beta\gamma - \cosh^2\phi\sinh^2\phi' - \cosh^2\phi'\sinh^2\phi \\ &- 2\cosh\phi\cosh\phi'\sinh\phi\sinh\phi'\cos\overline{\beta\alpha}\ \overline{\alpha\gamma};\end{aligned}$$

but $\cos\overline{\beta\alpha}\ \overline{\alpha\gamma} = -\cos\beta\gamma$, and $\cosh^2 = 1 + \sinh^2$, therefore,

$$(\cos\xi\eta)^2 + (\operatorname{Sin}\xi\eta)^2 = 1.$$

As the symbol i does not affect the geometrical composition, $\operatorname{Sin}\xi\eta$ must be normal to the plane of ξ and η; hence, if we analyze it into $\sin\xi\eta\cdot\overline{\xi\eta}$, we must have $\sin\xi\eta = \sqrt{1-(\cos\xi\eta)^2}$, and $\overline{\xi\eta} = \dfrac{\operatorname{Sin}\xi\eta}{\sqrt{1-(\cos\xi\eta)^2}}$.

Consider the special case, when $\gamma = \beta$. Then

$$\cos\xi\eta = \cosh\phi\cosh\phi' - \sinh\phi\sinh\phi',$$

and $\operatorname{Sin}\xi\eta = -i(\cosh\phi\sinh\phi' - \cosh\phi'\sinh\phi)\,\overline{\beta\alpha}.$

Hence $\xi\eta$ becomes an excircular versor. Consider next the special case where γ is perpendicular to β. Then

$$\cos\xi\eta = -\sinh\phi\sinh\phi',$$

and $\operatorname{Sin}\xi\eta = \cosh\phi\cosh\phi'\cdot\alpha + i(\cosh\phi\sinh\phi'\cdot\gamma + \cosh\phi'\sinh\phi\cdot\beta)$.

It appears that the locus of the poles of all the axes is the equilateral hyperboloid of one sheet. (*v.* page 27.)

FUNDAMENTAL THEOREM FOR THE EQUILATERAL HYPERBOLOID OF ONE SHEET.

To find the product of a circular and an excircular versor, when they have a common plane.

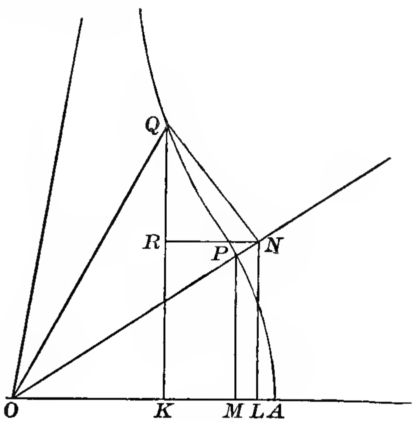

FIG. 13.

Let AOP represent a circular, and POQ an excircular, versor (Fig. 13); and let them be denoted by β^u and β^{iv}. We have

$$\begin{aligned}\beta^u\beta^{iv} = \beta^{u+iv} &= (\cos u + \sin u\cdot\beta^{\frac{\pi}{2}})(\cosh v + i\sinh v\cdot\beta^{\frac{\pi}{2}})\\ &= \cos u\cosh v - i\sin u\sinh v\\ &\quad + (\cosh v\sin u + i\cos u\sinh v)\cdot\beta^{\frac{\pi}{2}}.\end{aligned}$$

What is the meaning of the i which occurs in these scalar functions? Is the magnitude of the cosine

$$\sqrt{(\cos u\cosh v)^2 - (\sin u\sinh v)^2},$$

or is it $\qquad \cos u\cosh v - \sin u\sinh v$?

At page 48 of *Definitions of the Trigonometric Functions,* I show that

$$\cos(u+iv)=\frac{OK}{OA}, \text{ and } \sin(u+iv)=\frac{KQ}{OA},$$

and that the ordinary proof for the cosine and the sine of the sum of two angles gives

$$\frac{OK}{OA}=\frac{OM}{OA}\frac{ON}{OP}-\frac{MP}{OA}\frac{NQ}{OP};$$

that is, $\cos(u+iv)=\cos u\cosh v-\sin u\sinh v,$

and $$\frac{KQ}{OA}=\frac{MP}{OA}\frac{ON}{OP}+\frac{OM}{OA}\frac{NQ}{OP};$$

that is, $\sin(u+iv)=\sin u\cosh v+\cos u\sinh v.$

What, theu, is the function of the i? It shows that if you form the two squares, taking account of it, their sum will be equal to unity. Also, in forming the products of versors, it must be taken into account. When it is preserved, the rules for circular versors apply without change to excircular versors.

Here we have the true geometric meaning of a *bi-versor,* and consequently of a *bi-quaternion;* for the latter is only the former multiplied by a line.

As a special case, let $u=\frac{\pi}{2}$; we then have

$$\beta^{\frac{\pi}{2}+iv}=-i\sinh v+\cosh v\cdot\beta^{\frac{\pi}{2}};$$

this versor evidently refers to the conjugate hyperbola.

Again, let $u=\pi$; we have

$$\beta^{\pi+iv}=-(\cosh v+i\sinh v\cdot\beta^{\frac{\pi}{2}}),$$

which refers to the opposite hyperbola.

In the following table, the related excircular versors are placed in the same line with their circular analogues, and the diagram (Fig. 14) shows the related versors graphically.

Circular.			Excircular.			
β^{u}	$=$	$\cos u+\sin u\cdot\beta^{\frac{\pi}{2}}$	β^{iu}	$=$	$\cosh u+i\sinh u\cdot\beta^{\frac{\pi}{2}}$	AOP_1
$\beta^{\frac{\pi}{2}-u}$	$=$	$\sin u+\cos u\cdot\beta^{\frac{\pi}{2}}$	$\beta^{\frac{\pi}{2}-iu}$	$=$	$i\sinh u+\cosh u\cdot\beta^{\frac{\pi}{2}}$	AOP_2
$\beta^{\frac{\pi}{2}+u}$	$=-$	$\sin u+\cos u\cdot\beta^{\frac{\pi}{2}}$	$\beta^{\frac{\pi}{2}+iu}$	$=-$	$i\sinh u+\cosh u\cdot\beta^{\frac{\pi}{2}}$	AOP_3
$\beta^{\pi-u}$	$=-$	$\cos u+\sin u\cdot\beta^{\frac{\pi}{2}}$	$\beta^{\pi-iu}$	$=-$	$\cosh u+i\sinh u\cdot\beta^{\frac{\pi}{2}}$	AOP_4
$\beta^{\pi+u}$	$=-$	$\cos u-\sin u\cdot\beta^{\frac{\pi}{2}}$	$\beta^{\pi+iu}$	$=-$	$\cosh u-i\sinh u\cdot\beta^{\frac{\pi}{2}}$	AOP_5
$\beta^{-\frac{\pi}{2}-u}$	$=-$	$\sin u-\cos u\cdot\beta^{\frac{\pi}{2}}$	$\beta^{-\frac{\pi}{2}-iu}$	$=-$	$i\sinh u-\cosh u\cdot\beta^{\frac{\pi}{2}}$	AOP_6
$\beta^{-\frac{\pi}{2}+u}$	$=$	$\sin u-\cos u\cdot\beta^{\frac{\pi}{2}}$	$\beta^{-\frac{\pi}{2}+iu}$	$=$	$i\sinh u-\cosh u\cdot\beta^{\frac{\pi}{2}}$	AOP_7
β^{-u}	$=$	$\cos u-\sin u\cdot\beta^{\frac{\pi}{2}}$	β^{-iu}	$=$	$\cosh u-i\sinh u\cdot\beta^{\frac{\pi}{2}}$	AOP_8

It is evident that AOP_2 is the complement, AOP_4 the supplement, and AOP_8 the reciprocal, of AOP_1. It is not the circular

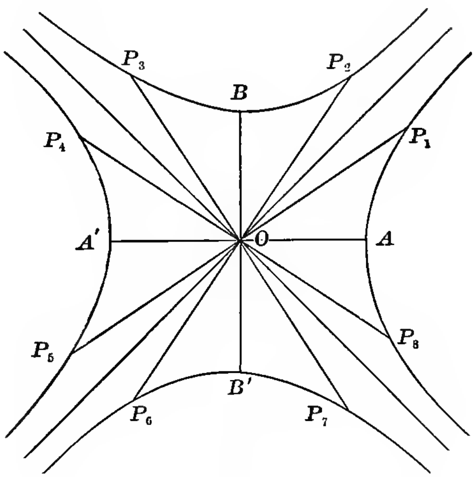

Fig. 14.

term of the complex exponent which is affected by the $\sqrt{-1}$, but the excircular term. Thus space analysis throws a new light upon the periodicity of the hyperbolic functions.

To find the product of two versors of the equilateral hyperboloid of one sheet, when each passes through the principal axis of the hyperboloid.

Let P be a point on the excircle of one sheet (Fig. 15), OP its radius; draw OB equal to OA, in the plane of OA and OP; AB is joined by a quadrant of a circle, and BOP by a sector of an excircle. Let u denote the ratio of twice the area of the sector POB to the square of OA; $\frac{\pi}{2}$ is the ratio of twice the area of BOA to the square of OA. Hence if β is a unit axis perpendicular to OB and OA, the expression for the versor POA is $\beta^{\frac{\pi}{2}+iu}$. Similarly, the expression for the versor AOQ is $\gamma^{\frac{\pi}{2}+iv}$.

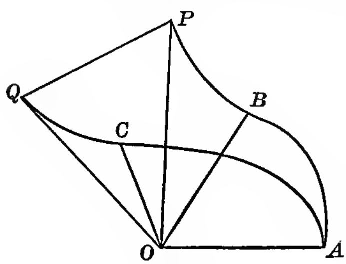

FIG. 15.

$$\text{Now } \beta^{\frac{\pi}{2}+iu}\gamma^{\frac{\pi}{2}+iv} = (-i\sinh u + \cosh u\cdot\beta^{\frac{\pi}{2}})(-i\sinh v+\cosh v\cdot\gamma^{\frac{\pi}{2}})$$
$$= -(\sinh u\sinh v + \cosh u\cosh v\cos\beta\gamma)$$
$$-\{i(\cosh u\sinh v\cdot\beta + \cosh v\sinh u\cdot\gamma) + \cosh u\cosh v\sin\beta\gamma\cdot\alpha\}^{\frac{\pi}{2}}.$$

Now the magnitude of $\cosh u\sinh v\cdot\beta + \cosh v\sinh u\cdot\gamma$ may be greater or less than $\cosh u\cosh v\sin\beta\gamma$. If it is greater, then the directed sine may be thrown into the form

$$-i\{(\cosh u\sinh v\cdot\beta+\cosh v\sinh u\cdot\gamma) - i\cosh u\cosh v\sin\beta\gamma\cdot\alpha\},$$

consequently, the ratio is excircular, and the axis excircular; hence the product takes the form

$$-\xi^{iw},\text{ where } \xi = \cosh\phi\cdot\epsilon - i\sinh\phi\cdot\alpha.$$

But if $\cosh u\cosh v\sin\beta\gamma$ is the greater, the directed sine takes the form

$$-\{\cosh u\cosh v\sin\beta\gamma\cdot\alpha + i(\cosh u\sinh v\cdot\beta+\cosh v\sinh u\cdot\gamma)\}.$$

The ratio of the product is circular, but the axis is excircular. Let w denote the ratio; the axis has the form $\cosh\phi\cdot\alpha - i\sinh\phi\cdot\epsilon$, so that the product is of the form

$$-\xi^{w} = -\cos w - \sin w(\cosh\phi\cdot\alpha - i\sinh\phi\cdot\epsilon)^{\frac{\pi}{2}}.$$

In the former case, the locus of the poles of the axes is the exsphere of one sheet; in the latter, the opposite sheet of the exsphere of two sheets.

To find the product of two general versors of the equilateral hyperboloid of one sheet.

The one versor may be represented by

$$-\{x+(iy\cdot\beta+z\cdot\alpha)^{\frac{\pi}{2}}\},$$

where $x^2-y^2+z^2=1$, and β is perpendicular to α. Similarly, the other versor may be represented by

$$-\{x'+(iy'\cdot\gamma+z'\cdot\alpha)^{\frac{\pi}{2}}\},$$

where $x'^2-y'^2+z'^2=1$, and γ is perpendicular to α.

The cosine of the product is

$$xx'+yy'\cos\beta\gamma-zz',$$

and the Sine of the product is

$$i(xy'\cdot\gamma+x'y\cdot\beta)+(xz'+x'z+yy'\sin\beta\gamma)\cdot\alpha.$$

As before, if $(xy')^2+(x'y)^2+2xx'yy'\cos\beta\gamma$ is greater than $(xz'+x'z+yy'\sin\beta\gamma)^2$, the ratio of the product is excircular; but if less, it is circular. In the former case the axis is an axis of the exsphere of one sheet, in the latter it is an axis of the exsphere of two sheets.

To find the product of two versors which pass through the principal axis, when the one belongs to the exsphere of two sheets, the other to the exsphere of one sheet.

Let the former versor be denoted by β^{iu}, and the latter by $\gamma^{\frac{\pi}{2}+iv}$. Then

$$^{u}\gamma^{\frac{\pi}{2}+iv}=(\cosh u+i\sinh u\cdot\beta^{\frac{\pi}{2}})(-i\sinh v+\cosh v\cdot\gamma^{\frac{\pi}{2}})$$

$$=-i(\cosh u\sinh v+\sinh u\cosh v\cos\beta\gamma)$$

$$+\{\cosh u\cosh v\cdot\gamma+\sinh u\sinh v\cdot\beta-i\sinh u\cosh v\sin\beta\gamma\cdot\alpha\}^{\frac{\pi}{2}}.$$

As the magnitude of $\cosh u\cosh v\cdot\gamma+\sinh u\sinh v\cdot\beta$ is by reasoning similar to that at page 23 seen to be greater than $\sinh u\cosh v\sin\beta\gamma$, we see that the axis is excircular; and the i before the scalar term shows that the ratio is excircular. From comparison of the table, page 27, we see that the product versor has the form

$$\xi^{\frac{\pi}{2}+iw}, \text{ where } \xi = \cosh\phi\cdot\epsilon - i\sinh\phi\cdot\alpha,$$

the equations being

$$\sinh w = \cosh u \sinh v + \sinh u \cosh v \cos\beta\gamma, \qquad (1)$$

$$\cosh w \sinh\phi = \cosh u \sinh v + \sinh u \cosh v \cos\beta\gamma, \qquad (2)$$

$$\cosh w \cosh\phi\cdot\epsilon = \cosh u \cosh v\cdot\gamma + \sinh u \sinh v\cdot\beta. \qquad (3)$$

FUNDAMENTAL THEOREM FOR THE HYPERBOLOID.

The theorems for the hyperboloid are obtained from the theorems for the exsphere in the same manner as the theorems for the ellipsoid are deduced from those for the sphere.

Two general versors for the hyperboloid of two sheets are expressed by ξ^{iu} and η^{iv}, where

$$\xi = \cosh\phi\ (\cos\psi\cdot k\beta + \sin\psi\cdot k'\gamma) - i\sinh\phi\cdot\alpha,$$

and $$\eta = \cosh\phi'(\cos\psi'\cdot k\beta + \sin\psi'\cdot k'\gamma) - i\sinh\phi'\cdot\alpha.$$

Now $$\xi^{iu}\eta^{iv} = (\cosh u + i\sinh u\cdot\xi^{\frac{\pi}{2}})(\cosh v + i\sinh v\cdot\eta^{\frac{\pi}{2}})$$

$$= \cosh u\cosh v + \sinh u \sinh v \cos\xi\eta$$

$$+\{i(\cosh v\sinh u\cdot\xi + \cosh u \sinh v\cdot\eta) + \sinh u\sinh v \operatorname{Sin}\xi\eta\}^{\frac{\pi}{2}}.$$

The problem is reduced to finding the versor $\xi\eta$. We apply the same principle as that employed in finding the versor between two elliptic axes (page 13), namely: Restore the axes to their excircular primitives, find the versor between these excircular axes (page 23), and change its axis according to the ratios of the contraction of the hyperboloid. This gives

$$\cos\xi\eta = \cosh\phi\cosh\phi'\{\cos(\psi-\psi')\} - \sinh\phi\sinh\phi',$$

$$\operatorname{Sin}\xi\eta = \cosh\phi\cosh\phi'\sin(\psi-\psi')\cdot\alpha$$

$$-i(\cosh\phi\sinh\phi'\sin\psi - \cosh\phi'\sinh\phi\sin\psi')\cdot k\beta$$

$$+i(\cosh\phi\sinh\phi'\cos\psi - \cosh\phi'\sinh\phi\cos\psi')\cdot k'\gamma.$$

In this manner, each theorem proved for the exsphere may be generalized for the hyperboloid.

DE MOIVRE'S THEOREM.

To find any integral power of a versor.

Let n denote any integral number. For the general spherical versor we have $(\xi^u)^n = \xi^{nu}$, because the axes of the factor versors are all the same. Hence

$$\cos nu + \sin nu \cdot \xi^{\frac{\pi}{2}}$$

$$= (\cos u + \sin u \cdot \xi^{\frac{\pi}{2}})^n$$

$$= \cos^n u + n \cos^{n-1} u \sin u \cdot \xi^{\frac{\pi}{2}} + \frac{n(n-1)}{2!} \cos^{n-2} u \sin^2 u \cdot \xi^{\pi} +,$$

from which it follows that

$$\cos nu = \cos^n u - \frac{n(n-1)}{2!} \cos^{n-2} u \sin^2 u +,$$

and $$\sin nu = n \cos^{n-1} u \sin u - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} u \sin^3 u +.$$

Similarly for the exspherical versor $(\xi^{iu})^n$, as the axes are all the same $(\xi^{iu})^n = \xi^{inu}$, and

$$\cosh nu + i \sinh nu \cdot \xi^{\frac{\pi}{2}} = (\cosh u + i \sinh u \cdot \xi^{\frac{\pi}{2}})^n$$

$$= \cosh^n u + ni \cosh^{n-1} u \sinh u \cdot \xi^{\frac{\pi}{2}} + \frac{n(n-1)}{2!} i^2 \cosh^{n-2} u \sinh^2 u \cdot \xi^{\pi} + ;$$

therefore

$$\cosh nu = \cosh^n u + \frac{n(n-1)}{2!} \cosh^{n-2} u \sinh^2 u +,$$

and $$\sinh nu = n \cosh^{n-1} u \sinh u + \frac{n(n-1)(n-2)}{3!} \cosh^{n-3} u \sinh^3 u +.$$

The only difference in the case of the general ellipsoidal versor is that u is measured elliptically and ξ is an ellipsoidal axis. So for the general hyperboloidal versor, u is measured hyperbolically and ξ is a hyperboloidal axis.

To find any integral root of a versor.

Consider first the case of an ellipsoidal versor. If u is defined as the ratio of twice the sector to the rectangle formed by the semi-axes, it cannot be greater than 2π. Then $(\xi^u)^{\frac{1}{n}}$ is unambiguously equal to $\xi^{\frac{u}{n}}$. Hence

$$\cos\frac{u}{n}+\sin\frac{u}{n}\cdot\xi^{\frac{\pi}{2}}=(\cos u+\sin u\cdot\xi^{\frac{\pi}{2}})^{\frac{1}{n}}.$$

If $\cos u$ is not less than $\sin u$, then

$$\cos\frac{u}{n}+\sin\frac{u}{n}\cdot\xi^{\frac{\pi}{2}}=(\cos u)^{\frac{1}{n}}\{1+\tan u\cdot\xi^{\frac{\pi}{2}}\}^{\frac{1}{n}}$$

$$=(\cos u)^{\frac{1}{n}}\left\{1+\frac{1}{n}\tan u\cdot\xi^{\frac{\pi}{2}}+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!}\tan^2 u\cdot\xi^{\pi}+\right\};$$

therefore

$$\cos\frac{u}{n}=(\cos u)^{\frac{1}{n}}\left\{1+\frac{(n-1)}{n^2\,2!}\tan^2 u\right.$$
$$\left.-\frac{(n-1)(2n-1)(3n-1)}{n^4\,4!}\tan^4 u+\right\},$$

and $\sin\frac{u}{n}=(\cos u)^{\frac{1}{n}}\left\{\frac{1}{n}\tan u-\frac{(n-1)(2n-1)}{n^3\,3!}\tan^3 u+\right\}.$

But if $\sin u$ is not less than $\cos u$, we have the complementary series

$$\xi^{\frac{u}{n}}=(\sin u)^{\frac{1}{n}}\xi^{\frac{\pi}{2n}}\{1+\cot u\cdot\xi^{-\frac{\pi}{2}}\}^{\frac{1}{n}}.$$

Consider next the case of a hyperboloidal versor. A versor for the hyperboloid of two sheets is denoted by ξ^{iu}. Now

$$(\xi^{iu})^{\frac{1}{n}}=\xi^{\frac{iu}{n}}=\{\cosh u+i\sinh u\cdot\xi^{\frac{\pi}{2}}\}^{\frac{1}{n}}$$

$$=(\cosh u)^{\frac{1}{n}}\{1+i\tanh u\cdot\xi^{\frac{\pi}{2}}\}^{\frac{1}{n}},$$

for $\cosh u$ is always greater than $\sinh u$; therefore

$$\cosh\frac{u}{n}=(\cosh u)^{\frac{1}{n}}\left\{1-\frac{n-1}{n^2\,2!}\tanh^2 u\right.$$
$$\left.-\frac{(n-1)(2n-1)(3n-1)}{n^4\,4!}\tanh^2 u+\cdots\right\},$$

and $\sinh\frac{u}{n} = (\cosh u)^n \left\{\frac{1}{n}\tanh u - \frac{(n-1)(2n-1)}{n^3\, 3\,!}\tanh^3 u + \right\}.$

But a versor for the hyperboloid of one sheet is expressed by $\xi^{\frac{\pi}{2}+iu}$. Now

$$\left(\xi^{\frac{\pi}{2}+iu}\right)^{\frac{1}{n}} = \xi^{\frac{\pi}{2n}+\frac{iu}{n}} = \left\{-i\sinh u + \cosh u \cdot \xi^{\frac{\pi}{2}}\right\}^{\frac{1}{n}}$$

$$= (\cosh u)^{\frac{1}{n}} \xi^{\frac{\pi}{2n}} \left\{1 - i\tanh u \cdot \xi^{-\frac{\pi}{2}}\right\}^{\frac{1}{n}},$$

which is expanded as before.

POLAR THEOREM.

To deduce in the trigonometry of the sphere the polar theorem corresponding to the fundamental theorem.

The cosine theorem, which is the fundamental theorem of spherical trigonometry, expresses the side of a spherical triangle in terms of the opposite sides and their included angle. In treatises on spherical trigonometry, it is shown how to deduce from the cosine theorem a polar or supplemental theorem which expresses an angle in terms of the other two angles and the opposite side. It is our object to find the polar theorem corresponding to the complete fundamental theorem.

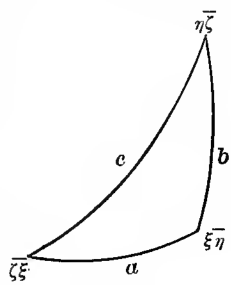

FIG. 16.

Let the versors of the three sides of the spherical triangle (Fig. 16), taken the same way round, be denoted by ξ^a, η^b, ζ^c, where ξ, η, ζ are unit axes, and a, b, c denote the ratio of twice the area of the sector to the area of the rectangle formed by the semi-axes of its circle (which, in this case, is simply the square of the radius). The angles included by the sides are usually denominated A, B, C, respectively, but what it is necessary to consider in view of further generalization is the angles between the planes, or rather the versors between the axes. These in accordance with our notation are denoted by $\eta\zeta$, $\zeta\xi$, and $\xi\eta$ respectively; the axes of these versors, which are also of unit length, are denoted by $\overline{\eta\zeta}$, $\overline{\zeta\xi}$, and $\overline{\xi\eta}$, respectively, and they correspond to the poles of the corners of the triangle as indicated by the figure.

The fundamental theorem is

$$\xi^a \eta^b = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{\cos b \sin a \cdot \xi + \cos a \sin b \cdot \eta - \sin a \sin b \sin \xi\eta \cdot \overline{\xi\eta}\}^{\frac{\pi}{2}};$$

but as ζ^c is taken in the opposite direction, we have

$$\zeta^c = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{-\cos b \sin a \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \sin \xi\eta \cdot \overline{\xi\eta}\}^{\frac{\pi}{2}}.$$

The polar theorem is obtained by changing each side into the supplement of the corresponding angle and the angle into the supplement of the corresponding side. Hence

$$\cos(\pi - \xi\eta) = \cos(\pi - \eta\zeta) \cos(\pi - \zeta\xi)$$
$$- \sin(\pi - \eta\zeta) \sin(\pi - \zeta\xi) \cos(\pi - c);$$

that is, $\cos \xi\eta = -\cos \eta\zeta \cos \zeta\xi - \sin \eta\zeta \sin \zeta\xi \cos c.$

When A, B, C, are used to denote the external angles between the sides, the above equation is written

$$\cos C = -\cos A \cos B - \sin A \sin B \cos c.$$

Apply the same rule of change to the Sine part, and we obtain

$$\text{Sin}(\pi - \xi\eta) = -\cos(\pi - \zeta\xi) \,\text{Sin}(\pi - \eta\zeta) - \cos(\pi - \eta\zeta) \,\text{Sin}(\pi - \zeta\xi)$$
$$+ \sin(\pi - \eta\zeta) \sin(\pi - \zeta\xi) \sin c \cdot \zeta;$$

that is, $\text{Sin}\, \xi\eta = \cos \zeta\xi \,\text{Sin}\, \eta\zeta + \cos \eta\zeta \,\text{Sin}\, \zeta\xi + \sin \eta\zeta \sin \zeta\xi \sin c \cdot \zeta.$

To deduce the polar theorem for the ellipsoid.

Let ξ^a, η^b, ζ^c denote the three versors of the original ellipsoidal triangle taken the same way round; then the corresponding versors of the polar triangle are $\eta\zeta$, $\zeta\xi$, and $\xi\eta$. The third versor of the original triangle is given in terms of the other two by the theorem

$$\zeta^c = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{-\cos b \sin a \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \,\text{Sin}\, \xi\eta\}^{\frac{\pi}{2}}.$$

The third versor of the polar triangle is obtained in terms of the other two by changing each versor into the supplement of its corresponding versor; hence

$$\cos \xi\eta = -\cos \eta\zeta \cos \zeta\xi - \sin \eta\zeta \sin \zeta\xi \cos c,$$

and $\operatorname{Sin} \xi\eta = \cos \zeta\xi \operatorname{Sin} \eta\zeta + \cos \eta\zeta \operatorname{Sin} \zeta\xi + \sin \eta\zeta \sin \zeta\xi \operatorname{Sin} \zeta^c$.

In form it is the same as for the sphere; the only difference is in the expressions for the ellipsoidal axes ξ, η, ζ, and the manner of deducing the cosine and Sine of the versor between two such axes. (See page 13.) The polar ellipsoid is not identical with the original ellipsoid; the ratios of the two minor axes are interchanged.

To deduce the polar theorem for the exsphere of two sheets.

Let ξ^{ia}, η^{ib}, ζ^{ic} denote the versors for the three sides of a triangle of the exsphere of two sheets, taken in the same order round. The axes ξ, η, ζ have their poles on the exsphere of two sheets (page 23); it is required to deduce the theorem for that polar triangle. For the original triangle, we have

$$\zeta^{ic} = \cos ia \cos ib - \sin ia \sin ib \cos \xi\eta$$
$$+ \{-\cos ib \sin ia \cdot \xi - \cos ia \sin ib \cdot \eta + \sin ia \sin ib \operatorname{Sin} \xi\eta\}^{\frac{\pi}{2}}.$$

By changing each versor into the supplement of the corresponding versor, we obtain

$$\xi\eta = -\cos \eta\zeta \cos \zeta\xi - \sin \eta\zeta \sin \zeta\xi \cosh c$$
$$+ \{\cos \zeta\xi \operatorname{Sin} \eta\zeta + \cos \eta\zeta \operatorname{Sin} \zeta\xi + i \sin \eta\zeta \sin \zeta\xi \sinh c \cdot \zeta\}^{\frac{\pi}{2}}.$$

The above cosine equation has a marked resemblance to the fundamental equation of non-euclidean geometry (see Dr. Günther's *Hyperbelfunctionen*, pages 306 and 322). It is true that $\eta\zeta$ and $\zeta\xi$ are not simple circular versors, but the functions are *cos* and *sin* in a generalized sense. I venture the opinion that non-euclidean geometry is nothing but trigonometry on the exsphere; and that the so-called elliptic and hyperbolic geometries are identical with the ellipsoidal and hyperboloidal trigonometry developed in this paper.

To deduce the general polar theorem for the exsphere.

Let ξ^a, η^b, ζ^c denote the three sides of an exspherical triangle; the axes ξ, η, ζ are exspherical, but the ratios a, b, c may be circular or excircular, or be compounded of π or $\frac{\pi}{2}$ and an excircular ratio. For the original triangle, we have

$$\zeta^c = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{-\cos a \sin b \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \operatorname{Sin} \xi\eta\}^{\frac{\pi}{2}},$$

and for the polar triangle,

$$\xi\eta = -\cos \eta\zeta \cos \zeta\xi - \sin \eta\zeta \sin \zeta\xi \cos c$$
$$+ \{\cos \zeta\xi \operatorname{Sin} \eta\zeta + \cos \eta\zeta \operatorname{Sin} \zeta\xi + \sin \eta\zeta \sin \zeta\xi \operatorname{Sin} \zeta^c\}^{\frac{\pi}{2}}.$$

Here the functions cos and sin are used in their most general meaning.

SINE THEOREM.

To prove that if ξ^a, η^b, ζ^c denote the three versors of a spherical triangle, then

$$\frac{\sin \eta\zeta}{\sin a} = \frac{\sin \zeta\xi}{\sin b} = \frac{\sin \xi\eta}{\sin c}.$$

We have $\cos c = \cos a \cos b - \sin a \sin b \cos \xi\eta$,

and $\quad \sin c \cdot \zeta = -\cos b \sin a \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \sin \xi\eta \cdot \overline{\xi\eta}$.

By squaring the second equation, we obtain

$$\sin^2 c = \cos^2 b \sin^2 a + \cos^2 a \sin^2 b + \sin^2 a \sin^2 b \sin^2 \xi\eta$$
$$+ 2 \cos a \cos b \sin a \sin b \cos \xi\eta;$$

then, by substituting for $\cos \xi\eta$ from the first equation, and reducing, we obtain

$$\sin a \sin b \sin \xi\eta = \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}.$$

Hence
$$\frac{\sin \xi\eta}{\sin c} = \frac{\sin \eta\zeta}{\sin a} = \frac{\sin \zeta\xi}{\sin b}.$$

This theorem is also true for an ellipsoid of revolution, for then

$$\sin a \sin b \sin \xi\eta = k \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}.$$

To find the analogue for the exsphere of the sine theorem.

Let ξ, η, ζ denote exspherical axes, and a, b, c versors which may be circular, or excircular, or both combined. Then, with the general meaning of the sin and cos functions,

$$\sin a \sin b \sin \xi\eta = \sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}.$$

Hence $$\frac{\sin \xi\eta}{\sin c} = \frac{\sin \eta\zeta}{\sin a} = \frac{\sin \zeta\xi}{\sin b}.$$

We have seen that, if a and b are both simply excircular, it does not follow that c is (page 28).

SUM AND DIFFERENCE THEOREMS.

The reciprocal of a given versor.

By the reciprocal of a given versor is meant the versor of equal index but of opposite axis. Let ξ^u denote the given spherical versor; its reciprocal is $(-\xi)^u$. But it may be shown that $\xi^{-u} = (-\xi)^u$. For

$$\begin{aligned} \xi^{-u} &= \cos(-u) + \sin(-u) \cdot \xi^{\frac{\pi}{2}} \\ &= \cos u - \sin u \cdot \xi^{\frac{\pi}{2}} \\ &= \cos u + \sin u \cdot (-\xi)^{\frac{\pi}{2}} \\ &= (-\xi)^u. \end{aligned}$$

Similarly the reciprocal of an exspherical versor ξ^{iu} is $(-\xi)^{iu}$ or ξ^{-iu}, and

$$\xi^{-iu} = \cosh u - i \sinh u \cdot \xi^{\frac{\pi}{2}}.$$

The reciprocal of an ellipsoidal versor ξ^u is also ξ^{-u}, the only difference being that ξ is no longer a spherical, but an ellipsoidal axis. So for the hyperboloidal versor.

To find the analogues of the sum and difference theorems of plane trigonometry.

At page 45 of "The Imaginary of Algebra," I have shown how to generalize for the sphere the following well-known theorems in plane trigonometry, namely,

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B,$$
$$\cos(A+B) - \cos(A-B) = -2\sin A \sin B,$$

$$\sin(A+B)+\sin(A-B)=2\cos B\sin A,$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B,$$

and
$$\cos C+\cos D=\quad 2\cos\frac{C+D}{2}\cos\frac{C-D}{2},$$
$$\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2},$$
$$\sin C+\sin D=\quad 2\sin\frac{C+D}{2}\cos\frac{C-D}{2},$$
$$\sin C-\sin D=\quad 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}.$$

The generalized formulæ of the first set for the sphere are, using general axes ξ and η,

$$\cos\xi^A\eta^B+\cos\xi^A\eta^{-B}=\quad 2\cos A\cos B,$$
$$\cos\xi^A\eta^B-\cos\xi^A\eta^{-B}=-2\cos(\mathrm{Sin}\,\xi^A\,\mathrm{Sin}\,\eta^B),$$
$$\mathrm{Sin}\,\xi^A\eta^B+\mathrm{Sin}\,\xi^A\eta^{-B}=\quad 2\cos B\,\mathrm{Sin}\,\xi^A,$$
$$\mathrm{Sin}\,\xi^A\eta^B-\mathrm{Sin}\,\xi^A\eta^{-B}=\quad 2\,\{\cos A\,\mathrm{Sin}\,\eta^B-\mathrm{Sin}(\mathrm{Sin}\,\xi^A\,\mathrm{Sin}\,\eta^B)\}.$$

Corresponding to the latter set of four equations we have

$$\cos\zeta^C+\cos\omega^D=\quad 2\cos\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\cos(\omega^{-D}\zeta^C)^{\frac{1}{2}},$$
$$\cos\zeta^C-\cos\omega^D=-2\cos[\mathrm{Sin}\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\,\mathrm{Sin}(\omega^{-D}\zeta^C)^{\frac{1}{2}}],$$
$$\mathrm{Sin}\,\zeta^C+\mathrm{Sin}\,\omega^D=\quad 2\cos(\omega^{-D}\zeta^C)^{\frac{1}{2}}\,\mathrm{Sin}\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\},$$
$$\mathrm{Sin}\,\zeta^C-\mathrm{Sin}\,\omega^D=\quad 2\cos\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\,\mathrm{Sin}(\omega^{-D}\zeta^C)^{\frac{1}{2}}$$
$$-2\,\mathrm{Sin}\,\mathrm{Sin}\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\,\mathrm{Sin}(\omega^{-D}\zeta^C)^{\frac{1}{2}}.$$

The corresponding theorems for the ellipsoid are the same, excepting that

$$\xi=\cos\phi\cdot k\beta-\sin\phi\cdot\alpha,\quad \eta=\cos\phi'\cdot k\gamma-\sin\phi'\cdot\alpha.$$

Consequently $\cos\xi\eta$ is the same as before, but

$$\mathrm{Sin}\,\xi\eta=\cos\phi\cos\phi'\sin\beta\gamma\cdot\alpha-k(\cos\phi\sin\phi'\cdot\overline{\beta\alpha}+\cos\phi'\sin\phi\cdot\overline{\alpha\gamma}).$$

For the general ellipsoid the only difference is in the expressions for ξ, η, and $\sin\xi\eta\cdot\overline{\xi\eta}$.

EXPONENTIAL THEOREM.

To find the exponential series for an ellipsoidal versor.

In the expression ξ^u for a spherical versor, the u and ξ are truly related as index to base, for $\log \xi^u = u \log \xi^1 = u \cdot \xi^{\frac{\pi}{2}}$, and therefore $\xi^u = e^{u \cdot \xi^{\frac{\pi}{2}}}$. Consequently

$$\xi^u = 1 - \frac{1}{2\,!} u^2 + \frac{1}{4\,!} u^4 -$$
$$+ \left\{ u - \frac{u^3}{3\,!} + \frac{u^5}{5\,!} - \right\} \cdot \xi^{\frac{\pi}{2}}.$$

In the case of the spherical versor, $\xi = \cos\phi \cdot \beta - \sin\phi \cdot \alpha$, or $\cos\phi(\cos\psi \cdot \beta + \sin\psi \cdot \gamma) - \sin\phi \cdot \alpha$, where α, β, γ are unit axes mutually rectangular.

The expansion for the ellipsoidal versor ξ^u differs only in the way in which u is measured, and in the expression for ξ, which is now $\cos\phi \cdot k\beta - \sin\phi \cdot \alpha$, or $\cos\phi(\cos\psi \cdot k\beta + \sin\psi \cdot k'\gamma) - \sin\phi \cdot \alpha$.

To find the exponential series for a hyperboloidal versor.

The expression for a versor on the exsphere of two sheets is ξ^{iu}. Now

$$\xi^{iu} = e^{iu \cdot \xi^{\frac{\pi}{2}}}$$
$$= 1 + iu \cdot \xi^{\frac{\pi}{2}} + \frac{(iu)^2}{2\,!} \cdot \xi^{\pi} + \frac{(iu)^3}{3\,!} \cdot \xi^{3\frac{\pi}{2}} +$$
$$= 1 + \frac{u^2}{2\,!} + \frac{u^4}{4\,!} +$$
$$+ i \left\{ u + \frac{u^3}{3\,!} + \frac{u^5}{5\,!} + \right\} \cdot \xi^{\frac{\pi}{2}}.$$

The expression for a mixed exspherical versor is ξ^{u+iv}. Now

$$\xi^{u+iv} = e^{(u+iv) \cdot \xi^{\frac{\pi}{2}}}$$
$$= 1 + (u + iv) \cdot \xi^{\frac{\pi}{2}} + \frac{(u+iv)^2}{2\,!} \cdot \xi^{\pi} + \frac{(u+iv)^3}{3\,!} \cdot \xi^{3\frac{\pi}{2}} +$$
$$= 1 - \frac{(u+iv)^2}{2\,!} + \frac{(u+iv)^4}{4\,!} -$$
$$+ \left\{ u + iv - \frac{(u+iv)^3}{3\,!} + \right\} \cdot \xi^{\frac{\pi}{2}}.$$

Both the cosine and the sine break up into two components, the one independent of i, and the other involving i. Here we have the sine and the cosine of the ordinary complex quantity.

As the ratio of a hyperboloidal versor may be circular or excircular, or both combined, the general versor may be expressed by ξ^a, where a is as general as stated. Then

$$\begin{aligned} \xi^a &= e^{a \cdot \xi^{\frac{\pi}{2}}} \\ &= 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \\ &\quad + \left\{ a - \frac{a^3}{3!} + \frac{a^5}{5!} - \right\} \cdot \xi^{\frac{\pi}{2}}. \end{aligned}$$

To find the exponential series for the product of two ellipsoidal versors.

In the paper on *The Fundamental Theorems of Analysis Generalized for Space* I have shown that if ξ^u and η^v denote any two spherical versors, then

$$\begin{aligned} \xi^u \eta^v &= e^{u \cdot \xi^{\frac{\pi}{2}} + v \cdot \eta^{\frac{\pi}{2}}} \\ &= 1 + (u \cdot \xi^{\frac{\pi}{2}} + v \cdot \eta^{\frac{\pi}{2}}) + \frac{1}{2!}(u \cdot \xi^{\frac{\pi}{2}} + v \cdot \eta^{\frac{\pi}{2}})^2 + \frac{1}{3!}(u \cdot \xi^{\frac{\pi}{2}} + v \cdot \eta^{\frac{\pi}{2}})^3 +, \end{aligned}$$

where the powers of the binomial are expanded according to the binomial theorem, but subject to the special proviso that the order of the axes ξ, η must be preserved in all the axial terms. Thus

$$\begin{aligned} \xi^u \eta^v = 1 &+ u \cdot \xi^{\frac{\pi}{2}} + v \cdot \eta^{\frac{\pi}{2}} \\ &+ \frac{1}{2!}\{u^2 \cdot \xi^{\pi} + 2uv \cdot \xi^{\frac{\pi}{2}} \eta^{\frac{\pi}{2}} + v^2 \cdot \eta^{\pi}\} \\ &+ \frac{1}{3!}\{u^3 \cdot \xi^{\frac{3\pi}{2}} + 3u^2 v \cdot \xi^{\pi} \eta^{\frac{\pi}{2}} + 3uv^2 \cdot \xi^{\frac{\pi}{2}} \eta^{\pi} + v^3 \cdot \eta^{\frac{3\pi}{2}}\} \\ &+ \text{etc.} \end{aligned}$$

$$\left.\begin{aligned} = 1 &- \frac{1}{2!}\{u^2 + 2uv \cos \xi\eta + v^2\} \\ &+ \frac{1}{4!}\{u^4 + 4u^3 v \cos \xi\eta + 6u^2v^2 + 4uv^3 \cos \xi\eta + v^4\} \\ &- \text{etc.} \end{aligned}\right\} \quad (1)$$

$$+\left\{u-\frac{1}{3!}(u^3+3uv^2)+\text{etc.}\right\}\cdot\xi^{\frac{\pi}{2}} \qquad (2)$$

$$+\left\{v-\frac{1}{3!}(3u^2v+v^3)+\text{etc.}\right\}\cdot\eta^{\frac{\pi}{2}} \qquad (3)$$

$$+\left\{-\frac{1}{2!}2uv+\frac{1}{4!}(4u^3v+4uv^3)-\text{etc.}\right\}\sin\xi\eta\cdot\overline{\xi\eta}^{\frac{\pi}{2}}. \qquad (4)$$

In the case of the sphere

$$\xi=\cos\phi\cdot\beta-\sin\phi\cdot\alpha,$$

and

$$\eta=\cos\phi'\cdot\gamma-\sin\phi'\cdot\alpha;$$

consequently $\cos\xi\eta=\cos\phi\cos\phi'\cos\beta\gamma+\sin\phi\sin\phi'$, and

$$\operatorname{Sin}\xi\eta=\cos\phi\cos\phi'\sin\beta\gamma\cdot\alpha-(\cos\phi\sin\phi'\cdot\overline{\beta\alpha}+\cos\phi'\sin\phi\cdot\overline{\alpha\gamma}).$$

For the ellipsoid of revolution the expansion is obtained by introducing ellipsoidal axes ξ and η; and the corresponding theorems for the hyperboloid are obtained by changing the axes and indices into hyperboloid axes and indices.

To find the exponential series for the product of two hyperboloidal versors.

Let ξ and η denote any two hyperboloidal axes, and u and v general hyperboloidal ratios (p. 40). Then the product is

$$\xi^u\eta^v=e^{u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}}}$$

$$=1+(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})+\frac{(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})^2}{2!}+\frac{(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})^3}{3!}+\cdots$$

The form of the theorem is the same as before.

LOGARITHMIC VERSORS.

In the paper on *The Fundamental Theorems of Analysis Generalized for Space*, page 16, I have shown that when the index of α, in $e^{A\cdot\alpha^{\frac{\pi}{2}}}$, is generalized, we obtain the expression for the versor corresponding to a sector of a logarithmic spiral. Let w denote the general angle, and α_w^A the generalized versor; then

$$\begin{aligned}
\alpha_w^A &= e^{A\cdot\alpha^w} \\
&= 1 + A\cdot\alpha^w + \frac{A^2\cdot\alpha^{2w}}{2!} + \frac{A^3\cdot\alpha^{3w}}{3!} + \frac{A^4\cdot\alpha^{4w}}{4!} + \\
&= 1 + A\cos w + \frac{A^2\cos 2w}{2!} + \frac{A^3\cos 3w}{3!} + \text{etc.} \\
&\quad + \left\{ A\sin w + \frac{A^2\sin 2w}{2!} + \frac{A^3\sin 3w}{3!} + \text{etc.} \right\}\cdot\alpha^{\frac{\pi}{2}} \\
&= e^{A\cos w}\, e^{A\sin w\,.\,\alpha^{\frac{\pi}{2}}}.
\end{aligned}$$

It is there shown that w is the constant angle between the radius vector and the tangent, or rather that it is the constant difference between the circular versor from the principal axis to the tangent, and that from the principal axis to the radius vector. It is also shown that $A\sin w$ gives the ratio of twice the area of the corresponding circular sector to the square of the radius, while $A\cos w$ gives the logarithm of the ratio of the radius vector to the principal axis.

I have there called such a logarithmic versor, when multiplied by a length, a *quinternion*. In his *Synopsis der Hoheren Mathematik*, Mr. Hagen has pointed out that the proper classical word is *quinion*. A quaternion means a ratio of three elements multiplied by a length; therefore, a ratio involving an additional element when multiplied by a length, is a quinion.

In the paper on *The Imaginary of Algebra*, an excircular analogue is deduced, namely, $\alpha_{iw}^A = e^{A\alpha^{iw}}$, but there are in reality three, according to whether A or w, or both, are affected by the $\sqrt{-1}$.

To deduce the four forms of logarithmic versor.

First: circular-circular. Let ξ^u denote a general spherical versor, then

$$\begin{aligned}
\xi_w^u &= e^{u\cdot\xi^w} = e^{u\cos w + u\sin w\cdot\xi^{\frac{\pi}{2}}} \\
&= 1 + u\xi^w + \frac{u^2}{2!}\xi^{2w} + \frac{u^3}{3!}\xi^{3w} + \text{etc.}
\end{aligned}$$

Here w denotes the constant difference between the versor from the principal axis to the tangent and that from the principal axis to the radius vector.

Second: circular-excircular. Let iw denote the constant difference between the excircular versor from the principal axis to the tangent, and that from the principal axis to the radius vector; then

$$\xi^u_{iw} = e^{u\cdot\xi^{iw}} = e^{u\cosh w + iu\sinh w\cdot\xi^{\frac{\pi}{2}}}$$

$$= 1 + u\cdot\xi^{iw} + \frac{u^2}{2!}\cdot\xi^{2iw} + \frac{u^3}{3!}\cdot\xi^{3iw} +$$

$$= 1 + u\cosh w + \frac{u^2}{2!}\cosh 2w + \frac{u^3}{3!}\cosh 3w +$$

$$+ i\left\{u\sinh w + \frac{u^2}{2!}\sinh 2w + \frac{u^3}{3!}\sinh 3w + \right\}\cdot\xi^{\frac{\pi}{2}}.$$

Third: excircular-circular. Let ξ^{iu} denote a general exspherical versor; it is equal to $e^{iu\cdot\xi^{\frac{\pi}{2}}}$, and here $\frac{\pi}{2}$ denotes the constant sum of the circular versors above mentioned. Let that constant sum be any other circular versor w. Then

$$\xi^{iu}_{w} = e^{iu\cdot\xi^{w}} = e^{iu\cos w + iu\sin w\cdot\xi^{\frac{\pi}{2}}}$$

$$= 1 + iu\cdot\xi^{w} + \frac{(iu)^2}{2!}\cdot\xi^{2w} + \frac{(iu)^3}{3!}\cdot\xi^{3w} + \text{etc.}$$

$$= 1 - \frac{u^2}{2!}\cdot\xi^{2w} + \frac{u^4}{4!}\cdot\xi^{4w} +$$

$$+ i\left\{u\cdot\xi^{w} - \frac{u^3}{3!}\cdot\xi^{3w} + \right\}$$

$$= 1 - \frac{u^2}{2!}\cos 2w + \frac{u^4}{4!}\cos 4w - \text{etc.}$$

$$+ i\left\{u\cos w - \frac{u^3}{3!}\cos 3w + \text{etc.}\right\}$$

$$+ \left\{-\frac{u^2}{2!}\sin 2w + \frac{u^4}{4!}\sin 4w - \right\}\cdot\xi^{\frac{\pi}{2}}$$

$$+ i\left\{u\sin w - \frac{u^3}{3!}\sin 3w + \text{etc.}\right\}\cdot\xi^{\frac{\pi}{2}}.$$

Here both the cosine and the sine consists of a real and an apparently imaginary part. The geometrical meaning has already been explained (page 25).

Fourth: excircular-excircular. Let iw denote the constant sum of the excircular versors mentioned in the second case. Then

$$\xi_{iw}^{iu} = e^{iu \cdot \xi^{iw}} = e^{iu \cosh w - u \sinh w \cdot \xi^{\frac{\pi}{2}}}$$

$$= 1 + iu \cdot \xi^{iw} + \frac{(iu)^2}{2!} \cdot \xi^{2iw} + \frac{(iu)^3}{3!} \cdot \xi^{3iw} +$$

$$= 1 - \frac{u^2}{2!}(\cosh 2w + i \sinh 2w \cdot \xi^{\frac{\pi}{2}}) +$$

$$+ iu(\cosh w + i \sinh w \cdot \xi^{\frac{\pi}{2}}) -$$

$$= 1 - \frac{u^2}{2!} \cosh 2w + \frac{u^4}{4!} \cosh 4w -$$

$$+ i \left\{ u \cosh w - \frac{u^3}{3!} \cosh 3w + \right\}$$

$$- \left\{ u \sinh w - \frac{u^3}{3!} \sinh 3w + \right\} \cdot \xi^{\frac{\pi}{2}}$$

$$+ i \left\{ - \frac{u^2}{2!} \sinh 2w + \frac{u^4}{4!} \sinh 4w - \right\} \cdot \xi^{\frac{\pi}{2}}.$$

To find the product of two logarithmic versors of the most general kind.

Let ξ and η denote general axes, and u, w, v, t general ratios; that is, each may be a sum of a circular and an excircular ratio. Then ξ_w^u and η_t^v each denote a general logarithmic versor. Then

$$\xi_w^u \eta_t^v = e^{u \cdot \xi^w + v \cdot \eta^t}$$

$$= 1 + (u \cdot \xi^w + v \cdot \eta^t) + \frac{(u \cdot \xi^w + v \cdot \eta^t)^2}{2!} + \frac{(u \cdot \xi^w + v \cdot \eta^t)^3}{3!} + \text{etc.}$$

The powers of the binomial are formed according to the same rule as before. (*Fundamental Theorems*, page 18.)

COMPOSITION OF ROTATIONS.

To find the resultant of two elliptic rotations round axes which pass through a common point.

Two circular rotations are compounded by the principle that the product of the half rotations is half of the resultant rotation.

Let any two circular rotations be denoted by ξ^u and η^v, and their resultant by $\xi^u \times \eta$; then

$$\xi^u \times \eta^v = (\xi^{\frac{u}{2}}\eta^{\frac{v}{2}})^2$$
$$= \left\{ \cos\frac{u}{2}\cos\frac{v}{2} - \sin\frac{u}{2}\sin\frac{v}{2}\cos\xi\eta \right.$$
$$\left. + \left(\cos\frac{v}{2}\sin\frac{u}{2}\cdot\xi + \cos\frac{u}{2}\sin\frac{v}{2}\cdot\eta - \sin\frac{u}{2}\sin\frac{v}{2}\,\mathrm{Sin}\,\xi\eta\right)^{\frac{\pi}{2}} \right\}^2.$$

Let $x = \cos\frac{u}{2}\cos\frac{v}{2} - \sin\frac{u}{2}\sin\frac{v}{2}\cos\xi\eta$,

$$y = \sqrt{1-x^2},$$

$$\zeta = \frac{\cos\frac{v}{2}\sin\frac{u}{2}\cdot\xi + \cos\frac{u}{2}\sin\frac{v}{2}\cdot\eta - \sin\frac{u}{2}\sin\frac{v}{2}\,\mathrm{Sin}\,\xi\eta}{\sqrt{1-x^2}};$$

then
$$\xi^u \times \eta^v = x^2 - y^2 + 2xy\cdot\zeta^{\frac{\pi}{2}}.$$

The elliptic generalization is obtained by generalizing the axes ξ and η and finding $\cos\xi\eta$ and $\mathrm{Sin}\,\xi\eta$, as at page 15.

To find the resultant of two hyperbolic rotations round axes which pass through a common point.

Let ξ^{iu} and η^{iv} denote two exspherical rotations which have a common principal axis; let their resultant be denoted by $\xi^{iu} \times \eta^{iv}$. By analogy we deduce that

$$\xi^{iu} \times \eta^{iv} = (\xi^{\frac{iu}{2}}\eta^{\frac{iv}{2}})^2$$
$$= \left\{ \cosh\frac{u}{2}\cosh\frac{v}{2} + \sinh\frac{u}{2}\sinh\frac{v}{2}\cos\xi\eta \right.$$
$$\left. + i\left(\cosh\frac{v}{2}\sinh\frac{u}{2}\cdot\xi + \cosh\frac{u}{2}\sinh\frac{v}{2}\cdot\eta - i\sinh\frac{u}{2}\sinh\frac{v}{2}\,\mathrm{Sin}\,\xi\eta\right)^{\frac{\pi}{2}} \right\}^2.$$

Let $x = \cosh\frac{u}{2}\cosh\frac{v}{2} + \sinh\frac{u}{2}\sinh\frac{v}{2}\cos\xi\eta$.

$$y = \sqrt{x^2-1},$$

$$\zeta = \frac{\cosh\frac{v}{2}\sinh\frac{u}{2}\cdot\xi + \cosh\frac{u}{2}\sinh\frac{v}{2}\cdot\eta - i\sinh\frac{u}{2}\sinh\frac{v}{2}\,\mathrm{Sin}\,\xi\eta}{\sqrt{x^2-1}}.$$

Then
$$\xi^{iu} \times \eta^{iv} = x^2 + y^2 + 2xy\cdot\zeta^{\frac{\pi}{2}}.$$

Suppose a fluid to move round the axis ξ, each particle describing a hyperbolic angle u, and then round the axis η by a hyperbolic angle v, the principal axes of the two motions coinciding; the resultant gives the angle, the plane, and the principal axes of the equivalent single motion of the same kind. The axis of that motion does not pass through the intersection of the axes of the components.

A more general result is obtained by supposing the ratios to be complex; the theorem is then expressed by the spherical theorem taken in a generalized sense, just as in ordinary algebra x may be positive or negative.

To find the effect of an elliptic rotation on a line.

The effect of a circular rotation ξ^u upon a unit axis ρ, is given by the equation

$$\xi^u\rho = \cos\xi\rho\cdot\xi + \sin u \operatorname{Sin}\xi\rho + \cos u \operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi.$$

(*Principles of the Algebra of Physics*, page 100.)

It was shown by Cayley that the effect of ξ^u upon ρ is given by the Sine of the product $\xi^{-\frac{u}{2}}\rho^{\frac{\pi}{2}}\xi^{\frac{\pi}{2}}$. For by the expansion of

$$\left(\cos\frac{u}{2} - \sin\frac{u}{2}\cdot\xi^{\frac{\pi}{2}}\right)\rho^{\frac{\pi}{2}}\left(\cos\frac{u}{2} + \sin\frac{u}{2}\cdot\xi^{\frac{\pi}{2}}\right),$$

the directed sine is found to be

$$\cos^2\frac{u}{2}\cdot\rho + \sin^2\frac{u}{2}\cos\xi\rho\cdot\xi + \cos\frac{u}{2}\sin\frac{u}{2}\operatorname{Sin}\xi\rho - \sin^2\frac{u}{2}\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi.$$

But $\quad \cos^2\frac{u}{2}\cdot\rho = \cos^2\frac{u}{2}\cos\xi\rho\cdot\xi + \cos^2\frac{u}{2}\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi,$

therefore the directed sine is

$$\cos\xi\rho\cdot\xi + \sin u\operatorname{Sin}\xi\rho + \cos u\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi.$$

To generalize for an elliptic rotation we substitute the more general value of ξ and form $\cos\xi\rho$, $\operatorname{Sin}\xi\rho$, and $\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi$, according to the rules stated at page 15. For example, let

$$\xi = k\cos\phi\cdot\beta - \sin\phi\cdot\alpha,$$

$$\rho = \sin\theta\cdot\gamma + \cos\theta\cdot\alpha;$$

then

$$\cos\xi\rho = \cos\phi\sin\theta\cos\beta\gamma - \sin\phi\cos\theta,$$

$$\operatorname{Sin}\xi\rho = \cos\phi\sin\theta\sin\beta\gamma\cdot\alpha + k(\cos\phi\cos\theta\cdot\overline{\beta\alpha} - \sin\phi\sin\theta\cdot\overline{\alpha\gamma}).$$

To find the effect of a hyperbolic rotation on a line.

Consider the simplest exspherical analogue of the spherical theorem of the preceding article; it is

$$\xi^{iu}\rho = \cos\xi\rho\cdot\xi + i\sinh u\operatorname{Sin}\xi\rho + \cosh u\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi.$$

But ξ is now an excircular axis of the form

$$\xi = \cosh\phi\cdot\beta - i\sinh\phi\cdot\alpha.$$

Let, as before, $\rho = \sin\theta\cdot\gamma + \cos\theta\cdot\alpha$;

then $\cos\xi\rho = \cosh\phi\sin\theta\cos\beta\gamma - i\sinh\phi\cos\theta,$

$$\operatorname{Sin}\xi\rho = \cosh\phi\sin\theta\sin\beta\gamma\cdot\alpha + \cosh\phi\cos\theta\cdot\overline{\beta\alpha} - i\sinh\phi\sin\theta\cdot\overline{\alpha\gamma},$$

$$\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi$$
$$= \cosh^2\phi\sin\theta\sin\beta\gamma\cdot\overline{\alpha\beta} + \cosh^2\phi\cos\theta\cdot\alpha - \sinh^2\phi\sin\theta\cdot\gamma$$
$$- i\cosh\phi\sinh\phi\cos\theta\cdot\beta - i\cosh\phi\sinh\phi\sin\theta\sin\overline{\alpha\gamma}\beta\cdot\alpha.$$

The effect of a hyperbolic rotation is obtained by taking the more general value of ξ and applying the hyperbolic rules of multiplication.

Utility of Quaternions in Physics. By A. McAulay, M.A., Lecturer in Mathematics and Physics in the University of Tasmania. pp. xiv, 107. London, Macmillan & Co.

The volume before us is an essay that was submitted in December, 1887, in competition for the Smith's Prizes at the University of Cambridge, under the title of "Quaternions as a Practical Instrument of Physical Research." An article bearing the original title of the essay appeared in the *Philosophical Magazine* for June, 1892, and another extract was printed in the *Proceedings of the Royal Society of Edinburgh*, 1890–91, p. 98, under the title of "Proposed Extension of the Powers of Quaternion Differentiation." The present volume contains the complete essay, with a short preface and some foot-notes in addition.

The essay opens as follows: "It is a curious phenomenon in the History of Mathematics that the greatest work of the greatest mathematician of the century which prides itself upon being the most enlightened the world has yet seen, has suffered the most chilling neglect." In further description of this phenomenon, it is stated that the work has been neglected alike by pure analysts and mathematical physicists with very few exceptions, the grand exception being Professor Tait; that it is not studied at the University of Cambridge except by a few, and only as a non-commutative algebra, not as a geometrical method; that there is a solid and well-nigh universal scepticism as to its utility in original physical investigations, and that the physicists who have studied it are satisfied with Maxwell's paradoxical position: "I am convinced that the introduction of the ideas, as distinguished from the operations and methods of quaternions, will be of great use to us in all parts of our subject." To complete the description of the phenomenon, I may add that a Scottish mathematician, on reading Hamilton's *Quaternions*, first formed the alternative conclusion that either he himself was a dull stupid or the book sheer nonsense, but on reading further was able to arrive at the more comforting alternative; that a German mathematician declared the method to be "an aberration of the human intellect"; and that a French mathematician gave the verdict, "Quaternions have no sense in them, and to try to find for them a geometrical interpretation is as if one were to turn out a well-rounded phrase, and were afterwards to bethink oneself about the meaning to be put into the words."

How does the author explain the curious phenomenon? As follows: "Workers naturally find themselves, while still inexperienced in the use of quaternions, incapable of clearly thinking through them and of making them do the work of Cartesian geometry, and they conclude that quaternions do not provide suitable treatment for what they have in hand. They then grow rather disgusted with these vexatious quaternions, and, consoling themselves with the reflection that Maxwell, before penning the above extract, had had more experience than themselves, decide that the subject only requires a superficial study to be rendered of as great utility as it is capable." But the author admits that there is a veritable stumbling-block in the way, and to remove it is the object of the essay. He says, p. 2: "The fact is that the subject requires a slight development in order readily to apply to the practical consideration of most physical subjects. The first steps of this, which consist chiefly in the invention of new symbols of operation and a slight examination of their chief properties, I have endeavored to give in the following pages." The author's development consists in an extension of quaternion differentiation, pp. 12–24. According to Hamilton and Tait, the symbol $\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$ is merely an operator, and therefore should be written immediately to the left of the operand; but, according to Mr. McAulay, it is a *symbolic vector*, and therefore is capable of any position, whether before or after the variable. He denotes the tie between the symbolic vector and the variable, not by juxtaposition to the left, but by a common suffix. In the third edition of his *Treatise*, Professor Tait allows separation, but he stickles for separation to the left only. A new symbolic vector Δ is introduced; it applies to all the variables in the term in which it appears. The symbol ∇ with a prefix, as $\sigma\Delta$, means $\frac{d}{du}i + \frac{d}{dv}j + \frac{d}{dw}k$, σ being $ui + vj + wk$. Similarly, ϕ being a linear vector function of any vector whose co-ordinates are $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $\phi\text{ロ}$ is defined as a symbolic linear vector function, whose co-ordinates are $\frac{d}{da_1}\frac{d}{db_1}\frac{d}{dc_1}$, $\frac{d}{da_2}\frac{d}{db_2}\frac{d}{dc_2}$, $\frac{d}{da_3}\frac{d}{db_3}\frac{d}{dc_3}$. Finally, a symbolic vector ζ is introduced, which is such that $Q\ (\alpha,\ \beta)$ being linear in each of the vectors α, β,

$$Q(\zeta,\ \zeta) = Q(\nabla_1,\ \rho_1) = Q(i,\ i) + Q(j, j) + Q(k,\ k).$$

The remainder of the essay consists of an application of the quaternion analysis so developed to the theories of elastic solids, electricity and magnetism, and hydrodynamics. It is almost wholly a translation into quaternion notation of known results: the author, however, has endeavored to advance each of the theories chosen in at least one direction. The work shown is designed to make good the following statements: *first*, that quaternions are in such a stage of development as already to justify the practically complete banishment of Cartesian geometry from physical questions of a general nature; and *second*, that quaternions will in physics produce many new results that cannot be produced by the rival and older theory. But in the preface, the author now states that he delayed publication until he could by a more striking example than any in the essay show the immense utility of quaternions; this he believes has been done by a paper published in the *Philosophical Transactions* for 1892. At the time of writing the essay he possessed little more than faith, and he felt that something more than faith was needed to convince scientists. In conclusion he exhorts mathematical physicists to study quaternions seriously, and he looks forward to the time when quaternions will appear in every physical text-book that assumes the knowledge of elementary plane trigonometry.

I agree with the author in his estimate of the value of Hamilton's quaternion researches: they constitute, in my opinion, the greatest mathematical work of the century. They contain what was long sought after — *a* veritable extension of algebra to space: I do not say *the*, for I believe that there is more than one. The Cartesian analysis is also an extension of algebra to space, but it is fragmentary and incomplete; whereas the quaternion analysis is the true spherical trigonometry in which the axis of an angle as well as its magnitude is considered.

But I cannot agree with the author in his explanation of the comparative neglect which the work has hitherto received. The notation has been a stumbling-block. Familiar functions, such as the cosine and sine, are replaced by new selective symbols S and V; Greek letters are used alike for axes, vectors, versors, and functions of these. As a consequence, the notation is contracted; to make it more expansive, Maxwell introduced German capitals for vectors, Mr. Heaviside used black letters instead, and in the work before us we have a rather incongruous mixture of Greek and black letters. The notation has divorced the method from the rest of analysis, so much so that Mr. McAulay believes that it is an independent plant which cannot be grafted on the old tree of analysis (*Philosophical Magazine*, Vol. 33, p. 479).

The identification of vectors and quadrantal versors has been a stumbling-block. Quaternion analysis is nothing but spherical trigonometry in which the axis of the angle is explicitly denoted; this will become clear to any one who studies the fundamental rules and the manner in which Hamilton arrived at them. It is the analysis of directed ratios. The identification mentioned assumes that it is also the analysis of lines, areas, volumes, and other physical products; it leads to the paradox that the square of a vector is essentially negative, and to a total disregard of the dimensions of physical quantities. Here no doubt we have the reason why Maxwell accepted the ideas but not the operations and methods. The analysis which the physicist mostly requires is not identical with, but complementary to, the analysis of ratios.

The largely *symbolic* character of the method has been a stumbling-block. Neither the definitions nor the rules have been placed on a clear and unambiguous basis. The fundamental rules are based, sometimes on the addition of quadrantal arcs, sometimes on the quadrantal rotation of an axis. The notation ∇ is presented as symbolic, so are the author's ∇, ζ, and Ɑ; at most they are shorthand rather than systematic and logical notation. Need we wonder that competent mathematicians cannot think clearly through quaternions, for the original writers do not pretend to do so.

The rival or antagonistic attitude towards the Cartesian analysis has been a stumbling-block. Not only does the quaternion plant, according to Mr. McAulay, require independent sowing, but he would have us pull up the old Cartesian tree with its multitude of branches and far-spreading roots, in order to make room for the new plant. But when he comes to consider more specifically how much should be pulled up, he encounters a difficulty. Thus, p. 3, he says, "For particular problems, such as the torsion problem for a cylinder of a given shape, we require of course the various theories specially constructed for the solution of particular problems, such as Fourier's theories, complex variables, spherical harmonics, etc. It will thus be seen that I do not propose to banish these theories, but merely Cartesian geometry." If, then, the quaternion analysis fails, and the problem is turned over to the theory of complex variables (as at p. 49), it is important that these two branches of analysis should be logically harmonious and free from contradiction in matters of convention. If they are logically harmonious, it will be easy for a student or analyst to pass from the one to the other; but, as a matter of fact, the conventions are contradictory. Is not this the very meaning of the author's metaphor of the independent plant that cannot be grafted on the already flourishing tree? In several papers recently published I have aimed at showing how this logical harmony may be brought about, and one space-analysis be developed which shall embrace algebra, trigonometry, complex numbers, Cartesian analysis, Grassmann's method, and quaternions. Till this harmony is established the ideas and methods of Hamilton will not bring forth the great results which exist in them potentially.

ALEXANDER MACFARLANE.