INTRODUCTION TO QUATERNIONS

INTRODUCTION
TO
QUATERNIONS

BY THE LATE PROFESSORS
PHILIP KELLAND, M.A., F.R.S.
AND
PETER GUTHRIE TAIT, M.A.

THIRD EDITION.

PREPARED BY
C. G. KNOTT, D.Sc.
LECTURER IN APPLIED MATHEMATICS IN THE UNIVERSITY OF EDINBURGH
FORMERLY PROFESSOR OF PHYSICS IN THE IMPERIAL UNIVERSITY OF JAPAN

London
MACMILLAN AND CO., LIMITED
NEW YORK: THE MACMILLAN COMPANY
1904

First Edition, 1873; Second Edition, 1882;
Third Edition, 1904.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO.

PREFACE TO THE THIRD EDITION.

In preparing the third edition of Kelland and Tait's *Introduction to Quaternions* I have been guided mainly by two considerations. In the first place, the average mathematical student of to-day attains either at school or in his early college courses a much higher standard than was possible in 1873 when Kelland wrote, or even in 1881, the date of the second edition. It seemed, therefore, desirable to delete many of the very simple geometrical illustrations which formed a large part of the text, indicating their nature by a word, or transferring them as exercises to the end of the appropriate chapter. In this way valuable space has been gained for the discussion of problems more fitted to bring out the power and beauty of the quaternion calculus.

It is right to mention, however, that Chapter I. has been left exactly as Kelland wrote it; and the greater part of Chapter II. is simply reproduced.

The second consideration was the necessity for presenting the main features of Hamilton's great calculus in a brief but yet logically complete form. This has led to the recasting of Chapters III. and IX. In the new Chapters III. and IV. the calculus in its essential features is developed systematically from the definition of a quaternion as the complex number which measures the ratio of two vectors, with the further assumption that the associative law holds in product combinations. From these two root principles the whole of Hamilton's powerful vector algebra evolves itself simply and naturally. It is hoped that the mode of presentation will remove the difficulty which some have experienced in accepting Hamilton's identification of vector and quadrantal versor. O'Brien, Hamilton's brilliant contemporary, confessed that the difficulty was to him insurmountable. But the difficulty is really created by the sceptic himself, who fails to see that, so far as the mathematical definition goes, a vector quantity in quaternions has a much wider significance than the step or displacement or velocity by means of which the simple summation principles are first illustrated. The law of vector addition, which is common to all kinds of vectors, including the Hamiltonian, determines nothing as to the laws of product combinations. These may be anything we please among vectors, so long as the law of vector addition is satisfied. Now it is proved in Chapter III., § 18, that quadrantal versors obey the vector law of addition. They are therefore true vectors; and hence follows, from the *geometrical* point of view, the analytical identification of vector and quadrantal versor. The identification, no doubt, requires every vector (whatever physical quantity it may symbolise) to be subject analytically to the quadrantal versor laws in product combinations; but this, as Hamilton himself proved, is tantamount to requiring that three or more vectors in product combinations obey the associative law. There is thus perfect consistency throughout.

From the point of view of pure analysis the difficulty mentioned above cannot, of course, present itself. The quaternion is then a quantity involving four units, which are defined as reproducing themselves in product combinations and as satisfying certain general laws. The mathematical properties of the quaternion being thus established, the utility of the calculus will depend simply upon the mode of interpretation. Thus Professor C. J. Joly, by a new interpretation of the quaternion, has recently developed an interesting treatment of projective geometry.

In Chapter IX. a completely new section has been introduced on dynamical applications. This seemed to be specially called for, inasmuch as vector ideas and notations are now a familiar feature of some of our best modern books on mathematical physics. It is to be hoped that they will become so more and more, and that the powerful Hamiltonian method which develops the ideas and underlies the notation will become equally familiar.

The last four articles of Chapter IX. have to do with the chief properties of the remarkable differential operator ∇. Differentiation in the ordinary sense was excluded from the earlier editions, although the method was implicitly used in the treatment of tangents. It was impossible, however, to give any true idea of the power of quaternions in dynamics without the explicit introduction of differentiation; and this consideration seemed to me to outweigh all considerations based on artificial distinctions as to what is or is not suitable in an elementary book. The mathematical student who is able to appreciate the exquisite beauties of the linear vector function as expounded in Chapter X. will have no difficulty in appreciating the significance of Nabla.

Tait's very instructive Chapter X. has been left practically untouched. It is the work of a recognised master, and has been a source of inspiration to many students of the subject. As a pupil of both Kelland and Tait, and as a colleague and friend of the latter, I have had peculiar pleasure in preparing this third edition of their joint work, and trust that it may draw the mathematical student into an attractive and largely unexplored field of mathematics. Analytically the quaternion is now known to take its place in the general theory of complex numbers and continuous groups; it is remarkable that it should have provided for the geometry and dynamics of our visible universe a calculus of great power and simplicity.

My thanks are due to Mr. Peter Ross, M.A., for his careful proof-reading of all but the very earliest Chapters.

C. G. KNOTT.

EDINBURGH UNIVERSITY,
October, 1903.

PREFACE TO SECOND EDITION.

In preparing this second edition for press I have altered as slightly as possible those portions of the work which were written entirely by Prof. Kelland. The mode of presentation which he employed must always be of great interest, if only from the fact that he was an exceptionally able teacher; but the success of the work, as an introduction to a method which is now rapidly advancing in general estimation, would of itself have been a sufficient motive for my refraining from any serious alteration.

A third reason, had such been necessary, would have presented itself in the fact that I have never considered with the necessary care those metaphysical questions connected with the growth and development of mathematical ideas, to which my late venerated teacher paid such particular attention.

My own part of the book (including mainly Chapter X. and worked out Examples 10—24 in Chapter IX.) was written hurriedly, and while I was deeply engaged with work of a very different kind; so that I had no hesitation in determining to re-cast it where I fancied I could improve it.

P. G. TAIT.

University of Edinburgh,
November, 1881.

PREFACE TO THE FIRST EDITION.

THE present Treatise is, as the title-page indicates, the joint production of Prof. Tait and myself. The preface I write in the first person, as this enables me to offer some personal explanations.

For many years past I have been accustomed, no doubt very imperfectly, to introduce to my class the subject of Quaternions as part of elementary Algebra, more with the view of establishing principles than of applying processes. Experience has taught me that to induce a student to think for himself there is nothing so effectual as to lay before him the different stages of the development of a science in something like the historical order. And justice alike to the student and the subject forbade that I should stop short at that point where, more simply and more effectually than at any other, the intimate connexion between principles and processes is made manifest. Moreover, in lecturing on the groundwork on which the mathematical sciences are based, I could not but bring before my class the names of great men who spoke in other tongues and belonged to other nationalities than their own—Diophantus, Des Cartes, Lagrange, for instance—and it was not just to omit the name of one as great as any of them, Sir William Rowan Hamilton, who spoke their own tongue and claimed their own nationality. It is true the name of Hamilton has not had the impress of time to stamp it with the seal of immortality. And it must be admitted that a cautious policy which forbids to wander from the beaten paths, and encourages converse with the past rather than interference with the present, is the true policy of a teacher. But in the case before us, quite irrespective of the nationality of the inventor, there is ample ground for introducing this subject of Quaternions into an elementary course of mathematics. It belongs to first principles and is their crowning and completion. It brings those principles face to face with operations, and thus not only satisfies the student of the mutual dependence of the two, but tends to carry him back to a clear apprehension of what he had probably failed to appreciate in the subordinate sciences.

Besides, there is no branch of mathematics in which results of such wide variety are deduced by one uniform process; there is no territory like this to be attacked and subjugated by a single weapon. And what is of the utmost importance in an educational point of view, the reader of this subject does not require to encumber his memory with a host of conclusions already arrived at in order to advance. Every problem is more or less self-contained. This is my apology for the present treatise.

The work is, as I have said, the joint production of Prof. Tait and myself. The preface I have written without consulting my colleague, as I am thus enabled to say what could not otherwise have been said, that mathematicians owe a lasting debt of gratitude to Prof. Tait for the singleness of purpose and the self-denying zeal with which he has worked out the designs of his friend Sir Wm. Hamilton, preferring always the claims of the science and of its founder to the assertion of his own power and originality in its development. For my own part I must confess that my knowledge of Quaternions is due exclusively to him. The first work of Sir Wm. Hamilton, *Lectures on Quaternions*, was very dimly and imperfectly understood by me and I dare say by others, until Prof. Tait published his papers on the subject in the

Messenger of Mathematics. Then, and not till then, did the science in all its simplicity develop itself to me. Subsequently Prof. Tait has published a work of great value and originality, *An Elementary Treatise on Quaternions.*

The literature of the subject is completed in all but what relates to its physical applications, when I mention in addition Hamilton's second great work, *Elements of Quaternions*, a posthumous work so far as publication is concerned, but one of which the sheets had been corrected by the author, and which bears all the impress of his genius. But it is far from elementary, whatever its title may seem to imply; nor is the work of Prof. Tait altogether free from difficulties. Hamilton and Tait write for mathematicians, and they do well, but the time has come when it behoves some one to write for those who desire to become mathematicians. Friends and pupils have urged me to undertake this duty, and after consultation with Prof. Tait, who from being my pupil in youth is my teacher in riper years, I have, in conjunction with him, and drawing unreservedly from his writings, endeavoured in the first nine chapters of this treatise to illustrate and enforce the principles of this beautiful science. The last chapter, which may be regarded as an introduction to the application of Quaternions to the region beyond that of pure geometry, is due to Prof. Tait alone. Sir W. Hamilton, on nearly the last completed page of his last work, indicated Prof. Tait as eminently fitted to carry on happily and usefully the applications, mathematical and physical, of Quaternions, and as likely to become in the science one of the chief successors of its inventor. With how great justice, the reader of this chapter and of Prof. Tait's other writings on the subject will judge.

PHILIP KELLAND.

UNIVERSITY OF EDINBURGH,
October, 1873.

CONTENTS.

CHAPTER I.

CHAPTER II.

VECTOR ADDITION AND SUBTRACTION. §§ 1-14.

CHAPTER III.

QUATERNIONS AND VERSORS OR QUOTIENTS AND PRODUCTS OF VECTORS. §§ 15-24.

CHAPTER IV.

QUATERNION PRODUCTS AND RELATED DEVELOPMENTS. §§ 25-34.

CHAPTER V.

SIMPLE GEOMETRICAL APPLICATIONS. §§ 35-42.

CHAPTER VI.

CONES AND THEIR SECTIONS. §§ 43-56.

CHAPTER VII.

CENTRAL SURFACES OF THE SECOND ORDER. §§ 57-65.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

INTRODUCTION TO QUATERNIONS.

CHAPTER I.

INTRODUCTORY.

THE science named Quaternions by its illustrious founder, Sir William Rowan Hamilton, is the last and the most beautiful example of extension by the removal of limitations.

The Algebraic sciences are based on ordinary arithmetic, starting at first with all its restrictions, but gradually freeing themselves from one and another, until the parent science scarce recognises itself in its offspring. A student will best get an idea of the thing by considering one case of extension within the science of Arithmetic itself. There are two distinct bases of operation in that science—addition and multiplication. In the infancy of the science the latter was a mere repetition of the former. Multiplication was, in fact, an abbreviated form of equal additions. It is in this form that it occurs in the earliest writer on arithmetic whose works have come down to us—Euclid. Within the limits to which his principles extended, the reasonings and conclusions of Euclid in his seventh and following Books are absolutely perfect. The demonstration of the rule for finding the greatest common measure of two numbers in Prop. 2, Book VIII., is identically the same as that which is given in all modern treatises. But Euclid dares not venture on fractions. Their properties were probably all but unknown to him. Accordingly we look in vain for any *demonstration* of the properties of fractions in the writings of the Greek arithmeticians. For that we must come lower down. On the revival of science in the West, we are presented with categorical treatises on arithmetic. The first printed treatise is that of Lucas de Burgo in 1494. The author considers a fraction to be a quotient, and thus, as he expressly states, the order of operations becomes the reverse of that for whole numbers—multiplication precedes addition, etc. In our own country we have a tolerably early writer on arithmetic, Robert Record, who dedicated his work to King Edward the Sixth. The ingenious author exhibits his treatise in the form of a dialogue between master and scholar. The scholar battles long with this difficulty—that multiplying a thing should make it less. At first, the master attempts to explain the anomaly by reference to proportion, thus: that the product by a fraction bears the same proportion to the thing multiplied that the multiplying fraction does to unity. The scholar is not satisfied; and accordingly the master goes on to say: "If I multiply by more than one, the thing is increased; if I take it but once, it is not changed; and if I take it less than once, it cannot be so much as it was before. Then, seeing that a fraction is less than one, if I multiply by a fraction, it follows that I do take it less than once," etc. The scholar thereupon replies, "Sir, I do thank you much for this reason; and I trust that I do perceive the thing."

Need we add that the same difficulty which the scholar in the time of King Edward experienced, is experienced by every thinking boy of our own times; and the explanation afforded him is precisely the same admixture of multiplication, proportion, and division which suggested itself to old Robert Record. Every schoolboy feels that to multiply by a fraction is not to multiply at all in the sense in which multiplication was originally presented to him, viz. as an abbreviation of equal additions, or of repetitions of the thing multiplied. A totally new view of the process of multiplication has insensibly crept in by the advance from whole numbers to fractions. So new, so different is it, that we are satisfied Euclid in his logical and unbending march could never have attained to it. It is only by standing loose for a time to logical accuracy that extensions in the abstract sciences—extensions at any rate which stretch from one science to another—are effected. Thus Diophantus in his Treatise on Arithmetic (*i.e.* Arithmetic extended to Algebra) boldly lays it down as a definition or first principle of his science that 'minus into minus makes plus.' The science he is founding is subject to this condition, and the results must be interpreted consistently with it. So far as this condition does not belong to ordinary arithmetic, so far the science extends beyond ordinary arithmetic: and this is the distance to which it extends—It makes subtraction to stand by itself, apart from addition; or, at any rate, not dependent on it.

We trust, then, it begins to be seen that sciences are extended by the removal of barriers, of limitations, of conditions, on which sometimes their very existence appears to depend. Fractional arithmetic was an impossibility so long as multiplication was regarded as abbreviated addition: the moment an extended idea was entertained, ever so illogically, that moment fractional arithmetic started into existence. Algebra, except as mere symbolized arithmetic, was an impossibility so long as the thought of subtraction was chained to the requirement of something adequate to subtract from. The moment Diophantus gave it a separate existence—boldly and logically as it happened—by exhibiting the law of *minus* in the forefront as the primary definition of his science, that moment algebra in its highest form became a possibility; and indeed the foundation-stone was no sooner laid than a goodly building arose on it.

The examples we have given, perhaps from their very simplicity, escape notice, but they are not less really examples of extension from science to science by the removal of a restriction. We have selected them in preference to the more familiar one of the extension of the meaning of an index, whereby it becomes a logarithm, because they prepare the way for a further extension in the same direction to which we are presently to advance. Observe, then, that in fractions and in the rule of signs, addition (or subtraction) is very slenderly connected with multiplication (or division). Arithmetic as Euclid left it stands on one support, addition only, inasmuch as with him multiplication is but abbreviated addition. Arithmetic in its extended form rests on two supports, addition and multiplication, the one different from the other. This is the first idea we want our reader to get a firm hold of; that multiplication is not necessarily addition, but an operation self-contained, self-interpretable—springing originally out of addition; but, when full-grown, existing apart from its parent.

The second idea we want our reader to fix his mind on is this, that when a science has been extended into a new form, certain limitations, which appeared to be of the nature of essential truths in the old science, are found to be utterly untenable; that it is, in fact, by throwing these limitations aside that room is made for the growth of the new science. We have instanced Algebra as a growth out of Arithmetic by the removal of the restriction that subtraction shall require something to subtract from. The word 'subtraction' may indeed be inappropriate, as the word multiplication appeared to be to Record's scholar, who failed to see how the multiplication of a thing could make it less. In the advance of the sciences the old terminology often becomes inappropriate; but if the mind can extract the right idea from the sound or sight of a word, it is the part of wisdom to retain it. And so all the old words have been retained in the science of Quaternions to which we are now to advance.

The fundamental idea on which the science is based is that of motion—of transference. Real motion is indeed not needed, any more than real superposition is needed in Euclid's Geometry. An appeal is made to mental transference in the one science, to mental superposition in the other.

We are then to consider how it is possible to frame a new science which shall spring out of Arithmetic, Algebra, and Geometry, and shall add to them the idea of motion—of transference. It must be confessed the project we entertain is not a project due to the nineteenth century. The Geometry of Des Cartes was based on something very much resembling the idea of motion, and so far the mere introduction of the idea of transference was not of much value. The real advance was due to the thought of severing multiplication from addition, so that the one might be the representative of a kind of motion absolutely different from that which was represented by the other, yet capable of being combined with it. What the nineteenth century has done, then, is to divorce addition from multiplication in the new form in which the two are presented, and to cause the one, in this new character, to signify motion forwards and backwards, the other motion round and round.

We do not purpose to give a history of the science, and shall accordingly content ourselves with saying, that the notion of separating addition from multiplication—attributing to the one, motion from a point, to the other motion about a point—had been floating in the minds of mathematicians for half a century, without producing many results worth recording, when the subject fell into the hands of a giant, Sir William Rowan Hamilton, who early found that his road was obstructed—he knew not by what obstacle—so that many points which seemed within his reach were really inaccessible. He had done a considerable amount of good work, obstructed as he was, when, about the year 1843, he perceived clearly the obstruction to his progress in the shape of an old law which prior to that time, had appeared like a law of common sense. The law in question is known as the *commutative* law of multiplication. Presented in its simplest form it is nothing more than this, 'five times three is the same as three times five'; more generally, it appears under the form of '$ab = ba$ whatever a and b may represent.' When it came distinctly into the mind of Hamilton that this law is not a necessity, with the extended signification of multiplication, he saw his way clear, and gave up the law. The barrier being removed, he entered on the new science as a warrior enters a besieged city through a practicable breach. The reader will find it easy to enter after him.

CHAPTER II.

VECTOR ADDITION AND SUBTRACTION.

1. DIRECTION AS A FUNDAMENTAL GEOMETRICAL CONCEPTION. The explicit recognition of direction as a fundamental geometrical conception is the distinguishing mark of quaternionic and other vectorial methods. A little consideration will soon convince us that the comparison of directions is more intuitive than the comparison of lengths. The eye has no difficulty in judging as to the parallelism of two lines, but has considerable difficulty in judging as to the equality of their lengths especially if the lines are not parallel. The similarity of two triangles of different size when set with their corresponding sides parallel is apparent at a glance; not so the equality of two triangles equal in all respects when they are set with their corresponding sides not parallel. These and other like illustrations show that the conception of direction is of a fundamental character.

When we wish to determine the position of one point with regard to another we must know not only their distance apart, but also the direction of the line joining them. In like manner the displacement of a point cannot be completely known unless both the direction and the amount of the displacement are given. .We may obviously fully represent this relative position or this displacement by drawing from any starting point or origin a line having the required direction and having a length (drawn to a convenient scale) numerically equal to the required distance apart or to the required amount. Such a representative line is called a VECTOR.

2. VECTORS AND SCALARS. The simplest example of a vector quantity is a directed line, the whole conception involving both length and direction. But it should be clearly understood from the outset that the word is descriptive of any quantity which may be represented by means of a directed line. Thus a vector quantity is one which possesses both direction and magnitude. As examples we may mention position, displacement, velocity, acceleration, momentum, force, moment of force, rotation, and so on.

On the other hand, there are quantities which possess no direction but only magnitude. Such for example are time, temperature, volume, mass, work, energy; and these are distinguished as SCALAR quantities. The magnitude of a vector is evidently a scalar quantity, and may be assigned quite independently of the direction. In ordinary algebra and analytical geometry the symbols used are all scalars, being indeed essentially numbers or ratios. In co-ordinate geometry certain fixed directions are assigned once for all, and the co-ordinates of a point referred to these directions are simply the number of units contained in the distances of the point measured parallel to these directions. They are essentially ratios of parallel vectors.

The importance of distinguishing clearly between vectors and scalars will appear as we proceed.

We shall first take the simplest conception of a vector as represented by transference through a given distance in a given direction. Thus if AB be a straight line, the idea to be attached to 'vector AB' is that of transference or "step" from A to B.

For the sake of definiteness we shall almost invariably represent vectors by Greek letters, retaining in the meantime the English letters to denote ordinary numerical or scalar quantities.

If we now start from B and advance to C in the same direction, BC being equal to AB, we may, as in ordinary geometry, designate 'vector BC' by the same symbol, which we adopted to designate 'vector AB.'

Further, if we start from any other point O in space, and advance from that point by the distance OX equal to and in the same direction as AB, we are at liberty to designate 'vector OX' by the same symbol as that which represents AB.

Other circumstances will determine the starting point, and individualize the line to which a specific vector corresponds. Our definition is therefore subject to the following condition :—*All lines which are equal and drawn in the same direction are represented by the same vector symbol.*

We have purposely employed the phrase 'drawn in the same direction' instead of 'parallel,' because we wish to guard the student against confounding 'vector AB' with 'vector BA.'

In order to apply algebra to geometry it is necessary to impose on geometry the condition that when a line measured in one direction is represented by a *positive* symbol, the same line measured in the opposite direction must be represented by the corresponding *negative* symbol.

In the science before us the same condition is equally requisite, and indeed the reason for it is even more manifest. For if a transference from A to B be represented by $+\alpha$, the transference which neutralizes this, and brings us back again to A, cannot be conceived to be represented by anything but $-\alpha$, provided the symbols $+$ and $-$ are to retain any of their old algebraic meaning. The vector AB, then, being represented by $+\alpha$, the vector BA will be represented by $-\alpha$.

3. PARALLEL VECTORS. Further it is abundantly evident that so far as addition and subtraction of parallel vectors are concerned, *all* the laws of Algebra must be applicable. Thus in last paragraph $AB+BC$ or $\alpha+\alpha$ produces the same result as AC which is twice as great as AB, and is therefore properly represented by 2α; and so on for all the rest. The *distributive* law of addition may then be assumed to hold in all its integrity so long at least as we deal with vectors which are parallel to one another. In fact there is no reason whatever, so far, why α should not be treated in every respect as if it were an ordinary algebraic quantity. It need scarcely be added that vectors in the same direction have the same proportion as the *lines* which correspond to them.

We have then advanced to the following—

LEMMA. *All lines drawn in the same direction are, as vectors, to be represented by numerical multiples of one and the same symbol, to which the ordinary laws of Algebra, so far as their addition, subtraction, and numerical multiplication are concerned, may be unreservedly applied.*

The converse is of course true, that if lines as vectors are represented by numerical or scalar multiples of the same vector symbol, they are parallel.

4. NON-PARALLEL VECTORS. It is only necessary to add to what has preceded, that if BC be a line *not* in the same direction with AB, then the vector BC cannot be represented by α or by any ~~scalar~~ multiple of α. The vector symbol α must be limited to express transference in a certain direction, and cannot, at the same time, express transference in any other direction. To express 'vector BC' then, another and quite independent symbol β must be introduced. This symbol, being united to α by the signs + and −, the laws of algebra will, of course, apply to the combination.

5. Vector Addition. If we now join AC, and thus form a triangle ABC, and if we denote vector AB by α, BC by β, AC by γ, it is clear that we shall be presented with the equation $\alpha + \beta = \gamma$, or, strictly speaking, with the *identity* $\alpha + \beta \equiv \gamma$.

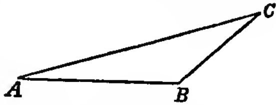

Fig. 1.

This equation appears at first sight to be a violation of Euclid I. 20: "Any two sides of a triangle are together greater than the third side." But it is not really so. The anomalous appearance arises from the fact that whilst we have extended the meaning of the symbol + beyond its arithmetical signification, we have said nothing about that of a symbol =. It is clearly necessary that the signification of this symbol shall be extended along with that of the other. It must now be held to designate, as it does perpetually in algebra, 'equivalent to.' This being premised, the equation above is freed from its anomalous appearance, and is perfectly consistent with everything in ordinary geometry. Expressed in words it reads thus: 'A transference from A to B followed by a transference from B to C is equivalent to a transference from A to C.'

6. Axiom. *If two vectors have not the same direction, it is impossible that the one can neutralize the other.*

This is quite obvious, for when a transference has been effected from A to B, it is impossible to conceive that any amount of transference whatever along BC can bring the moving point back to A.

It follows as a consequence of this axiom, that if α, β be *different* actual vectors, *i.e.* finite vectors not in the same direction, and if $m\alpha + n\beta = 0$, where m and n are numerical quantities; then must $m = 0$ and $n = 0$.

Another form of this consequence may be thus stated. If [still with the above assumption as to α and β] $m\alpha + n\beta = p\alpha + q\beta$, then must $m = p$, and $n = q$.

7. ELEMENTARY ILLUSTRATIONS. The properties of vectors must be based on the fundamental principles of geometry, and it may be well to show by a few elementary illustrations how these are involved.

Thus let AB, CD be equal and parallel lines and represented by the same vector α. Then, if β represents the vector CA, we have the vector equation

$$\text{vector } DB = -\alpha + \beta + \alpha = \beta.$$

In words, "the straight lines which join the extremities of equal and parallel straight lines are themselves equal and parallel."

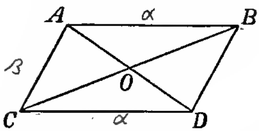

Fig. 2.

Again, if AB and CD are given parallel, and CA, DB given parallel, we may represent them by the vector α, $m\alpha$, β, $n\beta$ where m and n are scalar multiples. Then

$$\beta + \alpha = \text{vector } CB = n\alpha + m\beta,$$

whence by the last article $n = 1$, $m = 1$; and consequently the opposite sides of the parallelogram are equal.

Again as vectors
$$AB = AO + OB,$$
$$CD = CO + OD,$$

where O is the meeting point of the diagonals. But

$AB = CD$; hence $AO + OB = OD + CO$; $AO = OD$, $CO = OB$.

A few simple examples will show with what directness vector methods may be applied to plane geometry.

1. *The bisectors of the sides of a triangle meet in a point which trisects each of them.*

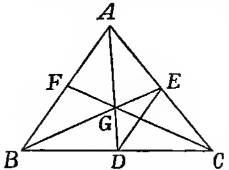

Fig. 3.

Let the sides of the triangle ABC be bisected in D, E, F; and let AD, BE meet in G.

Let vector BD or DC be α, CE or EA β, then, as vectors,

$$BA = BC + CA = 2\alpha + 2\beta = 2(\alpha + \beta),$$
$$DE = DC + CE = \alpha + \beta,$$

hence (§ 3) BA is parallel to DE, and equal to $2DE$.

Again,
$$BG+GA=BA=2DE$$
$$=2(DG+GE).$$

Now vector BG is along GE, and vector GA along DG.
$$\therefore\ (\S\ 6)\ BG=2GE,$$
$$GA=2DG,$$
whence the same is true of the lines.

Lastly,
$$BG=\frac{2}{3}BE=\frac{2}{3}(BC+CE)$$
$$=\frac{2}{3}(2\alpha+\beta);$$
$$\therefore\ CG=BG-BC=\frac{2}{3}(2\alpha+\beta)-2\alpha$$
$$=\frac{2}{3}(\beta-\alpha),$$
$$GF=BF-BG$$
$$=\frac{1}{2}BA-BG,$$
$$=\alpha+\beta-\frac{2}{3}(2\alpha+\beta)$$
$$=\frac{1}{3}(\beta-\alpha);$$
hence CG is in the same straight line with GF, and equal to $2GF$.

2. *When, instead of D, E, F being points taken within BC, CA, AB at distances equal to half those lines respectively, they*

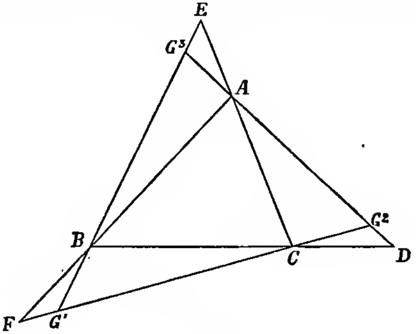

Fig. 4.

are points taken in BC, CA, AB produced, at the same distances respectively from C, A, and B; to find the intersections.

Let the points of intersection be respectively G_1, G_2, G_3.

Retaining the notation of the last example, we have

$$BD = 3\alpha,\ CE = 3\beta\,;$$

$$\text{and} \therefore\ BG_3 = xBE$$

$$= x(2\alpha + 3\beta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\begin{aligned} BG_3 &= BD + DG_3 \\ &= 3\alpha + yDA \\ &= 3\alpha + y(CA - CD) \\ &= 3\alpha + y(2\beta - \alpha)\,; \end{aligned}$$

$$\therefore\ 2x = 3 - y,\ 3x = 2y,\ \text{and}\ x = \frac{6}{7}\,;$$

$$\therefore\ \text{line}\ EG_3 = \frac{1}{7}EB$$

$$\text{Similarly line}\ FG_1 = \frac{1}{7}FC,$$

$$\text{line}\ DG_2 = \frac{1}{7}DA,$$

and from equation (1) $BG_3 = \frac{6}{7}(2\alpha + 3\beta)$.

But $$BG_3 = BA + AG_3 = 2\alpha + 2\beta + AG_3,$$

$$\therefore\ AG_3 = \frac{2}{7}(2\beta - \alpha)\,;$$

hence $$\text{line}\ AG_3 = \frac{2}{7}\ \text{line}\ DA$$

$$= 2DG_2,$$

and similarly of the others.

3. *If DEF be drawn cutting the sides of a triangle; then will* $AD \,.\, BF \,.\, CE = AE \,.\, CF \,.\, BD$.

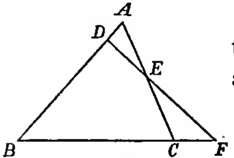

Fig. 5

Let $BD = \alpha$, $DA = p\alpha$, $AE = \beta$, $EC = q\beta$, then $BC = BA + AC = (1 + p)\,\alpha + (1 + q)\,\beta$ and CF is a multiple of BC.

Let $CF = xBC$

$$= x\{(1 + p)\alpha + (1 + q)\beta\}.$$

But $$CF = CE + EF = -EC + EF = -q\beta + y(p\alpha + \beta);$$

$\therefore$ equating, we have $x(1+p) = yp$, $x(1+q) = -q + y$,

whence $$x = (1+x)pq,$$

i.e. $$\frac{CF}{BC} = \frac{BF}{BC} \cdot \frac{AD}{BD} \cdot \frac{CE}{AE};$$

$$\therefore AD \,.\, BF \,.\, CE = AE \,.\, CF \,.\, BD.$$

4. *The points of bisection of the three diagonals of a complete quadrilateral are in a straight line.*

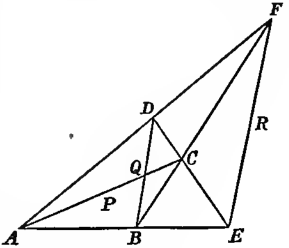

Fig. 6.

P, Q, R, the middle points of the diagonals of the complete quadrilateral $ABCD$, are in a straight line.

Let $AB = \alpha$, $AD = \beta$,

$AE = m\alpha$, $AF = n\beta$;

$\therefore$ $BF = n\beta - \alpha$ and $BC = x(n\beta - \alpha)$,

$ED = \beta - m\alpha$ and $CD = y(\beta - m\alpha)$.

Now $BC + CD = BD = AD - AB$

gives $$x(n\beta - \alpha) + y(\beta - m\alpha) = \beta - \alpha,$$

whence $$xn + y = 1, \; x + my = 1,$$

$$\therefore x = \frac{m-1}{mn-1},$$

and $$AP = \frac{1}{2}AC = \frac{1}{2}\left\{\alpha + \frac{m-1}{mn-1}(n\beta - \alpha)\right\}$$

$$= \frac{1}{2}\frac{m(n-1)\alpha + n(m-1)\beta}{mn-1},$$

$$AQ = \frac{1}{2}(\alpha + \beta),$$

$$AR = \frac{1}{2}(m\alpha + n\beta),$$

$$\therefore AQ - AP = \frac{1}{2(mn-1)}\{(m-1)\alpha + (n-1)\beta\},$$

$$AR - AP = \frac{mn}{2(mn-1)}\{(m-1)\alpha + (n-1)\beta\},$$

or vector PR is a multiple of vector PQ, and therefore they are in the same straight line.

COR. Line $PQ : PR :: 1 : mn$

$$:: AB . AD : AE . AF$$

$$:: \text{triangle } ABD : \text{triangle } AEF.$$

We shall presently exemplify a very elegant method due to Sir W. Hamilton of proving three points to be in the same straight line.

8. UNIT VECTOR AND TENSOR. It is often convenient to take a vector of the length of the unit, and to express the vector under consideration as a numerical multiple of this unit. Of course it is not necessary that the unit should have any specified value; all that is required is that when once assumed for any given problem, it must remain unchanged throughout the discussion of that problem.

If the line AB be supposed to be a units in length, and the *unit vector* along AB be designated by α, then will *vector* AB be $a\alpha$ (§ 3).

Sir William Hamilton has termed the length of the line in such cases, the TENSOR of the vector; so that the vector AB is the product of the tensor AB and the unit vector along AB. Thus if, as in the examples worked under the last article, we designate the vector AB by α, we may write $\alpha = T\alpha U\alpha$, where $T\alpha$ is an abbreviation for 'Tensor of the vector α'; $U\alpha$, for 'unit vector along α'.

Take the following example:

The three bisectors of the angles of a triangle meet in a point.

Let AD, BE bisect A, B and meet in G; CG bisects C.

Let unit vectors along BC, CA, AB be α, β, γ, and let a, b, c be the lengths of the corresponding sides.

Then $\quad a\alpha + b\beta + c\gamma = 0$ (see below, § 9).

$$AG = x(\gamma - \beta), \quad BG = y(\alpha - \gamma),$$

$$CG = CA + AG = b\beta + x\gamma - x\beta,$$

and also $\quad CG = CB + BG = -a\alpha + y\alpha - y\gamma.$

Hence $\quad (y-a)\alpha + (x-b)\beta = (x+y)\gamma$

$$= -(x+y)(a\alpha + b\beta)/c.$$

Equating coefficients of the α's and β's, and solving, we find

$$x = bc/(a+b+c), \ \ y = ca/(a+b+c),$$

and finally
$$CG = \frac{ab}{a+b+c}(\beta - \alpha),$$

and CG bisects the angle between the unit vectors β and $-\alpha$.

9. COPLANAR VECTORS. If α, β, γ are non-parallel vectors in the same plane, it is always possible to find numerical values of a, b, c so that $a\alpha + b\beta + c\gamma$ shall $= 0$.

For a triangle can be constructed whose sides shall be parallel respectively to α, β, γ.

Now if the vectors corresponding to those sides taken in order be $a\alpha$, $b\beta$, $c\gamma$ respectively, we shall have, by going round the triangle,

$$a\alpha + b\beta + c\gamma = 0.$$

If we multiply this equation by any quantity the right hand still remains zero, and the left hand represents a triangle similar and similarly situated to the original triangle but with its sides increased (or diminished) in a given ratio. Thus, though there is an infinity of values assignable to a, b, c, any one set is simply a multiple of any other. This may be proved directly as follows:

Let
$$a\alpha + b\beta + c\gamma = 0,$$
and also
$$p\alpha + q\beta + r\gamma = 0.$$

By eliminating γ we get

$$(ar - cp)\alpha + (br - cq)\beta = 0;$$
$$\therefore \ (\S\ 6)\ ar = cp, \ \ br = cq,$$
$$\text{or } a : b : c :: p : q : r,$$

so that the second equation is simply a multiple of the first.

10. COLLINEAR POINTS. If α, β, γ are coinitial, coplanar vectors terminating in a straight line, then the same values of a, b, c which render $a\alpha + b\beta + c\gamma = 0$ will also render $a + b + c = 0$.

Let vector $OA=\alpha$, $OB=\beta$, $OC=\gamma$, ABC being a straight line; then

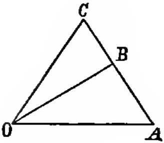

Fig. 7.

$$AB=\beta-\alpha,$$

$$AC=\gamma-\alpha.$$

But AC is a multiple of AB,

$$\text{or } \gamma-\alpha=p(\beta-\alpha),$$

i.e. $(p-1)\alpha-p\beta+\gamma=0.$

But $$(p-1)-p+1=0;$$

and as $p-1$, $-p$, $+1$ correspond to a, b, c and satisfy the condition required, the proposition is proved generally.

Conversely, if α, β, γ are coinitial coplanar vectors, and if both $a\alpha+b\beta+c\gamma=0$, and $a+b+c=0$, then do α, β, γ terminate in a straight line.

For $$a\gamma+b\gamma+c\gamma=0;$$

therefore by subtraction

$$a(\gamma-\alpha)+b(\gamma-\beta)=0,$$

i.e. $\gamma-\alpha$ is a multiple of $\gamma-\beta$, and therefore (§ 4) in the same straight line with it: i.e. AC is in the same straight line with BC.

This criterion for the collinearity of the extremities of three vectors drawn from the same origin has many elegant applications.

If ρ be any vector drawn from the same origin as α and β and terminating on the straight line passing through the extremities of α and β we may write

$$\rho=\frac{\alpha+m\beta}{1+m} \text{ or}=\frac{a\alpha+b\beta}{a+b},$$

where $m=b/a$. For, clearing of fractions, we have

$$a(\rho-\alpha)=b(\beta-\rho),$$

so that $\rho-\alpha$ and $\beta-\rho$ are collinear, being parallel with a point in common. Also the end of ρ divides $(\beta-\alpha)$ in the ratio b/a $(=m)$.

11. NON-COPLANAR VECTORS. If α, β, γ are three vectors neither parallel nor in the same plane, it is impossible to find numerical values of a, b, c, not equal to zero, which shall render $a\alpha + b\beta + c\gamma = 0$.

For (§ 5) $a\alpha + b\beta$ can be represented by a third vector in the plane which contains two lines parallel respectively to α, β. Now $c\gamma$ is not in that plane, therefore (§ 6) their sum cannot equal 0.

Or we may reason in this way. The equation may be written in the form

$$-c\gamma = a\alpha + b\beta,$$

so that, if a, b, c have finite values and if α and β are different vectors, γ must lie in the same plane with α and β. Hence with γ not coplanar with α and β, the above equation can hold only if a, b, and c all vanish.

Thus with a, b, c unrestricted, the equation $a\alpha + b\beta + c\gamma = 0$, means that α, β, γ are either parallel to one another or are in the same plane.

These theorems find illustration in the following examples:

1. *If two triangles are so situated that the lines which join corresponding angles meet in a point, then pairs of corresponding sides being produced will meet in a straight line.*

ABC, $A'B'C'$ are the triangles; O the point in which $A'A$, $B'B$, $C'C$ meet; P, Q, R the points in which BC, $B'C'$, etc., meet: PQR is a straight line.

Let $OA = \alpha$, $OB = \beta$, $OC = \gamma$,

$OA' = m\alpha$, $OB' = n\beta$, $OC' = p\gamma$,

then $\quad BA = \alpha - \beta$,

and $\quad BR = x(\alpha - \beta)$;

$\quad B'A' = m\alpha - n\beta$,

and $\quad B'R = y(m\alpha - n\beta)$.

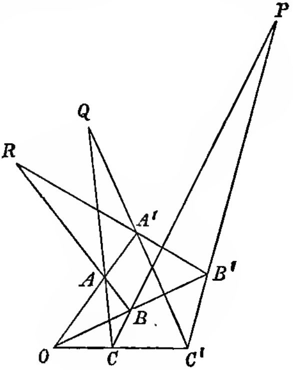

Fig. 8.

Now $BB' = BR - B'R$ gives

$$(n-1)\beta = x(\alpha-\beta) - y(m\alpha - n\beta);$$

$$\therefore\ n-1 = -x+ny,\ \ 0 = x-my,$$

and

$$x = -\frac{m(n-1)}{m-n};$$

whence $OR = OB + BR = \beta - \dfrac{m(n-1)}{m-n}(\alpha-\beta)$

$$= \frac{n(m-1)\beta - m(n-1)\alpha}{m-n}.$$

Similarly,

$$OP = \frac{p(n-1)\gamma - n(p-1)\beta}{p-m},$$

$$OQ = \frac{m(p-1)\alpha - p(m-1)\gamma}{p-m};$$

$$(m-n)(p-1)OR + (n-p)(m-1)OP + (p-m)(n-1)OQ = 0.$$

And also identically

$$(m-n)(p-1) + (n-p)(m-1) + (p-m)(n-1) = 0,$$

whence (§ 10) P, Q, R are in the same straight line.

2. *If AD, BE, CF be drawn cutting one another at any point G within a triangle, then FD, DE, EF shall meet the third sides of the triangle produced in points which lie in a straight line.*

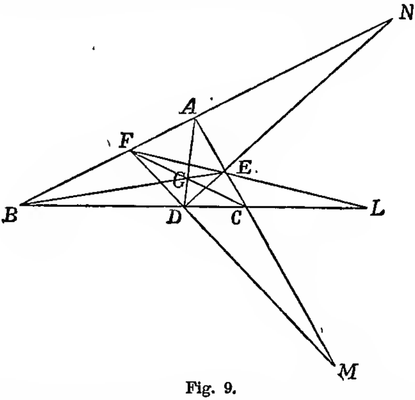

Fig. 9.

Also the produced sides of the triangle shall be cut harmonically.

Take G as origin, and let α, β, γ be the vectors to the angles A, B, C; then the vectors to D, E, F may be represented by $a\alpha$, $b\beta$, $c\gamma$. Let ρ, σ, τ be the vectors from G to the points L, M, N, the meeting points referred to in the enunciation.

The vector equations

$$\begin{aligned} GL &= GB + BL = GE + EL, \\ GM &= GC + CM = GF + FM, \\ GN &= GA + AN = GD + DN \end{aligned}$$

take the form

$$\left.\begin{aligned} \rho &= \beta + x\,(\gamma - \beta) = b\beta + y\,(c\gamma - b\beta), \\ \sigma &= \gamma + x'\,(\alpha - \gamma) = c\gamma + y'\,(a\alpha - c\gamma), \\ \tau &= \alpha + x''(\beta - \alpha) = a\alpha + y''(b\beta - a\alpha), \end{aligned}\right\}$$

whence we find from the first pair

$$\beta(1 - x - b + by) + \gamma(x - cy) = 0,$$

so that $\quad x = cy,\ (c - b)y = 1 - b,\ (c - b)x = c(1 - b),$

with similar expressions for $y'y''$, $x'x''$.

Substituting we get

$$\left.\begin{aligned} \rho(c - b) &= b(c - 1)\beta - c(b - 1)\gamma, \\ \sigma(a - c) &= c(a - 1)\gamma - a(c - 1)\alpha, \\ \tau(b - a) &= a(b - 1)\alpha - b(a - 1)\beta. \end{aligned}\right\}$$

Multiplying by $(a - 1)$, $(b - 1)$, $(c - 1)$ respectively, and adding we find

$$(a - 1)(c - b)\rho + (b - 1)(a - c)\sigma + (c - 1)(b - a)\tau = 0.$$

But evidently

$$(a - 1)(c - b) + (b - 1)(a - c) + (c - 1)(b - a) = 0,$$

hence ρ, σ, τ terminate on the same straight line.

Now $a\alpha$ may be written in the form $(\beta + m\gamma)/(1 + m)$ (§ 10), where m is to be found; and in like manner $b\beta$ and $c\gamma$ may be expressed, using n, p instead of m.

Thus we have the three equations

$$\left.\begin{aligned} (1 + m)a\alpha - \beta - m\gamma &= 0, \\ -n\alpha + (1 + n)b\beta - \gamma &= 0, \\ -\alpha - p\beta + (1 + p)c\gamma &= 0, \end{aligned}\right\}$$

and these (§ 9) must be simply multiples of one another.

Hence $$-\frac{1+m}{n}a=-\frac{1}{(1+n)b}=m,$$

$$n=-\frac{(1+n)b}{p}=\frac{1}{c(1+p)},$$

from which m, n, p may be found. Thus

$$m=\frac{c-cb}{b-bc}, \quad n=\frac{a-ac}{c-ca}, \quad p=\frac{b-ba}{a-ab}.$$

Now $$x=\frac{c-cb}{c-b} \text{ and } x-1=\frac{b-bc}{c-b}.$$

Hence $$m=\frac{x}{x-1}$$

or $$\frac{DC}{BD}=\frac{BL}{BL-BC}=\frac{BL}{CL},$$

and BL is cut harmonically.

Again, since $mnp=1$, we find

$$\frac{DC}{BD}\cdot\frac{EA}{CE}\cdot\frac{FB}{AF}=1.$$

12. The MEAN POINT of a group of points is that point whose vector position referred to any chosen origin is equal to the sum of the vector positions of the individual points divided by the number of points.

Thus for two points α, β, the mean point is

$$\mu=\frac{\alpha+\beta}{2}$$

or $$\mu-\alpha=\beta-\mu,$$

so that the point at the extremity of μ bisects the line $\beta-\alpha$.

For three points forming the triangle ABC,

$$\mu=\frac{1}{3}(\alpha+\beta+\gamma).$$

This point is the point G of Example 1, § 7; for it is the meeting point of the lines, which pass through A, D, and B, E. Let ρ be the vector of this meeting point. Then

$$\rho=\frac{\frac{1}{2}(\beta+\gamma)+m\alpha}{1+m}=\frac{\frac{1}{2}(\gamma+\alpha)+n\beta}{1+n},$$

where α, β, γ are vectors not necessarily in the same plane.

Hence rearranging we have

$$\left\{\frac{m}{1+m}-\frac{1}{2(1+n)}\right\}\alpha+\left\{\frac{1}{2(1+m)}-\frac{n}{1+n}\right\}\beta$$
$$+\frac{1}{2}\left\{\frac{1}{1+m}-\frac{1}{1+n}\right\}\gamma=0,$$

and the factors must vanish (§ 11). Hence

$$m=n=\frac{1}{2},$$

and
$$\rho=\frac{\frac{1}{2}(\beta+\gamma+\alpha)}{1+\frac{1}{2}}=\frac{1}{3}(\alpha+\beta+\gamma).$$

For four points A, B, C, D, with vector positions α, β, γ, δ,

$$\mu=\frac{1}{4}(\alpha+\beta+\gamma+\delta).$$

These four points are the corners of a quadrilateral not necessarily plane.

The middle point of the line joining the middle points of any pair of opposite sides is

$$\frac{1}{2}\left\{\frac{\alpha+\beta}{2}+\frac{\gamma+\delta}{2}\right\}=\mu,$$

the mean point. Hence the lines joining the points of bisection of the opposite sides of a quadrilateral in space meet and bisect each other.

Again the point of bisection of the line which joins the middle points of the diagonals is the same mean point; for

$$\mu=\frac{1}{2}\left\{\frac{\alpha+\gamma}{2}+\frac{\beta+\delta}{2}\right\}=\frac{\alpha+\beta+\gamma+\delta}{4},$$

and so on for any number of points.

13. CENTRE OF MASS. The properties of the centre of mass or centre of inertia of a system of particles are conveniently discussed here.

If we have a number of *equal* masses placed in given positions, then the centre of mass is simply the mean point.

To extend the discussion to the more general case, we shall first define what Maxwell has called the *mass-vector*. The mass-vector of the particle of mass m whose position is assigned by the vector ρ is the vector $m\rho$, that is the vector whose tensor is m times the tensor of the vector ρ.

Let there be two masses m_1, m_2 in positions ρ_1, ρ_2; then the vector

$$\sigma = \frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2}$$

is (§ 10) the vector of the point in the line joining m_1 and m_2 and dividing it in the ratio of m_2 to m_1. For

$$m_1(\sigma - \rho_1) = m_2(\rho_2 - \sigma),$$

so that the product of each mass and its distance from the centre of mass is the same.

Add to the system a third mass m_3 in position ρ_3. We may suppose $m_1 + m_2$ to be condensed at their centre of mass. Hence the centre of mass of $(m_1 + m_2)$ and m_3 got by the same process will be

$$\sigma = \frac{\frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2} \times (m_1 + m_2) + m_3\rho_3}{m_1 + m_2 + m_3}$$

$$= \frac{m_1\rho_1 + m_2\rho_2 + m_3\rho_3}{m_1 + m_2 + m_3}.$$

Generalizing we arrive at the definition of the centre of mass of a system of particles as that point whose vector position is the sum of the mass-vectors divided by the sum of the masses. Thus, generally

$$\sigma(m_1 + m_2 + m_3 + \ldots + m_n)$$

$$= m_1\rho_1 + m_2\rho_2 + m_3\rho_3 + \ldots + m_n\rho_n,$$

giving in every case a perfectly definite vector σ. This relation may be written briefly

$$\sigma\Sigma m = \Sigma(m\rho),$$

where Σ means the summation of terms of the type indicated.

Rearranging we find, since σ may be put under the summation symbol,

$$\Sigma m(\sigma - \rho) = 0,$$

or, the mass vectors referred to the centre of mass would if put end to end form a closed polygon.

14. VELOCITY A VECTOR QUANTITY. As already explained, the simplest conception of a vector is that of a transference or step in a definite direction, through a definite distance. When a moving particle changes position from, say, position ρ to position ρ', the *displacement* is measured by the vector $(\rho' - \rho)$.

In the very simplest case imagine the particle to be moving with constant speed along this vector line $(\rho' - \rho)$. This vector will represent by its direction the direction of motion and by its length the distance travelled through. Suppose the transference to be effected in t seconds of time. Then the vector $\frac{\rho' - \rho}{t}$, which is the tth part in length of the vector $\rho' - \rho$, will clearly represent in direction and magnitude the *velocity* of the moving particle. Velocity is therefore completely represented by a vector whose direction gives the direction of motion and whose tensor measures the speed or rate at which space is being described.

Velocity is defined quite generally as the rate of change of position, and is clearly a vector quantity. For when a particle moves it must move in a definite direction with a definite speed. The velocity is therefore fully symbolized by a vector line drawn in this direction and of a length measuring the speed on a convenient scale.

The *relative* velocity of two moving bodies is obtained at once by taking the vector *difference* of the vectors representing their velocities. If α, β represent the velocities of two particles A, B, we get their relative velocity by superposing on both such a velocity as will reduce one to rest. Thus, for example, let $-\alpha$ be superposed. This will annul the $+\alpha$ in the one case and produce with β a resultant velocity represented by $\beta-\alpha$ in the other case. The vector difference $\beta-\alpha$ is the velocity of B relatively to A.

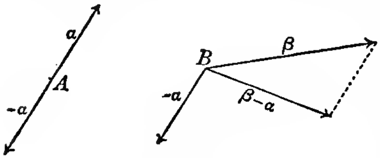

Fig. 10.

Any given velocity may be decomposed into any number of components, the sole condition being that the components drawn end to end form with the original velocity reversed a closed polygon.

If two directions be assigned coplanar with the given velocity, the components along these directions have determinate values. Thus in the figure on page 11, let AC be the given velocity and AB, BC parallel to the given directions. Then it is clear that when lines parallel to these directions are drawn through the extremities A and C, they will meet in a determinate point B, and the components AB, BC have determinate values.

Again, if any three *non-coplanar* directions are assigned, there is one way only in which a given velocity can be decomposed into components parallel to these directions.

Thus let OA, OB, OC be the required directions in space and OP the given velocity, and let OB, OC be in the plane of the paper. Through P draw a line parallel to OA till it meets the plane containing OB and OC. Let PM be this line. Through M draw a line parallel to OC till it meets OB in N. Then the velocity OP is decomposed into the components ON, NM, MP, which are all determinate in direction and magnitude.

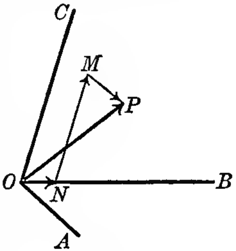

Fig. 11.

Long before the calculus of quaternions or any system of vector analysis was invented these vector properties of velocities were known; and they still form some of the most effective illustrations of the method.

EXAMPLES TO CHAPTER II

1. If P, Q, R, S be points taken in the sides AB, BC, CD, DA of a parallelogram, so that $AP : AB :: BQ : BC$, etc., $PQRS$ will form a parallelogram.

2. If the points be taken so that $AP = CR$, $BQ = DS$, the same is true.

3. The mean point of $PQRS$ is in both cases the same as that of $ABCD$.

4. The quadrilateral formed by bisecting the sides of a quadrilateral and joining the successive points of bisection is a parallelogram, with the same mean point.

5. If the same be true of any other equable division such as trisection, the original quadrilateral is a parallelogram.

6. If any line pass through the mean point of a number of points, the sum of the perpendiculars on this line from the different points, measured in the same direction, is zero.

7. From a point E in the common base AB of the two triangles ABC, ABD, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G; show that FG is parallel to CD.

8. From any point in the base of a triangle, straight lines are drawn parallel to the sides: show that the intersections of the diagonals of every parallelogram so formed lie in a straight line.

9 If the sides of a triangle be produced, the bisectors of the external angles meet the opposite sides in three points which lie in a straight line.

10. If straight lines bisect the interior and exterior angles at A of the triangle ABC in D and E respectively; prove that BD, BC, BE form an harmonical progression.

11. The mean point of a tetrahedron is the mean point of the tetrahedron formed by joining the mean points of the triangular faces; and also that of the mean points of the edges.

12. If through any point within the triangle *ABC*, three straight lines *MN*, *PQ*, *RS* be drawn respectively parallel to the sides *AB*, *AC*, *BC*; then will

$$\frac{MN}{AB}+\frac{PQ}{AC}+\frac{RS}{BC}=2.$$

13. *ABCD* is a parallelogram; *PQ* any line parallel to *CD*; *PD*, *QC* meet in *S*, *PA*, *QB* in *R*; prove that *AD* is parallel to *RS*.

14. If the vertical angle of a triangle be bisected by a straight line which cuts the base, the segments of the base shall have the same ratio that the other sides of the triangle have to one another.

15. Find the expression for the centre of mass of a uniform wire bent into the form of a triangle, the lengths of whose sides are *a*, *b*, *c*.

16. The mean point of a triangle trisects the line joining the point of intersection of the perpendiculars on the sides from the opposite angles, and the point of intersection of perpendiculars on the sides from their middle points.

CHAPTER III.

QUATERNIONS AND VERSORS OR QUOTIENTS AND PRODUCTS OF VECTORS.

15. THE QUATERNION AS A GEOMETRICAL OPERATOR In the preceding chapter the laws of addition and subtraction of vectors have been discussed; and broadly speaking these laws are common to all vectorial systems such as are met with in the Barycentrische Calcul of Möbius and the Ausdehnungslehre of Grassmann.

We now pass on to the discussion of products and quotients of vectors; and it is well at the outset to state distinctly what are the peculiar features of Hamilton's Quaternions as compared with other systems of vector analysis. It lies in this, that, whereas the commutative law in multiplication no longer holds, the distributive and associative laws are still retained. In symbols, $\alpha\beta$ is not the same as $\beta\alpha$; but $\alpha(\beta+\gamma)$ is the same as $\alpha\beta+\alpha\gamma$, and $\alpha\beta\gamma$ has the same value whether it is regarded as α multiplied by $\beta\gamma$ or as $\alpha\beta$ multiplied by γ. The whole system may be developed analytically from these fundamental restrictions.

Here, however, we shall develop the system geometrically, bearing in mind that the distributive and associative laws are to hold, and adopting the usual notations familiar to us in ordinary algebra, in so far as these are not inconsistent with the restrictions laid down.

Given any two vectors α, β there must be some multiplier or operator which changes β into α. Writing q for this operator we have symbolically

$$q\beta = \alpha.$$

If β and α were two scalar quantities of the same kind, q would be their ratio. When they are vector quantities of the same kind, we may still by analogy regard q as a ratio, and we may express it in the form α/β or $\alpha\beta^{-1}$. The relation

$$\alpha\beta^{-1} . \beta = \alpha$$

is obviously self-consistent, if we assume the associative law to hold, and if $\beta^{-1}\beta = 1$.

This involves the definition of the *reciprocal of a vector.* Thus if $\gamma\beta = 1$, γ is the reciprocal of β and simply undoes whatever effect may be produced by β. Evidently their tensors must be reciprocal in the ordinary arithmetical sense. Moreover, since the effect of a vector must depend in some way upon its direction it is reasonable to expect that the reciprocal vector will undo this effect in virtue of its having the opposite direction. That is to say, the most obvious interpretation of the reciprocal of a vector is a vector whose tensor is the reciprocal of the tensor of the original vector and whose direction is the reverse of that of the original vector. This we shall find to be its meaning in quaternion vector analysis.

Since a vector possesses both direction and magnitude, the process by which it is changed into another vector must involve the two distinct operations of change of direction and change of magnitude. When, for example, β is to be changed into α, it must first be rotated through a definite angle in a definite plane until it is parallel to α, and then its tensor or length must be altered in the proper ratio so as to make it equal to the tensor or length of α. Or, we may first effect the alteration in length, and then effect the rotation. The result is the same, so that these operations are commutative.

A further consideration shows that this process of changing one vector into another involves *four* numbers. (1) There is the change of tensor—one number; (2) there is the angle of rotation—one number; (3) there is the aspect of the plane in which the rotation takes place or the direction of the axis about which rotation takes place, and this requires two numbers for its determination. In all, four numbers. For this reason Hamilton called the multiplier q or $\alpha\beta^{-1}$ a *quaternion*.

16. THE CONSTITUENTS OF A QUATERNION. As already pointed out, the process of changing one vector into another consists in general of two separable operations—the one effecting the necessary change of length, the other the change of direction. These are distinguished as the *Tensor* and *Versor* parts of the quaternion, and are written Tq and Uq respectively.[1] Thus

$$q\beta = TqUq\beta = UqTq\beta,$$

or symbolically, $$q = TqUq = UqTq.$$

The operation represented by Tq is simply that of multiplying by a numerical factor, and requires no further discussion. Tq is in fact a scalar multiplier. When the value of Tq is unity, the quaternion is reduced to the expression Uq and is called a Versor, since its effect is simply one of turning.

A Versor involves an angle and a plane or direction of axis—three numbers. A Vector involves a length and a direction—also three numbers. Any two vectors, α and β, involve six numbers. But the quaternion, $\alpha\beta^{-1}$, as has been shown above, involves only *four* numbers. Hence, although $q\,(=\alpha\beta^{-1})$ is completely determined when α and β are given, α and β are not completely determined when

---

[1] The use of the selective symbol U in two senses as the unit of a vector and the versor of a quaternion will be found to lead to no confusion.

q is given. All we know, in this case, about α and β, are the aspect of the plane in which they lie, the angle between them, and the ratio of their tensors—in short, just the four constituents of the quaternion. Any other pair of vectors having these same relations will give the same quaternion. Thus a given quaternion can be expressed as the ratio of two vectors in an infinite number of ways, the conditions attaching to these vectors being, that they are perpendicular to a given direction known as the *axis* of the quaternion, that they contain an angle known as the *angle* of the quaternion, and that their tensors have a ratio equal to the *tensor* of the quaternion.

It should be mentioned that the axis of a quaternion is drawn in that direction which bears to the versor effect of the quaternion the same relation which the translational motion of a right-handed screw bears to its rotation.

17. THE QUATERNION AND ITS CONJUGATE. To each quaternion q there corresponds another quaternion Kq, called the *conjugate*, whose effect upon a vector operand is the same except that the angle of rotation is taken in the opposite direction. In other words, the axes of a quaternion q and its conjugate Kq are oppositely directed, the tensors and angles being the same.

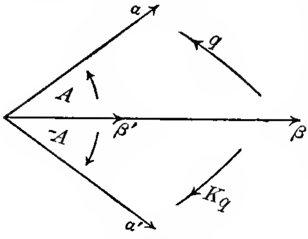

Fig. 12.

Thus, if $q\beta = \alpha$,
then $Kq\beta = \alpha'$,
where α' has the same tensor as α but lies on the opposite side of β, making the same angle A with it (see Figure 12).

Evidently if we operate on α by Kq, or on α' by q, we shall obtain a vector β' parallel to β, such that

$$T\beta' / T\alpha = T\alpha / T\beta,$$

or

$$T\beta' / T\beta = (T\alpha / T\beta)^2 = (Tq)^2.$$

Hence, $$Kqq\beta = Kq\alpha = (Tq)^2\beta,$$
and $$qKq\beta = q\alpha' = (Tq)^2\beta,$$
or symbolically, $$qKq = Kqq = (Tq)^2, \text{ a scalar quantity.}$$

From the fundamental equations given above we get by addition and subtraction

$$(q + Kq)\beta = \alpha + \alpha',$$
$$(q - Kq)\beta = \alpha - \alpha'.$$

But $\alpha + \alpha'$ is a vector parallel to β and having a length equal to $2T\alpha \cos A$; and $\alpha - \alpha'$ is a vector perpendicular to β, and having a length equal to $2T\alpha \sin A$. Let $U\gamma$ be unit vector perpendicular to β in the plane $\alpha\beta$. Then we may write

$$(q + Kq)\beta = 2T\alpha \cos A \,.\, U\beta,$$
$$(q - Kq)\beta = 2T\alpha \sin A \,.\, U\gamma,$$

or $$(q + Kq)\,U\beta = 2\frac{T\alpha}{T\beta}\cos A \,.\, U\beta = 2\,Tq \cos A \,.\, U\beta,$$

$$(q - Kq)\,U\beta = 2\frac{T\alpha}{T\beta}\sin A \,.\, U\gamma = 2\,Tq \sin A \,.\, U\gamma.$$

Thus $q + Kq$ is a scalar multiplier, while $q - Kq$ is a quaternion which rotates β through a right angle about the axis of the quaternion q and changes its tensor in the ratio of $2Tq \sin A$ to unity.

A quaternion which rotates the vector operand through a right angle is called a *quadrantal quaternion*; and a versor which does the same is called a *quadrantal versor.*

18. QUADRANTAL QUATERNIONS AND VERSORS. From the last paragraph we learn that any quaternion may be expressed as the sum of a scalar quantity and a quadrantal quaternion; and that when this is done the conjugate of the quaternion is then expressible as the difference of the same two quantities. In symbols $$q = S + Q,$$
$$Kq = S - Q,$$
where S is the appropriate scalar, and Q the appropriate quadrantal quaternion, whose meanings are given in the last paragraph.

If the scalar vanishes, q becomes the quadrantal quaternion Q, and we see that the conjugate of a quadrantal quaternion is simply the quadrantal quaternion with its sign changed. This is obvious from the geometry of the operation, for Q and KQ rotate the vector operand in the same plane through right angles measured in opposite directions.

In this case also

$$QKQ = Q(-Q) = -Q^2.$$

But

$$QKQ = (TQ)^2.$$

Hence the square of a quadrantal quaternion is equal to minus the square of its tensor.

The same conclusion may be readily established by direct consideration of the geometry of the operation, for UQ operating twice in succession simply reverses.

For simplicity of discussion let $TQ = 1$, so that Q becomes a quadrantal versor. The properties of the quadrantal versor being once established, those of the quadrantal quaternion are immediately obtained by introducing any scalar factor. Following Hamilton we shall symbolize quadrantal versors by one or other of the letters i, j, k; and these we shall represent geometrically by their axes, distinguishing them meanwhile from vectors by using two arrow heads instead of one.

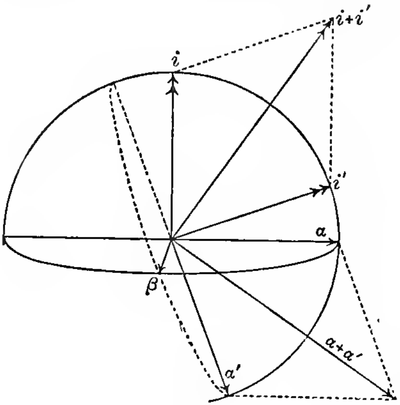

Fig. 13.

Let i, i' be two given quadrantal versors as shown in the figure, and let each act on the unit vector β perpendicular to both. Then

$$i\beta = \alpha,$$

a unit vector perpendicular to β and to i, the direction of rotation being right-handed with reference to the direction of the axis of i. Similarly,

$$i'\beta = \alpha',$$

a unit vector perpendicular to β and to i'. By the distributive law

$$\alpha + \alpha' = i'\beta + i'\beta = (i + i')\beta.$$

Hence $(i + i')$ is the operator which changes β into $(\alpha + \alpha')$. This operator is evidently a quadrantal quaternion, turning β through a right angle and changing its length to the value $T(\alpha + \alpha')$. This quadrantal quaternion will have its axis along the diagonal of the parallelogram formed by i and i', and its tensor will be equal to $T(i + i')$. It may therefore be completely symbolized by $(i + i')$, in which the *versors* i and i' are added like *vectors* to produce the quadrantal quaternion $(i + i')$. In other words, quadrantal versors, and (it is easy to show) quadrantal quaternions also, are compounded like vectors. Now, so far as our definitions go, any quantities which obey the vector law of addition may be regarded as vectors; and if no inconsistency results we may extend to vectors any analytical properties which these new quantities may possess. The explicit identification, so far as regards their properties in analytical combinations, of quadrantal quaternion and vector is one of the outstanding features of quaternions. It has been taken exception to by theorists; but there is no practical system of vector analysis in use in which the versorial character of a vector in product combinations is not either implicitly involved or explicitly assumed. The identification of versor and vector leads to no confusion and greatly facilitates transformations.

19. MULTIPLICATION OF QUADRANTAL QUATERNIONS. And now let us consider the result of operating with two quadrantal versors in succession. Let i, i' be these versors drawn from O as in Figure 14. Through O draw planes perpendicular to them and let γ be the unit vector along the line of intersection in the direction which is positive with regard to right-handed rotation from i' to i.

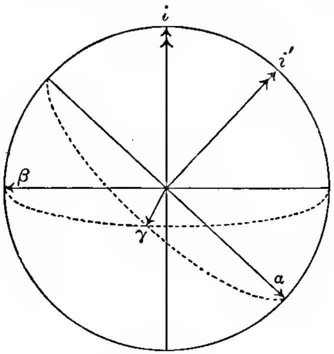

Fig. 14.

Take β perpendicular to γ and i, so that $i\beta = \gamma$; and take α perpendicular to γ and i' so that $i'\gamma = \alpha$. Then

$$i'i\beta = i'\gamma = \alpha,$$

or

$$i'i = \alpha/\beta.$$

Hence, $i'i$ is the quaternion (in this case, versor) which changes β into α. This versor $i'i$ has its axis perpendicular to the plane containing α and β, that is, to the plane containing the axes of the quadrantal versors i' and i; and its angle is equal to the complement of the angle between i' and i.

By introducing scalar multipliers we may pass from versors to quaternions; and thus any quaternion can be represented as the product of two quadrantal quaternions, the tensor of the product being the product of the tensors of the factors, the axis being perpendicular to the axes of the constituents, and the angle the complement of the angle between these axes.

By the original definition,

$$q = \alpha/\beta = \alpha\beta^{-1} = \alpha\beta',$$

where β' is the reciprocal of β. Hence a quaternion may be expressed as the product of two vectors. Thus we find that in their multiplication as well as in their addition, quadrantal quaternions and vectors obey the same laws.

The identification of vectors and quadrantal quaternions leads at once to the following conclusions.

The conjugate of a vector is its inverse. Thus

$$\alpha K\alpha = -\alpha^2 = (T\alpha)^2,$$

or the square of a vector is *minus* the square of its tensor.

If the vector is a unit vector,

$$aKa = 1 = aa^{-1};$$

$$\therefore \quad a^{-1} = Ka = -a,$$

or the reciprocal of a unit vector is equal to its inverse. Consequently, for any vector β we have

$$\beta^{-1} = (T\beta)^{-1}(U\beta)^{-1} = -\frac{U\beta}{T\beta} = -\frac{\beta}{(T\beta)^2}.$$

20. THE SCALAR AND VECTOR PARTS OF A QUATERNION.

It has been shown (§ 18) that any quaternion may be represented as the sum of an appropriate scalar and an appropriate quadrantal quaternion. For quadrantal quaternion we may now substitute the word vector, and write

$$q = Sq + Vq,$$

$$Kq = Sq - Vq,$$

when S and V are selective symbols separating out the scalar and vector parts of the quaternion. These parts have definite meanings, which have already been given. When q is a versor ($Tq = 1$), Sq is the cosine of the angle through which q turns a vector perpendicular to its axis, or it is *minus* the cosine of the angle between the axes of two quadrantal versors or unit vectors whose product gives q; and Vq is the vector (or quadrantal quaternion) measured along the axis of q and of length equal to the sine of the same angle.

The extension to quaternions is easily given. Let a and b be the lengths of the vectors a and β. Then

$$q = \frac{a}{\beta} = S\frac{a}{\beta} + V\frac{a}{\beta},$$

where

$$S\frac{a}{\beta} = \frac{a}{b}\cos A,$$

$$V\frac{a}{\beta} = \epsilon\frac{a}{b}\sin A,$$

in which A is the angle between a and β, and ϵ is the unit vector or quadrantal versor perpendicular to a and β.

Again, when $q=\alpha\beta$, where α and β may be regarded as quadrantal quaternions or as vectors, we have

$$\alpha\beta=S\alpha\beta+V\alpha\beta,\ \ K(\alpha\beta)=S\alpha\beta-V\alpha\beta,$$

where $$S\alpha\beta=-ab\cos A,$$

$$V\alpha\beta=\epsilon\,.\,ab\sin A,$$

ϵ being unit vector perpendicular to α and β, and A the angle between α and β.

$$K(\alpha\beta)\,.\,\alpha\beta=\{T(\alpha\beta)\}^2=(T\alpha)^2(T\beta)^2=a^2\beta^2.$$

Hence, $$K(\alpha\beta)\,.\,\alpha=a^2\beta=\beta a^2=\beta\alpha\alpha,$$

because α^2 is essentially a scalar quantity and fulfils the commutative law. Thus, finally, multiplying into α^{-1},

$$K(\alpha\beta)=\beta\alpha=S\beta\alpha+V\beta\alpha.$$

But $$K(\alpha\beta)\qquad=S\alpha\beta-V\alpha\beta,$$

from which we conclude that $S\alpha\beta=S\beta\alpha$, but $V\beta\alpha=-V\alpha\beta$. Altering the order of the factors in the product $\alpha\beta$ reverses the sign of the vector part but does not affect the scalar part. We also find

$$2S\alpha\beta=\alpha\beta+\beta\alpha,$$

$$2V\alpha\beta=\alpha\beta-\beta\alpha.$$

When $S\alpha\beta$ vanishes, the quaternion $\alpha\beta$ becomes reduced to its vector part; and this occurs when A is a right angle. The equation $S\alpha\beta=0$ means that α is perpendicular to β.

If we suppose α to be given, and ρ to be any vector satisfying the equation $S\alpha\rho=0$, we see at once that ρ may be any vector passing through the origin perpendicular to α. The equation therefore represents the plane passing through the origin and having its normal parallel to α.

When $V\alpha\beta$ vanishes, the quaternion becomes reduced to its scalar part; and this occurs when A is zero or equal to two right angles. Hence, $V\alpha\beta=0$ means that α and β are parallel. Conversely, when α and β are parallel, $V\alpha\beta=0$.

If we suppose α to be given, and ρ to be a vector satisfying the equation $V\alpha\rho=0$, then the sole condition is that $\rho\parallel\alpha$.

There is no limit to the length of ρ. Consequently, the equation $V\alpha\rho=0$ represents a straight line through the origin parallel to α.

21. UNIT VECTORS PERPENDICULAR TO ONE ANOTHER.

When α, β are two mutually perpendicular unit vectors, the product $\alpha\beta$ has no scalar part, but is wholly a vector. Hence we may write $\alpha\beta=\gamma$, where γ is the unit vector perpendicular to α and β. Taking the conjugates of both sides, we have $\beta\alpha=-\gamma$. Multiplying by β, we get

$$\beta^2\alpha=-\beta\gamma, \text{ or } \alpha=\beta\gamma, \text{ since } \beta^2=-1.$$

Or, multiplying into α, we get

$$\beta\alpha^2=-\gamma\alpha, \text{ or } \beta=\gamma\alpha.$$

These relations $\alpha\beta=\gamma$, $\beta\gamma=\alpha$, $\gamma\alpha=\beta$, necessarily hold among three rectangular unit vectors.

It has become customary to use for such a system of rectangular unit vectors or quadrantal versors the letters i, j, k; and, as Hamilton showed, from the properties of these space units the whole calculus may be analytically developed. The properties of i, j, k, as usually given, are

$$\left.\begin{aligned} ij=k&=-ji,\\ jk=i&=-kj,\\ ki=j&=-ik,\\ i^2=j^2=k^2=\quad ijk&=-1.\end{aligned}\right\}$$

It is instructive to see what relations among these quantities are necessary and sufficient for the purpose. Assume, to begin with, that

$$\begin{aligned} ij=k&=-ji,\\ jk=i&=-kj\\ ki=j&=-ik,\end{aligned}$$

and assume in addition that the associative law is to hold. That is to say, any combination, such as iij, is to have the same value whether it is regarded as made up of $(ii)j$ or $i(ij)$. On this assumption we have

$$i^2j = iij = i\,.\,ij = ik = -j\,; \quad \therefore\; i^2 = -1.$$

Similarly, $$j^2 = -1,\ k^2 = -1.$$

Again, $ijk = ii = -1$, and similarly for the products jki, kij, in which the same cyclical order is preserved. But

$$ikj = (-j)j = +1,$$

so that a change in the cyclical order changes the sign of the product.

It should be noted that the triple product ijk is a scalar quantity. Let us take vectors xi, yj, zk, where xyz are the tensors of the vectors parallel respectively to i, j, k. Then the product

$$xiyjzk = -xyz.$$

22. COMPARISON WITH CARTESIAN METHODS. Let ijk be unit vectors measured along a system of mutually perpendicular axes in space. Any other vector ρ may be expressed in the form

$$\rho = ix + jy + kz,$$

where xyz are the coordinates of the extremity of the vector ρ measured along the directions i, j, k.

Any other vector σ will have the corresponding form

$$\sigma = ix' + jy' + kz'.$$

Applying the distributive and associative laws we find for the product the form

$$\begin{aligned}\rho\sigma &= (ix + jy + kz)(ix' + jy' + kz')\\ &= i^2xx' + j^2yy' + k^2zz'\\ &\quad + ijxy' + jiyx' + \ldots\\ &= -xx' - yy' - zz'\\ &\quad + k(xy' - x'y) + i(yz' - y'z) + j(zx' - z'x).\end{aligned}$$

But $$\rho\sigma = S\rho\sigma + V\rho\sigma.$$

Thus the analytical expressions for the scalar and vector parts of the product of the two vectors are

$$S\rho\sigma = -(xx' + yy' + zz'),$$
$$V\rho\sigma = i(yz' - y'z) + j(zx' - z'x) + k(xy' - x'y).$$

From these expressions we may easily verify that

$$S\rho\sigma = S\sigma\rho, \quad V\rho\sigma = -V\sigma\rho.$$

If ρ and σ are unit vectors, xyz, $x'y'z'$ are direction cosines, and we infer at once that the common perpendicular to ρ and σ, being $UV\rho\sigma$, has its direction cosines proportional to the quantities $(yz' - y'z)$, $(zx' - z'x)$, $(xy' - x'y)$.

Again,
$$\begin{aligned}\rho^2 &= (ix + jy + kz)(ix + jy + kz)\\ &= -x^2 - y^2 - z^2\\ &\quad + ij(xy - xy) + \ldots\\ &= -(x^2 + y^2 + z^2),\end{aligned}$$

the vector part of the product $\rho\rho$ vanishing of necessity.

23. IMPORTANT GEOMETRICAL AND DYNAMICAL INTERPRETATIONS.

The quantities $S\rho\sigma$, $V\rho\sigma$ have interpretations of great importance in geometry and dynamics. Some of these have been already given, but their importance demands a further discussion. As proved above (§ 20)

$$-S\rho\sigma = T\rho T\sigma \cos A,$$

where A is the angle contained by ρ and σ. Hence, $-S\rho\sigma$ is the product of the tensor of either vector into the component of the other along its direction.

Let σ be a given constant vector. What is the locus of the extremity of ρ when $S\rho\sigma$ is constant, equal (say) to $c\sigma^2$? Since $S\rho\sigma = c\sigma^2 = S\sigma c\sigma$, we have $S\sigma(\rho - c\sigma) = 0$. Hence, $\rho - c\sigma$ is perpendicular to σ, and therefore ρ is the vector of any point in the plane which passes through the point $c\sigma$ and which is perpendicular to σ.

If ρ is a force and σ a displacement the quantity $-S\rho\sigma$ is the work done by the force during the displacement.

The geometrical meaning of $V\rho\sigma$ is the area of the parallelogram contained by ρ and σ, regarded as a vector quantity with direction perpendicular to the area. Just as a plane has an *aspect* in the direction of its normal, so any plane area has an aspect and may therefore be regarded as vectorial. The magnitude of the area is symbolized by the expression $TV\rho\sigma$, its direction by $UV\rho\sigma$.

Let σ be a constant vector, and let $V\rho\sigma=\gamma$, also a constant vector.

What is the interpretation of this equation? It means that the area contained by ρ and σ has a constant value and a constant aspect. Since γ is constant we may write it in the form $V\beta\sigma$, β being an appropriate constant vector. Then, since

$$V\rho\sigma=V\beta\sigma,$$

we have

$$V(\rho-\beta)\sigma=0$$

or $\rho-\beta$ is parallel to σ.

Hence the equation is that of a straight line passing through β and parallel to σ (see Fig. 15). The direct interpretation of the equation $V\rho\sigma=\gamma$ is that all triangles with vertex at the origin and with base $T\sigma$ taken anywhere in the line have the same area—in fact, they are all on equal bases and between the same parallels.

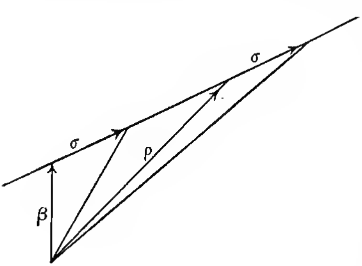

Fig. 15.

The equation $TV\rho\sigma=$ constant, σ as before being a constant vector, means that the area is constant in magnitude only, its aspect being undetermined. Hence it is the equation of a right cylinder with axis parallel to σ, the origin being on the axis; for if any triangle be formed with vertex at the origin and with base σ in any generating line of the cylinder, the area of this triangle will be always the same.

If σ be the momentum of a particle at the extremity of ρ,

$V\rho\sigma$ measures the moment of momentum about the origin. Or if σ be a force acting at the extremity of ρ, $V\rho\sigma$ is the moment of the force about the origin. Application of the distributive law at once gives us Varignon's Theorem of moments. Thus

$$V\rho\sigma + V\rho\sigma' = V\rho(\sigma + \sigma'),$$

or the sum of the moments of two concurrent forces is equal to the moment of their resultant.

24. We proceed to give a few simple examples in geometry and trigonometry, partly to show how directly quaternions supply us with well-known formulae, partly to illustrate the directness with which the quaternion method attacks any problem.

For the sake of reference we give the important relations already established together with others which are of frequent use.

1. $q = Sq + Vq; \qquad \alpha\beta = S\alpha\beta + V\alpha\beta.$
2. $Kq = Sq - Vq; \qquad \beta\alpha = S\alpha\beta - V\alpha\beta.$
3. $2Sq = q + Kq; \qquad 2S\alpha\beta = \alpha\beta + \beta\alpha.$
4. $2Vq = q - Kq; \qquad 2V\alpha\beta = \alpha\beta - \beta\alpha.$
5. $$\begin{aligned} qKq &= (Sq)^2 - (Vq)^2; & \left.\begin{matrix}\alpha\beta\beta\alpha \\ \text{or } \alpha^2\beta^2\end{matrix}\right\} &= (S\alpha\beta)^2 - (V\alpha\beta)^2 \\ &= (Sq)^2 + (TVq)^2; & &= (S\alpha\beta)^2 + (TV\alpha\beta)^2. \end{aligned}$$
6. $$\begin{aligned} q^2 &= (Sq)^2 + 2SqVq + (Vq)^2 \\ (\alpha\beta)^2 = \alpha\beta\alpha\beta &= (S\alpha\beta)^2 + (V\alpha\beta)^2 + 2S\alpha\beta V\alpha\beta. \end{aligned}$$
7. $$\begin{aligned} (\alpha \pm \beta)^2 = (\alpha \pm \beta)(\alpha \pm \beta) &= \alpha^2 \pm (\alpha\beta + \beta\alpha) + \beta^2 \\ &= \alpha^2 \pm 2S\alpha\beta + \beta^2. \end{aligned}$$

Since Sq is a scalar, the expression SSq can mean nothing but Sq. Hence we may adopt the notation S^2q to mean $(Sq)^2$. Similarly, VVq is simply Vq, so that we may use the notation V^2q with the meaning $(Vq)^2$. Note that by (6) above

$$Sq^2 = S^2q + V^2q,$$

and

$$Vq^2 = 2SqVq.$$

Also we may use T^2Vq in the sense $(TVq)^2$ without fear of ambiguity; for $TTVq$ cannot be other than TVq.

It is well to note that the symbols S, V, K are distributive so that for example $K(p+q)=Kp+Kq$. This is easily established. On the other hand, T and U are not distributive, except for coaxial quaternions and parallel vectors.

ILLUSTRATIVE EXAMPLES.

1. *To express the cosine of an angle of a triangle in terms of the sides.*

Let ABC be a triangle; and retaining the usual notation of Trigonometry, let

$$CB=\alpha, \quad CA=\beta;$$

then

$$(\text{vector } AB)^2=(\alpha-\beta)^2$$
$$=\alpha^2-2S\alpha\beta+\beta^2,$$

or, changing all the signs to pass from vectors to lines we get (§§ 19, 20),

$$c^2=a^2-2ab\cos C+b^2.$$

2. *To express the relations between the sides and opposite angles of a triangle.*

Let $$CB=\alpha, \quad CA=\beta, \quad BA=\gamma.$$

Then $CB+BA=CA$ gives

$$\alpha+\gamma=\beta,$$
$$\alpha=\beta-\gamma;$$
$$\therefore \alpha^2=\alpha(\beta-\gamma)=\alpha\beta-\alpha\gamma.$$

Take the vectors of each side.

Now $V\alpha^2=0$, for $\alpha^2=-a^2$ has no vector part,

$$\therefore V\alpha\beta=V\alpha\gamma;$$
$$\text{i.e. } ab\epsilon\sin C=ac\epsilon\sin B,$$
$$\text{or } b\sin C=c\sin B.$$

3. *On the sides AB, AC of a triangle are constructed any two parallelograms $ABDE$, $ACFG$: the sides DE, FG are produced to meet in H. Prove that the sum of the areas of the parallelograms $ABDE$, $ACFG$ is equal to the area of the parallelogram whose adjacent sides are respectively equal and parallel to BC and AH.*

Let $$BA=\alpha,\ AE=\beta,\ AC=\gamma,\ GA=\delta,$$

then $$AH=\beta+x\alpha, \text{ and } AH=-\delta-y\gamma;$$

$$\therefore\ V\alpha AH=V\alpha\beta \text{ and } V\gamma AH=-V\gamma\delta$$
$$=V\delta\lambda,$$

hence $$V(\alpha+\gamma)AH=V\alpha\beta+V\delta\gamma,$$

i.e. the parallelogram whose sides are parallel and equal to BC, AH, equals the two parallelograms whose sides are parallel and equal to BA, AE; GA, AC respectively.

[The reader is requested to notice that the *order* GA, AC is the same as the order BA, AE, and BA, AH: so that the vector ϵ is common to all.]

4. *If O be any point whatever either in the plane of the triangle ABC or out of that plane, the squares of the sides of the triangle fall short of three times the squares of the distances of the angular points from O, by the square of three times the distance of the mean point from O.*

Let $$OA=\alpha,\ OB=\beta,\ OC=\gamma,$$

then (§ 12), $$OG=\frac{1}{3}(\alpha+\beta+\gamma),$$

or $$\alpha^2+\beta^2+\gamma^2+2S(\alpha\beta+\beta\gamma+\gamma\alpha)=9OG^2.$$

Now $$AB=\beta-\alpha,\ BC=\gamma-\beta,\ CA=\alpha-\gamma,$$

$$\therefore\ AB^2+BC^2+CA^2=2(\alpha^2+\beta^2+\gamma^2)-2S(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$=3(\alpha^2+\beta^2+\gamma^2)-9OG^2,$$

and the lines

$$AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2)-(3OG)^2.$$

5. *The squares of the sides of any quadrilateral exceed the squares of the diagonals by four times the square of the line which joins the middle points of the diagonals.*

Let $\alpha\beta\gamma$ be the vectors to three of the corners drawn from the fourth corner. Then the vector sides are α, $\beta-\alpha$, $\gamma-\beta$, and γ; and the vector diagonals are β and $\gamma-\alpha$. The vector line joining the middle points of the diagonals is $\frac{\alpha+\gamma}{2}-\frac{\beta}{2}$; four times the square of this is

$$4\left\{\frac{(\alpha+\gamma)^2}{4}-\frac{2S(\alpha+\gamma)\beta}{4}+\frac{\beta^2}{4}\right\}$$
$$=\alpha^2+\beta^2+\gamma^2-2S\alpha\beta-2S\beta\gamma-2S\alpha\gamma.$$

But the sum of the squares of the vector sides is

$$\alpha^2+(\beta-\alpha)^2+(\gamma-\beta)^2+\gamma^2$$
$$=2(\alpha^2+\beta^2+\gamma^2)-2S(\alpha\beta-\beta\gamma),$$

and the sum of the squares of the vector diagonals is

$$\beta^2+(\gamma-\alpha)^2$$
$$=\alpha^2+\beta^2+\gamma^2-2S\alpha\gamma;$$

and the former sum exceeds the latter by

$$\alpha^2+\beta^2+\gamma^2-2S\alpha\beta-2S\beta\gamma+2S\alpha\gamma.$$

The theorem is thus proved, for by changing the signs throughout we pass from the vectors to the lines.

Note. The quadrilateral need not be in one plane.

6. *The lines which join the mean points of three equilateral triangles described outwards on the three sides of any triangle form an equilateral triangle whose mean point is the same as that of the given triangle.*

Let P, Q, R be the mean points of the equilateral triangles on BC, CA, AB; $PD=\alpha$, $DC=\beta$, $CE=\gamma$, $EQ=\delta$; and let the sides of the triangle ABC be $2a$, $2b$, $2c$.

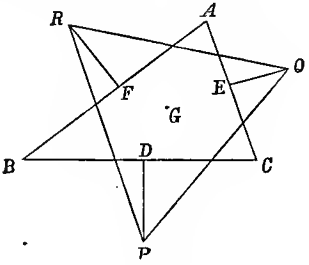

Fig. 16.

Then
$$PQ^2=(\alpha+\beta+\gamma+\delta)^2$$
$$=\alpha^2+\beta^2+\gamma^2+\delta^2+2S\alpha\beta+2S\alpha\gamma+2S\alpha\delta$$
$$+2S\beta\gamma+2S\beta\delta+2S\gamma\delta.$$

Changing all the signs and observing that

$$S\alpha\beta=0,\ \ S\alpha\gamma=-\frac{2}{\sqrt{3}}ab\sin C,\text{ etc.}$$

we have (writing the results in the same order),

$$\text{line } PQ^2=\frac{a^2}{3}+a^2+b^2+\frac{b^2}{3}+0$$

$$+\frac{2}{\sqrt{3}}ab\sin C+\frac{2}{3}ab\cos C-2ab\cos C+\frac{2}{\sqrt{3}}ab\sin C+0$$

$$=\frac{4}{3}(a^2+b^2-ab\cos C)+\frac{4}{\sqrt{3}}ab\sin C$$

$$=\frac{2}{3}(a^2+b^2+c^2)+\frac{2}{\sqrt{3}}\text{ area of } ABC,$$

which being symmetrical in a, b, c proves that PQR is equilateral.

Again, G being the mean point of ABC,

$$PG=PD+DG=\alpha+\frac{\beta}{3}+\frac{2\gamma}{3},$$

$$\therefore\ PG^2=\alpha^2+\frac{\beta^2}{9}+\frac{4\gamma^2}{9}+\frac{2}{3}S\alpha\beta+\frac{4}{3}S\alpha\gamma+\frac{4}{9}S\beta\gamma,$$

$$\text{and line } PG^2=\frac{a^2}{3}+\frac{a^2}{9}+\frac{4b^2}{9}+\frac{4}{3\sqrt{3}}ab\sin C-\frac{4}{9}ab\cos C$$

$$=\frac{2}{9}(a^2+b^2+c^2)+\frac{2}{3\sqrt{3}}\text{ area } ABC;$$

$$\therefore\ PG=QG=RG;$$

and G is the mean point of the equilateral triangle PQR.

7. *In any quadrilateral prism, the sum of the squares of the edges exceeds the sum of the squares of the diagonals by* *eight times the square of the straight line which joins the points of intersection of the two pairs of diagonals.*

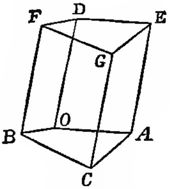

Fig. 17.

Let $OA = \alpha,\ OB = \beta,\ OC = \gamma,\ OD = \delta;$

sum of squares of edges

$$= 2\{\alpha^2 + \beta^2 + (\gamma - \alpha)^2 + (\gamma - \beta)^2 + 2\delta^2\}$$

$$= 2\{2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\delta^2 - 2S\alpha\gamma - 2S\beta\gamma\},$$

sum of squares of diagonals

$$= (\delta + \gamma)^2 + (\delta - \gamma)^2 + (\delta + \alpha - \beta)^2 + (\delta + \beta - \alpha)^2$$

$$= 2\{\alpha^2 + \beta^2 + \gamma^2 + 2\delta^2 - 2S\alpha\beta\}.$$

Also $$\frac{1}{2} OG = \frac{1}{2}(\delta + \gamma)$$

= vector to the point of bisection of CD, and therefore to the point of intersection of OG, CD, and vector from O to the point of bisection of AF, as also to that of BE, and therefore to the intersection of AF, BE

$$= \frac{1}{2}(\delta + \alpha + \beta),$$

hence vector which joins the points of intersection of diagonals

$$= \frac{1}{2}(\alpha + \beta - \gamma),$$

eight times the square of this vector

$$= 2(\alpha^2 + \beta^2 + \gamma^2 + 2S\alpha\beta - 2S\alpha\gamma - 2S\beta\gamma),$$

which, added to the sum of the squares of the diagonals, makes up the sum of the squares of the edges.

EXAMPLES TO CHAPTER III.

1. If in the figure of Euclid I. 47 DF, GH, KE be joined, the sum of the squares of the joining lines is three times the sum of the squares of the sides of the triangle.

The same is true whatever be the angle A.

2. If P, Q, R, S be points in the sides AB, BC, CD, DA of a rectangle, such that $PQ=RS$, prove that

$$AR^2+CS^3=AQ^2+CP^2.$$

3. The sum of the squares of the three sides of a triangle is equal to three times the sum of the squares of the lines drawn from the angles to the mean point of the triangle.

4. In any quadrilateral, the product of the two diagonals and the cosine of their contained angle is equal to the sum or difference of the two corresponding products for the pairs of opposite sides.

5. If a, b, c be three conterminous edges of a rectangular parallelepiped; prove that four times the square of the area of the triangle which joins their extremities is

$$=a^2b^2+b^2c^2+c^2a^2.$$

6. If two pairs of opposite edges of a tetrahedron be respectively at right angles, the third pair will be also at right angles.

7. Given that each edge of a tetrahedron is equal to the edge opposite to it. Prove that the lines which join the points of bisection of opposite edges are at right angles to those edges.

8. If from the vertex O of a tetrahedron $OABC$ the straight line OD be drawn to the base making equal angles with the faces OAB, OAC, OBC; prove that the triangles OAB, OAC, OBC are to one another as the triangles DAB, DAC, DBC.

9. The sum of the squares of the distances of any point O from the angular points of a triangle exceeds the sum of the squares of the distances from the middle points of the sides by the sum of the squares of half the sides.

10. Four times the sum of the squares of the distances of any point whatever from the angular points of a quadrilateral are equal to the sum of the squares of the sides, the squares of the diagonals, and the

square of four times the distance of the point from the mean point of the figure.

11. Interpret geometrically the following equations in which $\alpha\beta$ are given vectors:

$$S\rho\alpha = S\beta\alpha, \quad V\rho\alpha = V\beta\alpha,$$
$$K\rho\alpha = K\beta\alpha, \quad U\rho\alpha = U\beta\alpha,$$
$$T(\rho\alpha) = T(\beta\alpha), \quad TV(\rho\alpha) = TV(\beta\alpha),$$
$$TVU(\rho\alpha) = TVU(\beta\alpha), \quad SU(\rho\alpha) = SU\beta\alpha,$$
$$VU(\rho\alpha) = VU(\beta\alpha).$$

12. What vector is represented by the symbol $\alpha\beta\alpha^{-1}$?

13. Show by general reasoning without analytical transformations that $V\alpha V\beta\gamma$ is necessarily a vector in the plane $\beta\gamma$, and that $V(V\alpha\beta V\beta\gamma)$ is parallel to β.

14. Starting with the identity $\rho \equiv \alpha + \beta$, where β is perpendicular to α, and assuming that β may be expressed in the form $\epsilon\alpha$, where ϵ is a quadrantal versor, deduce the parts of the quaternion which changes α into ρ.

15. Assuming that the vector of the product of two parallel vectors is zero, prove by expansion of $V(\alpha+\beta)(\alpha+\beta)$ that $V\alpha\beta + V\beta\alpha = 0$.

CHAPTER IV.

QUATERNION PRODUCTS AND RELATED DEVELOPMENTS.

25. The Versor as the Power of a Vector. If $\alpha\beta\gamma$ are unit vectors in the same plane, and ϵ the unit vector perpendicular to that plane, we may write (§ 20)

$$\beta\alpha^{-1} = \cos A + \epsilon \sin A,$$
$$\gamma\beta^{-1} = \cos B + \epsilon \sin B,$$

where A is the angle between β and α, and B the angle between γ and β.

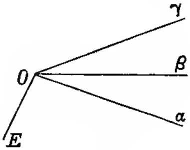

Fig. 18.

But clearly $\gamma\beta^{-1} . \beta\alpha^{-1} = \gamma\alpha^{-1}$

$$= \cos(A+B) + \epsilon \sin(A+B).$$

Hence, substituting, we have

$$(\cos A + \epsilon \sin A)(\cos B + \epsilon \sin B) = \cos(A+B) + \epsilon \sin(A+B),$$

which is Demoivre's Theorem, in which the unit vector ϵ takes the place of the imaginary $\sqrt{-1}$.

On the left hand side we have the product of two expressions involving the arguments A and B; on the right hand side we have the same kind of expression involving the argument $A+B$. The effect of each of these expressions regarded as operators acting on a vector perpendicular to ϵ is to turn the vector through the corresponding angle. Now the effect of ϵ is to turn the vector through one right angle; the effect of ϵ^2 is to turn it through two right angles; and generally the effect of ϵ^x is to turn the vector operand through x right angles. Also, the effect of the successive operations ϵ^x, ϵ^y is to turn the vector through $x+y$ right angles. Consequently the ordinary law of indices holds, namely,

$$\epsilon^x \cdot \epsilon^y = \epsilon^{x+y}.$$

These conclusions hold when x and y are integers; let us assume them to hold for all values of x and y, integral or fractional, positive or negative. Let the angle corresponding to x be $A = x \cdot \frac{\pi}{2}$, or $x = \frac{2A}{\pi}$. Similarly, let $y = 2B/\pi$. The expression $\epsilon^{2A/\pi}$ will then represent the versor which rotates the vector operand through the angle A, and will be a symbol for the expression $\cos A + \epsilon \sin A$. Evidently

$$\epsilon^{2A/\pi} \cdot \epsilon^{2B/\pi} = \epsilon^{2(A+B)/\pi}.$$

Hence we conclude that a versor may be represented by the *power* of the unit vector parallel to its axis, the power being the ratio of the angle of the versor to a right angle. A quaternion may similarly be represented by a power of the vector, whose tensor raised to that power is the tensor of the quaternion. Thus α^x represents a quaternion with axis parallel to α, tensor equal to $(T\alpha)^x$, and angle equal to $x\pi/2$.

26. TRIGONOMETRICAL APPLICATIONS. The identity

$$\gamma\alpha^{-1} = \gamma\beta^{-1} \cdot \beta\alpha^{-1},$$

leads with great ease to well-known trigonometrical formulae. For example, we have immediately

$$\begin{aligned}\cos(A+B) + \epsilon \sin(A+B) &= (\cos A + \epsilon \sin A)(\cos B + \epsilon \sin B) \\ &= \cos A \cos B - \sin A \sin B \\ &\quad + \epsilon(\sin A \cos B + \cos A \sin B),\end{aligned}$$

in which the scalar parts must be equal, and also the vector parts. Hence

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

If the actions of the versors be in opposite directions, β lying beyond γ, we obtain similarly the expressions for $\sin(A-B)$, $\cos(A-B)$.

As another example let us find the cosine of the angle of a spherical triangle in terms of the sides.

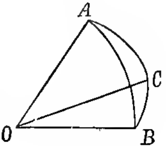

Fig. 19.

Let $\alpha\beta\gamma$ be unit vectors, OA, OB, OC, not on the same plane. The identity $\gamma\alpha^{-1}=\gamma\beta^{-1}\,.\,\beta\alpha^{-1}$ is still true, and this may be expanded in the form

$$\frac{\gamma}{\alpha}=\left(S\frac{\gamma}{\beta}+V\frac{\gamma}{\beta}\right)\left(S\frac{\beta}{\alpha}+V\frac{\beta}{\alpha}\right)$$

$$=S\frac{\gamma}{\beta}\,.\,S\frac{\beta}{\alpha}+S\frac{\gamma}{\beta}\,.\,V\frac{\beta}{\alpha}+S\frac{\beta}{\alpha}\,.\,V\frac{\gamma}{\beta}+V\frac{\gamma}{\beta}\,.\,V\frac{\beta}{\alpha}.$$

Taking the scalar part of both sides, we find

$$S\frac{\gamma}{\alpha}=S\frac{\gamma}{\beta}S\frac{\beta}{\alpha}+S\left(V\frac{\gamma}{\beta}\,V\frac{\beta}{\alpha}\right).$$

In the usual notation $S\frac{\gamma}{\alpha}=\cos b$, $S\frac{\gamma}{\beta}=\cos a$, $S\frac{\beta}{\alpha}=\cos c$,

and $$S\left(V\frac{\gamma}{\beta}\,V\frac{\beta}{\alpha}\right)=TV\frac{\gamma}{\beta}\,TV\frac{\beta}{\alpha}\,SUV\frac{\gamma}{\beta}\,UV\frac{\beta}{\alpha}$$
$$=\sin a\sin c\cos B,$$

B being the angle between the planes OBA, OBC.

Hence $$\cos b=\cos a\cos c+\sin a\sin c\cos B.$$

Again, let ϵ be the unit vector perpendicular to the plane of the triangle whose sides are parallel to the unit vectors $\alpha\beta\gamma$; and let A, B, C be the angles opposite sides. Then

$$\epsilon^{2A/\pi}\beta=-\gamma,\quad \epsilon^{2B/\pi}\gamma=-\alpha,\quad \epsilon^{2C/\pi}\alpha=-\beta,$$

whence $$-\beta=\epsilon^{2/C\pi}\alpha=-\epsilon^{2C/\pi}\,.\,\epsilon^{2B/\pi}\gamma=+\epsilon^{2(C+B+A)/\pi}\beta,$$

and $$\epsilon^{2(A+B+C)/\pi}=-1,$$

or $$A+B+C=\pi.$$

The angles of a triangle are together equal to two right angles.

27. VERSORS REPRESENTED BY ARCS ON A SPHERE. If α, β, γ, etc., be unit vectors drawn from a given origin, their extremities A, B, C, etc., lie on a sphere. The operation $\alpha\beta^{-1}$ which changes β into α may be represented by the arc of the great circle passing through BA. An equal arc taken anywhere on this great circle will represent the same versor. Let this versor be symbolized by p. Similarly, let q be the versor $\beta\gamma^{-1}$, represented by the arc CB. Then

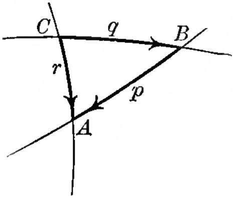

Fig. 20

$$pq = \alpha\beta^{-1} \,.\, \beta\gamma^{-1} = \alpha\gamma^{-1} = \overset{\frown}{CA} = r, \text{ say}$$

The conjugates of p, q, r are p^{-1}, q^{-1}, r^{-1} (§ 17). From the equation $pq = r$, we get by successive multiplications

$$pqr^{-1} = 1,$$

$$qr^{-1} = p^{-1},$$

$$r^{-1} = q^{-1}p^{-1},$$

so that
$$Kr = K(pq) = KqKp.$$

It is easy to see that this relation holds for quaternions as well as for versors; and that generally the conjugate of the product of any number of quaternions, versors, or vectors, is equal to the product of the individual conjugates taken in the reverse order.

Let us now find how the combination qp is to be represented. The versor arcs must be so arranged that the operation p is completed at the point where q begins. Hence p must end at B, and q must begin at B. p is therefore to be represented by $A'B$ (Fig. 21), an arc equal to BA and on the same great circle. Similarly, q is to be represented by $BC'(=CB)$. Then qp is represented by the arc $A'C'$, which is evidently equal *in magnitude* to the arc CA, but in general lies on quite a different great circle, that is in a different plane. It is therefore not the same versor. Let this versor qp

be symbolized by s, which, it must be remembered, may be represented by any arc equal to $A'C'$ in the same great circle, such for example as $C'D$.

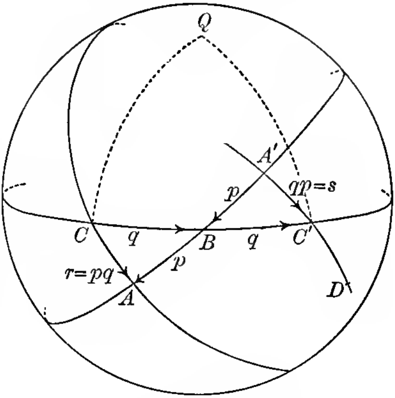

Fig. 21.

Since the associative law holds, we have

$$s = qp = qpqq^{-1} = qrq^{-1}.$$

Thus the complex operator $q(\quad)q^{-1}$ changes the versor r into the versor s, the great circle containing r being moved into the position of the great circle containing s.

It is obvious also because of the equality of the angles of the curvilinear triangles ACB, $A'C'B$ that the great circles containing s and r cut the great circle containing q at the same angle. Hence the motion by which r is changed into s may be effected by a rotation, C moving into the position C', and AC moving into the position DC'. Moreover any other great-circle arc drawn through C will be simultaneously rotated into a corresponding position with reference to C'. The particular great circle which meets q at C orthogonally will remain perpendicular to it at C' after the rotation; and the meeting point Q of these two great circles will be the pole of q, and will be the extremity of the axis about which all rotations are effected. Any given network of great circles, and therefore the corresponding vector lines drawn from the centre O, will, when operated on by $q(\quad)q^{-1}$, be rotated about the axis of q through an angle equal to twice the angle of q.

Since $$q = Tq\,Uq,$$
and $$Kq = Tq(Uq)^{-1},$$
we find $$q(\quad)Kq = (Tq)^2\,Uq(\quad)(Uq)^{-1}.$$

Hence the operator $q(\quad)Kq$ acting on any vector or collocation of vectors will have the same rotational effect as $q(\quad)q^{-1}$, but will increase the length of every vector in the ratio $(Tq)^2 : 1$. The operator $q(\quad)Kq$ therefore represents a simple rotation about the axis of q, accompanied by a uniform expansion (or contraction) of the system. It is a particular form of strain.

28. THE ROTATIONAL OPERATOR OTHERWISE DEDUCED. The operator $q(\quad)q^{-1}$ may be built up directly from the original definition of a quaternion. The problem is to find the quaternion operator which will rotate any vector about a given axis through a definite angle. The versor Q, acting on a vector perpendicular to its axis, turns that vector through the appropriate angle. But we may represent any vector ρ as composed of two parts, ϖ parallel to the axis of Q and ν perpendicular to it. Hence the vector

$$\rho' = \varpi + Q\nu$$

is what the vector ρ becomes when it is rotated conically about the axis of Q through the angle of Q. Taking conjugates of both sides we get

$$K\rho' = K\varpi + K\nu KQ,$$

or

$$-\rho' = -\varpi - \nu Q^{-1},$$

or

$$\rho' = \varpi + \nu Q^{-1}.$$

Now when ρ' is rotated about the axis of Q through the angle of Q, it becomes

$$\begin{aligned}\rho'' &= \varpi + Q\nu Q^{-1}\\ &= \varpi QQ^{-1} + Q\nu Q^{-1}\\ &= Q(\varpi + \nu)Q^{-1} = Q\rho Q^{-1},\end{aligned}$$

because Q and ϖ having parallel axes are commutative. Hence $Q(\quad)Q^{-1}$ rotates ρ through *twice* the angle of Q.

Another way of considering the effect of $q(\quad)Kq$ as an operator is to write

$$q = a + \alpha,\ \ Kq = a - \alpha,\ \ T^2q = a^2 - \alpha^2,$$

and expand the expression $q\rho Kq$. It becomes

$$\begin{aligned}\rho' = q\rho Kq &= (a+\alpha)\rho(a-\alpha)\\ &= a^2\rho + a(\alpha\rho - \rho\alpha) - \alpha\rho\alpha\\ &= a^2\rho + 2aV\alpha\rho - 2\alpha S\alpha\rho + \rho\alpha^2\\ &= (a^2+\alpha^2)\rho - 2\alpha S\alpha\rho + 2aV\alpha\rho.\end{aligned}$$

Now in all cases we may write

$$\rho = \alpha^{-1}\alpha\rho = \alpha^{-1}S\alpha\rho + \alpha^{-1}V\alpha\rho,$$

giving the components of ρ parallel to and perpendicular to α. But evidently

$$\begin{aligned}\alpha^{-1}S\alpha\rho' &= \alpha^{-1}\{(a^2+\alpha^2)S\alpha\rho - 2\alpha^2 S\alpha\rho\}\\ &= \alpha^{-1}(a^2-\alpha^2)S\alpha\rho = T^2q\alpha^{-1}S\alpha\rho,\end{aligned}$$

so that this component is increased in the ratio of T^2q or qKq to unity. The other component is

$$\begin{aligned}\alpha^{-1}V\alpha\rho' &= \alpha^{-1}\{(a^2+\alpha^2)V\alpha\rho + 2aV.\alpha V\alpha\rho\}\\ &= (a^2+\alpha^2)\alpha^{-1}V\alpha\rho + 2a\alpha\alpha^{-1}V\alpha\rho.\end{aligned}$$

Put $$\alpha^{-1}V\alpha\rho = \beta \text{ and } \alpha^{-1}V\alpha\rho' = \beta',$$

then $$\beta' = (a^2+\alpha^2)\beta + 2a\alpha\beta,$$

$$\beta'\beta^{-1} = a^2 + \alpha^2 + 2a\alpha = Q, \text{ say.}$$

Hence $$Q = (a+\alpha)^2 = q^2,$$

and β is turned into direction β' through twice the angle of q, and its tensor is increased in the ratio of T^2q or qKq to 1.

Thus the effect of $q(\quad)Kq$ on ρ is to change it into the vector ρ', whose projections along and perpendicular to the axis of q are greater than the projections of ρ in the ratio of qKq or T^2q to unity, so that the angle which ρ makes with the axis of q is unchanged, while at the same time the projection perpendicular to the axis of q is rotated through twice the angle of q.

The same result may be obtained with ease by use of the versor in the form α^x or by use of the expanded binomial form for a versor, namely, $\cos A + \alpha \sin A$. All give interesting exercises in quaternion transformations.

The strain symbolized by $q(\quad)Kq$ was noticed by Gauss, who saw that it involved four numbers. These are given by the scalar coefficients in the expanded quaternion form,

$$(w+xi+yj+zk)(\quad)(w-xi-yj-zk),$$

where i, j, k form a set of rectangular unit vectors.

29. COMPOSITION OF FINITE ROTATIONS. Let a rigid body be acted upon by a rotation $q(\quad)q^{-1}$ and then by a rotation $p(\quad)p^{-1}$, the resultant effect is

$$pq(\quad)q^{-1}p^{-1}=r(\quad)r^{-1} \text{ (§ 27).}$$

But r is a definite versor represented by the great-circle arc drawn from the beginning of the representative arc q to the end of the representative arc p, as shown in Fig. 20.

Hence, when a rigid body with one point fixed is subjected to a series of rotations about various axes, the final position can be arrived at by a single resultant rotation from the initial position about a definite axis through a definite angle. The consideration of the spherical triangle gives the position of the resultant rotation at a glance.

The resultant angle of rotation and the direction of the axis of rotation are determinate, and may easily be calculated. For example, let $p=\cos A+\alpha\sin A$, $q=\cos B+\beta\sin B$, $r=\cos C+\gamma\sin C$. Then the problem is to find C and γ from the equation

$$\begin{aligned}&\cos C+\gamma\sin C\\ &\quad=(\cos A+\alpha\sin A)(\cos B+\beta\sin B)\\ &\quad=\cos A\cos B+\alpha\sin A\cos B+\beta\cos A\sin B+\alpha\beta\sin A\sin B.\end{aligned}$$

Equating the scalar parts of both sides, we find

$$\begin{aligned}\cos C&=\cos A\cos B+\sin A\sin B\,S\alpha\beta\\ &=\cos A\cos B-\sin A\sin B\cos c,\end{aligned}$$

where c is the angle between the axes of p and q. Thus C is determined.

Then equating the vector parts of both sides, we get

$$\gamma \sin C = \alpha \sin A \cos B + \beta \cos A \sin B + V\alpha\beta \sin A \sin B,$$

so that the components of the vector γ along the directions α, β, and the directions perpendicular to these, are determined, and the position of γ is known.

Finite rotations are not in general commutative, for, as shown above, qp is not the same as pq. Hence the rotation $qp(\quad)p^{-1}q^{-1}$ is not in general the same as $pq(\quad)q^{-1}p^{-1}$. They are the same when the rotations are coaxial, so that p and q may be represented by arcs along the same great circle; also when each component rotates through four right angles.

30. COMPOSITION OF INFINITELY SMALL ROTATIONS.

If we write q in the form $a + \alpha$, Kq is $a - \alpha$, and

$$(Tq)^2 = qKq = a^2 + (T\alpha)^2.$$

We pass to the case of infinitely small rotations by taking $T\alpha$ very small, so that its square may be neglected in comparison with a^2, which in the present case may be taken as equal to unity. Hence, if we write q in the form $1 + \frac{1}{2}e\epsilon$, where e is a very small quantity and ϵ is a unit vector, the rotation is symbolized by

$$(1 + \tfrac{1}{2}e\epsilon)(\quad)(1 - \tfrac{1}{2}e\epsilon),$$

ϵ being unit vector along the axis of rotation, and e the measure of the (small) angle of rotation.

Any vector ρ becomes

$$\begin{aligned}\rho' &= (1 + \tfrac{1}{2}e\epsilon)\,\rho\,(1 - \tfrac{1}{2}e\epsilon)\\ &= \rho + \tfrac{1}{2}e(\epsilon\rho - \rho\epsilon) - \tfrac{1}{4}e^2\epsilon\rho\epsilon\\ &= \rho + eV\epsilon\rho, \text{ the term in } e^2 \text{ being negligible.}\end{aligned}$$

Thus $\rho' - \rho = eV\epsilon\rho$ is the displacement of the extremity of ρ. It is in a direction perpendicular to both ρ and ϵ, and its value is $eTV\epsilon\rho = eT\rho \sin A$, where A is the angle between ϵ and ρ.

If there are two simultaneous small rotations $e\,e'$ about axes $\epsilon\,\epsilon'$, the vector ρ becomes

$$\begin{aligned}\rho' &= (1+\tfrac{1}{2}e\epsilon)(1+\tfrac{1}{2}e'\epsilon')\,\rho\,(1-\tfrac{1}{2}e'\epsilon')(1-\tfrac{1}{2}e\epsilon)\\ &= \{1+\tfrac{1}{2}(e\epsilon+e\epsilon')\}\,\rho\,\{1-\tfrac{1}{2}(e\epsilon+e\epsilon')\},\end{aligned}$$

neglecting products of the small quantities e, e'.

Hence $$\rho' = \rho + V.\,(e\epsilon + e'\epsilon')\rho,$$

so that the resultant rotation is obtained by the same process of vector addition as resultant displacements and velocities are obtained.

Generally for any number of simultaneous small rotations $e_1 e_2 e_3 \ldots$ about axes $\epsilon_1\epsilon_2\epsilon_3 \ldots$ the displacement of any point ρ is

$$\rho' - \rho = V.\,(e_1\epsilon_1 + e_2\epsilon_2 + e_3\epsilon_3 + \ldots)\rho = V.\,\Sigma(e\epsilon)\rho.$$

There is no displacement when ρ is parallel to $\Sigma(e\epsilon)$.

This vector is therefore parallel to the axis of rotation, and the resultant angular displacement about this axis has the value

$$T(e_1\epsilon_1 + e_2\epsilon_2 + e_3\epsilon_3 + \ldots) = T\Sigma(e\epsilon).$$

The quantity $\rho V\epsilon\rho$ is evidently a vector, being the product of two perpendicular vectors; and the summation $(\Sigma m\rho V\epsilon\rho)$ of quantities of this kind in which ϵ is any vector, and ρ is one of a number of given vectors, is an example of what is called a linear vector function of the vector ϵ. In the present case its value depends upon the distribution of matter in the body. The linear vector function is one of the most beautiful of Hamilton's discoveries. Some of its properties are discussed in Chapters VI., VII., and X.

31. QUATERNION PRODUCTS. The product of any number of quaternions is a quaternion. This follows at once from the representation of versors on a sphere; for the passage from versors to quaternions requires simply the introduction of the scalar factors known as the tensors. Thus, in the equation $pq = r$, where p and q are given quaternions, r also is a quaternion whose tensor is equal to the product of the tensors of p and q, and whose versor is the resultant of the versors of p and q, as discussed in § 27.

The product pqs of three given quaternions is at once by the foregoing reduced to rs, a product of two quaternions, and this is a quaternion; and so on for any number.

Again, since $\qquad T(pq) = TpTq,$

we have $\qquad pqK(pq) = T^2(pq) = T^2pT^2q.$

Multiply by Kp, and then by Kq, and divide out the scalar factors. This gives

$$Kq(Tp)^2qK(pq) = KqKpT^2pT^2q,$$

and finally $\qquad K(pq) = KqKp.$

And generally the conjugate of the product of any number of quaternions is the product of the conjugates of the constituents taken in the reverse order; in symbols

$$K(pqrst) = KtKsKrKqKp.$$

32. PRODUCTS OF VECTORS. What is true of quaternion products in general will be true of particular types, such as quadrantal quaternions or vectors. Thus the continuous product of three or more vectors is in general a quaternion, degenerating in special cases to a vector or a scalar.

Consider the quaternion $q = \alpha\beta\gamma$, with its conjugate

$$K(\alpha\beta\gamma) = K\gamma K\beta K\alpha = (-\gamma)(-\beta)(-\alpha)$$
$$= -\gamma\beta\alpha.$$

From the general relations $2Vq = q - Kq$, $2Sq = q + Kq$, we have

$$2V . \alpha\beta\gamma = \alpha\beta\gamma + \gamma\beta\alpha,$$
$$2S . \alpha\beta\gamma = \alpha\beta\gamma - \gamma\beta\alpha.$$

$S\alpha\beta\gamma$ and $V\alpha\beta\gamma$ are the scalar and vector parts of the product $\alpha\beta\gamma$.

The geometrical meaning of $S . \alpha\beta\gamma$ is easily deduced. For

$$S . \alpha\beta\gamma = S . \alpha(S\beta\gamma + V\beta\gamma) = S . \alpha V\beta\gamma,$$

because $\alpha S\beta\gamma$ being a vector can have no scalar part. But $S . \alpha V\beta\gamma$ may be written in the form $TV\beta\gamma S . \alpha UV\beta\gamma$. Now

$TV\beta\gamma$ is the area of the parallelogram contained by β and γ, *i.e.* twice the area of the triangle OBC (fig. 22). Then $-S\,.\,\alpha UV\beta\gamma$ is the resolved part of OA perpendicular to OBC, *i.e.* the perpendicular from A upon the plane OBC. But the product of twice the area OBC and the height to A is evidently the volume of the parallelepiped whose base is the parallelogram contained by OB and OC and whose opposite face passes through A. In short, it is the volume of the parallelepiped whose edges are α, β, γ.

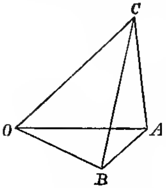

Fig. 22.

Since
$$S\,.\,\alpha\beta\gamma = S\alpha V\beta\gamma = S(V\beta\gamma)\alpha = S\beta\gamma\alpha$$
$$= -S\,.\,\alpha V\gamma\beta = -S\alpha\gamma\beta,$$
and so on, we see that so long as the cyclical order is unchanged the scalar of the product has the same value; but that if the order is changed the sign is changed.

If we express α, β, γ in terms of a set i, j, k of rectangular unit vectors, namely,

$$\alpha = a_1 i + a_2 j + a_3 k,$$
$$\beta = b_1 i + b_2 j + b_3 k,$$
$$\gamma = c_1 i + c_2 j + c_3 k,$$

and form the scalar of the product $\alpha\beta\gamma$, we notice that all terms of the form iij or jjk, being vectors, must vanish. Hence, only terms in ijk can exist. But since $ijk = -1$, we find

$$-S\,.\,\alpha\beta\gamma = \begin{vmatrix} a_1, & a_2, & a_3, \\ b_1, & b_2, & b_3, \\ c_1, & c_2, & c_3, \end{vmatrix}$$

the well-known determinant expression for six times the volume of the tetrahedron whose corners have the coordinates 000, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$.

The vector $V\,.\,\alpha\beta\gamma$ may, like any vector, be expressed linearly in terms of the non-coplanar vectors α, β, γ. This is most simply effected as follows:

$$\begin{aligned} 2V.\,\alpha\beta\gamma = \alpha\beta\gamma &+ \gamma\beta\alpha \\ +\alpha\gamma\beta \quad &- \alpha\gamma\beta \\ &+ \gamma\alpha\beta - \gamma\alpha\beta, \end{aligned}$$

adding and subtracting the quaternions $\alpha\gamma\beta$ and $\gamma\alpha\beta$. Combining in pairs, we get

$$\begin{aligned} 2V\alpha\beta\gamma &= \alpha(\beta\gamma+\gamma\beta)+\gamma(\beta\alpha+\alpha\beta)-(\alpha\gamma+\gamma\alpha)\beta \\ &= 2\alpha S\beta\gamma + 2\gamma S\alpha\beta - 2\beta S\alpha\gamma, \end{aligned}$$

or $$V.\,\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta.$$

From this form we see at once that α and γ may be interchanged without affecting the value of the quantity. Or

$$V.\,\alpha\beta\gamma = V.\,\gamma\beta\alpha.$$

Again, since $V.\,\alpha\beta\gamma = \alpha S\beta\gamma + V.\,\alpha V\beta\gamma$,
we obtain the further identity

$$V.\,\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\gamma\alpha,$$

an extremely important formula of frequent use in transformations.

When $S\alpha\beta\gamma = 0$, the volume of the parallelepiped becomes zero, which means that α, β, γ cannot form a parallelepiped. If they have different directions they must be in one plane. In fact, any one, say α, must be perpendicular to the common perpendicular to the other two, namely, $V\beta\gamma$. In other words, all three are perpendicular to the same line, and must therefore be coplanar when drawn from one point.

Under these circumstances the product $\alpha\beta\gamma$ must be a vector. Call it δ. Then $\delta = \alpha\beta\gamma$, or $\delta\gamma^{-1} = \alpha\beta$. Hence $\delta\gamma^{-1}$ and $\alpha\beta$ represent equal quaternions, showing that the operation which changes γ into δ will also change β^{-1} into α. In other words, $\alpha\beta\gamma\delta$ when drawn from one point or continuously end to end are coplanar vectors, and the angle between δ and γ is the same as that between α and β^{-1} (or $-\beta$). Thus we can draw the direction of δ at once, the vectors $\alpha\beta\gamma$ being given; and then the tensor of δ is equal to the product of the tensors of α, β, γ.

If $\alpha\beta\gamma\delta$ form the sides of a closed quadrilateral, then the interior angle between α and β is equal to the exterior angle between γ and δ; and the quadrilateral is inscribable in a circle.

If $\alpha\beta\gamma$ form the sides of a triangle, then $\delta (= \alpha\beta\gamma)$ is drawn in the direction of the tangent at the point (α, γ) to the circle circumscribing the triangle.

33. TRANSFORMATIONS OF SCALAR AND VECTOR PARTS OF PRODUCTS. The formulae of transformation for $V.\alpha\beta\gamma$ and $S.\alpha\beta\gamma$ are of great importance in applications to geometry and dynamics. We shall give a few of these.

In expressions of the form $S.\alpha\beta\gamma$, it is evident that the vector part only of the product of any pair is of importance, for $\alpha S\beta\gamma$ is necessarily a vector, and can have no scalar part.

Thus the expression $S.V\alpha\beta V\beta\gamma V\gamma\alpha$ may be written

$$\begin{aligned} SV\alpha\beta V(V\beta\gamma V\gamma\alpha) &= SV\alpha\beta(-\gamma S\alpha\beta\gamma + \alpha S\gamma\beta\gamma) \\ &= -S\alpha\beta\gamma S\alpha\beta\gamma + 0; \end{aligned}$$

$$\therefore \quad S.V\alpha\beta V\beta\gamma V\gamma\alpha = -(S\alpha\beta\gamma)^2.$$

This formula may be readily transformed into Cartesian coordinates; and occasionally practice of this kind is useful, if only to show how much more concise and expressive the quaternion notation is. Thus, with i, j, k as rectangular unit vectors, we have

$$\begin{aligned} \alpha &= a_1 i + a_2 j + a_3 k, \\ \beta &= b_1 i + b_2 j + b_3 k, \\ \gamma &= c_1 i + c_2 j + c_3 k, \\ V\alpha\beta &= (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k, \\ V\beta\gamma &= \text{etc.}, \qquad V\gamma\alpha = \text{etc.} \end{aligned}$$

By forming the products and taking the scalar parts, we readily find as the analytical equivalent of the formula given above the determinantal identity

$$\begin{vmatrix} a_2 b_3 - a_3 b_2, & a_3 b_1 - a_1 b_3, & a_1 b_2 - a_2 b_1 \\ b_2 c_3 - b_3 c_2, & b_3 c_1 - b_1 c_3, & b_1 c_2 - b_2 c_1 \\ c_2 a_3 - c_3 a_2, & c_3 a_1 - c_1 a_3, & c_1 a_2 - c_2 a_1 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}^2 .$$

When α, β, γ are unit vectors, the formula

$$V . V\alpha\beta V\beta\gamma = -\beta S\alpha\beta\gamma$$

has an immediate application in spherical trigonometry. Let A, B, C be the extremities of the vectors α, β, γ on the unit sphere; and let a, b, c be the arcs subtending the angles A, B, C respectively. Then throwing the above formula into the form

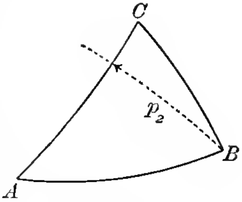

Fig. 23.

$$TV\alpha\beta TV\beta\gamma V . UV\alpha\beta UV\beta\gamma = -\beta TV\gamma\alpha S . \beta UV\gamma\alpha,$$

we deduce the relation

$$\sin c \sin a \sin B = \sin b \sin p_2,$$

where p_2 is the perpendicular arc from B, upon the arc AC, being the supplement of the arc whose cosine is $-S\beta UV\gamma\alpha$.

If, on the right hand, we write $-\beta TV\beta\gamma S . \alpha UV\beta\gamma$, $TV\beta\gamma$ divides out, and we get

$$\sin c \sin B = \sin p_1,$$

where p_1 is the perpendicular arc from A on CB. Similar expressions for p_2, p_3 may be written down at sight.

The transformation

$$\begin{aligned} S . V\alpha\beta V\beta\gamma &= S . \alpha V\beta V\beta\gamma \\ &= \beta^2 S\alpha\gamma - S\alpha\beta S\beta\gamma, \end{aligned}$$

when interpreted in the same way with $\alpha\beta\gamma$ as unit vectors, leads to the formula

$$\sin a \sin c \cos B = -\cos a \cos c + \cos b.$$

Again,

$$\tan B = \frac{\sin B}{\cos B} = \frac{TV . V\alpha\beta V\beta\gamma}{S . V\alpha\beta V\beta\gamma} = -\frac{S\alpha\beta\gamma}{S\alpha\gamma + S\alpha\beta S\beta\gamma},$$

giving
$$\begin{aligned} \tan B(-\cos b + \cos c \cos a) &= \sin a \sin p_1 \\ &= \sin b \sin p_2 \\ &= \sin c \sin p_3. \end{aligned}$$

These examples show with what peculiar readiness the calculus of quaternions attacks problems of spherical trigonometry.

The most immediate geometric interpretation of the formula

$$V\,.\,V\alpha\beta V\beta\gamma = -\beta S\alpha\beta\gamma$$

is that the line of intersection of two planes is perpendicular to the normals of these planes. For $V\alpha\beta$ is perpendicular to the plane containing α and β, and $V\beta\gamma$ is perpendicular to the plane containing β and γ. But these planes have the line β in common, and β is by the above formula perpendicular to its constituents $V\alpha\beta$, $V\beta\gamma$.

34. RELATION CONNECTING FOUR VECTORS. The expansion of $V\alpha\beta\gamma$ as a linear function of the three non-coplanar vectors α, β, γ (§ 32) is a particular case of the general truth that any vector may be so represented.

Let $$\rho = x\alpha + y\beta + z\gamma,$$
where xyz are the coordinates of the extremity of ρ referred to axes parallel to α, β, γ.

To express x in terms of the vectors, *operate by* $S\,.\,V\beta\gamma$, that is, multiply by $V\beta\gamma$ and take the scalar part. Then since $S\beta V\beta\gamma$ and $S\gamma V\beta\gamma$ both vanish, we get at once

$$S\beta\gamma\rho = xS\alpha\beta\gamma.$$

Similarly, $$S\gamma\alpha\rho = yS\alpha\beta\gamma,$$
$$S\alpha\beta\rho = zS\alpha\beta\gamma.$$

Hence, generally,

$$\rho S\alpha\beta\gamma = \alpha S\beta\gamma\rho + \beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho \;\ldots\ldots\ldots\ldots\ldots(1)$$

Now the vectors $V\alpha\beta$, $V\beta\gamma$, $V\gamma\alpha$ will be non-coplanar if $\alpha\beta\gamma$ are, for each is perpendicular to the plane containing its constituents. Hence ρ must be expressible in the form

$$\rho = x'V\alpha\beta + y'V\beta\gamma + z'V\gamma\alpha.$$

Operating by $S\,.\,\alpha$, we find $S\alpha\rho = y'S\alpha\beta\gamma$.

Similarly, $S\beta\rho = z'S\alpha\beta\gamma$, $S\gamma\rho = x'S\alpha\beta\gamma$;

and consequently,

$$\rho S\alpha\beta\gamma = V\alpha\beta S\gamma\rho + V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho \ldots\ldots\ldots\ldots(2)$$

The vector whose components are $V\alpha\beta$, $V\beta\gamma$, $V\gamma\alpha$ has an important property. Calling it δ, we have, operating by $S(\alpha-\beta)$,

$$\begin{aligned} S(\alpha-\beta)\delta &= S(\alpha-\beta)V(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= S\alpha\beta\gamma - S\beta\gamma\alpha \\ &= 0, \end{aligned}$$

the other products vanishing because they are of the form $S\rho V\rho\sigma$.

Similarly, $\quad S(\beta-\gamma)\delta=0, \;\; S(\gamma-\alpha)\delta=0.$

Hence δ is perpendicular to the plane passing through the extremities of $\alpha\beta\gamma$.

In like manner it may be shown that the vector $\alpha+\beta+\gamma$ is perpendicular to the plane passing through the extremities of $V\alpha\beta$, $V\beta\gamma$, $V\gamma\alpha$.

The condition that these two planes should meet at right angles to each other is

$$S(\alpha+\beta+\gamma)V(\alpha\beta+\beta\gamma+\gamma\alpha)=0,$$

or

$$3S\alpha\beta\gamma=0.$$

Hence $\alpha\,\beta\,\gamma$ are coplanar, and so are the vectors $V\alpha\beta$, $V\beta\gamma$, $V\gamma\alpha$.

The vector lines α, β, γ drawn from a point form in general three of the edges of a tetrahedron; and the perpendiculars on the faces from the opposite angles are parallel to the vectors

$$V\alpha\beta,\;\; V\beta\gamma,\;\; V\gamma\alpha,\;\; V(\alpha\beta+\beta\gamma+\gamma\alpha).$$

What is the condition that these perpendiculars meet in a point?

Evidently the edge α and the perpendicular $V\beta\gamma$ must lie in the same plane with $V(\alpha\beta+\beta\gamma+\gamma\alpha)$; and similarly for β and $V\gamma\alpha$, and for γ and $V\alpha\beta$. Hence,

$$S\,.\,\alpha V\beta\gamma V(\alpha\beta+\beta\gamma+\gamma\alpha)=0,$$

or

$$S(\gamma S\alpha\beta-\beta S\alpha\gamma)V(\alpha\beta+\beta\gamma+\gamma\alpha)=0.$$

This reduces at once to

$$S\alpha\beta\gamma(S\alpha\beta - S\alpha\gamma) = 0,$$

or

$$S\alpha\beta\gamma S\alpha(\beta - \gamma) = 0.$$

Hence, since α, β, γ are not coplanar, $S\alpha\beta\gamma$ has a finite value, and the other factor $S\alpha(\beta-\gamma)$ must vanish, or α is perpendicular to $(\beta-\gamma)$. Similarly, β is perpendicular to $(\gamma-\alpha)$, and γ perpendicular to $(\alpha-\beta)$.

Thus the six edges form three groups of perpendicular pairs. This also implies that the sum of the squares of any two opposite edges is the same for the three sets of pairs. For

$$\alpha^2 + (\beta-\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 - 2S\beta\gamma,$$

and

$$\beta^2 + (\gamma-\alpha)^2 = \alpha^2 + \beta^2 + \gamma^2 - 2S\gamma\alpha,$$

$$\gamma^2 + (\alpha-\beta)^2 = \alpha^2 + \beta^2 + \gamma^2 - 2S\alpha\beta,$$

and these have the same values because, as proved above,

$$S\beta\gamma = S\gamma\alpha = S\alpha\beta.$$

Having shown that $\delta = V(\alpha\beta + \beta\gamma + \gamma\alpha)$ is perpendicular to the plane passing through the extremities of α, β, and γ, let us next find the value of the perpendicular from the origin. It will be some scalar multiple, x, of δ, such that

$$x\delta - \alpha,\ \ \alpha - \beta,\ \ \beta - \gamma$$

will all lie in one plane, or

$$S(x\delta - \alpha)(\alpha - \beta)(\beta - \gamma) = 0.$$

This reduces to

$$xS\delta V(\alpha\beta + \beta\gamma + \gamma\alpha) = S\alpha\beta\gamma,$$

or

$$x\delta^2 = S\alpha\beta\gamma\,;$$

hence

$$x\delta = \delta^{-1}S\alpha\beta\gamma$$

Thus the vector perpendicular from the vertex of the tetrahedron α, β, γ upon the opposite face is

$$\frac{S\alpha\beta\gamma}{V(\alpha\beta + \beta\gamma + \gamma\alpha)},$$

and its length is

$$\frac{-S\alpha\beta\gamma}{TV(\alpha\beta + \beta\gamma + \gamma\alpha)}.$$

This suggests one form of the equation of a plane passing through the extremities of the coinitial vectors α, β, γ, namely, if ρ is the vector to any point on the plane,

$$S(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)=0,$$

or

$$S\rho V(\alpha\beta+\beta\gamma+\gamma\alpha)=S\alpha\beta\gamma.$$

And generally, if ϵ is a given vector,

$$S\rho\epsilon=-1$$

represents a plane perpendicular to ϵ.

Throwing it into the form

$$S\epsilon\left(\rho-\frac{\epsilon}{(T\epsilon)^2}\right)=0,$$

we see at once that the plane must pass through the point $(\epsilon/T\epsilon)^2=-\epsilon^{-1}$.

EXAMPLES TO CHAPTER IV.

1. Prove that $S.(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)=2S.\alpha\beta\gamma$.

2. $S.V\alpha\beta V\beta\gamma V\gamma\alpha=-(S\alpha\beta\gamma)^2$.

3. $S.V(V\alpha\beta V\beta\gamma)V(V\beta\gamma V\gamma\alpha)V(V\gamma\alpha V\alpha\beta)=-(S.\alpha\beta\gamma)^4$.

4. $S(V\beta\gamma V\gamma\alpha)=\gamma^2S\alpha\beta-S\beta\gamma S\gamma\alpha$.

5. $\alpha^2\beta^2\gamma^2=(V\alpha\beta\gamma)^2-(S\alpha\beta\gamma)^2$

6. $\qquad=\alpha^2(S\beta\gamma)^2+\beta^2(S\gamma\alpha)^2+\gamma^2(S\alpha\beta)^2-(S\alpha\beta\gamma)^2-2S\alpha\beta S\beta\gamma S\gamma\alpha$.

7. $S(\gamma V.\alpha\beta\gamma)=\gamma^2S\alpha\beta$.

8. $(\alpha\beta\gamma)^2=\alpha^2\beta^2\gamma^2+2\alpha\beta\gamma S.\alpha\beta\gamma$.

9. $S(V\alpha\beta\gamma V\beta\gamma\alpha V\gamma\alpha\beta)=4S\alpha\beta S\beta\gamma S\gamma\alpha S.\alpha\beta\gamma$.

10. The expression

$$V\alpha\beta V\gamma\delta+V\alpha\gamma V\delta\beta+V\alpha\delta V\beta\gamma$$

denotes a vector. What vector?

(Tait's *Quaternions*. Miscellaneous Ex. 1.)

11. $S\alpha\rho S.\beta\gamma\delta-S\beta\rho S.\gamma\delta\alpha+S\gamma\rho S.\delta\alpha\beta-S\delta\rho S.\alpha\beta\gamma=0$.

12. $(\alpha\beta\gamma)^2=2\alpha^2\beta^2\gamma^2+\alpha^2(\beta\gamma)^2+\beta^2(\alpha\gamma)^2+\gamma^2(\alpha\beta)^2-4\alpha\gamma S\alpha\beta S\beta\gamma$.

(Hamilton, *Elements*, p. 346.)

13. When A, B, C, D are in the same plane,

$$\alpha\,.\,BCD-\beta\,.\,CDA+\gamma\,.\,DAB-\delta\,.\,ABC=0,$$

where BCD, etc., are the areas of the triangles and $\alpha\beta\gamma\delta$ the vectors to $ABCD$ from any origin.

14. $\delta V\,.\,\alpha\beta\gamma+\alpha V\,.\,\beta\gamma\delta+\beta V\,.\,\gamma\delta\alpha+\gamma V\,.\,\delta\alpha\beta=4S\,.\,\alpha\beta\gamma\delta.$

15. $V\alpha\beta V\gamma\delta+V\beta\gamma V\delta\alpha+V\gamma\delta V\alpha\beta+V\delta\alpha V\beta\gamma$ is a scalar. What is its geometrical meaning?

16. If P be any point within the tetrahedron $ABCD$, and if a, b, c, d be the points in which the produced lines AP, BP, CP, DP meet the opposite faces, then

$$Pa/Aa+Bb/Bb+Pc/Cc+Pd/Dd=1.$$

17. Expand $S\,.\,\alpha\beta\gamma\delta$ and $V\,.\,\alpha\beta\gamma\delta$ in terms of scalars and vectors of the products of $\alpha\beta\gamma\delta$ in pairs.

18. Show that $V\,.\,\alpha V\beta\gamma$, $V\,.\,\beta V\gamma\alpha$, $V\,.\,\gamma V\alpha\beta$ are coplanar, and that their mutual perpendicular is

$$\frac{V\alpha\beta}{S\alpha\beta}+\frac{V\beta\gamma}{S\beta\gamma}+\frac{V\gamma\alpha}{S\gamma\alpha}.$$

19. Expand q^2 and q^3 in terms of the scalar and vector parts of q; and thence find $S\,.\,q^2$, $V\,.\,q^2$, $S\,.\,q^3$, $V\,.\,q^3$.

Give trigonometrical interpretations of the identities established.

20. Find a solution of the equation $Q^2=q^2$ in the form

$$Q=\pm\sqrt{-1}\,(Sq\,.\,UVq-TVq).$$

(Hamilton, *Lectures*, p. 673.)

21. Show that the equation

$$p(\quad)Kp+q(\quad)Kq+r(\quad)Kr=0,$$

where p, q, r are quaternions, is impossible except under very limited conditions. Find these conditions.

22. Show that for any three vectors α, β, γ, we have

$$(U\alpha\beta)^2+(U\beta\gamma)^2+(U\gamma\alpha)^2+(U\alpha\beta\gamma)^2+4U\alpha\gamma\,.\,SU\alpha\beta SU\beta\gamma=-2.$$

(Hamilton, *Elements*, p. 388.)

23. If $\alpha\beta\gamma=(\alpha-\gamma)\beta(\alpha-\gamma)^{-1}$, show that $\alpha\beta\gamma$ are coplanar unit vectors. Interpret the equation geometrically.

CHAPTER V.

SIMPLE GEOMETRICAL APPLICATIONS.

35. EQUATIONS OF STRAIGHT LINE AND PLANE. Let λ be a vector (unit or otherwise) parallel to or along the straight line; α the vector to a given point A in the line, ρ that to any point whatever P in the line, starting from the same origin O; then AP is a vector parallel to λ

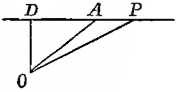

Fig. 24.

$$= x\lambda, \text{ say,}$$

and
$$OP = OA + AP$$

gives
$$\rho = \alpha + x\lambda$$

as the equation of the line.

Another form in which the equation of a straight line may be expressed is this: let $OA = \alpha$, $OB = \beta$ be the vectors to two given points in the line; then

$$AB = \beta - \alpha \text{ and } AP = x(\beta - \alpha);$$

$$\therefore \rho = \alpha + x(\beta - \alpha).$$

The first form of the equation supposes the direction of the line and the position of one point in it to be given, the second form supposes two points in it to be given.

Operating on the first equation by $V\lambda$, we get the equation $V\lambda(\rho - \alpha) = 0$ which means that $\rho - \alpha$ is parallel to λ. Also $\rho = \alpha$ is one value which ρ may have. Hence the equation is that of a straight line passing through α and parallel to λ.

A straight line may also be exhibited as the intersection of two planes (see below).

The equation of a plane is found thus:

Let ρ be the vector to any point in the plane, and ν the perpendicular to the plane, from any chosen origin. Then $\rho - \nu$ is perpendicular to ν. Hence

$$S\nu(\rho - \nu) = 0,$$

or

$$S\nu\rho = \nu^2 = -a^2$$

if a is the length of the perpendicular. Another form is

$$S\rho\nu^{-1} = 1.$$

COR. 1. If $S\nu\rho = -a$ be the equation of a plane, ν is a vector in the direction perpendicular to the plane.

COR. 2. If the plane pass through O, ρ can have the value zero;

$$\therefore\ S\nu\rho = 0 \text{ is the equation.}$$

COR. 3. If β be any vector in or parallel to the plane,

$$S\nu\beta = 0.$$

36. We proceed to exhibit certain modifications of the equations of a straight line and plane, and one or two results immediately deducible from the forms of those equations.

1. To find the equation of a straight line which is perpendicular to each of two given straight lines.

Let γ, γ', be vectors parallel respectively to the given lines.

Then the vector λ of last paragraph is parallel to $V\gamma\gamma'$, and the equation of the line becomes

$$V.\rho V\gamma\gamma' = V.\alpha V\gamma\gamma',$$

or

$$\rho = \alpha + xV\gamma\gamma'.$$

2. To find the length of the perpendicular from a given point on a given line.

Let γ be the given point, and ν the perpendicular from it on the line parallel to λ.

If β is the vector to a point in the line, then for any other point, vector ρ,

$$V(\rho-\beta)\lambda=0.$$

Hence for the particular value $\rho=\gamma+\nu$,

$$V(\gamma+\nu-\beta)\lambda=0,$$

or

$$V\lambda\nu=V(\gamma-\beta)\lambda.$$

But $S\lambda\nu=0$, so that we may write

$$V\lambda\nu=\lambda\nu.$$

Thus

$$\lambda\nu=V(\gamma-\beta)\lambda,$$

$$\nu=\lambda^{-1}V(\gamma-\beta)\lambda.$$

3. To find the length of the perpendicular from a given point on a given plane.

Let $S\nu\rho=-a$ be the equation of the plane, γ the vector to the given point.

Then if the vector perpendicular be represented by $x\nu$,

$$\rho=\gamma+x\nu$$

gives

$$S\nu\gamma+x\nu^2=-a,$$

and the vector perpendicular is

$$x\nu=\nu^{-1}(-a-S\nu\gamma);$$

the square of which with change of sign is the square of the perpendicular.

4. To find the length of the common perpendicular to each of two given straight lines.

Let β, β_1 be unit vectors along the lines; α, α_1 vectors to given points in the lines;

$$\rho=\alpha+x\beta,$$

$$\rho_1=\alpha_1+x_1\beta_1,$$

the vectors to the extremities of the common perpendicular ν.

Then $V\beta\beta_1$, being perpendicular to both lines, must be parallel to the common perpendicular ν; hence

$$\nu=yV\beta\beta_1.$$

But $$\nu = \rho - \rho_1 = \alpha + x\beta - \alpha_1 - x_1\beta_1,$$
hence $$S\,.\,\nu\beta\beta_1 = S\,.\,(\alpha - \alpha_1)\beta\beta_1\,;$$
i.e. $$S(yV\beta\beta_1\,.\,\beta\beta_1) = S\,.\,(\alpha - \alpha_1)\beta\beta_1,$$
or $$y(V\beta\beta_1)^2 = S\,.\,(\alpha - \alpha_1)\beta\beta_1\,;$$
$$\therefore\ \nu = yV\beta\beta_1 = \frac{S(\alpha - \alpha_1)\beta\beta_1}{V\beta\beta_1}.$$

5. To find the equation of a plane which passes through three given points, and the condition that four points lie in a plane. (See last chapter, § 34.)

Let α, β, γ be the vectors of the points.

Then $\rho - \alpha$, $\alpha - \beta$, $\beta - \gamma$ are in the same plane;
$$\therefore\ S\,.\,(\rho - \alpha)(\alpha - \beta)(\beta - \gamma) = 0,$$
or $$S\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) - S\,.\,\alpha\beta\gamma = 0$$
is the equation required. It may be written in the form
$$S\rho\alpha\beta + S\gamma\rho\beta + S\gamma\alpha\rho = S\alpha\beta\gamma,$$
and may be regarded as the condition that the four points $\alpha\beta\gamma\rho$ lie in one plane.

We may always express any vector ρ linearly in terms of the three non-coplanar vector $\alpha\beta\gamma$, namely,
$$x\rho + a\alpha + b\beta + c\gamma = 0\,;$$
whence
$$xS\rho\alpha\beta + cS\alpha\beta\gamma = 0,$$
$$xS\rho\beta\gamma + aS\alpha\beta\gamma = 0,$$
$$xS\rho\gamma\alpha + bS\alpha\beta\gamma = 0\,;$$
whence, adding, we find
$$x + a + b + c = 0.$$

$V\alpha\beta + V\beta\gamma + V\gamma\alpha$ is a vector in the direction perpendicular to the plane; and the perpendicular vector from the origin
$$= S\alpha\beta\gamma\,.\,(V\alpha\beta + V\beta\gamma + V\gamma\alpha)^{-1}.$$

6. To find the equation of a plane which shall pass through a given point and be parallel to each of two given straight lines.

Let γ be the vector to the given point, and $\beta\beta_1$ the vectors parallel to the given straight lines. Then $V\beta\beta_1$ is the normal to the plane, the equation of which is accordingly
$$S\,.\,\beta\beta_1(\rho - \gamma) = 0.$$

7. Given the two planes $S\alpha\rho = -a$, $S\beta\rho = -b$, then

$$\frac{S\alpha\rho}{a} = \frac{S\beta\rho}{b}, \text{ or } S\rho(b\alpha - a\beta) = 0.$$

This is the equation of a plane which passes through the origin and which is perpendicular to the vector $(b\alpha - a\beta)$. Also when $S\alpha\rho = -a$, then must $S\beta\rho = -b$; hence the plane must contain the line common to the two original planes.

Let it be required to find the equation of this line of intersection of the two planes. This line must be parallel to $V\alpha\beta$; hence its equation must be of the form $\rho = \gamma + xV\alpha\beta$, where γ may for simplicity be taken on the plane α, β. Hence we may write

$$\rho = m\alpha + n\beta + xV\alpha\beta,$$

where m, n are to be found.

Then $$S\alpha\rho = m\alpha^2 + nS\alpha\beta = -a,$$

since $V\alpha\beta$ is perpendicular to α, and similarly

$$S\beta\rho = mS\alpha\beta + n\beta^2 = -b;$$

$$\therefore\ m = \frac{-a\beta^2 + bS\alpha\beta}{\alpha^2\beta^2 - (S\alpha\beta)^2} = \frac{-bS\alpha\beta + a\beta^2}{(V\alpha\beta)^2},$$

$$n = \frac{-aS\alpha\beta + b\alpha^2}{(S\alpha\beta)^2 - \alpha^2\beta^2} = \frac{-aS\alpha\beta + b\alpha^2}{(V\alpha\beta)^2}.$$

37. We offer a few simple examples of loci.

1. *Planes cut off, from the three co-ordinate axes, pyramids of equal volume, to find the locus of the feet of perpendiculars on them from the origin.*

Let $\alpha\beta\gamma$ be unit vectors along the axes; and let $a\alpha$, $b\beta$, $c\gamma$ be the vectors to the points of section of the axes with the planes in any position. The volume of the pyramid is

$$\frac{1}{6}Sa\alpha b\beta c\gamma;$$

so that since $S\alpha\beta\gamma$ has always the same value, the condition requires that abc is constant. But the vector perpendicular to the plane is (§ 35)

$$\rho = abcS\alpha\beta\gamma \,/\, V(ab\alpha\beta + bc\beta\gamma + ca\gamma\alpha)$$
$$= S\alpha\beta\gamma \,/\, V\left(\frac{\alpha\beta}{c} + \frac{\beta\gamma}{a} + \frac{\gamma\alpha}{b}\right),$$

or

$$\frac{V\alpha\beta}{c} + \frac{V\beta\gamma}{a} + \frac{V\gamma\alpha}{b} = \rho^{-1}S\alpha\beta\gamma.$$

Operate by $S.\alpha$, $S.\beta$, $S.\gamma$, and we get

$$\frac{1}{a} = S\alpha\rho^{-1},\ \frac{1}{b} = S\beta\rho^{-1},\ \frac{1}{c} = S\gamma\rho^{-1};$$

whence

$$abcS\alpha\rho^{-1}S\beta\rho^{-1}S\gamma\rho^{-1} = 1,$$

or

$$CS\alpha\rho S\beta\rho S\gamma\rho = \rho^6,$$

the equation of the surface. If $\alpha\beta\gamma$ are perpendicular unit vectors, the Cartesian equation of this surface is easily seen to be

$$(x^2 + y^2 + z^2)^3 = Cxyz.$$

2. *To find the locus of a point such that the ratio of its distances from a given point and a given straight line is constant—all in one plane.*

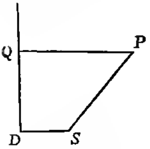

Fig. 25.

Let S be the given point, DQ the given straight line, $SP = ePQ$ the given relation.

Let vector $SD = \alpha$, $SP = \rho$, $DQ = y\gamma$, γ being the unit vector along DQ,

$$PQ = x\alpha;$$

then

$$T\rho = eT(PQ),$$

gives

$$\rho^2 = e^2PQ^2, \text{ where } PQ \text{ is a vector,}$$
$$= e^2(x\alpha)^2$$
$$= e^2x^2\alpha^2.$$

But

$$\rho + x\alpha = SQ = SD + DQ$$
$$= \alpha + y\gamma;$$

$$\therefore\ S\alpha\rho + x\alpha^2 = \alpha^2, \text{ for } S\alpha\gamma = 0;$$

and

$$x^2\alpha^4 = (\alpha^2 - S\alpha\rho)^2;$$

hence

$$\alpha^2\rho^2 = e^2(\alpha^2 - S\alpha\rho)^2,$$

a surface of the second order, whose intersection with the plane $S.\alpha\gamma\rho = 0$ is the required locus.

3. *The same problem when the points and line are not in the same plane.*

Retaining the same figure and notation, we see that PQ is no longer a multiple of α; but

$$PQ = SQ - SP$$
$$= \alpha + y\gamma - \rho;$$
$$\therefore \ \rho^2 = e^2(\alpha + y\gamma - \rho)^2,$$

and because PQ is perpendicular to DQ

$$S\gamma(\alpha + y\gamma - \rho) = 0;$$
$$\therefore \ (y\gamma^2, \text{ i.e.}) - y = S\gamma\rho,$$

and
$$\rho^2 = e^2(\alpha - \gamma S\gamma\rho - \rho)^2,$$

a surface of the second order.

COR. If $e = 1$, and the surface be cut by a plane perpendicular to DQ whose equation is $S\gamma\rho = c$, the equation of the section is

$$\alpha^2 + c^2 - 2S\alpha\rho = 0,$$

another plane, so that the section is a straight line.

4. *To find the locus of the middle points of lines of given length terminated by each of two given straight lines.*

Let AP, BQ be the given lines, AB the common perpendicular, and O its middle point. Let β, γ be unit vectors along AP, BQ, and let $OA = \alpha = -OB$. Then the vector ρ to the middle point R of PQ is given by the equation

$$2\rho = x\beta + \alpha + y\gamma - \alpha = x\beta + y\gamma \text{ ....}(1)$$

and
$$2RP = RP - RQ = 2\alpha + x\beta - y\gamma. \text{ ....}(2)$$

From equation (1), we have, since $\alpha \perp \beta$ and γ

$$S\alpha\rho = 0, \text{ ....}(22.\ 7)$$

and also
$$2S\beta\rho = -x + yS\beta\gamma,$$
$$2S\gamma\rho = xS\beta\gamma - y,$$

because β, γ are unit vectors.

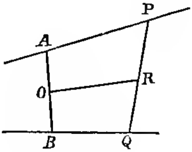

Fig. 26.

The first of these three equations shows that ρ lies in a plane through O perpendicular to AB.

The second and third equations give

$$x = \frac{2(S\beta\rho + S\beta\gamma S\gamma\rho)}{(S\beta\gamma)^2 - 1},$$

$$y = \frac{2(S\gamma\rho + S\beta\gamma S\beta\rho)}{(S\beta\gamma)^2 - 1}.$$

Now (2) gives, by squaring, if c = length QRP.

$$-4c^2 = 4\alpha^2 + x^2\beta^2 + y^2\gamma^2 - 2xyS\beta\gamma,$$

in which, if the values of x and y just obtained be substituted, there results an equation of the second order in ρ.

Hence the locus required is a plane curve of the second order, or a conic section, which by the very nature of the problem must be finite in extent, and therefore an ellipse.

38. EQUATIONS OF THE SPHERE AND CIRCLE. · The simplest equation of the sphere is $T(\rho - \alpha) = a$, a constant, where α is the vector of the centre of the sphere. This expresses the property of the radius of constant length.

Squaring both sides, we get

$$-(\rho - \alpha)^2 = T^2(\rho - \alpha) = a^2,$$

or

$$\rho^2 - 2S\alpha\rho + \alpha^2 + a^2 = 0,$$

which is the general scalar equation of the sphere.

When the origin lies on the surface, $T\alpha = a$, and

$$\alpha^2 + a^2 = -a^2 + a^2 = 0;$$

hence the equation becomes

$$\rho^2 - 2S\alpha\rho = 0,$$

or

$$S\rho(\rho - 2\alpha) = 0.$$

The immediate interpretation of the last form is that the vector ρ is perpendicular to the vector $\rho - 2\alpha$. But $\rho - 2\alpha$ is the vector joining the extremity of the diameter 2α with the point ρ. Consequently, every diameter subtends a right angle from every point on the sphere. In plane geometry, the angle in a semicircle is a right angle.

If we associate with the equation of the sphere the equation of any intersecting plane, we obtain the equations of the circle of intersection; and many properties established for the sphere will be true for the circle. From the general form of equation many of the ordinary properties are deduced with ease, such, for example, as the constancy of the product of the segments of a chord or secant drawn through a point, the perpendicularity of the plane (or line) of section of two intersecting spheres (or circles) to the line joining their centres, and so on. Some of these will be found in the examples at the end of the chapter.

39. TANGENT PLANES TO THE SPHERE, TANGENTS TO THE CIRCLE. The equation of the tangent plane at any point of a sphere is obtained at once if we assume that the tangent plane is perpendicular to the radius drawn to the point of contact. For this radius $(\rho-\alpha)$ is the normal to the plane, and therefore perpendicular to every line lying in the plane. If ϖ be the vector to any point in the plane, $\varpi-\rho$ must be perpendicular to $\rho-\alpha$. Hence

$$S(\varpi-\rho)(\rho-\alpha)=0,$$

which with the condition $T(\rho-\alpha)=\text{constant}$, represents the tangent plane.

Let us, however, derive the equation of the tangent plane directly from the definition that it is the plane determined by three contiguous points on the surface. That is, if we move along the surface through an infinitely short distance in any direction we move along the tangent plane.

Let τ be any such infinitely small arc on the surface drawn from the extremity of the vector ρ. Then ρ and $\rho+\tau$ both satisfy the equation of the sphere: in symbols

$$\left.\begin{aligned} \rho^2-2S\alpha\rho &= -a^2-\alpha^2, \\ (\rho+\tau)^2-2S\alpha(\rho+\tau) &= -a^2-\alpha^2. \end{aligned}\right\}$$

Subtracting, we find

$$(\rho+\tau)^2-\rho^2-2S\alpha\tau=0,$$

or, since

$$\beta^2-\gamma^2=S(\beta-\gamma)(\beta+\gamma),$$

$$S(\rho+\tau-\rho)(\rho+\tau+\rho)-2S\alpha\tau=0,$$

$$S\tau(2\rho-2\alpha+\tau)=0.$$

But in the limit, as τ is taken indefinitely small, squares of τ may be neglected. Hence

$$S\tau(\rho-\alpha)=0\,;$$

so that every tangent line is perpendicular to the radius at the points, and the tangent plane which contains all the tangent lines has the same property. The vector τ is a multiple of $\varpi-\rho$, where ϖ is vector to any point in τ produced. Hence the equation of the tangent plane at the point ρ is

$$S(\varpi-\rho)(\rho-\alpha)=0,$$

or

$$S\varpi(\rho-\alpha)=S\rho(\rho-\alpha)$$

$$=S\alpha\rho-a^2-\alpha^2$$

by the equation of the sphere.

When the origin is taken at the centre of the sphere, $\alpha=0$, and the equations of the sphere and of the tangent plane become

$$\rho^2=-a^2,$$

$$S\varpi\rho=\rho^2=-a^2.$$

When the origin is on the surface of the sphere, the equations of sphere and tangent plane are

$$\rho^2-2S\alpha\rho=0,$$

$$S\varpi(\rho-\alpha)=S\alpha\rho.$$

The perpendicular from the origin in the tangent plane must be parallel to $\rho-\alpha$. Let its value be $x(\rho-\alpha)$. Substituting this expression for ϖ, we find (§ 36)

$$\varpi=x(\rho-\alpha)=(\rho-\alpha)^{-1}S\alpha\rho,$$

or

$$\rho-\alpha=\varpi^{-1}S\alpha\rho.$$

If between this equation and the equation of the sphere we eliminate ρ, we get an equation in ϖ and α, which is the equation of the locus of the extremities of the perpendiculars from the origin upon tangent planes. The eliminations may be effected as follows:

Squaring the last equation, we have

$$\rho^2 - 2S\alpha\rho + \alpha^2 = \varpi^{-2}S^2\alpha\rho,$$

or

$$\alpha^2\varpi^2 = S^2\alpha\rho, \text{ since } \rho^2 - 2S\alpha\rho = 0.$$

Operating by $S.\alpha$, we find

$$S\alpha\rho - \alpha^2 = S\alpha\varpi^{-1}S\alpha\rho,$$

or

$$S\alpha\rho = \frac{\alpha^2}{1 - S\alpha\varpi^{-1}} = \frac{\alpha^2\varpi^2}{\varpi^2 - S\alpha\varpi}.$$

Hence

$$\alpha^2\varpi^2 = \left(\frac{\alpha^2\varpi^2}{\varpi^2 - S\alpha\varpi}\right)^2,$$

$$(\varpi^2 - S\alpha\varpi)^2 = \alpha^2\varpi^2.$$

This is a surface of revolution the section of which by a plane containing α has the polar equation

$$(r^2 - ar\cos\theta)^2 = a^2r^2$$

and the Cartesian equation

$$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2).$$

40. POLES AND POLAR PLANES. Referring to C the centre of the sphere, we have for the equations of the sphere and of the tangent plane at the point ρ,

$$\rho^2 = -a^2,$$

$$S\varpi\rho = -a^2.$$

If this tangent plane is to pass through a given point O (vector $CO = \gamma$), then

$$S\gamma\rho = -a^2.$$

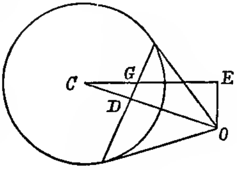

Fig. 27.

Now this is the equation of a plane perpendicular to γ; and the intersection of this plane with the sphere will give the circular line of contact of all tangent planes passing through O. The vector perpendicular on this plane is (§ 34) $-a^2\gamma^{-1}$. But $\gamma \times (-a^2\gamma^{-1}) = -a^2$, or $CD.CO = a^2$. The points O and D are what are called inverse points with regard to the sphere, the product of their distances from the centre being equal to the square of the radius.

The plane $S\gamma\rho = -a^2$ is called the *polar plane* of the point γ with reference to the sphere $T\rho = a$; and the point γ is the corresponding pole.

It is only when the point γ is outside the sphere that the polar plane is determined by the points of contact of all tangent planes passing through the point γ. When the point γ is inside the sphere, the plane $S\gamma\rho = -a^2$ has still a definite position, although that position can no longer be determined by drawing real tangent planes.

If we take any point δ on the polar plane of γ, the condition must be satisfied that $S\gamma\delta = -a^2$. But this may be written $S\delta\gamma = -a^2$ and δ regarded as constant. Then γ appears as a point on the polar plane of δ. The relation is a reciprocal one. If A is a point on the polar of B, B is a point on the polar of A.

Similar theorems hold for the circle if we substitute line or chord of contact for plane.

Again, let any point be chosen within the sphere, say G, vector δ, and let any plane GD be drawn through this point. If ν is the vector perpendicular CD, the equation of the plane is $S\nu(\rho - \nu) = 0$. Hence, since δ is a point on the plane,

$$S\nu\delta = \nu^2, \text{ or } S\nu^{-1}\delta = 1.$$

But the pole of this plane is at O, where

$$\varpi = \text{vector } CO = -a^2\nu^{-1}.$$

Hence, substituting, we find

$$S\varpi\delta = -a^2,$$

the equation of the polar plane of δ.

Thus the poles of all planes drawn through a given point lie on the polar plane of that point. Otherwise expressed, the vertices of all tangent cones whose lines of contact with the sphere lie in planes passing through a fixed point lie on a plane which is the polar of the given point.

41. INVERSION WITH REFERENCE TO THE UNIT SPHERE. If in any equation representing some curve or surface we substitute for ρ, the variable vector position, its reciprocal ρ^{-1}, we obtain the equation of the inverse curve or surface.

Beginning with the plane

$$S\nu\rho = -a,$$

let us put $\rho = \sigma^{-1}$. This gives

$$S\nu\sigma^{-1} = -a,$$

or

$$S\nu\sigma = -a\sigma^2,$$

or

$$\sigma^2 + 2S\frac{\nu}{2a}\sigma = 0,$$

which represents a sphere passing through the point of inversion, with centre at the point $-\nu/2a$.

It is interesting to note in passing that $S\rho\alpha^{-1} = 1$ represents a plane whose normal distance from the origin is α; and that $S\alpha\rho^{-1} = 1$ represents a sphere passing through the origin with diameter equal to $T\alpha$. Hence, the plane $S\rho\alpha^{-1} = 1$ is the tangent plane to the sphere $S\alpha\rho^{-1} = 1$, at the extremity of the diameter α.

If we invert the line $V.\rho\alpha = \gamma$, we get $V.\sigma^{-1}\alpha = \gamma$, which means in the first place that σ, as well as ρ, lies in the plane perpendicular to γ, so that the locus is a plane curve. Multiplying by σ^2, and operating by $S.\gamma^{-1}$, we find

$$S\gamma^{-1}\sigma\alpha = \sigma^2,$$

or

$$S\sigma(\sigma - \alpha\gamma^{-1}) = 0,$$

which is a sphere whose intersection with the plane $S\sigma\gamma = 0$ gives a circle. Thus $V.\sigma^{-1}\alpha = \gamma$ is the equation of a circle passing through the origin with its centre at the extremity of the vector $\frac{1}{2}\alpha\gamma^{-1}$. This may be thrown into the form $V.(\sigma^{-1} - \beta^{-1})\alpha = 0$, where β is the vector diameter of the circle drawn from the origin; for evidently $\sigma = \beta$ is one value of σ consistent with the equation.

Let us now invert the sphere

$$\rho^2 - 2S\alpha\rho = -a^2 - \alpha^2.$$

The result is

$$\sigma^{-2} - 2S\alpha\sigma^{-1} = -a^2 - \alpha^2,$$

or
$$(a^2+\alpha^2)\sigma^2 - 2S\alpha\sigma + 1 = 0,$$

or
$$\sigma^2 - 2S\frac{\alpha}{a^2+\alpha^2}\sigma = -\frac{1}{a^2+\alpha^2},$$

or
$$\left(\sigma - \frac{\alpha}{a^2+\alpha^2}\right)^2 = \frac{\alpha^2}{(a^2+\alpha^2)^2} - \frac{1}{a^2+\alpha^2} = -\frac{a^2}{(a^2+\alpha^2)^2},$$

the equation of a sphere the centre of which O' is at the extremity of $\alpha/(a^2+\alpha^2)$, and the radius of which is $a/(a^2+\alpha^2)$.

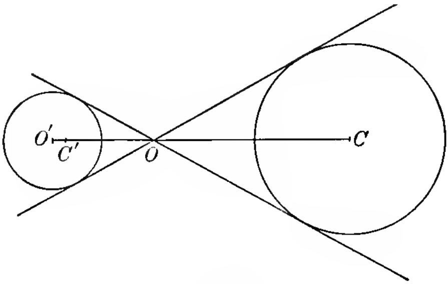

Fig. 28.

Let the origin be O, the centre of the original sphere C, the inversion of this centre C', and the centre of the inverted sphere O'.

The centre C of the original sphere inverts into position

$$\alpha^{-1} = \frac{\alpha}{\alpha^2}.$$

The vector distance $O'C'$ is

$$O'O - C'O = OC' - OO' = \frac{\alpha}{\alpha^2} - \frac{\alpha}{a^2+\alpha^2} = \alpha^{-1}\frac{a^2}{a^2+\alpha^2}.$$

Hence $O'C' \,.\, O'O = \dfrac{a^2}{(a^2+\alpha^2)^2} =$ square of radius of inverted sphere.

With reference to the inverted sphere, the origin and the inversion of the centre of the original sphere are inverse points.

42. EQUATION OF THE CONE OF CONTACT. If tangent lines be drawn from the surface of a sphere so as to pass through any point, these lines will lie on a right cone. It is required to find the equation of this cone.

With origin at the centre of the sphere, the two equations

$$\rho^2 = -a^2$$

and

$$S\gamma\rho = -a^2$$

determine the circle of contact of all tangent planes which can be drawn through the point γ. The cone with vertex at γ must pass through this circle. Shifting the origin to this point, the equations take the form

$$\left.\begin{aligned}(\gamma+\rho)^2 &= -a^2,\\ S\gamma(\gamma+\rho) &= -a^2,\end{aligned}\right\}$$

where ρ now means the radius vector from the vertex of the tangent cone. Subtracting, we get

$$S\rho(\gamma+\rho)=0,$$

or

$$S\rho\gamma = -\rho^2.$$

But also

$$S\rho\gamma = -a^2-\gamma^2.$$

Hence, multiplying these last two equations together, we find

$$S^2\rho\gamma = \rho^2(a^2+\gamma^2).$$

Here $T\rho$ may be divided out, so that *any length* of ρ in the proper direction satisfies the equation. It is therefore the equation of a cone referred to its vertex as origin. But when

$$S\rho\gamma = -\rho^2,\ S\rho\gamma \text{ also} = -a^2-\gamma^2.$$

Hence this cone must pass through the intersection of the sphere and plane represented by these equations. But we know that these equations determine the line of contact of all tangent planes through the origin. Hence the cone is the required tangent cone.

Since
$$\rho^2\gamma^2=\rho\gamma\gamma\rho$$
$$=(S\rho\gamma+V\rho\gamma)(S\rho\gamma-V\rho\gamma)$$
$$=S^2\rho\gamma-V^2\rho\gamma,$$
we may put the equation in the more concise form
$$a^2\rho^2-V^2\rho\gamma=0.$$

Returning to the equations $\rho^2=-a^2$, $S\gamma\rho=-a^2$, we obtain, by squaring the latter, and combining so as to have
$$a^2\rho^2+S^2\gamma\rho=0,$$
the equation of the cone which has its vertex at the centre of the sphere, and which cuts the sphere in the same circle. This cone is evidently at every point of section perpendicular to the tangent cone already found.

It is instructive to use this condition of orthogonality so as to obtain the equation of the one cone from that of the other.

The immediate interpretation of the first equation is obtained from the form
$$TV\gamma U\rho=a,$$
which means that the sine of the angle between the axis and any straight line drawn on the surface is equal to a a constant. For the cone which has the same axis and which cuts the surface of the first cone at right angles, the condition is evidently that the *cosine* of the angle between the axis and any straight line on its surface is equal to the same quantity a. Hence for its equation
$$-S\gamma U\rho=a,$$
$$-S\gamma\rho=aT\rho,$$
and $$S^2\gamma\rho=-a^2\rho^2,$$ the required equation.

The equation of the right cone might also be expressed in the form
$$\text{angle}\,\frac{\rho}{\gamma}=\text{constant}.$$

The equation of the right cylinder is written down at once from the condition that any triangle with its vertex on the surface and its base a definite length measured along the axis has the same area—in symbols

$$TV\rho\gamma = b,$$

which may be readily transformed into

$$V^2\rho\gamma + b^2 = 0,$$

$$S\,.\,\rho\gamma V\rho\gamma + b^2 = 0,$$

or

$$S^2\rho\gamma - \rho^2\gamma^2 + b^2 = 0.$$

Since the square of ρ is involved in the scalar equation, we recognize that we are dealing with a surface of the second degree.

EXAMPLES TO CHAPTER V.

1. Straight lines are drawn terminated by two given straight lines, to find the locus of a point in them whose distances from the extremities have a given ratio.

2. Two lines and a point S are given, not in one plane ; find the locus of a point P such that a perpendicular from it on one of the given lines intersects the other, and the portion of the perpendicular between the point of section and P bears to SP a constant ratio. Prove that the locus of P is a surface of the second order.

3. Prove that the section of this surface by a plane perpendicular to the line to which the generating lines are drawn perpendicular is a circle.

4. Prove that the locus of a point whose distances from two given straight lines have a constant ratio is a surface of the second order.

5. A straight line moves parallel to a fixed plane and is terminated by two given straight lines not in one plane ; find the locus of the point which divides the line into parts which have a constant ratio.

6. Required the locus of a point P such that the sum of the projections of OP on OA and OB is constant.

7. If the sum of the perpendiculars on two given planes from the point A is the same as the sum of the perpendiculars from B, this sum is the same for every point in the line AB.

8. If the sum of the perpendiculars on two given planes from each of three points A, B, C (not in the same straight line) be the same, this sum will remain the same for every point in the plane ABC.

9. A solid angle is contained by four plane angles. Through a given point in one of the edges to draw a plane so that the section shall be a parallelogram.

10. Through each of the edges of a tetrahedron a plane is drawn perpendicular to the opposite face. Prove that these planes pass through the same straight line.

11. ABC is a triangle formed by joining points in the rectangular coordinates OA, OB, OC; OD is perpendicular to ABC. Prove that the triangle AOB is a mean proportional beween the triangles ABC, ABD.

12. $V\alpha\rho V\beta\rho+(V\alpha\beta)^2=0$ is the equation of a hyperbola in ρ, the asymptotes being parallel to α, β.

13. If a plane be drawn through the points of bisection of two opposite edges of a tetrahedron it will bisect the tetrahedron.

14. Find the equation of the sphere circumscribing a given tetrahedron.

15. A straight line intersects a fixed line at right angles and turns uniformly about it while it slides uniformly along it. Find the equation of the surface described (1) when the fixed line is straight, (2) when it is a circle.

16. If two circles cut one another, and from one of the points of section diameters be drawn to both circles, their other extremities and the other point of section will be in a straight line.

17. If a chord be drawn parallel to the diameter of a circle, the radii to the points where it meets the circle make equal angles with the diameter.

18. The locus of a point from which two unequal circles subtend equal angles is a circle.

19. A line moves so that the sum of the perpendiculars on it from two given points in its plane is constant. Show that the locus of the middle point between the feet of the perpendiculars is a circle.

20. If O, O' be the centres of two circles, the circumference of the latter of which passes through O; then the point of intersection A of the circles being joined with O' and produced to meet the circles in C, D, we shall have

$$AC \, . \, AD=2AO^2.$$

21. If two circles touch one another in O, and two common chords be drawn through O at right angles to one another, the sum of their squares is equal to the square of the sum of the diameters of the circles.

22. A, B, C, are three points in the circumference of a circle; prove that if tangents at B and C meet in D, those at C and A in E, and those at A and B in F; then AD, BE, CF will meet in a point.

23. If A, B, C are three points in the circumference of a circle, prove that $V(AB \,.\, BC \,.\, CA)$ is a vector parallel to the tangent at A.

24. A straight line is drawn from a given point O to a point P on a given sphere: a point Q is taken in OP so that

$$OP \,.\, OQ = k^2.$$

Prove that the locus of Q is a sphere.

25. A point moves so that the ratio of its distances from two given points is constant. Prove that its locus is either a plane or a sphere.

26. A point moves so that the sum of the squares of its distances from a number of given points is constant. Prove that its locus is a sphere.

27. A sphere touches each of two given straight lines which do not meet; find the locus of its centre.

28. Any chord drawn from the point of intersection of two tangents to a circle are cut harmonically by the circle and the chord of contact.

29. If tangents be drawn at the angular points of a triangle inscribed in a circle, the intersections of these tangents with the opposite sides of the triangle lie in a straight line.

30. A fixed circle is cut by a number of circles, all of which pass through two given points, to prove that the lines of section of the fixed circle with each circle of the series all pass through a point whose distances from the two given points are proportional to the squares of the tangents drawn from these points to the fixed circle.

CHAPTER VI.

CONES AND THEIR SECTIONS.

43. THE CONE AND CYLINDER OF THE SECOND ORDER. In the preceding chapter the equation of the right cone was obtained, that is, the cone which is cut in a circle by a plane perpendicular to the axis. A more general case is when the perpendicular α to the plane of circular section is not parallel to the axis. Let the axis (γ) be cut by a plane perpendicular to α at a point $x\gamma$ from the vertex. The radius of the circle of section will be proportional to x. Let it be ax. Let ρ be the vector from the vertex of the cone to any point of the circumference of the circle.

Then the conditions are evidently

$$T(\rho - x\gamma) = ax, \text{ the circle of section ;}$$

$$S\alpha(\rho - x\gamma) = 0, \text{ the plane of section.}$$

These give

$$\rho^2 - 2xS\gamma\rho + x^2\gamma^2 = -a^2x^2,$$

$$S\alpha\rho = xS\alpha\gamma.$$

Eliminating x, we find

$$\rho^2S^2\alpha\gamma - 2S\alpha\rho S\gamma\rho S\alpha\gamma + (a^2 + \gamma^2)S^2\alpha\rho = 0,$$

the equation of a cone referred to the vertex, since any value of $T\rho$ satisfies the equation.

The equation of the cylinder is obtained in exactly the same way by using a instead of ax on the right-hand side of the first equation; for in the case of the cylinder the same size of circle is obtained at whatever distance from the chosen origin the plane of section is drawn. That is

$$T(\rho - x\gamma) = a,$$
$$S\alpha(\rho - x\gamma) = 0,$$

giving $$\rho^2 - 2xS\rho\gamma + x^2\gamma^2 = -a^2,$$

that is, substituting $S\alpha\rho / S\alpha\gamma$ for x,

$$\rho^2 S^2\alpha\gamma - 2S\rho\alpha S\rho\gamma S\alpha\gamma + \gamma^2 S^2\alpha\rho + a^2 S^2\alpha\gamma = 0,$$

in which $T\rho^2$ cannot be divided out.

By choice of appropriate vectors of reference these equations may be expressed in much simpler forms.

For instance, let a circular section of the cone be given by the intersection of the plane $S\beta\rho = -1$, with the sphere $S\alpha\rho^{-1} = 1$ passing through the origin. The product

$$\left.\begin{aligned} S\beta\rho S\alpha\rho^{-1} &= -1, \\ \text{or} \qquad \rho^2 + S\beta\rho S\alpha\rho &= 0, \end{aligned}\right\} \dots\dots\dots\dots(1)$$

represents a cone, since $T\rho$ may have any value, and this cone meets the plane $S\beta\rho = -1$ in the circle $S\rho(\rho - \alpha) = 0$.

But Equation (1) may be also thrown into the form

$$S\alpha\rho S\beta\rho^{-1} = -1,$$

so that the section with the plane $S\alpha\rho = -1$ is the circle

$$S\rho(\rho - \beta) = 0.$$

Hence α and β are normals to two sets of planes which cut the cone in circles. These circular or cyclic sections are known as the subcontrary sections; and the equation (1) above may be distinguished as the *cyclic* equation of the cone.

44. SPHERO-CONICS. The equation of the cone just given enables us to discuss with great elegance certain properties of the sphero-conics, that is, the curves of section of the cone with a sphere whose centre is at the vertex of the cone.

The condition is that $T\rho$ is constant. For simplicity, we may treat ρ as a unit vector; and the equation of the sphero-conic may be written in the various forms

$$1 = S\alpha\rho S\beta\rho = -S\alpha\rho^{-1}S\beta\rho = -S\alpha\rho S\beta\rho^{-1} = +S\alpha\rho^{-1}S\beta\rho^{-1}.$$

The equations $S\alpha\rho = 0$, $S\beta\rho = 0$ represent the so-called *cyclic planes.* They pass through the vertex of the cone, are parallel to the planes giving the subcontrary sections, intersect the unit sphere in two great circles or *cyclic arcs* which enclose the sphero-conic, and intersect each other along the line $V\alpha\beta$. $U\alpha$, $U\beta$ are vectors to the poles A, B of these cyclic arcs; and the arcs meet at the extremities of the diameter of the sphere which is parallel to $V\alpha\beta$.

Let P be the extremity of ρ on the sphere. Then $-S\rho U\alpha$ is the cosine of the arc AP, or the sine of PM, where PM

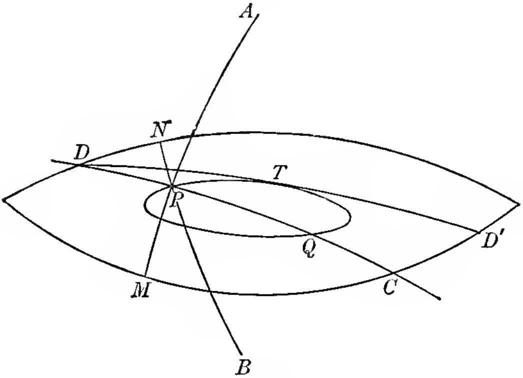

Fig. 29.

is the perpendicular drawn from P on the sphero-conic to the cyclic arc (α). Similarly, $-S\rho U\beta$ is the sine of the arc PN, the perpendicular on the other cyclic arc (β). But from the equation $S\alpha\rho S\beta\rho = 1$, we have

$$S\rho U\alpha S\rho U\beta = \frac{1}{T\alpha T\beta} = \text{constant}.$$

Hence $\sin PM \,.\, \sin PN = \text{constant}$, wherever P may be on the sphero-conic.

Let now the cyclic arcs and sphero-conic be intersected by any plane $S\gamma\rho = 0$ passing through the origin. $UV\alpha\gamma$, $UV\beta\gamma$

will be vectors to the points of intersection of this plane with the cyclic arcs, say C, D, in the figure; and let P, Q be the intersection of the same plane with the sphero-conic. Let ρ represent either of these points. We wish to compare the segments into which DC is divided by the sphero-conic. Now

$$V.\rho V\alpha\gamma = \gamma S\alpha\rho - \alpha S\gamma\rho = \gamma S\alpha\rho \text{ (for } S\gamma\rho = 0),$$

and $$V.\rho V\beta\gamma = \gamma S\beta\rho - \beta S\gamma\rho = \gamma S\beta\rho.$$

Hence $$V\rho V\alpha\gamma . V\rho V\beta\gamma = \gamma^2 S\alpha\rho S\beta\rho = -1,$$

or $$V\rho UV\alpha\gamma . V\rho UV\beta\gamma = -\frac{1}{TV\alpha\gamma TV\beta\gamma}.$$

This gives

$$\sin DP \sin PC = \sin DQ \sin QC = \frac{1}{\sin PCM . \sin PDN}.$$

But if $\sin L \sin (M+N) = \sin (L+M) \sin N$, of necessity $L = N$. Hence the intercepts DP and QC are equal.

As a corollary, if DD' be the points where the cyclic arcs are met by a tangent arc to the sphero-conic, that is, by the section of the sphere by a tangent plane to the cone, the point of contact T bisects the arc DD'.

45. TANGENT PLANE TO CONE OR CYLINDER; THE LINEAR VECTOR FUNCTION. To find the equation of the tangent plane to a cone, we proceed as in the case of the sphere. That is, we pass to a contiguous point $\rho + \tau$, and consider what the final form of the equation is as τ is taken indefinitely small. We have

$$\rho^2 + S\alpha\rho S\beta\rho = 0,$$

and $$(\rho+\tau)^2 + S\alpha(\rho+\tau) S\beta(\rho+\tau) = 0.$$

Expanding, subtracting, and neglecting terms involving τ twice, we find

$$2S\rho\tau + S\alpha\tau S\beta\rho + S\beta\tau S\alpha\rho = 0.$$

If we put $\varpi - \rho = x\tau$, ϖ is a vector to a point in the tangent plane.

This gives $S(\varpi - \rho)(2\rho + \alpha S\beta\rho + \beta S\alpha\rho) = 0$,

or $$S\varpi(2\rho + \alpha S\beta\rho + \beta S\alpha\rho) = 0,$$

since by the equation of the surface
$$S\rho(2\rho + \alpha S\beta\rho + \beta S\alpha\rho) = 0.$$

Half the expression in the bracket, namely,
$$\rho + \frac{\alpha S\beta\rho + \beta S\alpha\rho}{2},$$
involves ρ once in every term. It is a special case of what is known as the linear vector function of ρ. It is usual to represent it by the notation $\phi\rho$, where ϕ is a linear operator. In the present case
$$\phi\rho = \rho + \frac{\alpha S\beta\rho + \beta S\alpha\rho}{2},$$
and for another vector σ,
$$\phi\sigma = \sigma + \frac{\alpha S\beta\sigma + \beta S\alpha\sigma}{2}.$$

By addition we find
$$\begin{aligned}\phi\rho + \phi\sigma &= \rho + \sigma + \frac{\alpha S\beta(\rho + \sigma) + \beta S\alpha(\rho + \sigma)}{2} \\ &= \phi(\rho + \sigma),\end{aligned}$$
so that ϕ is distributive. In particular, $\phi(x\rho) = x\phi\rho$.

If we form the expressions $\sigma\phi\rho$ and $\rho\phi\sigma$, and take the scalar parts, we find
$$\begin{aligned}S\,.\,\sigma\phi\rho &= S\sigma\rho + \frac{S\alpha\sigma S\beta\rho + S\beta\sigma S\alpha\rho}{2} \\ &= S\,.\,\rho\phi\sigma,\end{aligned}$$
so that in this case ρ and σ may be interchanged without affecting the value of the expression. When this can be done for any linear vector function ϕ, ϕ is said to be self-conjugate. That this self-conjugate character is not a necessary property of the linear vector function may be seen at once by considering the very simple case $\phi\rho = \alpha S\beta\rho$, in which
$$S\sigma\phi\rho = S\alpha\sigma S\beta\rho = S\rho(\beta S\alpha\sigma) = S\rho\phi'\sigma, \text{ say.}$$
Evidently $\phi'\sigma = \beta S\alpha\sigma$ is not in general the same as $\phi\sigma = \alpha S\beta\sigma$. (See Chapter X.)

With $$\phi\rho = \rho + \frac{\alpha S\beta\rho + \beta S\alpha\rho}{2},$$
we have for the equation of the cone
$$S\rho\phi\rho = 0,$$
and for the equation of the tangent plane at any point
$$S\varpi\phi\rho = 0.$$

Since $S(\varpi - \rho)\phi\rho = 0$, we learn that $\phi\rho$ is perpendicular to the tangent plane. Thus $\phi\rho$ is the normal to the cone at the point ρ.

If we take the equation of the cylinder obtained in § 43, namely, $$\rho^2 - \frac{2S\alpha\rho S\gamma\rho}{S\alpha\gamma} + \frac{\gamma^2 S\alpha\rho S\alpha\rho}{S^2\alpha\gamma} + a^2 = 0,$$
and put $$\psi\rho = \rho - \frac{\alpha S\gamma\rho + \gamma S\alpha\rho}{S\alpha\gamma} + \frac{\gamma^2\alpha S\alpha\rho}{S^2\alpha\gamma}.$$
we see that $\psi\rho$ is like $\phi\rho$, a self-conjugate linear vector function of ρ, and that the equation of the surface is
$$S\rho\psi\rho + a^2 = 0.$$

Passing to a contiguous point $\rho + \tau$, we have
$$S(\rho + \tau)\psi(\rho + \tau) + a^2 = 0.$$

Expanding, subtracting, and neglecting squares of τ, we find
$$S\tau\psi\rho = 0.$$

Hence, for the equation of the tangents,
$$S(\varpi - \rho)\psi\rho = 0,$$
or $$S\varpi\psi\rho = S\rho\psi\rho = -a^2.$$

Here also, then, the equation of the tangent plane is obtained from the equation of the surface by substituting ϖ for one of the ρ's.

The sole conditions attached to ϕ and ψ are that they are *linear* vector functions of the variables and that they are *self-conjugate.* When we meet with any equation of the form
$$S\rho\phi\rho = -a^2,$$
which, being a scalar equation of the second degree in $T\rho$, must represent a surface of the second order; and if we know

that ϕ is self-conjugate *—then we may at once write down the equation of the tangent plane at the point ρ by simply changing one ρ into ϖ, the vector to the tangent plane, namely,

$$S\varpi\phi\rho = S\rho\phi\varpi = -a^2.$$

Also $\phi\rho$ must be a vector perpendicular to this plane, that is the vector normal to the surface at the point ρ.

46. THE CONIC SECTIONS. If we intersect the cone

$$\rho^2 + S\alpha\rho S\beta\rho = 0$$

by any plane $S\gamma\rho = -e$, we get one of the so-called conic sections, the circle, ellipse, parabola, or hyperbola, as the case may be.

It is usually more convenient, however, to develop the properties of each of these plane curves from some particular simple property which has no explicit relation to a cone, and which is then regarded as the definition of the curve.

A few examples of this mode of treatment will suffice to shew how simply and directly quaternion analysis may be applied to the geometry of the conic sections.

Thus, if we define a conic section as "the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line," we find the equation to be (§ 37, Ex. 2)

$$\alpha^2\rho^2 = e^2(\alpha^2 - S\alpha\rho)^2, \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

where $SP = ePQ$, vector $SD = \alpha$, $SP = \rho$

The nature of the curve depends upon the value of e, being and ellipse, parabola, or hyperbola, according as e is less than unity, equal to unity, or greater than unity.

Confining our attention to the ellipse ($e < 1$), let us find the values of ρ parallel to the axis α. Let SA be the required

---

*It may be mentioned, however, that even if we begin with a form ϕ which is not self-conjugate, it is its self conjugate part only which appears in the expression $S\rho\phi\rho$ (see Chapter X.).

value equal to $x\alpha$; then, by equation (1), putting $x\alpha$ for ρ, we get

$$x^2 = e^2(1-x)^2;$$

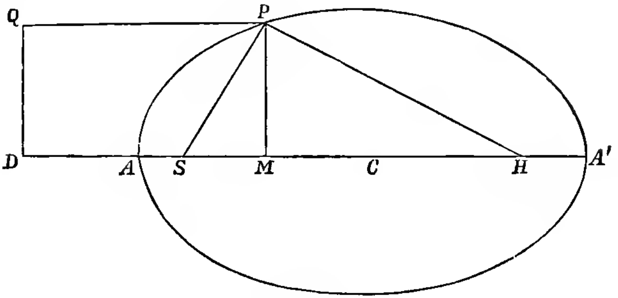

Fig. 30.

$$\therefore\ x = \frac{e}{1+e}, \text{ or } x = -\frac{e}{1-e}.$$

There are two values of x, one positive and the other negative. Therefore in addition to the point A, there is another point A' satisfying the same condition.

Thus $$SA = \frac{e}{1+e}SD,\ SA' = \frac{e}{1-e}SD;$$

$$\therefore\ AA' = \frac{2e}{1-e^2}SD = 2a,$$

the major axis of the ellipse.

If C be the centre of the ellipse

$$CS = SA' - CA' = \left(\frac{e}{1-e} - \frac{e}{1-e^2}\right)SD = \frac{e^2}{1-e^2}SD = ae.$$

The vector CS has the value $\frac{e^2}{1-e^2}\alpha = \alpha'$. To transfer the origin to the centre C, we must substitute for ρ in (1) the value $\rho = \rho' -$ vector $CS = \rho' - \alpha'$, and there results

$$a^2\rho'^2 + (S\alpha'\rho')^2 = -a^4(1-e^2),$$

which we may now write, CS being α and CP ρ,

$$a^2\rho^2 + (S\alpha\rho)^2 = -a^4(1-e^2). \quad \text{..................(2)}$$

This equation might have been obtained at once by referring the ellipse to the two foci, namely,

$$SP + HP = 2a,$$

or in vectors, if

$$CP = \rho, \quad CS = \alpha,$$

$$T(\rho + \alpha) + T(\rho - \alpha) = 2a;$$

i.e. $$\sqrt{-(\rho + \alpha)^2} + \sqrt{-(\rho - \alpha)^2} = 2a;$$

hence, squaring,

$$a\sqrt{-(\rho - \alpha)^2} = a^2 + S\alpha\rho;$$

i.e. $$a^2\rho^2 + (S\alpha\rho)^2 = -a^2(1 - e^2)$$

If now we write $\phi\rho$ for $+\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1 - e^2)}$, where $\phi\rho$ is a vector which coincides with ρ only in the cases in which either α coincides with ρ or when $S\alpha\rho = 0$, *i.e.* in the cases of the principal axes; the equation of the ellipse becomes

$$S\rho\phi\rho = -1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The same equation is, of course, applicable to the hyperbola, e being greater than 1.

It is evident that ϕ is of the same type of function as that already discussed in last section. It is distributive and self-conjugate; and many of the properties of the ellipse and hyperbola can be deduced with ease by its means.

The method is identical with that which will be used in the discussion of the more general properties of the ellipsoid and hyperboloid; and a few examples will suffice to show its power.

47. TANGENTS AND NORMALS. From demonstrations already given, we may at once write the equation of the tangent line to the ellipse (or hyperbola)

$$S\rho\phi\rho = -1$$

in the form $$S\varpi\phi\rho = -1,$$

where we must remember that ρ, $\phi\rho$, ϖ are for the present restricted class of problem all in one plane.

$\phi\rho$ is perpendicular to the tangent at ρ, that is, it is parallel to the normal at the point ρ.

The equation $V\rho\phi\rho=0$ means that ρ is parallel to $\phi\rho$, or the radius vector and normal are coincident. Given the function ϕ, we may set before us the problem to find the value or values of ρ which satisfy this equation. In the case of the ellipse and hyperbola there are *four* points but only *two* distances at which this condition is satisfied, namely, the extremities of the major and minor axes.

48. CARTESIAN EQUIVALENTS OF THE SCALAR EQUATIONS IN Φ. It is important at times to translate quaternion equations into their Cartesian equivalents; let us form from the equations just given the ordinary Cartesian equations of the ellipse and its tangent.

Let $CM=x$, $MP=y$ as usual (see Fig. 30, p. 97); then, taking i, j as unit vectors parallel and perpendicular respectively to CA, we have,

$$\text{vector } CM=xi,\ MP=yj,\ CS=aei;$$

$$\therefore\ \rho=xi+yj,$$

$$\phi\rho=+\frac{a^2\rho+\alpha S\alpha\rho}{a^4(1-e^2)}$$

$$=+\frac{a^2(1-e^2)xi+a^2yj}{a^4(1-e^2)}$$

$$=+\left(\frac{xi}{a^2}+\frac{yj}{b^2}\right),$$

where $\qquad b^2=a^2(1-e^2)=\text{square of semi-minor axis.}$

Hence $\quad S\rho\phi\rho=+S(xi+yj)\left(\frac{xi}{a^2}+\frac{yj}{b^2}\right)=-\frac{x^2}{a^2}-\frac{y^2}{b^2};$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the Cartesian form of $S\rho\phi\rho=-1$.

Again, if x', y' be the coordinates of T, a point in the tangent,

$$\varpi=x'i+y'j,$$

and
$$-S\varpi\phi\rho = -S(x'i+y'j)\left(\frac{xi}{a^2}+\frac{yj}{b^2}\right) = \frac{xx'}{a^2}+\frac{yy'}{b^2};$$
$$\therefore\ \frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$
the equation of the tangent.

49. POWERS OF THE LINEAR VECTOR FUNCTION. The values of ρ and $\phi\rho$ exhibited in the last article, viz.:
$$\rho = xi+yj,\ \ \phi\rho = +\left(\frac{xi}{a^2}+\frac{yj}{b^2}\right), \dots\dots\dots (1)$$
enable us to write
$$\phi\rho = -\left(\frac{iSi\rho}{a^2}+\frac{jSj\rho}{b^2}\right). \dots\dots\dots (2)$$

Applying the operation a second time, we have
$$\phi^2\rho = \phi\phi\rho = -\left(\frac{iSi\phi\rho}{a^2}+\frac{jSj\phi\rho}{b^2}\right)$$
$$= -\left(\frac{iSi\rho}{a^4}+\frac{jSj\rho}{b^4}\right). \dots\dots\dots (3)$$

Also it is easily verified that the inverse function
$$\phi^{-1}\rho = -a^2iSi\rho - b^2jSj\rho. \dots\dots\dots (4)$$
For, if so,
$$\rho = -a^2\phi iSi\rho - b^2\phi jSj\rho$$
$$= -iSi\rho - jSj\rho = xi+yj.$$

If, further, we write
$$\psi\rho = -\left(\frac{iSi\rho}{a}+\frac{jSj\rho}{b}\right), \dots\dots\dots (5)$$
we shall have
$$\psi^2\rho = \psi\psi\rho = -\left(\frac{iSi\rho}{a^2}+\frac{jSj\rho}{b^2}\right)$$
$$= \phi\rho. \dots\dots\dots (6)$$

Thus the operator ψ may be regarded as the square-root of the operator ϕ.

Also,
$$\psi^{-1}\rho = -(aiSi\rho + bjSj\rho),$$
$$\rho = \psi^{-1}\psi\rho$$
$$= -(aiSi\psi\rho + bjSj\psi\rho) \dots\dots\dots (7)$$

It is evident that the properties of $\phi\rho$ (p. 95) are possessed by all these functions.

Now $$S\rho\phi\rho = -1$$
gives $$S\rho\psi(\psi\rho) = -1.$$
But since $$S\rho\psi\sigma = S\sigma\psi\rho,$$
this becomes $$S\psi\rho\psi\rho = (\psi\rho)^2 = -1,$$
or $$T\psi\rho = 1\,;$$
which shows (1) that $\psi\rho$ is a unit vector; (2) that the equation of the ellipse may be expressed in the *form* of the equation of a circle, the vector which represents the radius being itself of variable length, deformed by the function ψ.

Lastly, when $\alpha\ \beta$ are such as to make
$$S\alpha\phi\beta = 0,$$
we have $$S\alpha\psi^2\beta = S\psi\alpha\psi = 0\,;$$
therefore $\psi\alpha$, $\psi\beta$ are vectors at right angles to one another.

This may be exhibited without use of the ψ, thus
$$S\,.\,\alpha\phi\beta = S\,.\,\alpha\phi^{\frac{1}{2}}\phi^{\frac{1}{2}}\beta = S\,.\,\phi^{\frac{1}{2}}\alpha\phi^{\frac{1}{2}}\beta = 0.$$

50. CONJUGATE DIAMETERS: PARALLEL CHORDS. The equation $S\,.\,\alpha\phi\beta = 0$ in the case of the ellipse or hyperbola means that α (or β) is perpendicular to the normal at the point on the curve whose vector is parallel to β (or α). In short, α is parallel to the tangent line at the point where β meets the curve; and β is parallel to the tangent line at the point where α meets the curve. Two diameters which have this property that either is parallel to the tangents at the extremities of the other are called *conjugate diameters.*

Let $\alpha\ \beta$ be two conjugate radii, so that
$$S\alpha\phi\beta = S\beta\phi\alpha = 0.$$

Any vector to the ellipse (or hyperbola) may be represented by $$\rho = x\alpha + y\beta,$$
hence $$\phi\rho = x\phi\alpha + y\phi\beta,$$
and
$$\begin{aligned} -1 &= S\rho\phi\rho \\ &= x^2 S\alpha\phi\alpha + y^2 S\beta\phi\beta \\ &= -x^2 - y^2 = -\frac{CN^2}{CP^2} - \frac{NQ^2}{CD^2}. \end{aligned}$$ (Fig 31, p. 103)

For each value of x, there correspond two equal and opposite values of y; and for each value of y there correspond two equal and opposite values of x.

Hence each diameter bisects chords parallel to its conjugate.

The converse that the locus of the middle points of parallel chords is the diameter conjugate to the diameter parallel to these chords is easily proved.

Let CP, CD be the conjugate semi-diameters α, β; and let DC be produced to meet the ellipse again in D'; then vector $DP = \alpha - \beta$, vector $D'P = \alpha + \beta$; and

$$\begin{aligned} S(\alpha+\beta)\phi(\alpha-\beta) &= S(\alpha+\beta)(\phi\alpha - \phi\beta) \\ &= S\alpha\phi\alpha - S\beta\phi\beta - S\alpha\phi\beta + S\beta\phi\alpha \\ &= 0, \end{aligned}$$

because $S\alpha\phi\alpha$ and $S\beta\phi\beta$ are equal quantities.

Therefore $\alpha+\beta$, $\alpha-\beta$ are parallel to conjugate diameters. This is the property of *Supplemental Chords.*

51. POLES AND POLARS. From any one point two tangents can be drawn to the curve.

Let ϖ be the vector CT, and ρ_1 ρ_2 vectors to the points of contact Q, R of the two tangents.

Then by the equation of the tangent

$$S\varpi\phi\rho_1 = -1,$$
$$S\varpi\phi\rho_2 = -1.$$

Hence $$S\varpi\phi(\rho_1 - \rho_2) = 0 = S(\rho_1 - \rho_2)\phi\varpi,$$

or the chord of contact $\rho_1 - \rho_2$ is perpendicular to the normal at the point where ϖ meets the curve. In the figure CT, QR are parallel to conjugate diameters.

The equation $$S\rho\phi\varpi = -1,$$

ϖ being constant, is (under present limitations) the equation of a straight line. ρ_1 ρ_2 are two values of ρ satisfying the equation. Hence the equation is that of the straight line passing through the points of contact.

The point N where this straight line cuts CT is found by putting

$$\rho = x\varpi,$$

whence

$$xS\varpi\phi\varpi = -1.$$

If $CP = y\varpi$, the equation of the curve gives

$$y^2 S\varpi\phi\varpi = -1,$$

whence

$$x = y^2,$$

or

$$CN \,.\, CT = CP^2$$

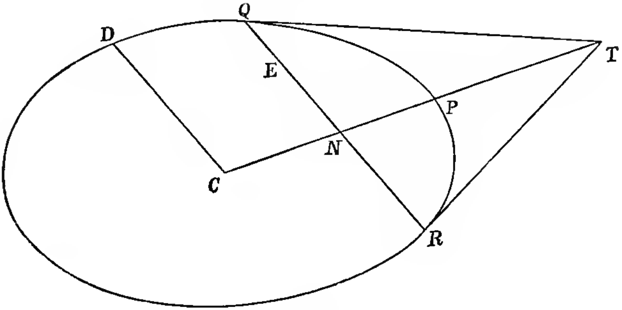

Fig. 31.

Suppose now that any chord such as QR is drawn through a fixed point E, vector σ, and that in all positions of this chord the meeting point T of the tangents at Q and R is found.

Then since

$$S\sigma\phi\varpi = -1,$$

we have also

$$S\varpi\phi\sigma = -1.$$

Hence with σ a fixed vector ϖ describes a straight line.

The point σ and the straight line $S\varpi\phi\sigma = -1$ are pole and polar; and reciprocally the points ϖ and the straight line $S\sigma\phi\varpi = -1$ are pole and polar. Thus, if A is a point in the polar of B, B is a point in the polar of A.

52. All parallelograms circumscribing an ellipse are equal in area.

These parallelograms are evidently bounded by lines parallel to conjugate diameters.

Let, as before, α β be two conjugate vector radii, and let us estimate the vector-area $V\alpha\beta$ of the parallelogram contained by them. This will evidently be one-fourth of the area of the circumscribing parallelogram.

One measure of the area $TV\alpha\beta$ will be the product of the length of α into the perpendicular from the centre on the tangent parallel to α. This perpendicular is parallel to $\phi\beta$, since $S\alpha\phi\beta=0$. Let its value be $x\phi\beta$. Then by the equation of the tangent we have

$$Sx\phi\beta\phi\beta=-1, \text{ or } x\phi\beta=-\frac{1}{\phi\beta}.$$

Consequently the length of the perpendicular on the tangent at the extremity of β is $1/T\phi\beta$.

Hence $$TV\alpha\beta=\frac{T\alpha}{T\phi\beta}=T\frac{\alpha}{\phi\beta},$$

and since $$UV\alpha\beta=U\frac{\alpha}{\phi\beta}=-U\frac{\beta}{\phi\alpha}$$

all being in one plane, we have finally

$$V\alpha\beta=\frac{\alpha}{\phi\beta}=-\frac{\beta}{\phi\alpha}.$$

To show that $V\alpha\beta$ is the same whatever pair of conjugate vector radii be chosen, let us express any other pair α', β' in terms of α, β.

We have already shown (§ 50) that if

$$\alpha'=x\alpha+y\beta,$$

we must have $$x^2+y^2=1.$$

Hence we may write

$$\alpha'=x\alpha\pm\sqrt{1-x^2}\beta.$$

Similarly, $$\beta'=x_1\alpha\pm\sqrt{1-x_1{}^2}\beta.$$

But $$S\beta'\phi\alpha'=0,$$

or $$xx_1S\alpha\phi\alpha\pm\sqrt{1-x^2 . 1-x_1{}^2}S\beta\phi\beta=0,$$

or $$xx_1=\pm\sqrt{1-x^2 . 1-x_1{}^2},$$

or $$0=1-x^2-x_1{}^2.$$

Hence $$x_1=\pm\sqrt{1-x^2}.$$

Bearing in mind the positions of any two self-conjugate diameters relatively to any other pair, we see that

$$\left.\begin{aligned}\alpha' &= x\alpha + \sqrt{1-x^2}\beta,\\ \beta' &= -\sqrt{1-x^2}\alpha + x\beta,\end{aligned}\right\}$$

represent a pair of self-conjugate radii. Hence

$$\begin{aligned}V\alpha'\beta' &= -(1-x^2)V\beta\alpha + x^2V\alpha\beta\\ &= (1-x^2)V\alpha\beta + x^2V\alpha\beta\\ &= V\alpha\beta.\end{aligned}$$

CP, CD are conjugate semi-diameters of an ellipse, as also CP′, CD′; PP′, DD′ are joined; to prove that the area of the triangle PCP′ equals that of the triangle DCD′.

Let α, β, α', β' be the vectors CP, CD, CP', CD'; k a unit vector perpendicular to the plane of the ellipse.

Since

$$\alpha = \psi^{-1}\psi\alpha = -(aiSi\psi\alpha + bjSj\psi\alpha), \text{ etc., etc.,} \qquad (\S\ 49.\ 7)$$

therefore

$$\begin{aligned}V\alpha\alpha' &= V(aiSi\psi\alpha + bjSj\psi\alpha)(aiSi\psi\alpha' + bjSj\psi\alpha')\\ &= abk(Si\psi\alpha Sj\psi\alpha' - Sj\psi\alpha Si\psi\alpha')\\ &= abkSi(\psi\alpha Sj\psi\alpha' - \psi\alpha' Sj\psi\alpha)\\ &= abkS.ijV\psi\alpha'\psi\alpha\\ &= -abkS.kV(\psi\alpha\psi\alpha').\end{aligned}$$

Similarly, $V\beta\beta' = -abkS.kV(\psi\beta\psi\beta')$.

Now $\psi\alpha$, $\psi\beta$ are unit vectors at right angles to one another; as are also $\psi\alpha'$, $\psi\beta'$; therefore the angle between $\psi\alpha$ and $\psi\alpha'$ is the same as that between $\psi\beta$ and $\psi\beta'$.

Hence $$S.kV(\psi\alpha\psi\alpha') = S.kV(\psi\beta\psi\beta'),$$

and $$V\alpha\alpha' = V\beta\beta',$$

i.e. area of PCP' = that of triangle DCD'.

We end this section with a few examples:

1. *The product of the perpendiculars from the foci on the tangent is equal to the square of the semi-axis minor.*

We have SY the vector perpendicular $=x\phi\rho$, and as Y is a point in the tangent, and

$$CY = CS + SY = \alpha + x\phi\rho,$$

$$S(\alpha + x\phi\rho)\phi\rho = -1,$$

$$x(\phi\rho)^2 = -1 - S\alpha\phi\rho,$$

$$-SY = -x\phi\rho = \frac{1 + S\alpha\phi\rho}{\phi\rho}.$$

Similarly, the perpendicular from the other focus H is the parallel vector

$$-HZ = \frac{1 - S\alpha\phi\rho}{\phi\rho};$$

$$\therefore\ SY.HZ = \frac{1 - S^2\alpha\phi\rho}{(\phi\rho)^2}.$$

Now (§ 46)

$$a^2\rho^2 = -S^2\alpha\rho - a^4(1 - e^2),$$

$$\phi\rho = +\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1 - e^2)};$$

$$\therefore\ (\phi\rho)^2 = \frac{S^2\alpha\rho - a^4}{a^6(1 - e^2)},$$

$$1 - S^2\alpha\phi\rho = \frac{a^4 - S^2\alpha\rho}{a^4};$$

$$\therefore\ SY.HZ = a^2(1 - e^2) = b^2.$$

2. *The perpendicular from the focus on the tangent intersects the tangent in the circumference of the circle described about the axis major.*

Retaining the notation of the last example, we have

$$CY = \alpha + x\phi\rho$$

$$= \alpha - \frac{\phi\rho(1 - S\alpha\phi\rho)}{(\phi\rho)^2};$$

and the square of this is easily found to be equal to $-a^2$.

3. *To find the locus of T when the perpendicular from the centre on the chord of contact is constant.*

If CT be π, the equation of QR, the chord of contact, is

$$S\sigma\phi\pi = -1,$$

and the vector perpendicular is $-\dfrac{1}{\phi\pi}$;

$$\therefore\ (\phi\pi)^2 = -c^2,$$

or

$$S\phi\pi\,.\,\phi\pi = -c^2,$$

or

$$S\pi\phi\phi\pi = -c^2;$$

the equation of an ellipse whose Cartesian equation is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = c^2.$$

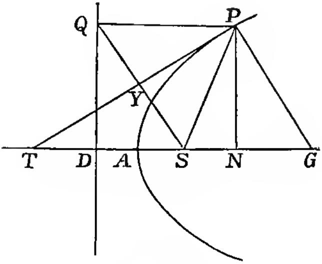

Fig. 32.

53. THE PARABOLA. Certain properties of the parabola may be discussed by the method employed for the ellipse.

Thus if S be the focus of a parabola, DQ the directrix, we have $SP = PQ$, $SA = AD = a$.

If $SP = \rho$, $SD = \alpha$, we have (§ 37, Ex. 2)

$$\alpha^2\rho^2 = (\alpha^2 - S\alpha\rho)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If

$$\phi\rho = \frac{\rho - \alpha^{-1}S\alpha\rho}{\alpha^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

to which the properties of $\phi\rho$ in § 45 evidently apply, the equation becomes $\quad S\rho(\phi\rho + 2\alpha^{-1}) = 1.$ ....(3)

If ρ' be another point in the parabola, $\rho' - \rho = \beta$, the limit to which β approaches is a vector along the tangent; so that if $x\beta = \pi - \rho$, π is the vector to a point in the tangent; this gives $\quad S(\pi - \rho)(\phi\rho + \alpha^{-1}) = 0;$ ....(4)

hence the equation of the tangent becomes

$$S\pi(\phi\rho + \alpha^{-1}) + S\alpha^{-1}\rho = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From (2) it is evident that

$$S\alpha\phi\rho = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

so that $\phi\rho$ is a vector perpendicular to the axis.

From the same equation

$$S\rho\phi\rho = \frac{\rho^2 - \alpha^{-2}S^2\alpha\rho}{\alpha^2}$$

$$= \frac{(\rho - \alpha^{-1}S\alpha\rho)^2}{\alpha^2}$$

$$= \alpha^2(\phi\rho)^2. \text{ ....} (7)$$

From (4) the normal vector is

$$\phi\rho + \alpha^{-1}; \text{ ....} (8)$$

therefore the equation of the normal is

$$\sigma = \rho + x(\phi\rho + \alpha^{-1}). \text{ ....} (9)$$

Equation (2) when exhibited as

$$\alpha^2\phi\rho = \rho - \alpha^{-1}S\alpha\rho,$$

reads by (6), 'vector along $NP = SP -$ vector along AN,' which requires that

$$NP = \alpha^2\phi\rho, \text{ ....} (10)$$

$$SN = \alpha^{-1}S\alpha\rho;$$

i.e.

$$= \alpha S\alpha^{-1}\rho. \text{ ....} (11)$$

For the subtangent AT, put $x\alpha$ for π in (5), and there results by (6)

$$x + S\alpha^{-1}\rho = 1,$$

whence

$$\left(x - \frac{1}{2}\right)\alpha = \frac{1}{2}\alpha - \alpha S\alpha^{-1}\rho;$$

i.e. vector $AT = -$ vector AN (by 11);

$\therefore$ line $AT = AN$;

By similar processes we may easily prove that $ST = SP$, $NG = SD$, vector $SD =$ vector GP, vector $AY = \frac{1}{2}$ vector NP.

The following examples may also be worked out by means of Equation (3), namely:

(*a*) Find the locus of the middle points of parallel chords.

(*b*) Find the locus of the point which divides a system of parallel chords into segments whose product is constant.

(*c*) Find the locus of the point in which the perpendicular from A on the tangent at P meets the line PQ produced.

(*d*) Find the locus of the intersection with the tangent of the perpendicular on it from the vertex.

To solve this last problem let π be the vector perpendicular on the tangent from *A*, then by (8)

$$\pi = x(\phi\rho + \alpha^{-1}),$$

and the equation of the tangent gives, putting $\pi + \frac{\alpha}{2}$ in place of π in (5) and multiplying by 2,

$$2S\pi\phi\rho + 2S\alpha^{-1}\pi + 2S\alpha^{-1}\rho = 1,$$

we have also

$$S\rho(\phi\rho + 2\alpha^{-1}) = 1.$$

From these three equations we have to eliminate x and ρ. The first gives

$$S\alpha\pi = x,$$

which gives x,

and
$$S\pi\phi\rho = x(\phi\rho)^2,$$

which substituted in the second gives

$$2x(\phi\rho)^2 + 2S\alpha^{-1}\pi + 2S\alpha^{-1}\rho = 1.$$

Also, substituting $\alpha^2(\phi\rho)^2$ for $S\rho\phi\rho$ (equation 7), the third equation gives

$$\alpha^2(\phi\rho)^2 + 2S\alpha^{-1}\rho = 1\,;$$

therefore by subtraction

$$(2x - \alpha^2)(\phi\rho)^2 + 2S\alpha^{-1}\pi = 0,$$

i.e.
$$(2S\alpha\pi - \alpha^2)(\phi\rho)^2 + S\alpha^{-1}\pi = 0.$$

Multiplying by $S^2\alpha\pi$ or x^2 and substituting for $(\phi\rho)^2$ we get

$$(2S\alpha\pi - \alpha)^2(\pi - \alpha^{-1}S\alpha\pi)^2 + 2S^2\alpha\pi S\alpha^{-1}\pi = 0.$$

This equation at once reduces to

$$2\pi^2 S\alpha\pi - \pi^2\alpha^2 + S^2\alpha\pi = 0,$$

an equation which represents the cissoid. We leave the verification to the reader, the usual definition of the cissoid being as follows: On *AD* as diameter a circle is described. Any chord *AC* is drawn, and *CM* drawn perpendicular to *AD*. *AM′* is taken equal to *MD*, and *M′U* is drawn perpendicular to *AD* to meet *AC* in *U*. *U* describes the cissoid.

54. It will probably have suggested itself to the reader, that there exists a large class of problems to which the processes we have illustrated are scarcely if at all applicable. Hence there may have arisen a contrast between the Cartesian Geometry and Quaternions unfavourable to the latter. To remove this unfavourable impression, all that is required in a reader familiar with the older Geometry is a little experience in combining the logic of the new analysis with the forms of the old. He will then see how simple and direct are the arguments which he can bring to bear on any individual problem, and consequently how little the memory is taxed. As a general rule, however, plane geometry is not fitted to bring out the peculiar power of the quaternion calculus.

We propose now to put the reader in the track of employing his old forms in conjunction with quaternion reasonings.

Thus, in general, we may represent any plane curve by the formula

$$\rho = x\alpha + y\beta,$$

where α and β are constant vectors, and x and y are variable scalars, between which some relation is assigned.

If $x+y=1$, $\rho=\beta+x(\alpha-\beta)$, a straight line passing through the extremities of α and β.

If $x^2+y^2=1$, the curve traced out by ρ is a circle; if $a^2x^2+b^2y^2=1$, the curve is an ellipse, and so on.

55. THE PARABOLA. If $y^2=4a'x$, the curve is a parabola passing through the origin, α being unit vector parallel to the diameter and β unit vector along the tangent at the origin. We then have

$$\rho = \frac{y^2}{4a'}\alpha + y\beta. \qquad \text{.............................(1)}$$

For the particular case in which the diameter in question is the axis, and the tangent at its extremity parallel to the directrix,

$$\rho = \frac{y^2}{4a}\alpha + y\beta, \qquad \text{...........................(2)}$$

where a is the distance between the focus and the vertex.

This is the most convenient form when the focus is referred to.

In other cases a somewhat simpler form may be obtained by supposing α, or if necessary both α and β, of equation (1) to be other than unit vectors.

The equation may then be written under the form

$$\rho = \frac{t^2}{2}\alpha + t\beta. \qquad \text{.............................}(3)$$

For the tangent, pass to the contiguous point

$$\rho + \tau = \frac{(t+e)^2}{2}\alpha + (t+e)\beta,\ e \text{ very small.}$$

Hence
$$\tau = (et + \tfrac{1}{2}e^2)\alpha + e\beta$$
$$= (t\alpha + \beta)e \text{ in the limit.}$$

The equation to the tangent line is therefore

$$\varpi = \rho + x(t\alpha + \beta)$$
$$= \frac{t^2}{2}\alpha + \beta + x(t\alpha + \beta), \qquad \text{...................}(4)$$

or retaining the form (2)

$$\varpi = \frac{y^2}{4a}\alpha + y\beta + x\left(\frac{y}{2a}\alpha + \beta\right). \qquad \text{............}(5)$$

If $\rho\ \rho'$ are vectors to the extremity of a focal chord, $a\alpha$ being the vector to the focus,

$$V(\rho - a\alpha)(\rho' - a\alpha) = 0,$$

or
$$V\rho\rho' + aV\rho'\alpha - aV\rho\alpha = 0,$$

or, by (2), $\left(\frac{y^2y'}{4a} - \frac{yy'^2}{4a}\right)V\alpha\beta + ay'V\alpha\beta - ayV\alpha\beta = 0\,;$

$$\therefore\ yy' + 4a^2 = 0,\ y' = -\frac{4a^2}{y}.$$

Hence
$$\rho = \frac{y^2}{4a}\alpha + y\beta,$$
$$\rho' = \frac{4a^3}{y^2}\alpha - \frac{4a^2}{y}\beta,$$

are vectors to the extremities of a focal chord. The tangents at these points are parallel to

$$\tau = \frac{y}{2a}\alpha + \beta,$$

$$\tau' = -\frac{2a}{y}\alpha + \beta.$$

Hence $$S\tau\tau' = -\alpha^2 + \beta^2 = +1 - 1 = 0,$$
that is the tangents are perpendicular to each other.

If ϖ is the meeting point of those two tangents, it is easily shown by the methods of Chapter II. that its value is

$$\varpi = -\alpha + \left(\frac{y^2 + 4a^2}{2y}\right)\beta,$$

the equation of the directrix. Closely connected with this result is the theorem that the circle described on a focal chord as diameter touches the directrix; and the circle described on any other chord does not reach the directrix.

We leave this as an exercise to the reader.

If a triangle be inscribed in a parabola, the three points in which the sides are met by the tangents at the angles lie in a straight line.

Let OPQ be the triangle.

Take O as the origin, then

$$\rho = \frac{t^2}{2}\alpha + t\beta,$$

$$\rho' = \frac{t'^2}{2}\alpha + t'\beta,$$

$$\pi = \frac{t^2}{2}\alpha + t\beta + x(t\alpha + \beta),$$

$$\pi' = \frac{t'^2}{2}\alpha + t'\beta + x'(t'\alpha + \beta),$$

are the vectors OP, OQ, and the equations of the tangents at P and Q.

If QO meet in A the tangent at P,

$$OA=\frac{t^2}{2}\alpha+t\beta+x(t\alpha+\beta)=yOQ=y\left(\frac{t'^2}{2}\alpha+t'\beta\right);$$

whence
$$y=\frac{t^2}{2tt'-t'^2},$$

and
$$OA=\frac{t^2}{2tt'-t'^2}\left(\frac{t'^2}{2}\alpha+t'\beta\right)=\frac{t^2}{2t-t'}\left(\frac{t'}{2}\alpha+\beta\right).$$

Similarly if the tangent at Q meets PO in B,

$$OB=\frac{t'^2}{2t'-t}\left(\frac{t}{2}\alpha+\beta\right).$$

If the tangent at O meets PQ in C,

$$OC=OP+z(PQ)$$
$$=\frac{t^2}{2}\alpha+t\beta+z\left\{\frac{t'^2-t^2}{2}\alpha+(t'-t)\beta\right\}.$$

But
$$OC=v\beta;$$

whence
$$v=\frac{tt'}{t+t'},$$

and
$$OC=\frac{tt'}{t+t'}\beta.$$

Now
$$\frac{2t-t'}{t}OA-\frac{2t'-t}{t'}OB-\frac{t^2-t'^2}{tt'}OC=0,$$

and also
$$\frac{2t-t'}{t}-\frac{2t'-t}{t'}-\frac{t^2-t'^2}{tt'}=0;$$

therefore (§ 10) A, B, C are in a straight line.

56. THE HYPERBOLA. If in the equation

$$\rho=x\alpha+y\beta,$$

the product xy is constant, we get the equation of a hyperbola referred to its asymptotes. The equation is

$$\rho=x\alpha+\frac{C}{x}\beta,$$

or, if α, β be not both units we may write the equation under the simpler form

$$\rho=t\alpha+\frac{\beta}{t}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find the equation of the tangent, we have as usual

$$\tau = \rho + \tau - \rho = (t+e)\,\alpha + \frac{\beta}{t+e} - t\alpha - \frac{\beta}{t}$$

$$= e\alpha - \frac{e}{t^2 + te}\beta$$

$$= \left(\alpha - \frac{\beta}{t^2}\right)e,$$

neglecting higher powers of e, and a vector parallel to the tangent may be written

$$t\alpha - \frac{\beta}{t}. \quad \text{..............................(2)}$$

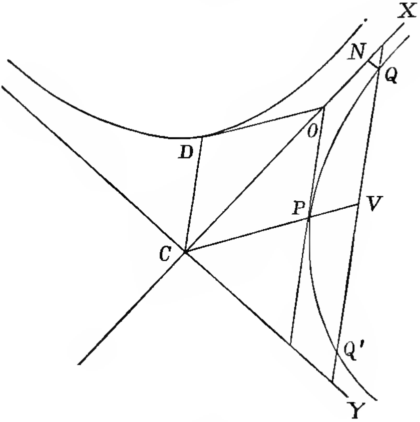

Fig. 33.

Hence the equation of the tangent is

$$\varpi = t\alpha + \frac{\beta}{t} + x\left(t\alpha - \frac{\beta}{t}\right). \quad \text{..................(3)}$$

It is evident that

$$t\alpha + \beta/t \text{ and } t\alpha - \beta/t$$

are conjugate semi-diameters, and the usual theorems regarding these and their parallel chords may be very easily worked out. For example,

$$V\rho\sigma = V\left(t\alpha + \frac{\beta}{t}\right)\left(t\alpha - \frac{\beta}{t}\right) = -2V\alpha\beta,$$

a constant area. Also if $\rho' \sigma'$ be another pair of conjugate radii

$$V\rho\sigma = V\rho'\sigma'.$$

It is easily shown that, if PO is the tangent at P, and QQ' a parallel chord, CPV bisects it in V; also that if CN, NQ be the coordinates of the point Q measured parallel to the asymptotes, they will form a parallelogram whose one diagonal is $CQ = t\alpha + \beta/t$, and whose other diagonal is $t\alpha - \beta/t$, and is therefore parallel to the tangent at Q.

If TQ, $T'Q'$ be two tangents to the hyperbola intersecting in R and terminated at T, T', Q, Q' by the asymptotes; then (1) *TQ' is parallel to $T'Q$;* (2) *area of triangle TRT' = area of triangle QRQ', and* (3) *CR bisects TQ' and $T'Q$.*

When the tangent

$$\varpi = t\alpha + \frac{\beta}{t} + x\left(t\alpha - \frac{\beta}{t}\right),$$

meets the asymptote α in T, the coefficient of β must vanish. Hence $x = 1$, and $\quad CT = 2t\alpha.$

Similarly, $\quad CQ = \dfrac{2\beta}{t}.$

In like manner $\quad CT' = 2t'\alpha,\ CQ' = \dfrac{2\beta}{t'};$

$$\therefore\ Q'T = 2\alpha t - \frac{2\beta}{t'} = \frac{2}{t'}(\alpha tt' - \beta),$$

and $\quad QT' = \dfrac{2}{t}(\alpha tt' - \beta);$

therefore $Q'T$ is parallel to QT'.

Again, $\quad CR = CQ + QR = \dfrac{2\beta}{t} + x2\left(\alpha t - \dfrac{\beta}{t}\right).$

Also $\quad CR = \dfrac{2\beta}{t'} + x'2\left(\alpha t' - \dfrac{\beta}{t'}\right);$

whence, in the usual way,

$$x = \frac{t'}{t+t'},\ x' = \frac{t}{t+t'},$$

and $\quad xx' = (1-x)(1-x'),$

i.e. $\quad QR \,.\, Q'R = RT \,.\, RT',$

and the triangles TRT', QRQ' are equal.

Lastly, $$CR = \frac{2\beta}{t} + \frac{2t'}{t+t'}\left(\alpha t - \frac{\beta}{t}\right) = \frac{t'}{t+t'}\left(2t\alpha + \frac{2\beta}{t'}\right),$$

or CR is in the direction of the diagonal of the parallelogram of which the sides are CT, CQ'; and therefore CR bisects TQ' and $T'Q$.

EXAMPLES TO CHAPTER VI.

1. Show that the locus of the points of bisection of chords to an ellipse, all of which pass through a given point, is an ellipse.

2. The locus of the middle points of all straight lines of constant length terminated by two fixed straight lines, is an ellipse whose centre bisects the shortest distance between the fixed lines; and whose axes are equally inclined to them.

3. If chords to an ellipse intersect one another in a given point, the rectangles by their segments are to one another as the squares of semi-diameters parallel to them.

4. If PCP', DCD' are conjugate diameters, then PD, PD' are proportional to the diameters parallel to them.

5. If Q be a point in the focal distance SP of an ellipse, such that SQ is to SP in a constant ratio, the locus of Q is a similar ellipse.

6. Diameters which coincide with the diagonals of the parallelogram on the axes are equal and conjugate.

7. Also diameters which coincide with the diagonals of any parallelogram formed by tangents at the extremities of conjugate diameters are conjugate.

8. The angular points of these parallelograms lie on an ellipse similar to the given ellipse and of twice its area.

9. If from the extremities of the axes of an ellipse four parallel lines be drawn, the points in which they cut the curve are the extremities of conjugate diameters.

10. If from the extremity of each of two semi-diameters ordinates be drawn to the other, the two triangles so formed will be equal in area.

11. Also if tangents be drawn from the extremity of each to meet the other produced, the two triangles so formed will be equal in area.

12. If on the semi-axes a parallelogram be described, and about it an ellipse similar and similarly situated to the given ellipse be constructed, any chord PQR of the larger ellipse, drawn from the further extremity of the diameter CD of the smaller ellipse, is bisected by the smaller ellipse at Q.

13. If TP, TQ be tangents to an ellipse, and PCP' be the diameter through P, then $P'Q$ is parallel to CT.

14. The sides of a parallelogram inscribed in an ellipse are parallel to conjugate diameters.

15. In the parabola $SY^2 = SP \,.\, SA$.

16. If the tangent to a parabola cut the directrix in R, SR is perpendicular to SP.

17. A circle has its centre at the vertex A of a parabola whose focus is S, and the diameter of the circle is $3AS$. Prove that the common chord bisects AS.

18. The tangent at any point of a parabola meets the directrix and latus rectum in two points equally distant from the focus.

19. The circle described on SP as diameter is touched by the tangent at the vertex.

20. Parabolas have their axes parallel and all pass through two given points. Prove that their foci lie in a conic section.

21. Two parabolas have a common directrix. Prove that their common chord bisects at right angles the line joining their foci.

22. The portion of any tangent to the parabola between tangents which meet in the directrix subtends a right angle at the focus.

23. If from the point of contact of a tangent to a parabola a chord be drawn, and another line be drawn parallel to the axis meeting the chord, tangent and curve; this line will be divided by them in the same ratio as it divides the chord.

24. The middle points of focal chords describe a parabola whose latus rectum is half that of the given parabola.

25. PSQ is a focal chord of a parabola: PA, QA meet the directrix in y, z. Prove that Pz, Qy are parallel to the axis.

26. The tangent at D to the conjugate hyperbola is parallel to CP.

27. The portion of the tangent to a hyperbola which is intercepted by the asymptotes is bisected at the point of contact.

28. The locus of a point which divides in a given ratio lines which cut off equal areas from the space enclosed by two given straight lines is a hyperbola of which these lines are the asymptotes.

29. The tangent to a hyperbola at P meets an asymptote in T, and TQ is drawn to the curve parallel to the other asymptote. PQ produced both ways meets the asymptotes in R, R': RR' is trisected in P, Q.

30. From any point R of an asymptote, RN, RM are drawn parallel to conjugate diameters intersecting the hyperbola and its conjugate in P and D. Prove that CP and CD are conjugate.

31. The intercepts on any straight line between the hyperbola and its asymptotes are equal.

32. If QQ' meet the asymptotes in R, r,

$$RQ \, . \, Qr = PO^2.$$

33. If the tangent at any point meet the asymptotes in X and Y, the area of the triangle XCY is constant.

34. If a chord of a hyperbola be one diagonal of a parallelogram whose sides are parallel to the asymptotes, the other diagonal passes through the centre.

35. If the tangents at the extremities Q, Q' of a diameter of a hyperbola meet the tangent at a point P in the points T, T'; and if CD, CD' are the semi-diameters conjugate to CP, CQ; then

$$(1) \quad PT/QT = PT'/Q'T' = CD/CD';$$
$$(2) \quad PT. \, PT' = CD^2$$

36. Straight lines move so that the triangular area which they cut off from two given straight lines which meet one another is constant: to find the locus of their ultimate intersections.

37. Eliminate t from the equations of the parabola and hyperbola as given in §§ 55, 56, and find their equations in terms of α, β only.

CHAPTER VII.

CENTRAL SURFACES OF THE SECOND ORDER.

57. The general scalar equation of the surface of the second order will contain terms involving the square of $T\rho$, terms involving $T\rho$ to the first power, and a term not involving $T\rho$ at all. If we limit ourselves to the *central* surfaces, namely, those which have a centre bisecting all diameters drawn through it, the equation must be satisfied by both $+\rho$ and $-\rho$, and hence no term of the form $AS\alpha\rho$ can exist. A little consideration of the forms of scalar functions of $T\rho^2$ will show that the required equation is

$$a\rho^2 + bS^2\alpha\rho + 2cS\alpha\rho S\beta\rho + \ldots = -1.$$

Now if we put

$$\phi\rho = a\rho + b\alpha S\alpha\rho + c(\alpha S\beta\rho + \beta S\alpha\rho) + \ldots$$

we shall have

$$\begin{aligned} S\rho\phi\rho &= a\rho^2 + bS^2\alpha\rho + 2cS\alpha\rho S\beta\rho + \ldots \\ &= -1, \end{aligned}$$

the equation required.

We easily see that ϕ is a linear vector function, distributive and self-conjugate, and fulfilling all the conditions given in § 45.

58. THE TANGENT PLANE. By reasoning identical with that employed several times already, it is easily shown that if τ be a vector in the tangent plane,

$$S\tau\phi\rho = 0,$$

or, if ϖ is a vector to a point on the tangent plane,

$$S(\varpi-\rho)\phi\rho=0,$$

$$S\varpi\phi\rho=S\rho\phi\rho=-1,$$

which is the equation of the tangent plane.

The vector $\phi\rho$ is perpendicular to the tangent plane at the extremity of the vector ρ.

59. PERPENDICULAR ON THE TANGENT. If OY be the perpendicular from the centre O on the tangent plane; then, since $\phi\rho$ is a vector perpendicular to that plane, $OY=x\phi\rho$ and $Sx(\phi\rho)^2=-1_2$ giving

$$\text{vector } OY=x\phi\rho=-\frac{1}{\phi\rho}=-(\phi\rho)^{-1}.$$

Sir W. R. Hamilton terms $\phi\rho$ the *vector of proximity.*

60. POLAR PLANES. If tangent planes all pass through a fixed point, the curve of contact is a plane curve.

Let T be the fixed point, vector α; ρ the vector to a point of contact.

Then (§ 58) $$S\alpha\phi\rho=-1;$$

i.e. $$S\rho\phi\alpha=-1,$$

which is the equation in ρ of a plane perpendicular to $\phi\alpha$.

Now $\phi\alpha$ is the normal vector of the point where OT cuts the ellipsoid; consequently the curve of contact lies in a plane parallel to the tangent plane at the extremity of the diameter drawn to the given point.

When α is vector to a point inside the ellipsoid, there can of course be no real tangent planes drawn; but in all cases the equation $S\rho\phi\alpha=-1$ represents a real plane, which is called the polar plane to the point.

61. Tangent planes are all parallel to a given straight line, to find the curve of contact.

Let α be a vector parallel to the given line; then

$$\varpi = \rho + x\alpha$$

is a point in the tangent plane;

$$\therefore \; S(\varpi + x\alpha)\phi\rho = -1;$$

and
$$S\alpha\phi\rho = 0,$$

or
$$S\rho\phi\alpha = 0,$$

the equation of a plane through the origin perpendicular to $\phi\alpha$: that is, the curve of contact lies in a plane through the centre parallel to the tangent plane at the extremity of the diameter which is parallel to the given line.

62. CONJUGATE DIAMETERS. Let us first find the locus of the middle points of parallel chords.

Let each of the chords be parallel to α, ϖ the vector to the middle point of one of them; then $\varpi + x\alpha$, $\varpi - x\alpha$ are points in the ellipsoid.

From the first,

$$S(\varpi + x\alpha)\phi(\varpi + x\alpha) = -1;$$

i.e.
$$S\varpi\phi\varpi + 2xS\varpi\phi\alpha + x^2 S\alpha\phi\alpha = -1.$$

From the second,

$$S\varpi\phi\varpi - 2xS\varpi\phi\alpha + x^2 S\alpha\phi\alpha = -1;$$

$$\therefore \text{ subtracting, } S\varpi\phi\alpha = 0,$$

i.e. the locus is a plane through the centre perpendicular to $\phi\alpha$, or parallel to the tangent plane at the extremity A of the diameter which is drawn parallel to α.

Let β be any vector in this plane $S\varpi\phi\alpha = 0$, then

$$S\beta\phi\alpha = 0,$$

and therefore
$$S\alpha\phi\beta = 0,$$

or α satisfies the equation $S\varpi\phi\beta = 0$ of the plane which bisects all chords parallel to β.

These two planes, bisecting chords parallel to α and β respectively, will intersect along a line through the centre. Let γ represent this line; then, since γ is a vector in both the planes $S\varpi\phi\alpha = 0$, and $S\varpi\phi\beta = 0$, we must have

$$S\gamma\phi\alpha = 0, \quad S\gamma\phi\beta = 0.$$

Hence $$S\alpha\phi\gamma=0, \quad S\beta\phi\gamma=0,$$
so that α and β are both vectors in the plane bisecting chords parallel to γ.

Let OA, OB, OC be the vector radii parallel to α, β, γ respectively; then these lines are such that all chords parallel to any one of them are bisected by the diametral plane which passes through the other two.

We may term these lines *conjugate* semi-diameters, and the corresponding diametral planes *conjugate diametral planes.*

It is evident that the number of sets of conjugate diameters is unlimited.

We have then the following equations:

$$S\alpha\phi\beta=0=S\beta\phi\alpha,$$
$$S\beta\phi\gamma=0=S\gamma\phi\beta,$$
$$S\alpha\phi\gamma=0=S\gamma\phi\alpha.$$

They show that γ is perpendicular to both $\phi\alpha$ and $\phi\beta$, and is therefore a vector perpendicular to their plane; hence

$$\gamma=xV\phi\alpha\phi\beta.$$

In the same way, since $\phi\gamma$ is perpendicular to both α and β, we have

$$\phi\gamma=yV\alpha\beta\,;$$

or we have the following pairs of parallel vectors:

$$\gamma \parallel V\phi\alpha\phi\beta, \quad \beta \parallel V\phi\alpha\phi\gamma, \quad \alpha \parallel V\phi\beta\phi\gamma,$$
$$\phi\gamma \parallel V\alpha\beta, \quad \phi\beta \parallel V\alpha\gamma, \quad \phi\alpha \parallel V\beta\gamma.$$

Note also

$$y\phi^{-1}V\alpha\beta=xV\phi\alpha\phi\beta,$$

upon which Hamilton founded his solution of linear equations.

63. SQUARE ROOT OF THE FUNCTION ϕ. If we write $\psi\psi\rho$ for $\phi\rho$, $\psi\rho$ being still a vector, the equation of the ellipsoid assumes the form

$$S\rho\psi(\psi\rho)=-1,$$

i.e. $$S\psi\rho\psi\rho=-1$$

$$(\psi\rho)^2=-T(\psi\rho)^2=-1, \;\ldots\ldots\ldots\ldots(1)$$

which, if we put $\sigma=\psi\rho$, becomes $T\sigma=1$, the equation of a sphere.

Hence the ellipsoid can be changed into the sphere, and *vice versâ*, by a linear deformation of each vector, the operator being the function ψ (or $\phi^{\frac{1}{2}}$) or its inverse.

The equations

$$S\alpha\phi\beta = 0, \text{ etc.},$$

now become
$$S\alpha\psi^2\beta = 0,$$

i.e.
$$S\psi\alpha\psi\beta = 0, \text{ etc., etc.} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(1) and (2) show that $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are unit vectors at right angles to one another.

Calling the sphere $T\sigma = 1$ the unit-sphere, we may enunciate this result by saying that any three vectors of the unit-sphere which correspond to a set of semi-conjugate diameters in the connected ellipsoid form a rectangular system.

64. CARTESIAN EQUIVALENTS. Let us now take i, j, k unit vectors along the principal axes of x, y, z; then we have

$$\rho = xi + yj + zk, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore\ Si\rho = -x, \text{ etc.},$$

so that for the sake of transformations in which it is desirable that the form of ρ should be retained, we may write

$$\rho = -(iSi\rho + jSj\rho + kSk\rho); \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and as $\phi\rho$ is a linear and vector function of ρ, its vector projections along the principal axes will be multiples of

$$iSi\rho,\ jSj\rho,\ kSk\rho;$$

we may therefore write

$$\phi\rho = -\left(\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}\right), \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

the particular multipliers having been chosen in order to make the equation $S\rho\phi\rho = -1$

coincide with the Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

As $$\phi\rho = \psi\psi\rho, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$
we require to take $\psi\rho$ so that ψ operating twice in succession on ρ shall give the same result as ϕ operating once.

Now a comparison of equations (2) and (3) will show that the latter operation introduces $\frac{1}{a^2}$, etc., into ρ; it is evident therefore that the former operation (ψ) is to introduce $\frac{1}{a}$, etc., or
$$\psi\rho = -\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right), \quad\ldots\ldots\ldots\ldots\ldots (5)$$

It may perhaps be worth while to verify this result. We have
$$\psi\psi\rho = -\left(\frac{iSi\psi\rho}{a} + \frac{jSj\psi\rho}{b} + \frac{kSk\psi\rho}{c}\right)$$
$$= iS\frac{i}{a}\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right) + \ldots$$
$$= i\frac{i^2Si\rho}{a^2} + \ldots$$
$$= -\left(\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}\right)$$
$$= \phi\rho.$$
$$\phi^2\rho = \phi\phi\rho = -\left(\frac{iSi\phi\rho}{a^2} + \frac{jSj\phi\rho}{b^2} + \frac{kSk\phi\rho}{c^2}\right)$$
$$= -\left(\frac{iSi\rho}{a^4} + \frac{jSj\rho}{b^4} + \frac{kSk\rho}{c^4}\right), \quad\ldots\ldots\ldots\ldots\ldots\ldots (6)$$
$$\phi^{-1}\rho = -(a^2iSi\rho + b^2jSj\rho + c^2kSk\rho), \quad\ldots\ldots\ldots\ldots\ldots (7)$$
because $\phi\phi^{-1}\rho$ produces ρ.
$$\psi^{-1}\rho = -(aiSi\rho + bjSj\rho + ckSk\rho), \quad\ldots\ldots\ldots\ldots (8)$$
$$\rho = \psi^{-1}\psi\rho = -(aiSi\psi\rho + bjSj\psi\rho + ckSk\psi\rho). \quad\ldots\ldots\ldots\ldots (9)$$

It is evident that the properties of Art. 44 apply to all these functions.

65. This section contains a series of examples, chiefly on the ellipsoid, chosen with a view to variety of treatment. Other examples will be found at the end of the chapter. The student will find it a very good exercise to work through by quaternionic method the theorems and problems given in any recognised treatise on solid geometry—such as Salmon's or Smith's.

1. *Find the point on an ellipsoid, the tangent plane at which cuts off equal portions from each of a given set of conjugate axes.*

Let $a\alpha$, $b\beta$, $c\gamma$ be the set of conjugate vector radii, α, β, γ unit vectors. Then the vector to the required point may be written

$$\rho = x\alpha + y\beta + z\gamma.$$

Let p be the length cut off from each axis; so that $p\alpha$, $p\beta$, $p\gamma$ will be vectors to the tangent plane. Hence

$$Sp\alpha\phi\rho = -1,$$

or

$$Sp\alpha(x\phi\alpha + y\phi\beta + z\phi\gamma) = -1,$$

or

$$px = -\frac{1}{S\alpha\phi\alpha} = -\frac{a^2}{-1} = a^2.$$

Similarly,

$$py = b^2,$$

$$pz = c^2.$$

and

$$\rho = \frac{1}{p}(a^2\alpha + b^2\beta + c^2\gamma),$$

and

$$-1 = S\rho\phi\rho$$

$$= \frac{1}{p^2}S(aa\alpha + bb\beta + cc\gamma)(a\phi a\alpha + b\phi b\beta + c\phi c\gamma)$$

$$= -\frac{a^2 + b^2 + c^2}{p^2};$$

$$\therefore\ p = \sqrt{a^2 + b^2 + c^2}.$$

Let x, y, z be the coordinates of the point, p the portion cut off, then

$$\rho = xi + yj + zk.$$

Now pi, pj, pk are points on the tangent plane;

$$\therefore\ Spi\phi\rho = 1,$$

which gives

$$pSi\left(\frac{iSi\rho}{a^2} + \ldots\right) = 1,$$

or

$$\frac{px}{a^2} = 1.$$

Similarly, $$\frac{py}{b^2}=1,$$

$$\frac{pz}{c^2}=1,$$

$$\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}=\frac{1}{p}=\frac{1}{\sqrt{a^2+b^2+c^2}}.$$

2. *To find the locus of a point when the perpendicular from the centre on its polar plane is of constant length.*

Let ϖ be the vector to the point, then

$S\rho\phi\varpi=-1$ is the equation of the polar plane,

and $-\frac{1}{\phi\varpi}$ is the vector perpendicular on it;

$\therefore$ $(\phi\varpi)^2=-C^2$, by the question.

But $$(\phi\varpi)^2=S\,.\,\phi\varpi\phi\varpi=S\varpi\phi\phi\varpi=S\varpi\phi^2\varpi;$$

hence $$S\varpi\phi^2\varpi=-C^2,$$

the equation of an ellipsoid whose Cartesian equation is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=C^2.$$

3. *To find the surface enveloped by the polar planes with respect to one ellipsoid of points which lie on another ellipsoid.*

Let $S\rho\phi\rho=-1$, $S\rho\psi\rho=-1$ be the two ellipsoids.

If α is a point on the latter, $S\alpha\psi\alpha=-1$, and the polar plane to α has the equation

$$-1=S\rho\phi\alpha=S\rho\phi\psi^{-1}\psi\alpha=S\psi^{-1}\phi\rho\psi\alpha.$$

Hence $$\alpha=\psi^{-1}\phi\rho,$$

and $-1=S\rho\phi\alpha=S\rho\phi\psi^{-1}\phi\rho$, an ellipsoid.

4. *The sum of the squares of three conjugate semi-diameters is constant.*

Let α, β, γ be the semi-diameters; $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are rectangular unit vectors (§ 63).

Now $$\alpha=-(aiSi\psi\alpha+bjSj\psi\alpha+ckSk\psi\alpha);\ \ldots\ldots\ldots\ldots(64.\ 9)$$

$$\therefore\ (T\alpha)^2=-\alpha^2=a^2(Si\psi\alpha)^2+b^2(Sj\psi\alpha)^2+c^2(Sk\psi\alpha)^2,$$

$$(T\beta)^2=\qquad a^2(Si\psi\beta)^2+b^2(Sj\psi\beta)^2+c^2(Sk\psi\beta)^2,$$

$$(T\gamma)^2=\qquad a^2(Si\psi\gamma)^2+b^2(Sj\psi\gamma)^2+c^2(Sk\psi\gamma)^2:$$

adding, and observing that

$$(Si\psi\alpha)^2 + (Si\psi\beta)^2 + (Si\psi\gamma)^2 = 1,$$

we get

$$(T\alpha)^2 + (T\beta)^2 + (T\gamma)^2 = a^2 + b^2 + c^2,$$

i.e.

$$a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2.$$

5. *The sum of the squares of the projections of three conjugate diameters on any of the principal axes is equal to the square of that axis.*

Let α, β, γ be conjugate semi-diameters; then, since

$$\alpha = -(aiSi\psi\alpha + bjSj\psi\alpha + ckSk\psi\alpha), \quad \ldots\ldots\ldots(63.\ 9)$$

$$Si\alpha = aSi\psi\alpha.$$

Similarly,

$$Si\beta = aSi\psi\beta,$$

$$Si\gamma = aSi\psi\gamma;$$

$$\therefore\ (Si\alpha)^2 + (Si\beta)^2 + (Si\gamma)^2 = a^2\{(Si\psi\alpha)^2 + (Si\psi\beta)^2 + (Si\psi\gamma)^2\}$$

$$= a^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31.\ \text{Cor.})$$

because $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are at right angles to on another. ......(62)

But $-Si\alpha$ is the projection of $T\alpha$ along the axis of x; and similarly of the others. Hence the proposition.

6. *The sum of the reciprocals of the squares of the three perpendiculars from the centre on tangent planes at the extremities of conjugate diameters is constant.*

Let Oy_1, Oy_2, Oy_3 be the perpendiculars.

$$\frac{1}{Oy_1^{\ 2}} = -(\phi\alpha)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58)$$

$$= \frac{(Si\alpha)^2}{a^4} + \frac{(Sj\alpha)^2}{b^4} + \frac{(Sk\alpha)^2}{c^5}; \quad \ldots\ldots\ldots\ldots(63.\ 3)$$

$$\frac{1}{Oy_2^{\ 2}} = \frac{(Si\beta)^2}{a^4} + \frac{(Sj\beta)^2}{b^4} + \frac{(Si\beta)^2}{c^4};$$

$$\frac{1}{Oy_3^{\ 2}} = \frac{(Si\gamma)^2}{a^4} + \frac{(Sj\gamma)^2}{b^4} + \frac{(Sk\gamma)^2}{c^4};$$

$$\therefore\ \frac{1}{Oy_1^{\ 2}} + \frac{1}{Oy_2^{\ 2}} + \frac{1}{Oy_3^{\ 2}} = \frac{1}{a^4}\left\{(Si\alpha)^2 + (Sj\beta)^2 + (Si\gamma)^2\right\} + \text{etc}$$

$$= \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{Ex. } 7)$$

7. *A, B, and C are three similar and similarly situated ellipsoids; A and B are concentric, and C has its centre on the surface of B To show that the tangent plane to B at this point is parallel to the plane of intersection of A and C.*

Let α be the vector to the centre of C.

$$S\rho\phi\rho = -a \text{ the equation of } A,$$
$$S\rho\phi\rho = -b \ldots\ldots\ldots\ldots\ldots\ldots\ldots B,$$
$$S(\rho-\alpha)\phi(\rho-\alpha) = -c \ldots\ldots\ldots C.$$

Now at the intersection of A and C, ρ is the same for both; therefore the equation of the plane of intersection is to be found by subtracting the one from the other.

It is therefore $2S\rho\phi\alpha = S\alpha\phi\alpha - a + c$;

and the equation of the tangent plane to B at the centre of C is $S\pi\phi\alpha = -b$;

∴ both planes are perpendicular to $\phi\alpha$, and are consequently parallel.

8. *Two similar and similarly situated ellipsoids are cut by a series of ellipsoids similar and similarly situated to the two given ones; and in such a manner that the planes of intersection are at right angles to one another. Show that the centres of the cutting ellipsoids lie on another ellipsoid.*

Let
$$S\rho\phi\rho = -1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$S(\rho-\alpha)\phi(\rho-\alpha) = -C, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

be the given ellipsoids;

$$S(\rho-\pi)\phi(\rho-\pi) = -x, \ldots\ldots\ldots\ \ldots\ldots\ldots(3)$$

one of the cutting ellipsoids.

ϕ is the same for all because the ellipsoids are similar.

The plane of intersection of (1) and (3) is found by subtracting the equations; and is therefore

$$2S\rho\phi\pi = S\pi\phi\pi - 1 + x.$$

The plane of intersection of (2) and (3) is

$$2S\rho(\phi\pi - \phi\alpha) = S\pi\phi\pi - S\alpha\phi\alpha - C + x.$$

The former of these planes is perpendicular to $\phi\pi$ and the latter to $\phi\pi - \phi\alpha$; and, since by the question, the former is perpendicular to the latter, $\phi\pi$ is perpendicular to $\phi\pi - \phi\alpha$,

$$\therefore \; S\phi\pi(\phi\pi - \phi\alpha) = 0,$$

or

$$S(\pi - \alpha)\phi^2\pi = 0,$$

the equation of the locus of the centres of the cutting ellipsoids.

It represents an ellipsoid of which the semi-axes are proportional to the squares of the semi-axes of (1).

The Cartesian equation is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} - \left(\frac{xx'}{a^4} + \frac{yy'}{b^4} + \frac{zz'}{c^4}\right) = 0.$$

9. *Find the equation of the curve described by a given point in a line of given length whose extremities move in fixed straight lines.*

This is a good example of the ease with which the quaternion method attacks the most general case.

Suppose the lines not to meet, and take the origin half way along the line perpendicular to both. Let γ, $-\gamma$ be the vectors perpendicular to the lines, and let α, β be unit vectors along the lines. Then the extremities of the line of given length are determined by the vectors

$$\gamma + x\alpha, \quad -\gamma + y\beta,$$

and the vector of the point which divides this line in the ratio of m to 1 is

$$\rho = \frac{(\gamma + x\alpha) + m(-\gamma + y\beta)}{1 + m},$$

or

$$(1 + m)\rho = x\alpha + my\beta + (1 - m)\gamma.$$

Also constancy of length gives

$$T(2\gamma + x\alpha - y\beta) = l.$$

From the first of these two equations, we get

$$S\gamma\left(\rho - \frac{1 - m}{1 + m}\gamma\right) = 0,$$

so that the extremity of ρ lies in a plane perpendicular to γ.

We also get, by operating by $S.V\beta\gamma$, $S.V\gamma\alpha$ the values of x and y, namely

$$xS\alpha\beta\gamma = (1+m)S\beta\gamma\rho,$$
$$yS\alpha\beta\gamma = (1+m)S\gamma\alpha\rho,$$

and finally

$$\{2\gamma S\alpha\beta\gamma + (1+m)\alpha S\beta\gamma\rho - (1+m)\beta S\gamma\alpha\rho\}^2 = -l^2 S^2\alpha\beta\gamma,$$

or, if $4\gamma^2 = -c^2$,

$$S^2\beta\gamma\rho + S^2\gamma\alpha\rho + 2S\alpha\beta S\beta\gamma\rho S\gamma\alpha\rho = \frac{l^2 - c^2}{(1+m)^2} S^2\alpha\beta\gamma.$$

Since γ occurs in the second degree in every term, we may multiply it throughout by any number. We may therefore put in its place $V\alpha\beta$, and so obtain an equation involving only α, β, l, c, and m as constants.

If we write

$$\phi\rho S^2\alpha\beta\gamma = \beta\gamma S\rho(\beta\gamma + \gamma\alpha S\alpha\beta) + \gamma\alpha S\rho(\gamma\alpha + \beta\gamma S\alpha\beta),$$

the equation becomes

$$S\rho\phi\rho = (l^2 - c^2)/(1+m).$$

But $\phi\rho$ is always perpendicular to $V.\beta\gamma\gamma\alpha$, that is to $V\alpha\beta$ or γ. Hence the above equation is the equation of a cylinder whose axis is parallel to γ; and the intersection of this cylinder with the plane

$$S\gamma\rho = \frac{1-m}{1+m}\gamma^2,$$

gives the required elliptic locus. The dimensions of the ellipse will depend upon l, c, and m. When $c = 0$, the path is still an ellipse in the plane $\alpha\beta$.

10. *Tangent planes are drawn to an ellipsoid from a given external point, to find the cone which has its vertex at the origin [the centre of the ellipsoid], and which passes through all the points of contact of the tangent planes with the ellipsoid.*

Let α be the vector to the external point, ρ a point in the ellipsoid where a tangent plane through α touches it.

Then the equation of the ellipsoid is

$$S\rho\phi\rho = -1,$$

and the equation of the tangent plane

$$S\alpha\phi\rho = -1, \quad i.e. \; S\rho\phi\alpha = -1.$$

The equation $$S\rho\phi\rho = (S\rho\phi\alpha)^2,$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right)^2,$$

represents a surface passing through the points of contact; and is the cone required. [For it is homogeneous in $T\rho$.]

11. *The equation* $\rho = t^2\alpha + u^2\beta + (t+u)^2\gamma$ *is that of a cone of the second order touched by each of the three planes through* OAB, OBC, OCA, *and the section* ABC *through the extremities of* α, β, γ *is an ellipse touched at their middle points by* AB, BC, CA.

(1) If the surface be referred to oblique co-ordinates parallel to α, β, γ respectively, we shall have

$$\rho = x\alpha + y\beta + z\gamma,$$

therefore $$x = t^2, \; y = u^2, \; z = (t+u)^2,$$

or $$z = (\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy},$$

which gives $$(z - x - y)^3 = 4xy,$$

a cone of the second order.

(2) If $t = -u$, the equation becomes

$$\rho = t^2(\alpha + \beta),$$

the equation of a straight line bisecting the base AB, which, since it satisfies the equation relative to t, shows that this line coincides with the cone in all its length; *i.e.* the cone is touched in this line by the plane OAB.

Similarly, by putting $t = 0$, $u = 0$ respectively, we can show that the cone is touched by the plane BOC, COA in the lines which bisect AC, CA.

(3) Restricting ourselves to the plane ABC, we have the section of a cone of the second order enclosed by the triangle

ABC, which triangle is itself the section of three planes each of which touches the cone.

12. *The equation* $\rho = a\alpha + b\beta + c\gamma$ *with the condition*

$$ab + bc + ca = 0$$

is a cone of the second order, and the lines OA, OB, OC *coincide throughout their length with the surface.*

(1) It is evident that the equation gives

$$xy + yz + zx = 0.$$

(2) That if $b = 0$, $c = 0$, the question is satisfied by

$$\rho = a\alpha,$$

whatever be a, therefore, etc.

13. *The lines which divide proportionally the pairs of opposite sides of a gauche quadrilateral, are the generating lines of a hyperbolic paraboloid.*

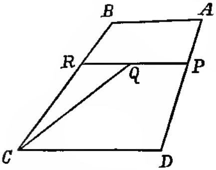

Fig. 34.

Let $ABCD$ be the quadrilateral.

AD, BC are divided proportionally in P and R.

Let $CA = \alpha$, $CB = \beta$, $CD = \gamma$;

$$CR = m\beta, \quad DP = mDA;$$

i.e. $$CP - \gamma = m(\alpha - \gamma);$$

therefore $$RP = CP - CR = \gamma + m(\alpha - \gamma) - m\beta,$$

$$\rho = CQ = CR + pRP$$

$$= m\beta + p\{\gamma + m(\alpha - \gamma) - m\beta\}$$

$$= x\alpha + y\beta + z\gamma, \text{ say};$$

therefore $$x = pm, \; y = m - pm, \; z = p(1 - m);$$

therefore $$m = x + y, \; p = \frac{x}{x + y},$$

$$z = \frac{x}{x + y} - x,$$

or $$(x + z)(x + y) = x,$$

the equation referred to oblique co-ordinates parallel to α, β, γ.

EXAMPLES TO CHAPTER VII.

1. Find the locus of a point, the ratio of whose distances from two given straight lines is constant.

2. Find the locus of a point the square of whose distance from a given line is proportional to its distance from a given plane.

3. Prove that the locus of the foot of the perpendicular from the centre on the tangent plane of an ellipsoid is

$$(ax)^2+(by)^2+(cz)^2=(x^2+y^2+z^2)^2.$$

4. The sum of the squares of the reciprocals of any three radii at right angles to one another is constant.

5. If Oy_1, Oy_2, Oy_3 be perpendiculars from the centre on tangent planes at the extremities of conjugate diameters, and if Q_1, Q_2, Q_3 be the points where they meet the ellipsoid; then

$$\frac{1}{OY_1^2 \, . \, OQ_1^2}+\frac{1}{OY_2^2 \, . \, OQ_2^2}+\frac{1}{OY_3^2 \, . \, OQ_3^2}=\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}.$$

6. If tangent planes to an ellipsoid be drawn from points in a plane parallel to that of xy, the curves which contain all the points of contact will lie in planes which all cut the axis of z in the same point.

7. Two similar and similarly situated ellipsoids intersect in a plane curve whose plane is conjugate to the line which joins the centres of the ellipsoids.

8. If points be taken in conjugate semi-diameters produced, at distances from the centre equal to p times those semi-diameters respectively; the sum of the squares of the reciprocals of the perpendiculars from the centre on their polar planes is equal to p^2 times the sum of the squares of the perpendiculars from the centre on tangent planes at the extremities of those diameters.

9. If P be a point on the surface of an ellipsoid, PA, PB, PC any three chords at right angles to each other, the plane ABC will pass through a fixed point, which is the normal to the ellipsoid at P; and distant from P by

$$\frac{2}{p}\bigg/\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),$$

where p is the perpendicular from the centre on the tangent plane at P.

10. Find the equation of the cone which has its vertex in a given point, and which touches and envelopes a given ellipsoid.

11. Find the locus of the points of contact of tangent planes to an ellipsoid, when the tangent planes make a given angle with one of the principal axes.

12. The sum of the squares of the three perpendiculars from the centre on three tangent planes at right angles to one another is constant.

13. If through a fixed point within an ellipsoid three chords be drawn mutually at right angles, the sum of the reciprocals of the products of their segments will be constant.

14. Establish for the central surface of the second order the theorems of Poles and Polars corresponding to those established for the sphere (§ 39).

15. If through a given point chords be drawn to an ellipsoid, the intersection of pairs of tangent planes at their extremities all lie in a plane parallel to the tangent plane at the extremity of the diameter which passes through the point.

16. If a tangent plane be drawn to the inner of two similar concentric and similarly situated ellipsoids, the point of contact is the centre of the elliptic section of the outer ellipsoid.

17. If two of a system of three rectangular vectors are confined to given planes, show that the third lies in a cone of the second order whose circular sections are parallel to the given planes.

18. Find the locus of a point, the sum of the squares of whose distances from a number of given planes is constant.

CHAPTER VIII.

MISCELLANEOUS GEOMETRICAL APPLICATIONS.

66. PASCAL'S HEXAGRAM. Let O be the origin, OA, OB, OC, OD, OE five given vectors lying on the surface of a cone, and terminated in a plane section of the cone $ABCDEF$, not passing through O; OX *any* vector lying on the same surface.

Let $OA=\alpha$, $OB=\beta$, $OC=\gamma$, $OD=\delta$, $OE=\epsilon$, $OX=\rho$.

The equation

$$S\,.\,V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha)=0 \text{ ....(1)}$$

is the equation of a cone of the second order whose vertex is O and vector ρ along the surface. For

1. It is a cone whose vertex is O because it is not altered by writing $x\rho$ for ρ. Also it is of the second order in ρ, since ρ occurs in it twice and twice only.

2. All the vectors OA, OB, OC, OD, OE lie on its surface. This we shall prove by showing that if ρ coincide with any one of them the equation (1) is satisfied.

If ρ coincide with α, the last term of the left-hand side of the equation viz. $V\rho\alpha$, becomes $V\alpha\alpha=V\alpha^2=0$, and the equation is satisfied.

If ρ coincide with β, the left-hand side of the equation becomes

$$S\,.\,V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\beta)V(V\gamma\delta V\beta\alpha). \text{ ....(2)}$$

Now $V(V\beta\gamma V\epsilon\beta) = -V(V\epsilon\beta V\beta\gamma)$, (§ 30), is a vector parallel to β (§ 33), call it $m\beta$; and

$$\begin{aligned} & V.\{V(V\alpha\beta V\delta\epsilon)V(V\gamma\delta V\beta\alpha)\} \\ &\qquad = V.\{V(V\alpha\beta V\delta\epsilon)V(V\alpha\beta V\gamma\delta)\} \\ &\qquad = \text{a multiple of } V\alpha\beta \\ &\qquad = nV\alpha\beta, \text{ say.} \end{aligned}$$

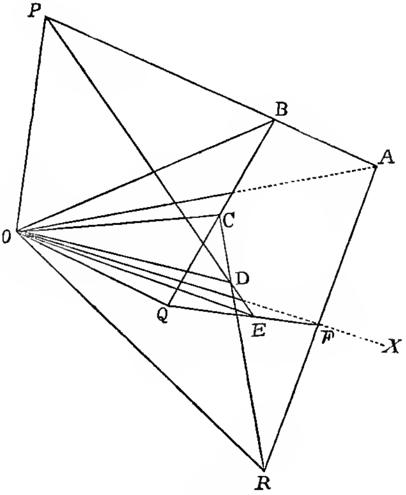

Fig. 35.

Hence the product of the first and third vectors in expression (2) becomes

$$\text{scalar} + nV\alpha\beta,$$

and the second is $m\beta$; therefore expression (2) becomes (§ 32)

$$\begin{aligned} & S.(\text{scalar} + nV\alpha\beta)m\beta \\ &\qquad = mnS\beta V\alpha\beta \\ &\qquad = 0, \end{aligned}$$

because $V\alpha\beta$ is a vector perpendicular to β.

Equation (1) is therefore satisfied when ρ coincides with β.

If ρ coincide with γ both the second and third vectors are parallel to β (§ 33); therefore their product is a scalar, and equation (1) is satisfied.

The other cases are but repetitions of these.

Hence equation (1) is satisfied if ρ coincide with any one of the five vectors α, β, γ, δ, ϵ; i.e. OA, OB, OC, OD, OE are vectors on the surface of the cone.

3. Let F be the point in which OX cuts the plane $ABCDE$; then $ABCDEF$ are the angular points of a hexagon inscribed in a conic section.

4. Let the planes OAB, ODE intersect in OP; OBC, OEF in OQ; OCD, OFA in OR; then

$$V.V\alpha\beta V\delta\epsilon = mOP,$$
$$V.V\beta\gamma V\epsilon\rho = nOQ,$$
$$V.V\gamma\delta V\rho\alpha = pOR;$$

therefore

$$S.V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha) = mnpS(OP.OQ.OR);$$

hence the equation (1) gives

$$S(OP.OQ.OR) = 0,$$

or (§ 32) OP, OQ, OR are in the same plane.

Hence PQR, the intersection of this plane with the plane $ABCDEF$, is a straight line. But P is the point of intersection of AB, ED, etc.

Therefore, the opposite sides (1st and 4th, 2nd and 5th, 3rd and 6th) of a hexagon inscribed in a conic section being produced meet in the same straight line.

COR. It is evident that the demonstration applies to any six points in the conic, whether the lines which join them form a hexagon or not.

67. CONFOCAL SURFACES OF THE SECOND ORDER. Two surfaces are said to be confocal when their principal axes have the same directions, and when the squares of the lengths of the corresponding axes differ by the same quantity. Thus, if a^2, b^2, c^2 are the squares of the semi-axes of one surface, then a^2+h, b^2+h, c^2+h are the squares of the semi-axes of another confocal with it. Here h may be positive or negative.

Let ω, ϖ be the appropriate linear vector functions, so that

$$\omega\rho = -\frac{iSi\rho}{a^2} - \frac{jSj\rho}{b^2} - \frac{kSk\rho}{c^2},$$

$$\varpi\rho = -\frac{iSi\rho}{a^2+h} - \frac{jSj\rho}{b^2+h} - \frac{kSk\rho}{c^2+h}.$$

We have
$$\omega i = +\frac{i}{a^2},\ \omega j = +\frac{j}{b^2},\ \omega k = +\frac{k}{c^2},$$

$$\varpi i = +\frac{i}{a^2+h},\ \text{etc., etc.,}$$

or
$$\omega^{-1}i = a^2 i,\ \omega^{-1}j = b^2 j,\ \omega^{-1}k = c^2 k,$$

$$\varpi^{-1}i = (a^2+h)i,\ \varpi^{-1}j = (b^2+h)j,\ \varpi^{-1}k = (c^2+h)k\,;$$

and, by subtraction,

$$(\varpi^{-1}-\omega^{-1})i = hi,\ (\varpi^{-1}-\omega^{-1})j = hj,\ (\varpi^{-1}-\omega^{-1})k = hk,$$

and generally
$$(\varpi^{-1}-\omega^{-1})\rho = h\rho.$$

Let
$$\omega^{-1} = \phi,\ \text{then}\ \omega = \phi^{-1},$$

and
$$\varpi = (\phi+h)^{-1}.$$

Hence $S\rho\phi^{-1}\rho = -1$ and $S\rho(\phi+h)^{-1}\rho = -1$ represent two confocal quadric surfaces. Or generally

$$S\rho(\phi+h)^{-1}\rho = -1$$

represents a series of confocal quadric surfaces with parameter h.

From the expanded semi-Cartesian form or from the theory of Chapter X., we see that, if a point ρ be chosen, there are three distinct surfaces passing through it corresponding to the three roots of the cubic in h.

Certain other properties of confocal surfaces may be established with great ease by means of the quaternion form of the equation.

1. Let
$$S\rho\phi^{-1}\rho+1=0,$$
$$S\rho(\phi+h)^{-1}\rho+1=0,$$
be two confocals having a common tangent joining the point ρ on the one to the point σ on the other. The vector $\rho-\sigma$ is perpendicular to the two normals $\phi^{-1}\rho$, $(\phi+h)^{-1}\sigma$. Hence
$$S(\rho-\sigma)\phi^{-1}\rho=0,\ \ S(\rho-\sigma)(\phi+h)^{-1}\rho=0,$$
or
$$S\rho\phi^{-1}\sigma=-1=S\rho(\phi+h)^{-1}\sigma,$$
and consequently
$$0=S\rho\left\{\frac{1}{\phi}-\frac{1}{\phi+h}\right\}\sigma=hS\rho\phi^{-1}(\phi+h)^{-1}\sigma$$
$$=hS\phi^{-1}\rho(\phi+h)^{-1}\sigma.$$

Hence the normals are perpendicular to one another, and form with the common tangent a rectangular system of vectors.

The theorem will hold however close the two points are, and will still hold when the points coalesce. Consequently any two confocals which meet at a point intersect at right angles; and generally the three confocals which meet at a point form a set of orthogonal surfaces.

This theorem also follows at once from the identity
$$S(\phi+h)^{-1}\rho(\phi+h')^{-1}\rho=S\rho\frac{1}{\phi+h}\,\frac{1}{\phi+h'}\rho$$
$$=\frac{1}{h'-h}S\rho\left\{\frac{1}{\phi+h}-\frac{1}{\phi+h'}\right\}\rho$$
$$=\frac{1}{h'-h}(-1+1)=0.$$

2. Let $S\rho\phi^{-1}\rho+1=0$ be the equation to one of three confocals meeting at a point α. The central section by a plane parallel to the tangent plane at α is given by the two equations
$$\left.\begin{aligned}S\varpi\phi^{-1}\alpha&=\ \ \,0,\\ S\varpi\phi^{-1}\varpi&=-1.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let ϖ be one of the principal axes of the section, and therefore perpendicular to the tangent line τ, drawn from its extremity. This same tangent line being in the plane conjugate to α is perpendicular to $\phi^{-1}\alpha$, and being in the tangent plane to the surface is also perpendicular to $\phi^{-1}\varpi$. Hence $\varpi\phi^{-1}\varpi$ and $\phi^{-1}\alpha$ lie in one plane, and we may write

$$\varpi + h\phi^{-1}\varpi = y\phi^{-1}\alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where h and y are arbitrary constants. This gives at once

$$\phi\varpi + h\varpi = y\alpha,$$

or

$$\varpi = y(\phi + h)^{-1}\alpha.$$

To find h, operate on (2) by $S.\varpi$. The result is

$$\varpi^2 = +h, \text{ or } h = -(T\varpi)^2.$$

Now $(\phi + h)^{-1}\alpha$ is the normal to the confocal passing through α. Hence the principal axis ϖ of the central section conjugate to α in the surface ϕ^{-1} is parallel to the normal to the confocal $\{\phi - (T\varpi)^2\}^{-1}$ which meets the surface ϕ^{-1} at the point α. The other principal axis of the same central section will be parallel in like manner to the normal to the other confocal.

Let ρ be the vector to any point common to the surfaces ϕ^{-1}, $(\phi + h)^{-1}$, and let σ be a vector radius to the surface ϕ^{-1}, parallel to $(\phi + h)^{-1}\rho$, and therefore perpendicular to $\phi^{-1}\rho$. That is

$$(\phi + h)\sigma \parallel \rho,$$

or

$$\sigma + h\phi^{-1}\sigma \parallel \phi^{-1}\rho.$$

Hence, operating by $S.\sigma$, we find

$$\sigma^2 + hS\sigma\phi^{-1}\sigma = 0, \text{ or } h = +\sigma^2.$$

In words, a diameter (2σ) in the quadric ϕ^{-1} drawn parallel to the normal to the confocal $(\phi + h)^{-1}$ at any common point of the two is of constant length.

If $(\phi + h')^{-1}$ represent the other confocal, then $-h$ and $-h'$ are the squares of the semi-axes of the section of ϕ^{-1} conjugate to the vector to the point common to the three confocals. Hence if p is the perpendicular on the tangent plane to ϕ^{-1} at the point ρ,

$$p\sqrt{hh'} = \frac{1}{8} \text{ area of circumscribing parallelepiped, or}$$

$$= abc.$$

The quantity $p\sqrt{h'}$ is the product of the perpendicular to the tangent plane to ϕ^{-1} and the semi-diameter of ϕ^{-1} parallel to the tangent to the line of intersection of ϕ^{-1} and $(\phi+h)^{-1}$ at the point common to the three surfaces. Its value is $abc/\sqrt{h}$ and depends only on the two confocals ϕ^{-1} and $(\phi+h)^{-1}$. Hence it is the same for all points in the line of section of these two surfaces.

3. Given the confocal system $(\phi+h)^{-1}$. The equation $S\rho(\phi+h)^{-1}\varpi+1=0$ is the equation of the polar plane with reference to the point ϖ.

Let this plane be given in position, and let its equation be given in the form $S\rho\alpha+1=0$. Then

$$(\phi+h)^{-1}\varpi=\alpha \text{ or } \varpi=(\phi+h)\alpha.$$

Hence $$V\alpha\varpi=V\alpha\phi\alpha=\gamma,$$

a constant vector.

That is, the poles with reference to a series of confocals of a given plane lie on a straight line parallel to α.

Thus, when a particular plane is regarded as the polar plane with reference to a series of confocals, the corresponding poles lie on a straight line which is perpendicular to the plane, and which is therefore the normal at the point of contact to the particular surface which touches the plane.

4. The equation $S\rho\phi^{-1}\rho=-1$ may be written in the form

$$S\rho\phi^{-\frac{1}{2}}\phi^{-\frac{1}{2}}\rho=S\phi\rho^{-\frac{1}{2}}\phi^{-\frac{1}{2}}\rho=(\phi^{-\frac{1}{2}}\rho)^2=-1,$$

so that $$T\phi^{-\frac{1}{2}}\rho=1.$$

For a confocal surface,

$$T(\phi+h)^{-\frac{1}{2}}\rho=1.$$

Let us consider the relation between the points on these two surfaces for which

$$\phi^{-\frac{1}{2}}\rho=(\phi+h)^{-\frac{1}{2}}\rho_1=\alpha,$$

a constant unit vector.

When this relation is satisfied, the points ρ, ρ_1 on the two confocal surfaces are called *corresponding points.*

Let σ, σ_1 be another pair of corresponding points on the same two surfaces, so that

$$\phi^{-\frac{1}{2}}\sigma = (\phi+h)^{-\frac{1}{2}}\sigma_1 = \beta,$$

another constant unit vector.

These give

$$\rho = \phi^{\frac{1}{2}}\alpha, \quad \rho_1 = (\phi+h)^{\frac{1}{2}}\alpha,$$

$$\sigma = \phi^{\frac{1}{2}}\beta, \quad \sigma_1 = (\phi+h)^{\frac{1}{2}}\beta,$$

whence $$S\rho\sigma_1 = S\sigma\rho_1 = S\alpha\phi^{\frac{1}{2}}(\phi+h)^{\frac{1}{2}}\beta.$$

Also $$\rho^2 - \rho_1{}^2 = S\alpha\phi\alpha - S\alpha(\phi+h)\alpha = h,$$

and $$\sigma^2 - \sigma_1{}^2 = h.$$

Consequently, $$\rho^2 + \sigma_1{}^2 = \rho_1{}^2 + \sigma^2.$$

Adding or subtracting from each side the quantity $2S\rho\sigma_1$ or $2S\rho_1\sigma$, we find

$$T(\rho \pm \sigma_1) = T(\rho_1 \pm \sigma).$$

If we take the negative sign we get Ivory's theorem that the distance between two points, one on each of two confocal ellipsoids, is equal to the distance between the two corresponding points.

If we take the positive sign we have the relation that the other diagonal of the parallelogram formed by the vectors to two points, one on each of two confocal ellipsoids, is equal to the similar diagonal of the parallelogram formed by the vectors to the corresponding points.

We may regard the equation

$$\rho = (\phi+h)^{\frac{1}{2}}\alpha, \text{ with } T\alpha = 1,$$

as representing a system of confocal surfaces.

If α is a constant unit vector, then, as h varies, ρ traces out the locus of a set of corresponding points. Let h change by a small quantity e, while ρ changes by a small vector increment τ, which in the limit will give the direction of the tangent to the curve traced out by ρ. We have

$$\begin{aligned}\rho+\tau &= (\phi+h+e)^{\frac{1}{2}}\alpha\\ &= (\phi+h)^{\frac{1}{2}}\left\{1+\frac{e}{\phi+h}+\dots\right\}^{\frac{1}{2}}\alpha\\ &= (\phi+h)^{\frac{1}{2}}\left\{1+\frac{1}{2}\frac{e}{\phi+h}\right\}\alpha,\end{aligned}$$

neglecting higher powers of e.

Hence $$\tau=\tfrac{1}{2}e(\phi+h)^{-\frac{1}{2}}\alpha=\tfrac{1}{2}e(\phi+h)^{-1}\rho.$$

Thus τ is parallel to the normal to the surface

$$S\rho(\phi+h)^{-1}\gamma=-1.$$

In other words the locus of corresponding points on a series of confocal ellipsoids cuts the ellipsoids orthogonally, and is therefore the line of intersection of two of the confocal hyperboloids.

68. VERSOR EQUATION OF THE ELLIPSE. We shall indicate briefly another quaternion mode of discussing the ellipse, leaving the student to fill in the steps.

As shown in Chapter III., § 25, the operator α^x, where $T\alpha=1$, is a versor which acts on any vector perpendicular to α so as to turn it through the angle $\frac{1}{2}\pi x$. When it operates on any vector β, the product $\alpha^x\beta$ is a quaternion whose vector part, $\rho=V.\alpha^x\beta$, will trace out an ellipse.

Since α^x is of the form $S\alpha^x+V\alpha^x$, ρ is of the form

$$m\beta+nV\alpha\beta,$$

and lies in the plane whose normal is parallel to

$$V\beta V\alpha\beta=\beta S\alpha\beta-\alpha\beta^2.$$

Expanding α^x in the form

$$\cos\frac{\pi x}{2}+\alpha\sin\frac{\pi x}{2},$$

we find $$\rho=\cos\frac{\pi x}{2}.\beta+\sin\frac{\pi x}{2}V\alpha\beta,$$

an ellipse referred to conjugate vector radii.

From last section we may at once write down—or we may deduce from the equation of the tangent—the expression for the vector radius conjugate to ρ, namely

$$\sigma = -\sin\frac{\pi x}{2}\,.\,\beta + \cos\frac{\pi x}{2}V\alpha\beta = V\,.\,\alpha^{x+1}\beta.$$

The vectors β and $V\alpha\beta$ are perpendicular to each other; no value of $T\rho$ can be greater than $T\beta$; hence β and $V\alpha\beta$ are the major and minor semi-axes.

Evidently $V\alpha^x \parallel \alpha = p\alpha$, say.

Then $S\alpha V\alpha^x = -p = S\alpha^{x+1}$,

and $V\alpha^x = -\alpha S\alpha^{x+1}$.

By use of this equality it is easily shown that

$$V\rho\sigma = V\,.\,\beta V\alpha\beta,$$

as in § 52.

69. 1. *The sum of the squares of the areas of the faces of all parallelepipeds, constructed on the semi-conjugate diameters of an ellipsoid, is constant.*

By § 64. 9, $\alpha = -(aiSi\psi\alpha + bjSj\psi\alpha + ckSk\psi\alpha)$

$\beta = -(aiSi\psi\beta + bjSj\psi\beta + ckSk\psi\beta)$;

therefore

$$\begin{aligned} V\alpha\beta = {} & abk(Si\psi\alpha Sj\psi\beta - Si\psi\beta Sj\psi\alpha) \\ & + acj(Si\psi\alpha Sk\psi\beta - Si\psi\beta Sk\psi\alpha) \\ & + bci(Sj\psi\alpha Sk\psi\beta - Sj\psi\beta Sk\psi\alpha). \end{aligned}$$

Now

$$\begin{aligned} Si\psi\alpha Sj\psi\beta - Si\psi\beta Sj\psi\alpha &= SVijV\psi\beta\psi\alpha \\ &= -Sk\psi\gamma; \quad \ldots\ldots\ldots\ldots\ldots\ldots(\S\ 63) \end{aligned}$$

therefore

$$\begin{aligned} V\alpha\beta &= -(abkSk\psi\gamma + acjSj\psi\gamma + bciSi\psi\gamma), \\ V\gamma\alpha &= -(abkSk\psi\beta + acjSj\psi\beta + bciSi\psi\beta), \\ V\beta\gamma &= -(abkSk\psi\alpha + acjSj\psi\alpha + bciSi\psi\alpha). \end{aligned}$$

If now we square and add these expressions, observing that because $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are unit vectors at right angles to one another,

$$(Si\psi\alpha)^2 + (Si\psi\beta)^2 + (Si\psi\gamma)^2 = 1,$$

we shall have

$$(V\alpha\beta)^2 + (V\alpha\gamma)^2 + (V\beta\gamma)^2 = -\{(ab)^2 + (ac)^2 + (bc)^2\},$$

which is the proposition to be proved.

2. *To find the locus of the intersections of tangent planes at the extremities of conjugate diameters of an ellipsoid.*

Let π be the vector to the point of intersection of tangent planes at the extremities of α, β, γ: then

$$\pi = \alpha + \beta + \gamma,$$
$$\phi\pi = \phi\alpha + \phi\beta + \phi\gamma,$$

and consequently

$$\begin{aligned} S\pi\phi\pi &= S(\alpha + \beta + \gamma)(\phi\alpha + \phi\beta + \phi\gamma) \\ &= S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma \\ &= -3, \end{aligned}$$

for all terms of the form $S\alpha\phi\beta$ vanish. This is an ellipsoid similar to the given ellipsoid.

3. *If O, A, B, C, D, E are any six points in space, OX any given direction, OA', OB', OC', OD', OE' the projections of OA, OB, OC, OD, OE on OX; $BCDE$, $CDEA$, $DEAB$, $EABC$, $ABCD$ the volumes of the pyramids whose vertices are B, C, D, E, A, with a positive or negative sign according as the order of the letters naming the angles at the base is right-handed or left-handed as seen from the vertex; then*

$$OA' . BCDE + OB' . CDEA + OC' . DEAB + OD' . EABC + OE' . ABCD = 0.$$

Let OA, OB, OC, OD, OE be α, β, γ, δ, ϵ respectively.

Write for $\alpha S(\gamma - \beta)(\delta - \beta)(\epsilon - \beta)$ its value

$$\alpha(S . \gamma\delta\epsilon - S . \delta\epsilon\beta + S . \epsilon\beta\gamma - S . \beta\gamma\delta),$$

and similar expressions for $\beta S(\alpha - \gamma)(\delta - \gamma)(\epsilon - \gamma)$, etc., and there will result, by addition,

$$\begin{aligned} \alpha S(\gamma - \beta)(\delta - \beta)(\epsilon - \beta) + \beta S(\alpha - \gamma)(\delta - \gamma)(\epsilon - \gamma) \\ + \gamma S(\alpha - \delta)(\beta - \delta)(\epsilon - \delta) + \delta S(\alpha - \epsilon)(\beta - \epsilon)(\gamma - \epsilon) \\ + \epsilon S(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha) = 0, \end{aligned}$$

or, using the notation explained above,

$$OA . BCDE + OB . CDEA + OC . DEAB + OD . EABC + OE . ABCD = 0.$$

Now let π be a vector along OX; then the operation by $S . \pi$ on the above expression gives the result required.

70. In some of the examples which follow, we will endeavour to show how a problem should *not*, as well as how it should, be attacked.

1. *Given any three planes, and the direction of the vector perpendicular to a fourth, to find its length so that they may meet in one point.*

Let $S\alpha\rho = a$, $S\beta\rho = b$, $S\gamma\rho = c$ be the three planes, and let δ be the vector perpendicular to the new plane. Then, if its equation be

$$S\delta\rho = d,$$

we must find the value of d that these four equations may all be satisfied by one value of ρ.

Formula (2), § 34, gives

$$\begin{aligned}\rho S\,.\,\alpha\beta\gamma &= V\alpha\beta S\gamma\rho + V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho\\ &= cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha,\end{aligned}$$

by the equations of the first three. Operate by $S\,.\,\delta$, and use the fourth equation, and we have the required value

$$dS\,.\,\alpha\beta\gamma = aS\,.\,\beta\gamma\delta + bS\,.\,\gamma\alpha\delta + cS\,.\,\alpha\beta\delta.$$

2. *The sum of the (vector) areas of the faces of any tetrahedron, and therefore of any polyhedron, is zero.*

Take one corner as origin, and let α, β, γ be the vectors of the other three. Then the vector areas of the three faces meeting in the origin are

$$\frac{1}{2}V\alpha\beta, \quad \frac{1}{2}V\beta\gamma, \quad \frac{1}{2}V\gamma\alpha, \text{ respectively.}$$

That of the fourth may be expressed in any of the forms

$$\frac{1}{2}V(\gamma-\alpha)(\beta-\alpha), \quad \frac{1}{2}V(\alpha-\beta)(\gamma-\beta), \quad \frac{1}{2}V(\beta-\gamma)(\alpha-\gamma).$$

But all of these have the common value

$$\frac{1}{2}V(\gamma\beta+\beta\alpha+\alpha\gamma),$$

which is obviously the sum of the three other vector-areas taken negatively. Hence the proposition, which is an elementary one in Hydrostatics.

Now any polyhedron may be cut up by planes into tetrahedra, and the faces exposed by such treatment have vector-areas equal and opposite in sign. Hence the extension.

3. *If the pressure be uniform throughout a fluid mass, an immersed tetrahedron (and therefore any polyhedron) experiences no couple tending to make it rotate.*

This is supplementary to the last example. The pressures on the faces are fully expressed by the vector-areas above given, and their points of application are the centres of inertia of the areas of the faces. The co-ordinates of these points are

$$\frac{1}{3}(\alpha+\beta),\quad \frac{1}{3}(\beta+\gamma),\quad \frac{1}{3}(\gamma+\alpha),\quad \frac{1}{3}(\alpha+\beta+\gamma),$$

and the sum of the couples is

$$\frac{1}{6}V.\{V\alpha\beta.(\alpha+\beta)+V\beta\gamma.(\beta+\gamma)+V\gamma\alpha.(\gamma+\alpha)$$
$$+V(\gamma\beta+\beta\alpha+\alpha\gamma).(\alpha+\beta+\gamma)\}$$
$$=-\frac{1}{6}V(V\alpha\beta.\gamma+V\beta\gamma.\alpha+V\gamma\alpha.\beta)=0.$$

4. *What are the conditions that the three planes*

$$S\alpha\rho=a,\quad S\beta\rho=b,\quad S\gamma\rho=c,$$

shall intersect in a straight line?

There are many ways of attacking such a question, so we will give a few for practice.

(*a*)
$$\rho S.\alpha\beta\gamma=V\alpha\beta S\gamma\rho+V\beta\gamma S\alpha\rho+V\gamma\alpha S\beta\rho$$
$$=cV\alpha\beta+aV\beta\gamma+bV\gamma\alpha$$

by the given equations. But this gives a single definite value of ρ unless both sides vanish, so that the conditions are

$$S.\alpha\beta\gamma=0,$$

and
$$cV\alpha\beta+aV\beta\gamma+bV\gamma\alpha=0,$$

which *includes* the preceding.

(*b*)
$$S(l\alpha-m\beta)\rho=al-bm$$

is the equation of any plane passing through the intersection of the first two given planes. Hence, if the three intersect in a straight line there must be values of l, m such that

$$l\alpha - m\beta = \gamma,$$
$$la - mb = c.$$

The first of these gives, as before,

$$S \,.\, \alpha\beta\gamma = 0,$$

and it also gives

$$V\gamma\alpha = mV\alpha\beta, \quad V\beta\gamma = -lV\alpha\beta,$$

so that if we multiply the second by $V\alpha\beta$,

$$laV\alpha\beta - mbV\alpha\beta = cV\alpha\beta$$

becomes $$-aV\beta\gamma - bV\gamma\alpha = cV\alpha\beta\,;$$

the second condition of (a).

(c) Again, suppose ρ to be given by the first two in the form $$\rho = p\alpha + q\beta + xV\alpha\beta,$$

we find $$a = p\alpha^2 + qS\alpha\beta, \text{ because } S\alpha V\alpha\beta = 0,$$
$$b = pS\alpha\beta + q\beta^2\,;$$

therefore

$$(\rho - xV\alpha\beta)\begin{vmatrix} \alpha^2, & S\alpha\beta \\ S\alpha\beta, & \beta^2 \end{vmatrix} = \alpha\begin{vmatrix} a, & S\alpha\beta \\ b, & \beta^2 \end{vmatrix} + \beta\begin{vmatrix} \alpha^2, & a \\ S\alpha\beta, & b \end{vmatrix}$$

so that the third equation gives, operating by $S \,.\, \gamma$,

$$(c - xS\alpha\beta\gamma)\begin{vmatrix} \alpha^2, & S\alpha\beta \\ S\alpha\beta, & \beta^2 \end{vmatrix} = S\alpha\gamma\begin{vmatrix} a, & S\alpha\beta \\ b, & \beta^2 \end{vmatrix} + S\beta\gamma\begin{vmatrix} \alpha^2, & a \\ S\alpha\beta, & b \end{vmatrix}$$

Now a determinate value of x would mean intersection in one point only; so, as before,

$$S \,.\, \alpha\beta\gamma = 0,$$
$$c(\alpha^2\beta^2 - S^2\alpha\beta) = a(\beta^2S\alpha\gamma - S\alpha\beta S\beta\gamma) - b(S\alpha\beta S\alpha\gamma - \alpha^2S\beta\gamma).$$

The latter may be written

$$S \,.\, \alpha[c(\alpha\beta^2 - \beta S\alpha\beta) - a(\gamma\beta^2 - \beta S\beta\gamma) - b(\alpha S\beta\gamma - \gamma S\alpha\beta)] = 0.$$

Now $$S \,.\, \alpha(\alpha\beta^2 - \beta S\alpha\beta) = S\alpha(\beta \,.\, \beta\alpha - \beta S\beta\alpha)$$
$$= S \,.\, \alpha(\beta V\beta\alpha)$$
$$= -S \,.\, \alpha(\beta V\alpha\beta) = -S(\alpha\beta V\alpha\beta)$$

Similarly, $S.\alpha(\gamma\beta^2 - \beta S\beta\gamma) = S(\alpha\beta V\beta\gamma)$,
and
$$S.\alpha(\alpha S\beta\gamma - \gamma S\alpha\beta) = S.\alpha(V.\beta V\gamma\alpha)$$
$$= S(\alpha\beta V\gamma\alpha).$$

The equation now becomes
$$S.\alpha\beta(cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha) = 0.$$

Now since $S.\alpha\beta\gamma = 0$, α, β, γ are vectors in the same plane; therefore γ may be written $m\alpha + n\beta$,

and
$$cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha$$
assumes the form $eV\alpha\beta$, which, unless $e = 0$, gives
$$S(\alpha\beta V\alpha\beta) = 0,$$
or $V\alpha\beta$ is in the same plane with α, β; but it is also perpendicular to the plane, which is absurd; therefore $e = 0$, or
$$cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha = 0;$$
thus the third and prolix method leads to the same conclusion as the first.

5. *Find the surface traced out by a straight line which remains always perpendicular to a given line while intersecting each of two fixed lines.*

Let the equations of the fixed lines be
$$\varpi = \alpha + x\beta, \quad \varpi_1 = \alpha_1 + x_1\beta_1.$$
Then if ρ be the vector of the new line in any position,
$$\rho = \varpi + y(\varpi_1 - \varpi)$$
$$= (1 - y)(\alpha + x\beta) + y(\alpha_1 + x_1\beta_1).$$
This is not, as yet, the equation required. For it involves essentially *three* independent constants, x, x_1, y; and may therefore in general be made to represent any point whatever of infinite space. The reader may easily see this if he reflects that two lines which are not parallel must appear, from every point of space, to intersect one another. We have still to introduce the condition that the new line is perpendicular to a fixed vector, γ suppose, which gives
$$S.\gamma(\varpi_1 - \varpi) = 0 = S.\gamma[(\alpha_1 - \alpha) + x_1\beta_1 - x\beta].$$

This gives x_1 in terms of x, so that there are now but two indeterminates in the equation for ρ, which therefore represents a surface, which, it is not difficult to see, is one of the second order.

6. *Find the condition that the equation*

$$S.\rho\phi\rho = -1$$

may represent a surface of revolution.

The expression $\phi\rho$ here stands for something more general than that employed in Chap. VII. above, in fact it may be written

$$\phi\rho = \alpha S\alpha_1\rho + \beta S\beta_1\rho + \gamma S\gamma_1\rho,$$

where α, α_1, β, β_1, γ, γ_1 are any six vectors whatever. This will be more carefully examined in the next chapter.

If the surface be one of revolution then, since it is central and of the second degree, it is obvious that any sphere whose centre is at the origin will cut it in two equal circles in planes perpendicular to the axis, and that these will be equidistant from the origin. Hence, if r be the radius of one of these circles, ϵ the vector to its centre, ρ the vector to any point in its circumference, it is evident that we have the following equation,

$$S\rho\phi\rho + 1 - C(\rho^2 + r^2) = (S\epsilon\rho)^2 - e^2,$$

where C and e are constants. This, being an identity, gives

$$\left.\begin{array}{l} 1 + e^2 - Cr^2 = 0 \\ S\rho\phi\rho - C\rho^2 = (Se\rho)^2 \end{array}\right\}.$$

The form of these equations shows that C is an absolute constant, while r and e are related to one another by the first; and the second gives

$$\phi\rho = C\rho + \epsilon S\epsilon\rho.$$

This shows simply that $S.\epsilon\rho\phi\rho = 0$,

i.e. ϵ, ρ, and $\phi\rho$ are coplanar, *i.e.* all the normals pass through a given straight line; or that the expression

$$V\rho\phi\rho,$$

whatever be ρ, expresses always a vector parallel to a particular plane.

7. *If three mutually perpendicular vectors be drawn from a point to a plane, the sum of the reciprocals of the squares of their lengths is independent of their directions.*

Let
$$S\epsilon\rho = -1$$
be the equation of the plane, and let α, β, γ be any set of mutually perpendicular unit-vectors. Then, if $x\alpha$, $y\beta$, $z\gamma$ be points in the plane, we have

$$xS\alpha\epsilon = -1,\ \ yS\beta\epsilon = -1,\ \ zS\gamma\epsilon = -1,$$

whence $\epsilon = -(\alpha S\alpha\epsilon + \beta S\beta\epsilon + \gamma S\gamma\epsilon)$ (64. 2) $= \frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z}$.

Taking the tensor, we have

$$T\epsilon^2 = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}.$$

8. *Find the equation of the straight line which meets, at right angles, two given straight lines.*

Let
$$\varpi = \alpha + x\beta,\ \ \varpi = \alpha_1 + x_1\beta_1,$$
be the two lines; then the equation of the required line must be of the form

$$\varpi = \alpha_2 + x_2 V\beta\beta_1,$$

where α_2 only needs to be determined.

Since the first and third equations denote lines having one point in common, we have

$$S\,.\,\beta\ V\beta\beta_1(\alpha - \alpha_2) = 0.$$

Similarly
$$S\,.\,\beta_1 V\beta\beta_1(\alpha_1 - \alpha_2) = 0.$$

Let
$$\alpha_2 = y\beta + y_1\beta_1$$
(it is obviously superfluous to add a term in $V\beta\beta_1$), then

$$S\,.\,\alpha\beta V\beta\beta_1 = y_1 T^2 V\beta\beta_1,$$
$$S\,.\,\alpha_1\beta_1 V\beta\beta_1 = -yT^2 V\beta\beta_1,$$

and, finally,

$$\varpi = \frac{1}{T^2 V\beta\beta_1}(\beta_1 S\,.\,\alpha\beta V\beta\beta_1 - \beta S\,.\,\alpha_1\beta_1 V\beta\beta_1) + x_2 V\beta\beta_1.$$

9. *If* $T\rho = T\alpha = T\beta = 1$, *and* $S.\alpha\beta\rho = 0$, *show that*

$$S.U(\rho-\alpha)U(\rho-\beta) = \sqrt{\frac{1}{2}(1 - S\alpha\beta)}.$$

Interpret this theorem geometrically.

We have, from the given equations, the following, which are equivalent to them,

$$\left.\begin{array}{r}\rho^2 = \alpha^2 = \beta^2 = -1 \\ \rho = x\alpha + y\beta\end{array}\right\}.$$

Hence $$-x^2 - y^2 + 2xyS\alpha\beta = -1,$$

$$U(\rho-\alpha) = \frac{(x-1)\alpha + y\beta}{\sqrt{(x-1)^2 - 2(xy-y)S\alpha\beta + y^2}},$$

$$U(\rho-\beta) = \frac{x\alpha + (y-1)\beta}{\sqrt{x^2 - 2(xy-x)S\alpha\beta + (y-1)^2}},$$

$$S.U(\rho-\alpha)U(\rho-\beta)$$

$$= \frac{-x(x-1) + [xy + (x-1)(y-1)]S\alpha\beta - y(y-1)}{\sqrt{x^2+y^2-2x+1-2(xy-y)S\alpha\beta}\sqrt{x^2+y^2-2y+1-2(xy-x)S\alpha\beta}}$$

$$= \frac{x+y-(x+y-1)S\alpha\beta - 1}{\sqrt{2-2x+2yS\alpha\beta}\sqrt{2-2y+2xS\alpha\beta}}$$

$$= \frac{(x+y-1)(1-S\alpha\beta)}{2\sqrt{(1-x-y)(1-S\alpha\beta) + xy\{1-(S\alpha\beta)^2\}}}$$

$$= \frac{x+y-1}{2}\sqrt{\frac{1-S\alpha\beta}{1-x-y+xy(1+S\alpha\beta)}}$$

$$= \frac{x+y-1}{2}\sqrt{\frac{1-S\alpha\beta}{1-x-y+\frac{1}{2}(2xy+x^2+y^2-1)}}$$

$$= \frac{x+y-1}{\sqrt{2}}\sqrt{\frac{1-S\alpha\beta}{1-2(x+y)+x^2+y^2+2xy}}$$

$$= \pm\sqrt{\frac{1}{2}(1-S\alpha\beta)}.$$

Of course there are far simpler solutions. Thus, for instance, the given equations show that ρ, α, β are radii of some unit *circle.* Hence the expression is the cosine of the supplement of the angle between two chords of a circle drawn from the same point in the circumference. This is obviously half the angle subtended at the centre by radii drawn to the other ends of the chords. The cosine of this angle is

$$-S\alpha\beta,$$

and therefore the cosine of its half is

$$\sqrt{\frac{1}{2}(1-S\alpha\beta)}.$$

10. *Find the relative position, at any instant, of two points, which are moving uniformly in straight lines.*

If α', β' be their velocities, t the time elapsed since their vector positions were α, β, their relative vector is

$$\begin{aligned}\rho &= \alpha + t\alpha' - \beta - t\beta' \\ &= (\alpha - \beta) + t(\alpha' - \beta'),\end{aligned}$$

so that relatively to one another the motion is rectilinear, and the relative velocity is

$$\alpha' - \beta'.$$

To find the time at which the mutual distance is least.

Here we may write

$$\begin{aligned}\rho &= \gamma + t\delta, \\ T\rho^2 &= -\gamma^2 - 2tS\gamma\delta - t^2\delta^2 \\ &= \frac{(S\gamma\delta)^2}{\delta^2} - \gamma^2 - \delta^2\left(t + \frac{S\gamma\delta}{\delta^2}\right)^2.\end{aligned}$$

As the last term is positive, this expression is least when it vanishes, *i.e.* when

$$t = -S\,.\,\gamma\delta^{-1}.$$

This gives

$$\begin{aligned}\rho &= \gamma - \delta S\gamma\delta^{-1} \\ &= \gamma V\delta^{-1}\gamma,\end{aligned}$$

the vector perpendicular drawn to the relative path; as is, of course, self-evident.

11. *Find the locus of a given point in a line of given length, when the extremities of the line move in circles in one plane.* (Watt's *Parallel Motion.*)

Let σ and τ be the vectors of the ends of the line, drawn from the centres α, β of the circles. Then if ρ be the vector of the required point

$$\rho = (\alpha + \sigma)(1 - e) + e(\beta + \tau),$$

subject to the conditions

$$\{\alpha + \sigma - (\beta + \tau)\}^2 = -l^2,$$
$$S\gamma\sigma = 0, \quad S\gamma\tau = 0,$$
$$\sigma^2 = -a^2, \quad \tau^2 = -b^2.$$

From these equations σ and τ must be eliminated. We leave the work to the reader. There is obviously an equation of condition

$$S \, . \, \gamma(\beta - \alpha) = 0.$$

12. *Classify the curves represented by an equation of the form*

$$\rho = \frac{\alpha + x\beta + x^2\gamma}{a + bx + cx^2},$$

where α, β, γ *are given vectors, and* a, b, c *given scalars.*

In the first place we remark that x^2 in the numerator merely adds a constant vector to the value of ρ, unless $c = 0$.

Thus, if c do not vanish, the equation may be written, with a change of α and β and in general a change of origin,

$$\rho = \frac{\alpha + x\beta}{a + bx + cx^2};$$

and this again, by change of x and of α and β, as

$$\rho = \frac{\alpha + x\beta}{a + cx^2}.$$

It is obvious that this represents a plane curve.

Also $$\frac{S\alpha\rho}{S\beta\rho} = \frac{\alpha^2 + xS\alpha\beta}{S\alpha\beta + x\beta^2}.$$

Hence both numerator and denominator of x are of the first degree in $S\alpha\rho$, $S\beta\rho$; and therefore

$$S\alpha\rho = \frac{a^2 + xS\alpha\beta}{a + cx^2}$$

gives an equation of the third degree in ρ by the elimination of x.

When we have $$S\alpha\beta = 0,$$

$$S\alpha\rho = \frac{a^2}{a + cx^2},$$

$$S\beta\rho = \frac{x\beta^2}{a + cx^2},$$

whence $$x = \frac{a^2 S\beta\rho}{\beta^2 S\alpha\rho},$$

and $$a(S\alpha\rho)^2 + c\frac{a^4}{\beta^4}(S\beta\rho)^2 = a^2 S\alpha\rho,$$

a conic section.

If $c = 0$, then with a change of x, α, β, γ, the equation may be written

$$\rho = \frac{\alpha}{x} + \beta + x\gamma,$$

a hyperbola—so long at least as b does not also vanish.

If b and c both vanish, the equation is obviously that of a parabola.

If a and b both vanish, whilst c has a real value, we have again a parabola.

If a vanish while b and c have real values, we have again a hyperbola.

13. *Find the locus of a point at which a given finite straight line subtends a given angle.*

Take the middle point of the line as origin, and let $\pm\alpha$ be the vectors of its ends. At ρ it subtends an angle whose cosine is

$$-SU(\rho - \alpha)U(\rho + \alpha).$$

This, equated to a constant, gives the locus required. We may write the equation

$$\alpha^2 - \rho^2 = cT(\rho - \alpha)T(\rho + \alpha).$$

This is, obviously, a surface of the fourth order; a ring or tore formed by the rotation of a circle about a chord. When $c = 0$, *i.e.* when the angle is a right angle, the two sheets of this surface close up into the sphere

$$\rho^2 = \alpha^2.$$

A plane section (in the plane α, β (suppose) where $T\beta = T\alpha$ and $S\alpha\beta = 0$) gives

$$\rho = x\alpha + y\beta,$$

$$\{\alpha^2(1 - x^2) - y^2\alpha^2\}^2 = c^2\{(x-1)^2 + y^2\}\{(x+1)^2 + y^2\}\alpha^4,$$

or
$$\{1 - (x^2 + y^2)\}^2 = c^2\{(x^2 + y^2 + 1)^2 - 4x^2\},$$

or, finally,
$$1 - (x^2 + y^2) = \pm\frac{2cy}{\sqrt{1 - c^2}},$$

which, of course, denotes two equal circles intersecting at the ends of the fixed line.

14. *A ray of light falls on a thin reflecting cylinder, show that it is spread over a right cone.*

Let α be the ray, τ a normal to the cylinder, ρ a reflected ray, β the axis of the cylinder.

Then τ is perpendicular to β, or

$$S\beta\tau = 0. \quad \dots\dots\dots\dots\dots\dots(1)$$

Again ρ and α make equal angles with τ, on opposite sides of it, in one plane; therefore

$$\rho \parallel \tau\alpha\tau$$

or
$$V.\tau\alpha\tau\rho = 0. \quad \dots\dots\dots\dots\dots\dots(2)$$

Eliminating τ between (1) and (2) we have

$$\frac{\rho^2}{\alpha^2} = \left(\frac{S\beta\rho}{S\alpha\beta}\right)^2,$$

the equation of the right cone of which β is the axis, and α a side.

EXAMPLES TO CHAPTER VIII.

1. Find the equation of the surface described by a straight line which rotates about a fixed axis, the axis and straight line not being in the same plane.

2. Find the locus of a point whose shortest distances from two straight lines have a constant ratio.

3. Find the equation of a sphere circumscribing a given tetrahedron.

4. A straight line intersects a fixed line at right angles and turns uniformly about it while it slides uniformly along it. Find the equation of the surface described (1) when the fixed line is straight, (2) when it is circular.

5. Find the equation of the surface described by a circle which is made to rotate about any chosen axis in its plane.

6. Show that the equation $S\rho\phi\rho = -1$ may be expressed in the following forms:

$$S\rho(g\rho + V\lambda\rho\mu) = -1,$$

λ and μ being normals to the circular sections, and g a scalar constant;

$$a(V\alpha\rho)^2 + b(S\beta\rho)^2 = -1,$$

where a and b are constant scalars, and α, β constant vectors; and

$$T^2(\iota\rho + \rho\kappa) = (\kappa^2 - \iota^2)^2,$$

ι, κ being two vector constants, which are real only when the equation is that of the ellipsoid.

7. Show that the equation of the surface generated by lines drawn through the origin parallel to the normals to $S\rho\phi^{-1}\rho = -1$ along its lines of intersection with the confocal surface $(\phi + h)^{-1}$ is

$$\varpi^2 - hS\varpi(\phi + h)^{-1}\varpi = 0.$$

8. Show that the equation

$$l^2(e^2 - 1)(e + S\alpha\alpha') = (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2 + (1 - e^2)\rho^2,$$

where e is a variable scalar parameter, α α' unit vectors, and l a given scalar, represents a system of confocal surfaces.

9. Find the positions of the generating lines through any point of the hyperboloid $S\rho\phi\rho = -1$.

10. Find the locus of all points on $S\rho\phi\rho = -1$ where the normals meet the normal at the point α.

CHAPTER IX.

DYNAMICAL APPLICATIONS.

71. DIFFERENTIATION OF QUATERNIONS. In the following dynamical applications we shall assume the simpler processes of differentiation and integration as in ordinary analysis. In general, time will be the independent variable flowing continuously; and in terms of it the rates of change of other varying quantities are expressed. When a scalar quantity, such as x, is varying continuously its rate of variation at any instant will have a definite value, and this we shall, following Newton, represent by the notation $\dot{x}$. The more usual notation dx will also be used when necessary.

There is no difficulty in extending the methods of the Differential Calculus to quaternions and functions of quaternions if we bear in mind the non-commutative character of quaternion products.

For example, if $q = \alpha\beta$, then the rate of change is

$$\dot{q} = \dot{\alpha}\beta + \alpha\dot{\beta} = V(\dot{\alpha}\beta + \alpha\dot{\beta}) + S(\dot{\alpha}\beta + \alpha\dot{\beta}).$$

Hence it follows that the symbol of differentiation is commutative with the selective symbols V and S. Thus

$$dV\rho\sigma = V(d\rho \, . \, \sigma) + V\rho d\sigma.$$

Again the rate of change of the product pq is $\dot{p}q + p\dot{q}$.

An interesting case is the rate of change of q^2 or qq. Its value is

$$q\dot{q} + \dot{q}q = 2Sq\dot{q}.$$

Similarly $$\frac{d}{dt}(\rho^2) = 2S\rho\dot{\rho}.$$

But since $$\rho^2 = -(T\rho)^2,$$
its rate of change may also be expressed in the form
$$-2T\rho\frac{d}{dt}(T\rho) = -2T\rho\dot{T}\rho.$$

Hence $$\dot{T}\rho = -\frac{1}{T\rho}S\rho\dot{\rho} = -S\dot{\rho}U\rho.$$

This symbolizes the obvious geometrical truth that the rate at which the *length* of ρ changes is the resolved part in the direction of ρ of its complete vectorial rate of change (see Fig. 36).

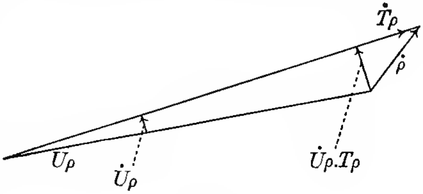

Fig. 36.

We may also derive geometrically the value of $\frac{d}{dt}(U\rho)$ or $\dot{U}\rho$. For by comparison of two different expressions for the vector area of the triangle shown in the figure, we obtain in the limit
$$V.\rho\dot{U}\rho T\rho = V.\rho\dot{\rho}.$$

But $\dot{U}\rho$ being tangential to the sphere traced by $U\rho$ is obviously perpendicular to ρ; hence we may drop the symbol V on the left hand side, and the result is
$$\rho\dot{U}\rho = -V.\dot{\rho}U\rho,$$
$$\dot{U}\rho = -\rho^{-1}V.\dot{\rho}U\rho = \frac{V\dot{\rho}U\rho}{\rho}.$$

This result ($\dot{T}\rho$ being assumed) may also be obtained from the identity
$$\dot{\rho} = \frac{d}{dt}(T\rho U\rho) = \dot{T}\rho U\rho + T\rho\dot{U}\rho.$$

These examples will suffice to indicate the precautions that must be taken in differentiating quaternion quantities.

72. DYNAMICS OF THE CENTRE OF MASS. If A, B are successive positions of a moving particle, the change of position is the vector AB, and the velocity will be the limiting ratio of this vector to the time taken as the distance AB is made smaller and smaller. Hence if ρ represents the vector position of a particle, its velocity $\dot{\rho}$ will be a tangent to the path.

Draw from any origin the quantities $\dot{\rho}$ in every position ρ. Then as the particle describes its path, the end of the velocity $\dot{\rho}$ will describe a curve. This curve is called the Hodograph, and its radius vector σ is equal to $\dot{\rho}$.

Now just as $\dot{\rho}$ represents the rate of change of ρ, so will $\dot{\sigma}$ or $\ddot{\rho}$ represent the rate of change of $\dot{\rho}$. The same process which gives $\dot{\rho}$ from ρ gives $\ddot{\rho}$ from $\dot{\rho}$. The quantity $\ddot{\rho}$ is the acceleration.

Introducing the mass of the particle, we have

$m\rho$, the mass vector,

$m\dot{\rho}$, the mass velocity or momentum,

$m\ddot{\rho}$, the mass acceleration or the force.

When there are a number of particles forming a system, free or connected in any way, the vector position of the centre of mass is given by the equation (§ 13)

$$\sigma\Sigma m = \Sigma(m\rho), \text{ ....(1)}$$

and its velocity and acceleration are

$$\dot{\sigma} = \frac{\Sigma m\dot{\rho}}{\Sigma m}, \quad \ddot{\sigma} = \frac{\Sigma m\ddot{\rho}}{\Sigma m}.$$

Each particle may be supposed to be acted upon by an externally-applied force γ, and to be subject to an internal force γ' due to the stresses between it and the other particles of the system. For each particle

$$m\ddot{\rho} = \gamma + \gamma'. \text{ ....(2)}$$

Hence for the whole system

$$\ddot{\sigma}\Sigma m = \Sigma(m\ddot{\rho}) = \Sigma(\gamma + \gamma').$$

But since between any two particles the mutual action consists of equal and opposite forces (Newton's Lex III.), it follows that when the whole system is taken into account

$$\Sigma\gamma' = 0.$$

Hence $$\ddot{\sigma}\Sigma m = \Sigma\gamma, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

or the centre of mass moves as if the whole mass were condensed there and acted upon by the vector sum (or resultant) of all the external forces acting on the system.

Again, if we operate on (2) by $V\,.\,\rho$, we get

$$mV\rho\ddot{\rho} = V\,.\,\rho(\gamma + \gamma'),$$

and summing for the whole system

$$\Sigma mV\rho\ddot{\rho} = \Sigma\,.\,V\rho\gamma,$$

$\Sigma V\rho\gamma'$ vanishing if we suppose that the forces between the particles are in the lines joining them. Putting $\rho = \sigma + \varpi$, where ϖ is the position referred to the centre of mass, we find

$$\Sigma mV(\sigma + \varpi)(\ddot{\sigma} + \ddot{\varpi}) = \Sigma\,.\,V(\sigma + \varpi)\gamma.$$

Since σ, $\ddot{\sigma}$ refer to a definite point they may be taken outside the summation symbol, and since the quantities $\Sigma m\varpi$, $\Sigma m\ddot{\varpi}$ vanish by (1), we find

$$V\sigma\ddot{\sigma}\Sigma m + \Sigma mV\varpi\ddot{\varpi} = V\sigma\Sigma\gamma + \Sigma V\varpi\gamma,$$

whence by (3) $$\Sigma mV\varpi\ddot{\varpi} = \Sigma V\varpi\gamma. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

But $$\frac{d}{dt}V\varpi\dot{\varpi} = V\,.\,\dot{\varpi}\dot{\varpi} + V\,.\,\varpi\ddot{\varpi} = V\varpi\ddot{\varpi},$$

since $\dot{\varpi}\dot{\varpi}$ or $\dot{\varpi}^2$ is essentially scalar. Hence we may write (4) in the form

$$\frac{d}{dt}\Sigma mV\varpi\dot{\varpi} = \Sigma V\varpi\gamma. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In words, the moment of the applied forces about the centre of mass is equal to the rate of change of the moment of momentum of the system about the same centre.

73. RIGID BODY WITH ONE POINT FIXED. Let the system be a rigid body with one point fixed. Then (§ 30) the displacement of any point due to the resultant (small) angular displacement e about axis ϵ is

$$\rho' - \rho = Ve\epsilon\rho.$$

Dividing by the short interval of time and passing to the limit, we find for the velocity of the point ρ the expression

$$\dot{\rho} = V\omega\rho,$$

where ω represents the angular velocity, that is the angular speed $T\omega$ about the axis parallel to $U\omega$.

The momentum of the mass m at this point is $mV\omega\rho$, and the moment of momentum is $V.m\rho V\omega\rho$, or simply $m\rho V\omega\rho$, since identically $S.\rho V\omega\rho = 0$. Thus the moment of momentum of the whole mass is

$$\mu = \Sigma m\rho V\omega\rho = \phi\omega, \qquad \text{.......................(1)}$$

where ϕ is evidently a self-conjugate linear vector function depending on the distribution of matter in the body (see §§ 45, 57, 64, but especially next chapter).

To find its significance, operate by $S.\omega$, and we find

$$S\omega\phi\omega = \Sigma m(V\omega\rho)^2 = \Sigma m(VU\omega\rho)^2T^2\omega.$$

Now $\Sigma m(TVU\omega\rho)^2$ is the moment of inertia about the axis ω and $T\omega$ is the angular speed. Hence $-S\omega\phi\omega$ represents twice the kinetic energy of rotation.

If no couples act on the body, the kinetic energy and moment of momentum are each constant; hence

$$\left.\begin{aligned} S\omega\phi\omega &= -e, \\ S\omega\phi^2\omega &= \mu^2 \end{aligned}\right\}, \qquad \text{.......................(2)}$$

and the intersection of these two ellipsoids gives the cone in space described by the axis of spin. Its equation is

$$S\omega\phi(\mu^2 + e\phi)\omega = 0.$$

The second equation of (2) may also be written, $T\mu = T\phi\omega$; so that the perpendicular on the tangent plane at the extremity of ω to the ellipsoid ϕ is constant. Hence this ellipsoid rolls on a fixed plane perpendicular to μ.

74. THE SPINNING TOP. In the case of the ordinary spinning top the couple acting on the body may be written $V\beta\alpha$, where β is a unit vector drawn along the axis which passes through the centre of mass, and α the vector drawn vertically downwards with tensor equal to the product of the weight of the body and the distance of the centre of gravity from the origin.

Since β is the vector position of a point in the body

$$\dot{\beta} = V\omega\beta. \qquad\qquad (1)$$

The dynamic equation is

$$V\beta\alpha = \dot{\mu} = \frac{d}{dt}\Sigma m V\rho\dot{\rho} = \Sigma m V\rho\ddot{\rho}.$$

Hence, since $\quad \dot{\rho} = V\omega\rho, \;\; \ddot{\rho} = V\dot{\omega}\rho + V\omega\dot{\rho},$

we find, after a slight transformation,

$$\begin{aligned} V\beta\alpha &= \Sigma m\rho V\dot{\omega}\rho + \Sigma m V\,.\,\omega V\rho V\omega\rho \\ &= \phi\dot{\omega} + V\omega\phi\omega. \qquad\qquad (2) \end{aligned}$$

Operating by $S\,.\,\omega$, and using (1), we get

$$S\dot{\beta}\alpha = S\omega\beta\alpha = S\omega\phi\dot{\omega} = \tfrac{1}{2}\frac{d}{dt}S\omega\phi\omega,$$

because the term $V\omega\phi\omega$ in $\frac{d}{dt}(\phi\omega)$ vanishes when operated on by $S\,.\,\omega$. Thus we get the energy equation in the form

$$-\tfrac{1}{2}S\omega\phi\omega + S\beta\alpha = h^2, \text{ a constant.} \qquad\qquad (3)$$

This of course could have been written down at once.

Since ϕ is a self-conjugate linear vector function, we may put

$$\phi\omega = -AiSi\omega - BjSj\omega - CkSk\omega,$$

where $i\ j\ k$ are unit vectors fixed in the body; and from the meaning of $S\omega\phi\omega$ we readily deduce that these unit vectors are parallel to the principal axes of inertia and that $A\ B\ C$ are the corresponding moments of inertia.

For simplicity, as in the case of the ordinary symmetrical spinning top, let $A = B$; then since

$$\omega = -iSi\omega - jSj\omega - kSk\omega,$$

we get, multiplying by A and subtracting from the former expression, the particular form for $\phi\omega$, namely,

$$\begin{aligned}\phi\omega &= A\omega - (C-A)kSk\omega \\ &= A\omega - (C-A)\beta S\beta\omega,\end{aligned}$$

if we take $$k=\beta.$$

Equation (2) becomes, by differentiation of $\phi\omega$, and by use of (1),

$$V\beta\alpha = A\dot{\omega} - (C-A)\beta S\beta\dot{\omega} - (C-A)V\omega\beta S\beta\omega. \ \ldots\ldots(2)$$

Operate by $S\,.\,\beta$, and there results

$$0 = AS\beta\dot{\omega} + (C-A)S\beta\dot{\omega} = CS\beta\dot{\omega}.$$

But by (1) $S\dot{\beta}\omega = 0$, hence

$$\frac{d}{dt}(S\beta\omega) = S\dot{\beta}\omega + S\beta\dot{\omega} = 0,$$

and $S\beta\omega$, which measures the angular speed about the axis of figure, is constant.

Equation (2) takes the simplified form,

$$\begin{aligned}V\beta\alpha &= A\omega - (C-A)V\omega\beta S\beta\omega \\ &= A\dot{\omega} + (C-A)\dot{\beta}\,.\,c, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)\end{aligned}$$

where c $(= -S\beta\omega)$ is the constant angular speed about the axis of figure.

By use of $$\omega\beta = V\omega\beta + S\omega\beta = \dot{\beta} - c, \ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

we may eliminate either β or ω, and find the equation satisfied by ω or β. The ω equation is somewhat complex (see Tait's *Scientific Papers*, Vol. I., p. 126); but that in β is comparatively simple and is easily obtained. For, multiplying into β and taking the vector part, we find

$$\omega = -V\dot{\beta}\beta + \beta c,$$

and $$\dot{\omega} = -V\ddot{\beta}\beta + \dot{\beta}c.$$

Hence substituting in (4), we get

$$AV\beta\ddot{\beta} + Cc\dot{\beta} = V\beta\alpha. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

If the second term be omitted, the equation becomes identical in form with the equation of motion of the conical pendulum.

Operate on (6) in succession by $S\,.\,V\beta\dot{\beta}$, $S\,.\,\alpha$, and $S\,.\,V\alpha\beta$, and integrate the first two. There result

$$\tfrac{1}{2}AV^2\beta\dot{\beta} = S\alpha\beta + H^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$AS\alpha\beta\dot{\beta} = H' - CcS\alpha\beta, \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$AS\alpha(-\ddot{\beta} - \beta S\beta\ddot{\beta}) + CcS\alpha\beta\dot{\beta} + V^2\beta\alpha = 0. \quad \ldots\ldots\ldots(9)$$

Eliminating $S\alpha\beta$ between (7) and (8), we find that the vector $V\beta\dot{\beta}$ describes a curve on a spherical surface whose centre lies in the vertical line α.

Equation (7) is one form of the energy equation (3), and may be written

$$+\tfrac{1}{2}A\dot{\beta}^2 = S\alpha\beta + H^2, \text{ since } S\beta\dot{\beta} = 0.$$

Also, differentiating $S\beta\dot{\beta} = 0$, we find

$$S\beta\ddot{\beta} = -\dot{\beta}^2 = +\frac{2S\alpha\beta + 2H^2}{A}.$$

Hence substituting in (9) from (7) and (8), and making a few transformations, we get

$$AS\alpha\ddot{\beta} = 3S^2\alpha\beta + 2S\alpha\beta\left(H^2 - \frac{C^2c^2}{2A}\right) + \frac{Cc}{A} + \alpha^2,$$

and finally, multiplying by $S\alpha\dot{\beta}$ and integrating,

$$\tfrac{1}{2}AS^2\alpha\dot{\beta} = S^3\alpha\beta + S^2\alpha\beta\left(H^2 - \frac{C^2c^2}{2A}\right) + S\alpha\beta\left(\frac{Cc}{A} + \alpha^2\right) + K.$$

If we put $S\alpha\dot{\beta} = 0$, we get the usual cubic for determining the limiting positions of the top.

Returning to the fundamental equation (6), let us study the simple case in which the precessional motion is steady, the axis β describing a right cone about α.

Evidently $\dot{\beta} \parallel V\beta\alpha$; and if a is the precessional angular speed,

$$T\dot{\beta} = aTV\beta U\alpha = \frac{a}{T\alpha}TV\beta\alpha.$$

Hence
$$\dot\beta = \frac{a}{T\alpha}V\beta\alpha,$$

$$\ddot\beta = \frac{a}{T\alpha}V\dot\beta\alpha = -\frac{a^2}{T^2\alpha}V\alpha V\beta\alpha$$
$$= -\frac{a^2}{T^2\alpha}(\alpha S\alpha\beta - \beta\alpha^2),$$

and
$$V\beta\ddot\beta = -\frac{a^2}{T^2\alpha}V\beta\alpha S\alpha\beta.$$

Hence equation (6) becomes in this case
$$-A\frac{a^2}{T^2\alpha}S\alpha\beta + Cc\frac{a}{T\alpha} = 1,$$

or
$$-Aa^2\cos\theta + Cca - T\alpha = 0,$$

where θ is the inclination between β and $-\alpha$. Writing $T\alpha = mgh$, we get the usual quadratic equation expressing the precessional angular speed a in terms of the rate of rotation c about the principal axis, namely,
$$Aa^2\cos\theta - Cca + mgh = 0.$$

75. MUTUAL ACTION OF MAGNETS. When a magnet with pole-strength m and vector length μ/m is placed in any position in a uniform field of force β, the couple acting on it is
$$mV\beta\frac{\mu}{m} = V\beta\mu,$$
tending to bring μ parallel to β.

The quantity μ, the product of the pole strength into the distance between the poles, is called the magnetic moment of the magnet.

The work done in moving the magnet so that the positive pole moves against β, and the negative pole with β, is
$$-mS\beta\frac{d\mu}{m} = -S\beta d\mu.$$

But the work done in bringing from infinity the positive and negative poles to such positions that the vector μ lies perpendicular to β is evidently zero. Hence the integral of $S\beta d\mu$, namely, $S\beta\mu$, measures the potential energy of the magnet μ in the field β, being equal to the minimum value $-T\beta T\mu$ when μ is co-directional with β, to the maximum value $+T\beta T\mu$, when μ is turned the other way, and to zero when μ is perpendicular to β.

Let the field β be due to a second magnet λ with its centre at the origin; and let ρ be the vector position of the centre μ. Both magnets are supposed to be short compared to their distance apart. Considering the action of the individual positive and negative poles at the extremities of the short vector λ/n, n being the strength of the positive pole, and assuming the law of the inverse square, we have for the force at the point ρ,

$$\beta = n\left\{\frac{U(\rho-\lambda/2n)}{T^2(\rho-\lambda/2n)} - \frac{U(\rho+\lambda/2n)}{T^2(\rho+\lambda/2n)}\right\}$$

$$= n\left\{\frac{\rho-\lambda/2n}{(-\rho^2+S\rho\lambda/n)^{\frac{3}{2}}} + \frac{\rho+\lambda/2n}{(\rho^2+S\rho\lambda/n)^{\frac{3}{2}}}\right\},$$

neglecting $(\lambda/2n)^2$ in comparison with ρ^2. Expanding each denominator by the binomial theorem, we find

$$\beta = \frac{n}{T^3\rho}\left\{\left(\rho-\frac{\lambda}{2n}\right)\left(1+\frac{3}{2n}S\lambda\rho^{-1}\right)-\left(\rho+\frac{\lambda}{2n}\right)\left(1-\frac{3}{2n}S\lambda\rho^{-1}\right)\right\}$$

$$= \frac{1}{T^3\rho}(-\lambda+3\rho S\lambda\rho^{-1}). \quad\text{.......................................(1)}$$

Hence the couple acting on μ because of λ is

$$V\beta\mu = \frac{1}{T^3\rho}(V\mu\lambda + 3V\rho\mu S\lambda\rho^{-1}). \quad\text{...............(2)}$$

Similarly, the couple acting on λ because of μ is

$$\frac{1}{T^3\rho}(V\lambda\mu + 3V\rho\lambda S\mu\rho^{-1}). \quad\text{..................(3)}$$

The mutual potential energy of the two magnets is

$$S\beta\mu = \frac{1}{T^3\rho}(-S\lambda\mu + 3S\mu\rho S\lambda\rho^{-1})$$

$$= \frac{1}{T^3\rho}(-S\lambda\mu - 3S\mu U\rho S\lambda U\rho)$$

$$= -\frac{S\lambda\mu}{T^3\rho} - \frac{3S\mu\rho S\lambda\rho}{T^5\rho}.$$

To find the translation force acting on either magnet, calculate the increment of this expression when ρ becomes $\rho + d\rho$. This is equivalent to differentiating the expression, ρ being the only variable. We find for the work done in effecting this displacement against the force, the expression

$$+\frac{3dT\rho S\lambda\mu}{T^4\rho}+\frac{15S\mu\rho S\lambda\rho\,.\,dT\rho}{T^6\rho}-\frac{3S\mu d\rho S\lambda\rho+3S\mu\rho S\lambda d\rho}{T^5\rho}$$

$$=Sd\rho\left\{-\frac{3U\rho S\lambda\mu}{T^4\rho}-\frac{15U\rho S\mu U\rho S\lambda U\rho}{T^4\rho}-\frac{3\mu S\lambda U\rho}{T^4\rho}-\frac{3\lambda S\mu U\rho}{T^4\rho}\right\}$$

$$=Sd\rho\left\{\frac{3U\rho(-S\lambda\mu-5S\mu U\rho S\lambda U\rho)}{T^4\rho}-\frac{3\mu S\lambda U\rho}{T^4\rho}-\frac{3\lambda S\mu U\rho}{T^4\rho}\right\}.$$

The part in the brackets represents the total force against which work is done during the small displacement $d\rho$. It consists of three parts, one parallel to ρ, and the others parallel to the axes of the two magnets—all varying inversely as the fourth power of the distance. The couples acting on the magnets vary inversely as the cube of the distance.

As a particular case, let λ be set parallel to ρ, and μ perpendicular to ρ, and the two magnets to be in the same plane. Then the couples are

$$\text{on } \mu,\ \frac{1}{T^3\rho}(\mu\lambda-3\mu\lambda)=-\frac{2\mu\lambda}{T^3\rho},$$

$$\text{on } \lambda,\ \frac{1}{T^3\rho}\lambda\mu \qquad =-\frac{\mu\lambda}{T^3\rho},$$

so that the couple acting on the one is twice the value of the couple acting on the other. If we suppose the two rigidly fixed together, the system seems to be acted upon by a couple equal to $-3\mu\lambda/T^3\rho$. But then the translational force acting on either is equal to $+\frac{3\mu T\lambda}{T^4\rho}$; and these two equal and opposite forces give rise to a couple $+3\mu\lambda/T^3\rho$ acting on the system. Thus the system is held in equilibrium.

The general proposition may be easily established that, if the magnets are fixed relatively to each other by a rigid framework, the sum of the couples acting on the two magnets is balanced by the moment of the translational forces acting on them.

76. FIELD OF FORCE AND POTENTIAL; PROPERTIES OF NABLA (∇). In a field of force, gravitational, electric, or magnetic, the force β at any point has a definite magnitude and direction. The work done by the force β acting through distance $d\rho$ is $-S\beta d\rho$; and the integral of this along any path connecting two points is defined as measuring the difference of potential between the points. If $d\rho$ is perpendicular to β, this expression vanishes and no work is done, $d\rho$ is then an element in an equipotential surface.

The field may be imagined as mapped out by a series of surfaces $u=c$, where u is a scalar function of the position ρ and c a parameter which is constant for any one surface and varies as we pass from one surface to another. Let c be chosen so as to measure the work done against the forces in bringing up from infinity unit mass of the matter acted upon. Then if we pass from one surface to another near it the change $du=dc$ will measure the difference of potential between the two surfaces, that is, the work done in passing from one to the other.

Since u is a scalar function of ρ, its differential will consist of terms, each of which contains $d\rho$ once. We may write du in the form $S\nu d\rho$, where ν is a vector function of ρ. If $d\rho$ lies in the equipotential surface, $du=0$, and therefore $S\nu d\rho=0$, so that ν is a vector parallel to the normal to the equipotential surface at the point ρ. Hence we may write

$$du=dc=-S\beta d\rho. \qquad (1)$$

where β ($=-\nu$) is the force associated with the equipotential system of surfaces $u=c$.

It is clear that β the force is derived from u the potential by a definite analytical process involving differentiation. Let ∇ be the operator which derives β from u, so that $\beta = \nabla u$. It is defined by the equation

$$du = -Sd\rho\nabla u. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The vector quantity ∇u is such that when resolved in direction $d\rho$ and multiplied by the length of $d\rho$ it gives a quantity which measures the work done in passing from the one to the other equipotential surface passing through the extremities of $d\rho$. It is in fact the force due to the potential u and acts in the direction of u diminishing.

Let $d\rho$ be written in the form idx where i is unit vector parallel to $d\rho$ and dx is the tensor of $d\rho$. Then (2) may be written

$$du = -Sidx\nabla u,$$

or

$$\frac{du}{dx} = -Si\nabla u,$$

giving the rate of change of u per unit distance in any assigned direction i.

For three perpendicular directions i, j, k, we have

$$\frac{du}{dx} = -Si\nabla u, \quad \frac{du}{dy} = -Sj\nabla u, \quad \frac{du}{dz} = -Sk\nabla u.$$

But the vector

$$\nabla u = -iSi\nabla u - jSj\nabla u - kSk\nabla u$$
$$= \left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)u, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

which assigns the analytical expression for ∇ in terms of the rates of change along any three perpendicular directions. It was in this form that Hamilton first defined the operator ∇.

From (3) we can at once verify that $Si\nabla u = -du/dx$, and so on; and we see that we may form the operator $Si\nabla$ first and then operate on the scalar function u, or we may form ∇u first and then operate by $S\,.\,i$; symbolically

$$Si\nabla u = Si\nabla\,.\,u. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Since any vector σ may be written in the form

$$\sigma = ui + vj + wk$$

where u, v, w, are scalar functions of ρ, we find

$$\begin{aligned} d\sigma &= du\,.\,i + dv\,.\,j + dw\,.\,k \\ &= -Sd\rho\nabla\,.\,(ui + vj + wk) \\ &= -Sd\rho\nabla\,.\,\sigma, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5) \end{aligned}$$

a vector quantity which is necessarily different from the scalar quantity $-Sd\rho\nabla\sigma$. Thus the identity (4) does not hold when a vector is the operand.

Returning now to the discussion of the potential let us take the equipotential surfaces to be parallel planes, or

$$u = -S\alpha\rho = c,$$

then $$du = -S\alpha d\rho = -S\nabla u d\rho,$$

or $$\nabla Sd\rho = -\alpha,$$

giving α a constant force perpendicular to the planes.

Let u be a function of the distance $(T\rho = r)$ from the origin, say, $u = f(T\rho)$. Then

$$du = f'(T\rho)\,.\,dT\rho = -f'(T\rho)\,SU\rho d\rho = -S\nabla u d\rho.$$

Hence $$\nabla f(r) = U\rho f'(r).$$

For example let $f(Tr) = ar = aT\rho$. We find $f'(r) = a$, and $f = aU\rho$, so that the force is radial and constant in magnitude throughout all space.

Again let $u = f(r) = ar^{-1}$, $f'(r) = -ar^{-2}$, so that

$$\nabla u = -\frac{aU\rho}{r^2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

the important law of the inverse square, including the dynamic theories of gravitation, electricity, and magnetism.

Again let u depend upon the distance from a given axis. If α is unit vector along this axis, $TV\alpha\rho$ is the distance of any point from the axis, and

$$u = f(TV\alpha\rho),$$

$$du = -SUV\alpha\rho V\alpha d\rho \,.\, f'(TV\alpha\rho)$$

$$= -Sd\rho\alpha^{-1}UV\alpha\rho \,.\, f'(TV\alpha\rho).$$

The force will have the value $a\alpha^{-1}UV\alpha\rho/TV\alpha\rho$ if $f(TV\alpha\rho) = a \log \rho TV\alpha\rho$, the potential for cylindrical distributions.

77. POTENTIAL DUE TO DISTRIBUTIONS OF MATTER. The potential of a continuous distribution of matter may be written in the form

$$u = \iiint \frac{mdv}{T\rho},$$

where m is the mass in unit volume, dv is the element of volume, and the integration is taken through the region occupied by the attracting matter. $T\rho$ is the distance between the element dv and the point P at which u is the potential. The force acting at P is

$$\nabla u = \nabla \iiint \frac{mdv}{T\rho} = \iiint mdv \nabla \frac{1}{T\rho},$$

since ∇ may be taken inside the integral or summation symbol and act on each term separately. Hence

$$\nabla u = -\iiint mdv \frac{U\rho}{T\rho^2} = -\iiint mdv \frac{\rho}{T\rho^3}.$$

Apply ∇ a second time. Then

$$\nabla^2 u = -\iiint mdv \left(\frac{\nabla\rho}{T\rho^3} - \frac{3}{T\rho^4} \nabla T\rho . \rho\right)$$

$$= -\iiint mdv \left(-\frac{3}{T\rho^3} - \frac{3U\rho \,.\, \rho}{T\rho^4}\right)$$

$$= -\iiint mdv \left(-\frac{3}{T\rho^3} + \frac{3}{T\rho^3}\right) \quad \text{.................} \quad (7)$$

This vanishes for all finite values of ρ. Hence if P is wholly outside the attracting matter $\nabla^2 u = 0$. If P is a point occupied by attracting matter $\nabla^2 u$ may have a value, which must depend solely upon the matter at P. For we may draw a small closed surface round P and consider the potential at P to consist of two parts u_1 and u_2, the former being due to matter within the small closed surface, the latter to matter without it. But, since $u = u_1 + u_2$ we have $\nabla^2 u = \nabla^2 u_1 + \nabla^2 u_2$, of which the latter necessarily vanishes. Hence $\nabla^2 u = \nabla^2 u_1$.

The value of $\nabla^2 u_1$ is most easily found by considering the value of $\nabla\beta$, where β is any vector function of ρ. Let β be the value at ρ the centre of the small parallelepiped, whose edges are idx, jdy, kdz. At the face $\rho + \frac{1}{2}\,idx$, the value of β changes to $\beta - \frac{1}{2}\,dxSi\nabla\,.\,\beta$, and multiplying by $+idydz$, the vector area, we get the whole value over the surface element. Similarly, on the opposite end, looking the other way, the value of the corresponding quantity is

$$(\beta + \tfrac{1}{2}\,dxSi\nabla\,.\,\beta) \times -idydz.$$

Hence adding we obtain $-iSi\nabla\,.\,\beta dxdydz$. Similar expressions are obtained for the surface integrals on the other faces; and adding all three together we find for the surface integral over the parallelepiped the value

$$\begin{aligned}\iint \beta d\nu &= -(iSi\nabla + jSj\nabla + kSk\nabla)\,.\,\beta dxdydz\\ &= +\nabla\beta dv,\end{aligned}$$

where $d\nu$ is the vector area of the surface looking outwards and dv is the enclosed volume. This may be at once extended to finite volumes and enclosing surfaces in the form

$$\iiint \nabla\beta dv = \iint \beta d\nu. \quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Now if there is matter at the point P the force β is outwards over the surface of any small enclosing sphere of radius $Td\rho$; and so far as it depends on the matter at P its value is

$$\beta = -\frac{Ud\rho}{(Td\rho)^2}\,mdv.$$

Let the area be divided into n equal parts, where n is very large; then

$$\nabla\beta dv = \iint \beta d\nu = \Sigma - \frac{Ud\rho}{(Td\rho)^2} mdv \,.\, Ud\rho \,.\, \frac{4\pi(Td\rho)^2}{n}$$

$$= 4\pi mdv.$$

But $$\nabla\beta = \nabla\nabla u = \nabla^2 u.$$

Hence $$\nabla^2 u = 4\pi m. \qquad\qquad (9)$$

This includes (7), for when there is no attracting matter at P, m vanishes and $\nabla^2 u = 0$.

78. CONVERGENCE AND CURL. From (8) we may derive very simply the important physical meanings of $S\nabla\beta$ and $V\nabla\beta$. Take first the scalar part of (8), namely:

$$\iiint S\nabla\beta dv = \iint S\beta d\nu.$$

Let β be the flow of fluid. Then the surface integral

$$\iint S\beta d\nu$$

represents the amount of fluid which has entered the region; and thus $S\nabla\beta$ represents the convergence or increase of density of the flowing fluid. If the fluid be incompressible

$$S\nabla\beta = 0.$$

Secondly, take the vector part of (8). This gives

$$\iiint V\nabla\beta dv = \iint V\beta d\nu.$$

Draw from any origin the vector areas $d\nu$ for all points of the surface, and from their extremities draw the corresponding β's. Then if we consider β to be a force the surface integral will represent a couple or moment of force. Hence $V\nabla\beta$ is the measure of this moment per unit volume. Maxwell has called it the curl of the vector β. If $V\nabla\beta = 0$, there is no curl, there is no molecular couple, or there is no vorticity in fluid of which β is the displacement.

When β is a force derived from a potential

$$\beta = \nabla u,$$

and $$\nabla\beta = \nabla^2 u,$$

essentially a scalar quantity. Hence $V\nabla\beta = 0$, or there is no curl when the vector quantity can be derived by differentiation from a scalar function.

79. ELECTRICAL DISTRIBUTIONS. An electrically charged conductor is at the same potential throughout. There is no electric force within it, and the charge is wholly on the surface. Let us apply theorem (8) to a region enclosing a small part of the charged surface and bounded in the field outside the conductor laterally by lines of force which of course spring normally from the charged surface, and terminally by a small area parallel and very close to the element of the surface. Since in this case $V\nabla\beta = 0$, equation (8), becomes

$$\iint S\beta d\nu = \iiint \nabla^2 u dv.$$

On the sides of the region considered $S\beta d\nu$ vanishes because β is perpendicular to $d\nu$. Within the conductor β has no value. On the end of the region β is parallel to $d\nu$, which in this case is ultimately equal to the vector area element on the surface. But

$$\nabla^2 u = 4\pi \times \text{volume density}.$$

Hence

$$\begin{aligned}\nabla^2 u dv &= 4\pi \times \text{volume density} \times \text{area of element} \\ &\qquad \times \text{thickness of electrified layer} \\ &= 4\pi \times \text{surface density} \times \text{area of element}.\end{aligned}$$

Hence we obtain at once for the electric force just outside the surface the expression

$$T\beta = 4\pi \times \text{surface density}.$$

This in fact is the dynamical definition of the surface density of the charge.

As a final example consider the distribution on an ellipsoidal conductor.

Let $$u = -S\rho\phi\rho = c$$

represent the ellipsoidal equipotential surface. Then

$$du = -2Sd\rho\phi\rho = -Sd\rho\nabla u.$$

Hence the force at the point ρ of the surface is

$$\nabla u = 2\phi\rho.$$

But (§ 59) the perpendicular (p) on the tangent at the point ρ is equal to $c/T\phi\rho$; hence

$$T\nabla u = 2c/p,$$

and the surface density is $c/2\pi p$, that is inversely as the perpendicular from the centre on the tangent plane at the point.

These are some of the simple applications of the important differential operator ∇, the theory of which was developed by Tait. For further discussion the reader is referred to the works of Tait and M'Aulay, and to Joly's Appendix to the second edition of Hamilton's *Elements*.

EXAMPLES TO CHAPTER IX.

1. A particle is moving under the action of a constant force. Prove that the hodograph is a straight line and that the path is a parabola.

2. Two equal and opposite magnetic poles are placed at A and A' (vector $AA' = 2\alpha$). Show that the equation giving the direction of the line of force at any point P (vector distance ρ from the middle point of AA') leads to the result

$$S\alpha\{U(\rho+\alpha) \mp U(\rho-\alpha)\} = \text{const.}$$

3. Show that in uniplanar motion, the motion of any rigid figure may in general be represented by a rotation about a determinate point; and that if the motion is continuous, the velocity of any point is given by $\dot{\rho} = ci(\rho - \sigma)$, where σ is the vector of the instantaneous centre of rotation. Find the acceleration of the point in the body which momentarily coincides with the instantaneous centre, and interpret the result. Find the position of the point of zero configuration, and the locus of points having the same acceleration.

4. Let $q(\quad)q^{-1}$ be the rotation which changes the rectangular system $i\,j\,k$ into $\alpha\,\beta\,\gamma$. If we write

$$q = w + xi + yj + zk, \text{ and } q^{-1} = w - xi - yj - zk$$

(an assumption which determines Tq), find expressions for $\alpha\,\beta\,\gamma$ in terms of $i\,j\,k$ and the scalar quantities $w\;x\;y\;z$.

5. A particle moves so that its radius vector describes equal areas in equal times. Prove that the force is directed towards the origin.

6. A particle describes an ellipse (1) about the centre, (2) about a focus as a centre of force. Find the law of the force in each case.

7. The equation $\rho = V\alpha^{nt}\beta$, where α is not of unit length, represents a spiral. Find the linear differential equation in ρ, $\dot{\rho}$, $\ddot{\rho}$, the constant vectors $U\alpha$ and β being eliminated. Interpret the equation dynamically.

8. Given a system of forces $\beta_1\beta_2\beta_3\ldots$ acting at the points $\rho_1\rho_2\rho_3\ldots$, show that if we write the quaternion $\Sigma(\rho\beta) = (c + \varpi)\Sigma\beta$, the vector ϖ is a point on the line of action of the resultant force when the resultant couple has its axis parallel to this line, and that c is the ratio of the resultant couple to the resultant force.

9. The resultant angular velocity ϖ has a component angular velocity $\rho^{-1}S\varpi\rho$ about the axis parallel to ρ. Prove that the angular acceleration about the instantaneous axis of rotation is the same whether we regard that axis as fixed or as moving with the body.

10. The instantaneous position of any vector ρ of a rigid body can be expressed in terms of its original position α in the form $\rho = q\alpha q^{-1}$. By differentiation find expressions for the instantaneous angular velocity and acceleration.

11. By reasoning similar to that on p. 173 establish the identity

$$\int d\rho q = \int\!\!\int V d\nu \nabla .\, q,$$

where the line integral is taken round the curve (ρ) which bounds the surface of which the vector surface element is $d\nu$ and over which the surface integral is taken, and where q is any continuous quaternion function of the position.

CHAPTER X.

VECTOR EQUATIONS OF THE FIRST DEGREE.

WITH the object of giving the student an idea of one of the physical applications of Quaternions, we will treat the solution of linear and vector equations from an elementary kinematical point of view. For this purpose we choose the problem of the deformation of a solid or fluid body, when all its parts are similarly and equally deformed.

DEF. *Homogeneous Strain* is such that portions of a body, originally equal, similar, and similarly placed, remain after the strain equal, similar, and similarly placed.

Thus straight lines remain straight lines, parallel lines remain parallel, equal parallel lines remain equal, planes remain planes, parallel planes remain parallel, and equal areas on parallel planes remain equal. Also the volumes of *all* portions of the body are increased or diminished in the same proportion, as is easily seen by supposing the body originally divided into small equal cubes by series of planes perpendicular to each other. After the strain, these cubes are all changed into similar, similarly placed, and equal parallelepipeds.

It is thus obvious that a homogeneous strain is entirely determined if we know into what vectors three given (non-coplanar) vectors are changed by it. Thus if α, β, γ become

α', β', γ' respectively: any other vector, which may of course be expressed as

$$\rho = \frac{1}{S\,.\,\alpha\beta\gamma}(\alpha S\,.\,\beta\gamma\rho + \beta S\,.\,\gamma\alpha\rho + \gamma S\,.\,\alpha\beta\rho),$$

is changed to

$$\rho' = \frac{1}{S\,.\,\alpha\beta\gamma}(\alpha' S\,.\,\beta\gamma\rho + \beta' S\,.\,\gamma\alpha\rho + \gamma' S\,.\,\alpha\beta\rho).$$

No needful generality is lost, while much simplification is gained, by taking α, β, γ as unit vectors at right angles to one another. This is, in fact, the method already spoken of, *i.e.* the imaginary division of the body into small equal cubes, by three mutually perpendicular series of equidistant planes. We thus have

$$\rho = -(\alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho),$$
$$\rho' = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho).$$

Comparing these expressions we see that *Homogeneous Strain* alters a vector into a definite linear and vector function of its original value.

In abbreviated notation, we may write (as in § 57, though our symbol, as will soon be seen, is more general than that there employed)

$$\phi\rho = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho),$$

where ϕ itself depends upon *nine* independent constants involved in the three equations

$$\left.\begin{aligned}\phi\alpha &= \alpha'\\ \phi\beta &= \beta'\\ \phi\gamma &= \gamma'\end{aligned}\right\}.$$

For α', β', γ' may of course be expressed in terms of α, β, γ: and, as they are quite independent of one another, the nine coefficients in the following equations may have absolutely any values whatever;

$$\left.\begin{aligned}\phi\alpha &= \alpha' = A\alpha + c\beta + b'\gamma\\ \phi\beta &= \beta' = c'\alpha + B\beta + a\gamma\\ \phi\gamma &= \gamma' = b\alpha + a'\beta + C\gamma\end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

In discussing the particular form of ϕ which occurs in the treatment of surfaces of the second order we found, § 45, that it possessed the property

$$S.\sigma\phi\rho = S.\rho\phi\sigma, \qquad \text{......................} (b)$$

whatever vectors are represented by ρ and σ. Remembering that α, β, γ form a rectangular unit system, we find from (a)

$$\left.\begin{aligned} S.\beta\phi\alpha &= -c \\ S.\alpha\phi\beta &= -c' \end{aligned}\right\},$$

with other similar pairs; so that our new value of ϕ satisfies (b) if, and only if, we have in (a)

$$\left.\begin{aligned} a &= a' \\ b &= b' \\ c &= c' \end{aligned}\right\}. \qquad \text{............................} (c)$$

The physical meaning of this condition, as will be seen immediately, is that the distortion expressed by ϕ takes place *without rotation.* In this case the nine constants are reduced to six.

But, although (b) is not generally true, we have

$$\begin{aligned} S.\sigma\phi\rho &= -(S\alpha'\sigma S\alpha\rho + S\beta'\sigma S\beta\rho + S\gamma'\sigma S\gamma\rho) \\ &= -S.\rho(\alpha S\alpha'\sigma + \beta S\beta'\sigma + \gamma S\gamma'\sigma), \end{aligned}$$

where the expression in brackets is a linear and vector function of σ, depending upon the same *nine* scalars as those in ϕ; and which we may therefore express by ϕ', so that

$$\phi'\sigma = -(\alpha S\alpha'\sigma + \beta S\beta'\sigma + \gamma S\gamma'\sigma). \qquad \text{..............} (d)$$

And with this we have obviously

$$S.\sigma\phi\rho = S.\rho\phi'\sigma, \qquad \text{.......................} (e)$$

which is the general relation, of which (b) is a mere particular case.

By putting α, β, γ in succession for σ in (d) and referring to (a), we have

$$\left.\begin{aligned} \phi'\alpha &= A\alpha + c'\beta + b\gamma \\ \phi'\beta &= c\alpha + B\beta + a'\gamma \\ \phi'\gamma &= b'\alpha + a\beta + C\gamma \end{aligned}\right\}. \qquad \text{.....................} (f)$$

Comparing (f) with (a), we see that

$$\phi\rho = \phi'\rho,$$

whatever be ρ, provided the conditions (c) be fulfilled. This agrees with the result already obtained.

Either of the functions ϕ and ϕ', thus defined together, is called the *Conjugate* of the other: and when they are equal (*i.e.* when (c) is satisfied) ϕ is called a *Self-Conjugate* function. As we employed it in Chap. VI., ϕ was self-conjugate; and, even had it not been so, it was involved (as we shall presently see) in such a manner that its non-conjugate part was necessarily absent.

We may now write, as before,

$$\phi\rho = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho),$$

and, by (d),

$$\phi'\rho = -(\alpha S\alpha'\rho + \beta S\beta'\rho + \gamma S\gamma'\rho).$$

From these we have by subtraction,

$$\begin{aligned}(\phi - \phi')\rho &= \phi\rho - \phi'\rho \\ &= \alpha S\alpha'\rho - \alpha' S\alpha\rho + \beta S\beta'\rho - \beta' S\beta\rho + \gamma S\gamma'\rho - \gamma' S\gamma\rho \\ &= -V\rho V\alpha\alpha' - V\rho V\beta\beta' - V\rho V\gamma\gamma' \\ &= 2V.\epsilon\rho\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots (g)\end{aligned}$$

if we agree to write

$$2\epsilon = V(\alpha\alpha' + \beta\beta' + \gamma\gamma'). \quad \ldots\ldots\ldots\ldots (h)$$

We may now express that ϕ is self-conjugate by writing

$$\epsilon = 0,$$

the physical interpretation of which equation is of the highest importance, as will soon appear.

If we form by means of (a) the value of ϵ as in (h), we get

$$\begin{aligned}2\epsilon &= (c\gamma - b'\beta) + (a\alpha - c'\gamma) + (b\beta - a'\alpha) \\ &= (a - a')\alpha + (b - b')\beta + (c - c')\gamma,\end{aligned}$$

which obviously cannot vanish unless (as before) the three conditions (c) are satisfied.

By adding the values of $\phi\rho$ and $\phi'\rho$ above, we obtain

$$\begin{aligned}(\phi+\phi')\rho &= \phi\rho+\phi'\rho \\ &= -(\alpha S\alpha'\rho+\alpha' S\alpha\rho+\beta S\beta'\rho+\beta' S\beta\rho+\gamma S\gamma'\rho+\gamma' S\gamma\rho).\end{aligned}$$

Operating by $S.\sigma$ we see at once that this new function of ρ is self-conjugate.

Hence we may write

$$(\phi+\phi')\rho=2\overline{\varpi}\rho, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

where the bar over ϖ signifies that it is self-conjugate, and the factor 2 is introduced for convenience.

From (g) and (i), we have

$$\left.\begin{aligned}\phi\rho&=\overline{\varpi}\rho+V\epsilon\rho\\ \phi'\rho&=\overline{\varpi}\rho-V\epsilon\rho\end{aligned}\right\}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(j)$$

If instead of $\phi\rho$ in any of the above investigations we write $(\phi+g)\rho$, it is obvious that $\phi'\rho$ becomes $(\phi'+g)\rho$: and the only change in the coefficients in (a) and (f) is the addition of g to each of the main series A, B, C.

We now come to Hamilton's grand proposition with regard to linear and vector functions. If ϕ be such that, in general, the vectors

$$\rho,\ \phi\rho,\ \phi^2\rho$$

(where $\phi^2\rho$ is an abbreviation for $\phi(\phi\rho)$) are not in one plane, then any fourth vector such as $\phi^3\rho$ (a contraction for $\phi(\phi(\phi\rho))$) can be expressed in terms of them as in 31. 5.

Thus
$$\phi^3\rho=m_2\phi^2\rho-m_1\phi\rho+m\rho, \quad\ldots\ldots\ldots\ldots\ldots\ldots(k)$$

where m, m_1, m_2 are scalars whose values will be found immediately. That they are independent of ρ is obvious, for we may put α, β, γ in succession for ρ, and thus obtain three equations of the form

$$\phi^3\alpha=m_2\phi^2\alpha-m_1\phi\alpha+m\alpha, \quad\ldots\ldots\ldots\ldots\ldots\ldots(l)$$

from which their values can be found. For by repeated applications of (a) we can express (l) in the form

$$\mathfrak{A}\alpha+\mathfrak{B}\beta+\mathfrak{C}\gamma=0.$$

This gives
$$\mathfrak{A}=0,\ \ \mathfrak{B}=0,\ \ \mathfrak{C}=0.$$

These are three equations connecting m, m_1, m_2, with the nine coefficients in (a). The other two groups of three equations, furnished by the other two equations of the form (l), are merely *consistent* with these; and involve no farther limitations. This method, however, is very inferior to one which will shortly be given.

Conversely, if quantities m, m_1, m_2 can be found which satisfy (l), we may reproduce (k) by putting

$$\rho = x\alpha + y\beta + z\gamma$$

and adding together the three expressions (l) multiplied by x, y, z respectively. For it is obvious from the expression for ϕ that

$$x\phi\rho = \phi\,(x\rho), \quad x\phi^2\rho = \phi^2\,(x\rho), \text{ etc.,}$$

whatever scalar be represented by x.

If ρ, $\phi\rho$, and $\phi^2\rho$ are in the same plane, then applying the strain ϕ again, we find $\phi\rho$, $\phi^2\rho$, $\phi^3\rho$ in one plane; and thus equation (k) holds for this case also. And it of course holds if $\phi\rho$ is parallel to ρ, for then $\phi^2\rho$ and $\phi^3\rho$ are also parallel to ρ.

We will prove that scalars can be found which satisfy the three equations (l) (equivalent to *nine* scalar equations, of which, however, as we have seen, six depend upon the other three) by actually determining their values.

The volume of the parallelepiped whose three conterminous edges are λ, μ, ν is (§ 32)

$$-S\,.\,\lambda\mu\nu.$$

After the strain its volume is

$$-S\,.\,\phi\lambda\phi\mu\phi\nu,$$

so that the ratio

$$\frac{S\,.\,\phi\lambda\phi\mu\phi\nu}{S\,.\,\lambda\mu\nu}$$

is the same whatever vectors λ, μ, ν may be; and depends therefore on the constants of ϕ alone. We may therefore assume

$$\left.\begin{aligned}\lambda &= \rho,\\ \mu &= \phi\rho,\\ \nu &= \phi^2\rho,\end{aligned}\right\}$$

and by inspection of (k), we find

$$\frac{S.\phi\lambda\phi\mu\phi\nu}{S.\lambda\mu\nu} = \frac{S.\phi\rho\phi^2\rho\phi^3\rho}{S.\rho\phi\rho\phi^2\rho} = m, \quad \ldots\ldots\ldots\ldots\ldots (m)$$

which gives the physical meaning of this constant in (k). As we may put if we please

$$\left.\begin{aligned} \lambda &= \alpha, \\ \mu &= \beta, \\ \nu &= \gamma, \end{aligned}\right\}$$

we see by (a) that

$$m = \frac{S.\phi\alpha\phi\beta\phi\gamma}{S.\alpha\beta\gamma} = \begin{vmatrix} A, & c, & b' \\ c', & B, & a \\ b, & a', & C \end{vmatrix},$$

which is the expression for the ratio in which the volume of each portion has been increased. This is unchanged by putting ϕ' for ϕ, for it becomes, by (f),

$$m = \begin{vmatrix} A, & c', & b \\ c, & B, & a' \\ b', & a, & C \end{vmatrix}.$$

Hence *conjugate strains produce equal changes of volume.*

Recurring to (m), we may write it by (e) as

$$S.\lambda\phi' V\phi\mu\phi\nu = mS.\lambda V\mu\nu,$$

from which, as λ is *absolutely any vector*, we have

$$\left.\begin{aligned} \phi' V\phi\mu\phi\nu &= mV\mu\nu \\ \text{or}\qquad \phi V\phi'\mu\phi'\nu &= mV\mu\nu \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (n)$$

[In passing we may notice that (n) gives us the complete solution of a linear and vector equation such as

$$\phi\sigma = \delta,$$

where δ and ϕ are given and σ is to be found. We have in fact only to take any two vectors μ and ν which are perpendicular to δ, and such that

$$V\mu\nu = \delta,$$

and we have for the unknown vector

$$\sigma = \frac{1}{m} V\phi'\mu\phi'\nu,$$

which can be calculated, as ϕ is given.]

If in (n) we put $\phi + g$ for ϕ, we must do so for the value of m in (m). Calling the latter M_g, we have

$$\begin{aligned} M_g &= \frac{S.(\phi+g)\lambda(\phi+g)\mu(\phi+g)\nu}{S.\lambda\mu\nu} \\ &= m + g\frac{S.\lambda\phi\mu\phi\nu + S.\mu\phi\nu\phi\lambda + S.\nu\phi\lambda\phi\mu}{S.\lambda\mu\nu} \\ &\quad + g^2\frac{S.\lambda\mu\phi\nu + S.\nu\lambda\phi\mu + S.\mu\nu\phi\lambda}{S.\lambda\mu\nu} \\ &\quad + g^3, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (o) \end{aligned}$$

and by (n) $\quad (\phi+g)V(\phi'+g)\mu(\phi'+g)\nu = M_g . V\mu\nu, \quad\ldots\ldots\ldots\ldots (p)$

or

$$\left.\begin{aligned} M_g &= m + \mu_1 g + \mu_2 g^2 + g^3 \\ (\phi+g)[m\phi^{-1}V\mu\nu + g(V\phi'\mu\nu + V\mu\phi'\nu) + g^2V\mu\nu] &= M_g V\mu\nu \end{aligned}\right\}\ldots\ldots (q)$$

From the latter of these equations it is obvious that

$$V\phi'\mu\nu + V\mu\phi'\nu$$

must be a linear and vector function of $V\mu\nu$, since all the other terms of the equation are such functions.

As practice in the use of these functions we will solve a problem of a little greater generality. The vectors

$$V\mu\nu, \quad V\phi'\mu\nu, \text{ and } V\mu\phi'\nu$$

are not generally coplanar. In terms of these (§ 34), let us express $\phi V\mu\nu$.

Let $\qquad \phi V\mu\nu = xV\mu\nu + yV\phi'\mu\nu + zV\mu\phi'\nu.$

Operate by $S.\lambda$, $S.\mu$, $S.\nu$ successively, then

$$\begin{aligned} S.\mu\nu\phi'\lambda &= xS.\lambda\mu\nu + yS.\nu\lambda\phi'\mu + zS.\lambda\mu\phi'\nu, \\ S.\mu\nu\phi'\mu &= yS.\nu\mu\phi'\mu, \\ S.\mu\nu\phi'\nu &= zS.\nu\mu\phi'\nu. \end{aligned}$$

The two last equations give (§ 32)

$$y = -1, \ z = -1,$$

and therefore the first gives

$$x = \frac{S.\mu\nu\phi'\lambda + S.\nu\lambda\phi'\mu + S.\lambda\mu\phi'\nu}{S.\lambda\mu\nu}$$

$$= \mu_2, \text{ by } (o) \text{ and } (q).$$

Hence, finally,

$$\phi V\mu\nu = \mu_2 V\mu\nu - V\phi'\mu\nu - V\mu\phi'\nu. \ \ldots\ldots\ldots\ldots\ldots (\boldsymbol{r})$$

Substituting this in (q), and putting σ for $V\mu\nu$, which is any vector whatever, we have

$$(\phi + g)[m\phi^{-1} + g(\mu_2 - \phi) + g^2]\sigma = (m + \mu_1 g + \mu_2 g^2 + g^3)\sigma,$$

or, multiplying out,

$$(m - g\phi^2 + \mu_2 g\phi - g^2\phi + gm\phi^{-1} + g^2\phi + g^2\mu_2 + g^3)\sigma$$
$$= (m + \mu_1 g + \mu_2 g^2 + g^3)\sigma\,;$$

that is $$(-\phi^2 + \mu_2\phi + m\phi^{-1})\sigma = \mu_1\sigma,$$

or $$(\phi^3 - \mu_2\phi^2 + \mu_1\phi - m)\sigma = 0.$$

Comparing this with (k), we see that

$$\left.\begin{aligned} m_2 = \mu_2 &= \frac{S.\lambda\mu\phi\nu + S.\nu\lambda\phi\mu + S.\mu\nu\phi\lambda}{S.\lambda\mu\nu} \\ m_1 = \mu_1 &= \frac{S.\lambda\phi\mu\phi\nu + S.\mu\phi\nu\phi\lambda + S.\nu\phi\lambda\phi\mu}{S.\lambda\mu\nu} \end{aligned}\right\}, \ \ldots\ldots\ldots (s)$$

and thus the determination is complete.

We may write (k), if we please, in the form

$$m\phi^{-1}\rho = m_1\rho - m_2\phi\rho + \phi^2\rho, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (k')$$

which gives another, and more direct, solution of the equation (above mentioned)

$$\phi\sigma = \delta.$$

Physically, the result we have arrived at is the solution of the problem, "By adding together scalar multiples of any vector of a body, of the corresponding vector of the same strained homogeneously, and of that of the same twice over strained, to represent the state of the body which would be produced by supposing the strain to be reversed or inverted."

These properties of the function ϕ are sufficient for many applications, of which we proceed to give a few.

(I.) Homogeneous strain converts an originally spherical portion of a body into an ellipsoid.

For if ρ be a radius of the sphere, σ the vector into which it is changed by the strain, we have

$$\sigma = \phi\rho,$$

and $$T\rho = C,$$

from which we obtain

$$T\phi^{-1}\sigma = C,$$

or $$S \, . \, \phi^{-1}\sigma\phi^{-1}\sigma = -C^2,$$

or, finally, $$S \, . \, \sigma\phi'^{-1}\phi^{-1}\sigma = -C^2.$$

This is the equation of a central surface of the second degree; and, therefore, of course, from the nature of the problem, an ellipsoid.

(II.) To find the vectors whose direction is unchanged by the strain.

Here $\phi\rho$ must be parallel to ρ or

$$\phi\rho = g\rho.$$

This gives $$\phi^2\rho = g^2\rho, \text{ etc.},$$

so that by (k), we have

$$g^3 - m_2 g^2 + m_1 g - m = 0.$$

This must have one real root, and may have three. Suppose g_1 to be a root, then

$$\phi\rho - g_1\rho = 0,$$

and therefore, whatever be λ,

$$S\lambda\phi\rho - g_1 S\lambda\rho = 0,$$

or $$S \, . \, \rho(\phi'\lambda - g_1\lambda) = 0.$$

Thus it appears that the operator $\phi' - g_1$ cuts off from any vector λ the part which is parallel to the required value of ρ, and therefore that we have

$$\rho \parallel MV.(\phi' - g_1)\lambda(\phi' - g_1)\mu$$
$$\parallel \{m\phi^{-1} - g_1(m_2 - \phi) + g_1{}^2\}\zeta,$$

where ζ is absolutely any vector whatever. This may be written as

$$\rho \parallel \left\{\frac{m}{g_1} - (m_2 - g_1)\phi + \phi^2\right\}\zeta$$
$$\parallel \frac{\phi^3 - m_2\phi^2 + m_1\phi - m}{\phi - g_1}\zeta.$$

The same result may more easily be obtained thus:

The expression

$$(\phi^3 - m_2\phi^2 + m_1\phi - m)\rho = 0,$$

being true for all vectors whatever, may be written

$$(\phi - g_1)(\phi - g_2)(\phi - g_3)\rho = 0,$$

and it is obvious that each of these factors deprives ρ of the portion corresponding to it: *i.e.* $\phi - g_1$ applied to ρ cuts off the part parallel to the root of

$$(\phi - g_1)\sigma = 0, \text{ etc., etc.,}$$

so that the operator $(\phi - g_2)(\phi - g_3)$ when applied to a vector leaves only that part of it which is parallel to σ where

$$(\phi - g_1)\sigma = 0.$$

(III.) Thus it appears that there is always one vector, and that there may be three vectors, whose direction is unchanged by the strain.

DEF. *Pure, or non-rotational, strain consists in altering the lengths of three lines at right angles to one another, without altering their directions.*

Hence, if
$$\phi\rho_1 = g_1\rho_1,$$
$$\phi\rho_2 = g_2\rho_2,$$
$$\phi\rho_3 = g_3\rho_3,$$

the strain ϕ is pure if, and not unless, ρ_1, ρ_2, ρ_3 form a rectangular system. [There is a qualification if two or more of g_1 g_2 g_3 be equal.]

Hence, for a pure strain, we have

$$S\rho_2\phi\rho_1 = g_1 S\rho_2\rho_1 = 0,$$

and

$$S\rho_1\phi\rho_2 = g_2 S\rho_1\rho_2 = 0,$$

or

$$S\rho_1\phi\rho_2 = S\rho_2\phi\rho_1.$$

But we have, generally,

$$S\rho_1\phi\rho_2 = S\rho_2\phi'\rho_1.$$

As we have two other pairs of equations like these, we see that

$$\phi = \phi'$$

when the strain is pure.

Conversely, if

$$\phi = \phi',$$

the three unchanging directions ρ_1, ρ_2, ρ_3 are perpendicular to one another.

For, in this case, the roots of

$$M_g = 0$$

are real. Let them be such that

$$\left.\begin{aligned}(\phi - g_1)\rho_1 = 0\\ (\phi - g_2)\rho_2 = 0\\ (\phi - g_3)\rho_3 = 0\end{aligned}\right\},$$

then

$$\begin{aligned} g_1 g_2 S\rho_1\rho_2 &= S\phi\rho_1\phi\rho_2\\ &= S\rho_1\phi\phi\rho_2\end{aligned}$$

(because, by hypothesis, the strain is pure)

$$= g_2{}^2 S\rho_1\rho_2,$$

for

$$\phi\rho_2 = g_2\rho_2 \text{ and } \phi^2\rho_2 = g_2{}^2\rho_2.$$

Hence, except in the particular case of

$$g_1 = g_2,$$

we must have

$$S\rho_1\rho_2 = 0,$$

whence the proposition.

When g_1 and g_2 are equal, ρ_1 and ρ_2 are each perpendicular to ρ_3, but *any* vector in their plane satisfies

$$\phi\sigma - g_1\sigma = 0.$$

When all three roots are equal, *every* vector satisfies

$$\phi\sigma - g_1\sigma = 0.$$

(IV.) Thus we see that when the strain is unaccompanied by rotation the three values of g are real. [But we must take care to notice that the converse does not hold. This will be discussed later.] If these values be real and *different*, there are three vectors at right angles to one another which are the only lines in the body whose directions remain unchanged. When two are equal, every vector parallel to a given plane, and all vectors perpendicular to it, are unchanged in direction. When all three are equal no vector has its direction changed.

(V.) There is, however, a peculiarity to be noticed, which distinguishes true physical strain from the results of our mathematical analysis. When one or more of the values of g has a *negative* sign, we cannot interpret *physically* the result without introducing the idea of a pure strain which shall, as it were, pull the parts of an originally spherical portion of the body through the centre of the sphere, and so form an ellipsoid by turning a part of the body outside in. When two, only, are negative we can represent physically the result by introducing the conception of a rotation through two right angles about the third axis. But we began by assuming that there is no rotation! Hence, for the case considered, all three roots must be positive. See end of next section (VI.).

(VI.) This will appear more clearly if we take the case of a rigid body, for here we must have, whatever vectors be represented by ρ and σ,

$$\left.\begin{array}{l} T\phi\rho = T\rho \\ S\rho\sigma = S\,.\,\phi\rho\phi\sigma \end{array}\right\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(t)$$

i.e. the lengths of vectors, and their inclinations to one another, are unaltered. In this case, therefore, the strain can be nothing but a rotation. It is easy to see that the second of these equations includes the first; so that if, for variety, we take ϕ as represented in equations (a), and write

$$\rho = x\alpha + y\beta + z\gamma,$$

$$\sigma = \xi\alpha + \eta\beta + \zeta\gamma,$$

we have, for *all* values of the six scalars x, y, z, ξ, η, ζ, the following identity:

$$\begin{aligned}-(x\xi + y\eta + z\zeta) &= S.(x\alpha' + y\beta' + z\gamma')(\xi\alpha' + \eta\beta' + \zeta\gamma')\\ &= \alpha'^2 x\xi + \beta'^2 y\eta + \gamma'^2 z\zeta\\ &\quad + (x\eta + y\xi)S\alpha'\beta' + (y\zeta + z\eta)S\beta'\gamma' + (z\xi + x\zeta)S\gamma'\alpha'.\end{aligned}$$

This necessitates

$$\left.\begin{aligned}\alpha'^2 = \beta'^2 = \gamma'^2 = -1\\ S\alpha'\beta' = S\beta'\gamma' = S\gamma'\alpha' = 0\end{aligned}\right\}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(u)$$

i.e. the vectors α', β', γ' form, like α, β, γ, a rectangular unit system. And it is evident that *any* and *every* such system satisfies the given conditions. But the system α', β', γ' must be similar to α, β, γ, *i.e.* if a quadrant of *positive* rotation round α changes β to γ, etc., a quadrant of *positive* rotation about α' must change β' to γ', etc.

When this is not the case, the system α', β', γ' is the *perversion* of α, β, γ, *i.e.* its image in a plane mirror; and the strain is impossible from a physical point of view.

This is easily seen from another point of view. The volume of the parallelepiped whose edges are rectangular unit vectors α, β, γ is

$$-S.\alpha\beta\gamma$$

if a positive quadrant of rotation round α brings β to coincide with γ, etc. But, in the perverted system, *the volume has changed sign* and is expressed by

$$S.\alpha\beta\gamma.$$

(VII.) It may be interesting to form, for this particular case, the equation giving the values of g. We have

$$M_g = \frac{S.(\phi+g)\alpha(\phi+g)\beta(\phi+g)\gamma}{S.\alpha\beta\gamma}$$

$$= \frac{S.(\alpha'+g\alpha)(\beta'+g\beta)(\gamma'+g\gamma)}{S.\alpha\beta\gamma}.$$

$$= 1 - gS(\alpha\beta'\gamma' + \alpha'\beta\gamma' + \alpha'\beta'\gamma)$$

$$- g^2S(\alpha\beta\gamma' + \alpha\beta'\gamma + \alpha'\beta\gamma) + g^3.$$

Recollecting that α, β, γ; α', β', γ' are systems of rectangular unit vectors, we find that this may be written

$$M_g = 1 - (g+g^2)S(\alpha\alpha' + \beta\beta' + \gamma\gamma') + g^3$$

$$= (g+1)[g^2 - g\{1 + S(\alpha\alpha' + \beta\beta' + \gamma\gamma')\} + 1].$$

Hence the roots of $M_g = 0$

are in this case; first and always,

$$g_1 = -1,$$

which refers to the axis about which the rotation takes place: secondly, the roots of

$$g^2 - g\{1 + S(\alpha\alpha' + \beta\beta' + \gamma\gamma')\} + 1 = 0.$$

Now the roots of this equation are imaginary so long as the coefficient of the first power of g lies *between* the limits ± 2.

Also the values of the several quantities $S\alpha\alpha'$, $S\beta\beta'$, $S\gamma\gamma'$ can never exceed the limits ± 1. When the system α, β, γ coincides with α', β', γ', the value of each of the scalars is -1, and the coefficient of the first power of g is $+2$. When two of them are equal to $+1$ and the third to -1 we have the coefficient of the first power of $g = -2$. These are the only two cases in which the three values of g are all real.

In the first, all three values of g are equal to -1, *i.e.*

$$\phi\rho = \rho$$

for all values of ρ, and there is no rotation whatever. In the second case there is a rotation through two right angles about the axis of the -1 value of g.

VIII. It is an exceedingly remarkable fact that, however a body may be homogeneously strained, there is always at least one vector whose direction remains unchanged. The proof is simply based on the fact that the strain-function depends on a cubic equation (with real coefficients) which must have at least one real root.

IX. As an illustration of what precedes (though one which must be approached cautiously), suppose a body to be strained so that three vectors, α'', β'', γ'' (not coplanar, and not necessarily at right angles to one another), preserve their direction, becoming $e_1\alpha''$, $e_2\beta''$, $e_3\gamma''$. Then we have

$$\phi\rho S\,.\,\alpha''\beta''\gamma'' = e_1\alpha'' S\,.\,\beta''\gamma''\rho + e_2\beta'' S\,.\,\gamma''\alpha''\rho + e_3\gamma'' S\,.\,\alpha''\beta''\rho.$$

By the formulae (m, s) we have

$$m = \frac{S\,.\,\phi\alpha''\phi\beta''\phi\gamma''}{S\,.\,\alpha''\beta''\gamma''} = e_1e_2e_3,$$

$$m_1 = \frac{S(\alpha''\phi\beta''\phi\gamma'' + \beta''\phi\gamma''\phi\alpha'' + \gamma''\phi\alpha''\phi\beta'')}{S\,.\,\alpha''\beta''\gamma''} = e_2e_3 + e_3e_1 + e_1e_2,$$

$$m_2 = \frac{S(\alpha''\beta''\phi\gamma'' + \beta''\gamma''\phi\alpha'' + \gamma''\alpha''\phi\beta'')}{S\,.\,\alpha''\beta''\gamma''} = e_1 + e_2 + e_3\,;$$

so that we have by (k)

$$(\phi - e_1)(\phi - e_2)(\phi - e_3)\rho = 0.$$

Though the values of g are here all real, we must not rashly adopt the conclusions of (IV.), for we must remember that α'', β'', γ'' do not, like α, β, γ, necessarily form a rectangular system.

In this case we have

$$\phi'\rho S\,.\,\alpha''\beta''\gamma'' = e_1V\beta''\gamma''S\alpha''\rho + e_2V\gamma''\alpha''S\beta''\rho + e_3V\alpha''\beta''S\gamma''\rho.$$

So that, by (g) and (h),

$$2\epsilon S\,.\,\alpha''\beta''\gamma'' = V\,.\,(e_1\alpha''V\beta''\gamma'' + e_2\beta''V\gamma''\alpha'' + e_3\gamma''V\alpha''\beta'')$$

$$= (\overline{e_2 - e_3}\alpha''S\beta''\gamma'' + \overline{e_3 - e_1}\beta''S\gamma''\alpha'' + \overline{e_1 - e_2}\gamma''S\alpha''\beta'').$$

This vanishes, or the strain is pure, if either

1. $$S\alpha''\beta'' = S\beta''\gamma'' = S\gamma''\alpha'' = 0,$$

i.e. if α'', β'', γ'' are rectangular, in which case e_1, e_2, e_3 may have any values; or

2. $e_1 = e_2 = e_3$, in which case

$$\begin{aligned}\phi'\rho S\,.\,\alpha''\beta''\gamma'' &= e_1\{V\beta''\gamma''S\alpha''\rho + V\gamma''\alpha''S\beta''\rho + V\alpha''\beta''S\gamma''\rho\}\\ &= e_1\rho S\,.\,\alpha''\beta''\gamma'' \text{ by } (\S\ 34.\ 2),\end{aligned}$$

so that $$\phi'\rho = e_1\rho = \phi\rho$$

for every vector · a general uniform dilatation unaccompanied by change of direction.

3. $e_1 = e_2$, and α'' and β'' both perpendicular to γ''.

From what precedes it is evident that for the complete study of a strain we must endeavour to distinguish in each case between the *pure* strain and the merely *rotational* part. If a strain be capable of being decomposed into, first, a pure strain, second, a rotation, it is obvious that the vectors which in the altered state of the body become the axes of the strain-ellipsoid (I.) must have been originally at right angles to one another.

The equation of the strain-ellipsoid is

$$S\rho\phi^{-2}\rho = -c^2,$$

and in this it is obvious that ϕ^{-2} is self-conjugate, or at least is to be treated as such: for a non-conjugate term in $\phi^{-2}\rho$ would be (g) of the form

$$V\epsilon\rho,$$

and would therefore not appear in the equation.

For the proper treatment of rotations, the following simple but excessively important proposition, due to Hamilton, forms the best starting-point.

If q be any quaternion, the operator $q(\ \ )q^{-1}$ turns the vector, quaternion, or body operated on round an axis perpendicular to the plane of q and through an angle equal to double that of q.

For the proof we refer the reader to Hamilton's *Lectures*, § 282, *Elements*, § 179 (1), or Tait, § 353. It is obvious that the tensor of q may be taken to be unity, *i.e.* q may be considered as a mere versor, because the value of its tensor does not affect that of the operator.*

[A very simple but important example of this proposition is given by supposing q and r to be both vectors, α and β let us say. Then

$$\alpha\beta\alpha^{-1}$$

is the result of turning β conically through two right angles about α, *i.e.* if α be the normal to a reflecting surface and β the incident ray, $-\alpha\beta\alpha^{-1}$ is the reflected ray.]

Now let the strain ϕ be effected by (1), a pure strain $\overline{\varpi}$ (self-conjugate of course) followed by the rotation $q(\quad)q^{-1}$. We have, for all values of ρ,

$$\phi\rho = q(\overline{\varpi}\rho)q^{-1}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

whence

$$\phi'\rho = \overline{\varpi}(q^{-1}\rho q).$$

The interpretation is that, under the above definition, *the conjugate to any strain consists of the reversed rotation, followed by the pure strain.*

We may of course put, as in Chap. VII.,

$$\overline{\varpi}\rho = e_1\alpha S\alpha\rho + e_2\beta S\beta\rho + e_3\gamma S\gamma\rho,$$

where α, β, γ form a rectangular system. Hence

$$\phi\rho = e_1 q\alpha q^{-1} S\alpha\rho + e_2 q\beta q^{-1} S\beta\rho + e_3 q\gamma q^{-1} S\gamma\rho.$$

Here the axes are parallel to

$$q\alpha q^{-1}, \quad q\beta q^{-1}, \quad q\gamma q^{-1},$$

and we have

$$S \,.\, q\alpha q^{-1} q\beta q^{-1} = S \,.\, q\alpha\beta q^{-1} = S\alpha\beta = 0, \text{ etc.}$$

So far the matter is nearly self-evident, but we now come to the important question of the *separation of the pure strain from the rotation.* By the formulae above we see that

$$\begin{aligned}\phi'\phi\rho &= \overline{\varpi} q^{-1}\phi\rho q \\ &= \overline{\varpi} q^{-1}(q\overline{\varpi}\rho q^{-1})q \\ &= \overline{\varpi}^2\rho,\end{aligned}$$

---

* The proof is now given above, Chap. IV., § 27.

so that we have in symbols, for the determination of ϖ, the equation

$$\phi'\phi = \varpi^2.$$

That is, as we see at once from the statements above, *any strain, followed by its conjugate, gives a pure strain, which is the square (or the result of two applications) of the pure part of either.*

To solve this equation we employ expressions like (k). $\phi'\phi$ being a known function, let us call it ω, and form its equation as

$$\omega^3 - m_2\omega^2 + m_1\omega - m = 0.$$

Here the coefficients are perfectly determinate.

Also *suppose* that the corresponding equation in ϖ is

$$\varpi^3 - g_2\varpi^2 + g_1\varpi - g = 0,$$

where g, g_1, g_2 are unknown scalars. By the help of the given relation $\varpi^2 = \omega$,

we may modify this last equation as follows:

$$\varpi\omega - g_2\omega + g_1\varpi - g = 0,$$

whence $$\varpi = \frac{g + g_2\omega}{g_1 + \omega};$$

i.e. ϖ is given definitely in terms of the known function, ω, as soon as the quantities g are found. But our given equation $\varpi^2 = \omega$

may now be written

$$\left(\frac{g + g_2\omega}{g_1 + \omega}\right)^2 = \omega,$$

or $$\omega^3 - (g_2{}^2 - 2g_1)\omega^2 + (g_1{}^2 - 2gg_2)\omega - g^2 = 0.$$

As this is an equation between ω and constants it must be equivalent to that already given; so that, comparing coefficients, we have

$$g_2{}^2 - 2g_1 = m_2,$$
$$g_1{}^2 - 2gg_2 = m_1,$$
$$g^2 = m;$$

from which, by elimination of g and g_2, we have

$$\left(\frac{g_1{}^2 - m_1}{2\sqrt{m}}\right)^2 = m_2 + 2g_1.$$

The solution of the problem is therefore reduced to that of this biquadratic equation; for, when g_1 is found, g_2 is given linearly in terms of it.

[A neat way of arriving at the same result is to throw the equation in ϖ into the form

$$\varpi = \frac{g + g_2\varpi^2}{g_1 + \varpi^2},$$

square both sides and multiply up. The result is

$$\varpi^6 - (g_2{}^2 - 2g_1)\varpi^4 + (g_1{}^2 - 2gg_2)\varpi^2 - g^2 = 0,$$

or
$$\omega^3 - (g_2{}^2 - 2g_2)\omega^2 + (g_1{}^2 - 2gg_2)\omega - g^2 = 0].$$

It is to be observed that in the operations above we have not been particular as to the arrangement of factors. This is due to the fact that any functions of the *same* operator are commutative in their application.

Having thus found the pure part of the strain we have at once the rotation, for (v) gives

$$\phi\varpi^{-1}\rho = q\rho q^{-1},$$

or, as it may more expressively be written,

$$\frac{\phi}{\sqrt{\phi'\phi}} = q(\quad)q^{-1}.$$

If instead of (v) we write

$$\phi\rho = \varpi(r\rho r^{-1}), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v')$$

we assume that the rotation takes place first, and is succeeded by the pure strain. This form gives

$$\phi'\rho = r^{-1}(\varpi\rho)r,$$

and
$$\phi\phi'\rho = \varpi^2\rho,$$

whence ϖ is found as above. And then (v') gives

$$\varpi^{-1}\phi = r(\quad)r^{-1}.$$

Thus, to recapitulate, a strain ϕ is equivalent to the pure strain $\sqrt{\phi'\phi}$ followed by the rotational strain $\phi\dfrac{1}{\sqrt{\phi'\phi}}$, or to the rotational strain $\dfrac{1}{\sqrt{\phi\phi'}}\phi$ followed by the pure strain $\sqrt{\phi\phi'}$.

This leads us, as an example, *to find the condition that a given strain is rotational only*, i.e. that a quaternion q can be found such that

$$\phi = q(\quad)q^{-1}.$$

Here we have $$\phi' = q^{-1}(\quad)q,$$

or $$\phi' = \phi^{-1}. \qquad\qquad (w)$$

But $$m\phi^{-1} = m_1 - m_2\phi + \phi^2,$$

or $$\left.\begin{array}{r}m\phi' = m_1 - m_2\phi + \phi^2,\\ m\phi = m_1 - m_2\phi' + \phi'^2,\end{array}\right\}$$

whose conjugate is (the second line above)

and the elimination of ϕ' between these two equations gives

$$m\phi = m_1 - \frac{m_2}{m}(m_1 - m_2\phi + \phi^2) + \frac{1}{m^2}(m_1 - m_2\phi + \phi^2)^2,$$

i.e.

$$0 = \left|\begin{array}{l}(m^2m_1 - mm_1m_2 + m_1{}^2)\\ -(m^3 - mm_2{}^2 + 2m_1m_2)\phi\\ -(mm_2 - 2m_1 - m_2{}^2)\phi^2\\ -2m_2\phi^3\\ +\phi^4\end{array}\right. = \left|\begin{array}{l}(m^2m_1 - mm_1m_2 + m_1{}^2)\\ -(m^3 - mm_2{}^2 + 2m_1m_2 - m)\phi\\ +(2m_1 + m_2{}^2 - mm_2 - m_1)\phi^2\\ -m_2\phi^3\\ \end{array}\right.$$

by using the expression for ϕ^4 from the cubic in ϕ.

Now this last expression can be nothing else than the cubic in ϕ itself, else ϕ would have *two different* sets of constants in the form (k), which is absurd, as these constants, from the mode in which they are determined, can have but single values. Thus we have, by comparing coefficients,

$$\left.\begin{array}{r}m_2{}^2 = 2m_1 + m_2{}^2 - mm_2 - m_1\\ m_1m_2 = m^3 - mm_2{}^2 + 2m_1m_2 - m\\ mm_2 = m^2m_1 - mm_1m_2 + m_1{}^2\end{array}\right\}.$$

The first gives $m_1 = mm_2$,
by the help of which the second and third each become

$$m^3 - m = 0.$$

The value $m = 0$
is to be rejected, as otherwise we should have been working with non-existent terms; and m, as the ratio of the volumes of two tetrahedra, is positive, so that finally

$$m = 1,$$

$$m_1 = m_2,$$

and the cubic for a rotational strain is, therefore,

$$\phi^3 - m_2\phi^2 + m_2\phi - 1 = 0,$$

or

$$(\phi - 1)\{\phi^2 + (1 - m_2)\phi + 1\} = 0,$$

where m_2 is left undetermined.

By comparison with the result of (VII.) we see that in the notation there employed

$$m_2 = -S(\alpha\alpha' + \beta\beta' + \gamma\gamma').$$

The student will perhaps here require to be reminded that in the section just referred to we employed the positive sign in operators such as $\phi + g$. In the one case the coefficients in the cubic are all positive, in the other they are alternately positive and negative. The example we have given is a particularly valuable one, as it gives a glimpse of the extent to which the separation of symbols can be safely carried in dealing with these questions.

DEF. A *simple shear* is a homogeneous strain in which all planes parallel to a fixed plane are displaced in the same direction parallel to that plane, and therefore through spaces proportional to their distances from that plane.

Let α be normal to the plane, β the direction of displacement, the former being considered as an unit-vector, and the tensor of the latter being the displacement of points at unit distance from the plane.

We obviously have, by the definition,

$$S\alpha\beta=0.$$

Now if ρ be the vector of any point, drawn from an origin in the fixed plane, the distance of the point from the plane is

$$-S\alpha\rho.$$

Hence, if σ be the vector of the point after the shear,

$$\sigma=\phi\rho=\rho-\beta S\alpha\rho.$$

This gives $$\phi'\rho=\rho-\alpha S\beta\rho,$$
which may be written as

$$=\rho-T\beta\,.\,\alpha S\,.\,U\beta\rho,$$

so that the conjugate of a simple shear is another simple shear equal to the former. But the direction of displacement in each shear is perpendicular to the unaltered planes in the other.

The equation for ϕ is easily found (by calculating m, m_1, m_2 from (m), (s)) to be *

$$\phi^3-3\phi^2+3\phi-1=0.$$

Putting $\phi'\phi=\psi$, we easily find (with $b=T\beta$)

$$\psi^3-(3+b^2)\psi^2+(3+b^2)\psi-1=0.$$

Solving by the process lately described, we find

$$\left(\frac{g_1{}^2-3-b^2}{2}\right)^2=3+b^2+2g_1.$$

If $b=2$, this gives $g_1=1$, and the farther equation

$$g_1{}^3+g_1{}^2-13g_1-21=0,$$

---

* In many cases the most expeditious way of finding the cubic in ϕ is to find the (same) cubic in g, where g corresponds to the roots defined by the relation $g\rho=\phi\rho$. In the case discussed $g\rho=\rho-\beta S\alpha\rho$, or $(g-1)\rho+\beta S\alpha\rho=0$. Operate in succession by $S\alpha$, $S\beta$, $SV\alpha\beta$; then since $S\alpha\beta=0$ we find

$$\left.\begin{array}{lr}(g-1)S\alpha\rho & =0,\\ \quad\beta^2S\alpha\rho+(g-1)S\beta\rho & =0,\\ & (g-1)S\alpha\beta\rho=0.\end{array}\right\}$$

Eliminating $S\alpha\rho$, $S\beta\rho$, $S\alpha\beta\rho$, we find $(g-1)^3=0$, which gives $(\phi-1)^3=0$ for the cubic in ϕ.

of which $g_1 = -3$ is a root, so that

$$g_1^2 - 2g_1 - 7 = 0,$$

and

$$g_1 = 1 \pm 2\sqrt{2}.$$

We leave to the student the selection (by trial) of the proper root, and the formation of the complete expressions for the pure and rotational parts of the strain in this simple and yet very interesting case.

As a simple example of the case in which two of the roots of the cubic are unreal, take the vector function when the strain is equivalent to a rotation θ about the unit vector α; the others of the rectangular system being β, γ.

Here we have, obviously,

$$\phi\alpha = \alpha,$$

$$\phi\beta = \beta\cos\theta + \gamma\sin\theta,$$

$$\phi\gamma = \gamma\cos\theta - \beta\sin\theta,$$

whence at once

$$-\phi\rho = \alpha S\alpha\rho + (\beta\cos\theta + \gamma\sin\theta)S\beta\rho + (\gamma\cos\theta - \beta\sin\theta)S\gamma\rho$$

$$= (1 - \cos\theta)\alpha S\alpha\rho - \rho\cos\theta - V\alpha\rho\sin\theta.$$

Forming the quantities m, m_1, m_2 as usual, we have *

$$\phi^3 - (1 + 2\cos\theta)\phi^2 + (1 + 2\cos\theta)\phi - 1 = 0,$$

or

$$(\phi - 1)(\phi^2 - 2\cos\theta\phi + 1) = 0,$$

or $$(\phi - 1)(\phi - \cos\theta - \sqrt{-1}\sin\theta)(\phi - \cos\theta + \sqrt{-1}\sin\theta) = 0.$$

---

* Solving for g as in the preceding example we have

$$(g - \cos\theta)\rho + (1 - \cos\theta)\alpha S\alpha\rho - \sin\theta\, V\alpha\rho = 0$$

whence operating by $S\alpha,\ S\beta,\ SV\alpha\beta,$

we find $$\left.\begin{array}{r}(g-1)S\alpha\rho \qquad\qquad\qquad\qquad = 0,\\ (g - \cos\theta)S\beta\rho + \sin\theta\, S\alpha\beta\rho \quad = 0,\\ \sin\theta\, S\beta\rho + (g - \cos\theta)\, S\alpha\beta\rho = 0,\end{array}\right\}$$

and finally $$\begin{vmatrix} g-1 & 0 & 0 \\ 0 & g - \cos\theta & \sin\theta \\ 0 & \sin\theta & g - \cos\theta \end{vmatrix} = 0$$ as in the text.

Now $$-(\phi-1)\rho=(1-\cos\theta)(\alpha S\alpha\rho+\rho)-\sin\theta\, V\alpha\rho,$$
$$-(\phi-\cos\theta-\sqrt{-1}\sin\theta)\rho=(1-\cos\theta)\alpha S\alpha\rho+\sin\theta(\rho\sqrt{-1}-V\alpha\rho),$$
$$-(\phi-\cos\theta+\sqrt{-1}\sin\theta)\rho=(1-\cos\theta)\alpha S\alpha\rho-\sin\theta(\rho\sqrt{-1}+V\alpha\rho).$$

To detect the components which are destroyed by each of these factors separately, we have, by (II.), for $(\phi-1)$, the vector

$$(\phi^2-2\cos\theta\phi+1)\rho=-2\alpha S\alpha\rho(1-\cos\theta);$$

so that $$(\phi-1)\alpha=0,$$

which is, of course, true. Again

$$(\phi-1)(\phi-\cos\theta-\sqrt{-1}\sin\theta)\rho$$
$$=-\sin\theta(1-\epsilon^{-\theta\sqrt{-1}})(\sqrt{-1}\,\alpha+1)V\alpha\rho,$$

which we leave to the student to verify. The imaginary directions which correspond to the unreal roots are thus, in this case, parallel to the *Bivectors*

$$(\alpha\pm\sqrt{-1})V\alpha\rho.$$

Here, however, we reach notions which, though by no means difficult, cannot well be called elementary.

A very curious case, whose special interest however is rather mathematical than physical, is presented by the assumptions $$\alpha'=\beta+\gamma,$$
$$\beta'=\gamma+\alpha,$$
$$\gamma'=\alpha+\beta,$$

for then $$\phi\rho=(\beta+\gamma)S\alpha\rho+(\gamma+\alpha)S\beta\rho+(\alpha+\beta)S\gamma\rho$$
$$=(\alpha+\beta+\gamma)S(\alpha+\beta+\gamma)\rho-(\alpha S\alpha\rho+\beta S\beta\rho+\gamma S\gamma\rho)$$
$$=3\delta S\delta\rho+\rho,$$

where δ is a known unit vector. This function is obviously self-conjugate. Its cubic is *

$$\phi^3-3\phi+2=0=(\phi-1)^2(\phi+2),$$

---

* Operating on $g\rho=(\alpha+\beta+\gamma)S(\alpha+\beta+\gamma)\rho+\rho$ by $S\alpha$, $S\beta$, $S\gamma$, we find the cubic

$$0=\begin{vmatrix} g & 1 & 1 \\ 1 & g & 1 \\ 1 & 1 & g \end{vmatrix}=(g-1)^2(g+2).$$

which might easily have been seen from the facts that

1st, $\phi\delta = -2\delta$,

2nd, $\phi\alpha = \alpha$, if $S\alpha\delta = 0$.

The case is but slightly altered when the *signs* of α', β', γ' are changed. Then

$$\phi\rho = -3\delta S\delta\rho - \rho,$$

and the cubic is

$$\phi^3 - 3\phi - 2 = (\phi + 1)^2(\phi - 2) = 0.$$

These are mere particular cases of extension parallel to the single axis δ. The general expression for such extension is obviously

$$\phi\rho = \rho - e\delta S\delta\rho,$$

and we have for its cubic

$$(\phi - 1)^2\{\phi - (1 + e)\} = 0.$$

We will conclude our treatment of strains by solving the following problem: *Find the conditions which must be satisfied by a simple shear which is capable of reducing a given strain to a pure strain.*

Let ϕ be the given strain, and let the shear be, as above,

$$\psi = 1 + \beta S\,.\,\alpha,$$

then the resultant strain is

$$\psi\phi = \phi + \beta S\,.\,\alpha\phi$$
$$= \phi + \beta S\,.\,\phi'\alpha.$$

Taking the conjugate and subtracting, we must have

$$0 = \psi\phi - \phi'\psi' = \phi - \phi' + \beta S\,.\,\phi'\alpha - \phi'\alpha S\,.\,\beta$$
$$= 2V\,.\,\epsilon - V\,.\,(V\phi'\alpha\beta),$$

so that the requisite conditions are contained in the sole equation

$$2\epsilon = V\phi'\alpha\beta.$$

This gives (1) $S\,.\,\beta\epsilon = 0$,

(2) $S\phi'\alpha\epsilon = 0 = S\alpha\phi\epsilon$.

But (3) $S\alpha\beta = 0$ (by the conditions of a shear),

so that $x\alpha = V\,.\,\beta\phi\epsilon$.

Again, (4) $2\epsilon^2 = S\,.\,\phi'\alpha\beta\epsilon = S\,.\,\alpha\phi(\beta\epsilon)$,

$$2x\epsilon^2 = S\,.\,\beta\phi\epsilon\,\phi(\beta\epsilon) = -m\beta^2\epsilon^2,$$

or $$-m\alpha = 2V\,.\,\beta^{-1}\phi\epsilon.$$

Hence we may assume any vector perpendicular to ϵ for β, and α is immediately determined.

When two of the roots of the cubic in ϕ are imaginary let us suppose the three roots to be

$$e_1,\ e_2 \pm e_3\sqrt{-1}.$$

Let β and γ be such that

$$\phi(\beta + \gamma\sqrt{-1}) = (e_2 + e_3\sqrt{-1})(\beta + \gamma\sqrt{-1}).$$

Then it is obvious that, by changing throughout the sign of the imaginary quantity, we have

$$\phi(\beta + \gamma\sqrt{-1}) = (e_2 - e_3\sqrt{-1})(\beta - \gamma\sqrt{-1}).$$

These two equations, when expanded, unite in giving by equating the real and imaginary parts the values

$$\left.\begin{aligned}\phi\beta &= e_2\beta - e_3\gamma\\ \phi\gamma &= e_2\gamma + e_3\beta\end{aligned}\right\}.$$

To find the values of α, β, γ we must, as before, operate on any vector by two of the factors of the cubic.

As an example, take the very simple case

$$\phi\rho = eVi\rho.$$

Here it is easily seen by (m), (s), that $m = 0$, $m_1 = +e^2$, $m_2 = 0$,

so that * $$\phi^3 + e^2\phi = 0,$$

that is $$\phi(\phi + e\sqrt{-1})(\phi - e\sqrt{-1}) = 0.$$

---

* Operating on $g\rho = eVi\rho$ by Si, Sj, Sk, we find

$$0 = \begin{vmatrix} g & 0 & 0 \\ 0 & g & e \\ 0 & -e & g \end{vmatrix} = g(g^2 + e)$$

As operand take $\rho = ix + jy + kz$,

then
$$\begin{aligned} \alpha &\parallel V(\phi + e\sqrt{-1})(\phi - e\sqrt{-1})\rho \\ &\parallel eV.(\phi + e\sqrt{-1})(ky - jz - \rho\sqrt{-1}) \\ &\parallel (-jy - kz + \rho) \\ &\parallel i \end{aligned}$$

Again
$$\begin{aligned} \beta - \gamma\sqrt{-1} &\parallel \phi(\phi + e\sqrt{-1})\rho \\ &\parallel e\phi(ky - jz + \sqrt{-1}\rho) \\ &\parallel -jy - kz + \sqrt{-1}(ky - jz) \\ &\parallel jy + kz - \sqrt{-1}(jz - ky). \end{aligned}$$

With a change of sign in the imaginary part, this will represent
$$\beta + \gamma\sqrt{-1},$$
so that
$$\beta = jy + kz,$$
$$\gamma = jz - ky.$$

Thus, as the student will easily find by trial, β and γ form with α a rectangular system. But for all that the system of principal vectors of ϕ, viz.
$$\alpha,\ \beta \pm \gamma\sqrt{-1}$$
does not satisfy the conditions of rectangularity. In fact we see by the above values of β and γ that
$$S.(\beta + \gamma\sqrt{-1})(\beta - \gamma\sqrt{-1}) = \beta^2 + \gamma^2 = -2(y^2 + z^2).$$

It may be well to call the student's attention at this point to the fact that the tensors of these imaginary vectors vanish, for
$$T^2(\beta \pm \gamma\sqrt{-1}) = -S(\beta \pm \gamma\sqrt{-1})(\beta \pm \gamma\sqrt{-1}) = \gamma^2 - \beta^2 = 0.$$

This gives a simple example of the new and very curious modifications which our results undergo when we pass to *Bivectors;* or, more generally, to *Biquaternions.*

As a pendant to the last problem we may investigate the relation of two vector-functions whose successive application produces rotation merely.

Here
$$\phi = \psi\chi^{-1}$$
is such that by (w)
$$\phi' = \phi^{-1},$$
$$\text{i.e. } \chi'^{-1}\psi' = \chi\psi^{-1},$$
or
$$\chi'\chi = \psi'\psi = \varpi^2,$$
since each of these functions is evidently self-conjugate. This shows that the pure parts of the strains ψ and χ are the same, which is the sole condition.

One solution is, obviously,
$$\chi' = \chi^{-1},\ \psi' = \psi^{-1},$$
i.e. each of the two is itself a rotation; and a new proof that any number of successive rotations can be compounded into a single one may easily be given from this.

But we may also suppose either of ψ, χ, suppose the latter, to be self-conjugate, so that
$$\chi' = \chi = \bar{\chi},$$
or
$$\psi'\psi = \bar{\chi}^2,$$
which leads to previous results.

EXAMPLES TO CHAPTER X.

1. If α, β, γ be a rectangular unit system
$$S . V\alpha\phi\alpha V\beta\phi\beta V\gamma\phi\gamma = -mS . \beta\phi'^{-1}\alpha S . \beta(\phi - \phi')\alpha,$$
and therefore vanishes if ϕ be self-conjugate. State in words the theorem expressed by its vanishing.

2. With the same supposition find the values of
$$\Sigma V . V\alpha\phi\alpha . V\beta\phi\beta \text{ and of } \Sigma S . V\alpha\phi\alpha V\beta\phi\beta.$$
Also of
$$\Sigma . \alpha S\alpha\phi\alpha.$$

3. When are two simple shears commutative?

4. Expand $\frac{1}{1-e\varphi}$ in powers of ϕ, and reduce the result to three terms by the cubic in ϕ.

5. Show that $\phi' V . \phi\rho\phi^2\rho = \frac{S . \phi\rho\phi^2\rho\phi^3\rho}{S . \rho\phi\rho\phi^2\rho} \; V . \rho\phi\rho$

$$= m V \rho\phi\rho.$$

6. Why cannot we expand ϕ' in terms of ϕ^0, ϕ, ϕ^2?

7. Express $V\rho\phi\rho$ in terms of ρ, $\phi\rho$, $\phi^2\rho$, and from the result find the conditions that $\phi\rho$ shall be parallel to ρ.

8. Given the coefficients of the cubic in ϕ, find those of the cubics in ϕ^2, ϕ^3, etc. ϕ^n.

9. Prove
$$\phi V . \alpha\phi'\alpha - m V . \alpha\phi'^{-1}\alpha = 0,$$
$$(\phi + m_2) V . \alpha\phi'\alpha = V\alpha\phi'^2\alpha.$$

10. If $m = \begin{vmatrix} A, & b, & c \\ a, & B, & c' \\ a', & b', & C \end{vmatrix}$ show that $M_g = 0$ may be written as

$$\left\{ g^3 \frac{d^3}{dAdBdC} + g^2 \left(\frac{d^2}{dAdB} + \dots \right) + g \left(\frac{d}{dA} + \dots \right) + 1 \right\} m = 0,$$

$$\text{or } \epsilon^{g(d/dA + ..)} m = 0.$$

11. Interpret the invariants m_1 and m_2 in connexion with Homogeneous Strain.

12. The cubics in $\phi\psi$ and $\psi\phi$ are the same.

13. Find the unknown strains ϕ and χ from the equations

$$\phi + \chi = \varpi,$$
$$\phi\chi = \theta.$$

14. Show that the value of $V(\phi\alpha\chi\alpha + \phi\beta\chi\beta + \phi\gamma\chi\gamma)$ is the same, whatever rectangular unit system is denoted by α, β, γ.

15. Find a system of simple shears whose successive application results in a pure strain.

16. Show that, if ϕ be self-conjugate, and ξ, η two vectors, the two following equations are consequences one of the other:

$$\frac{\xi}{S^{\frac{1}{3}} . \xi\phi\xi\phi^2\xi} = \frac{V . \eta\phi\eta}{S^{\frac{2}{3}} . \eta\phi\eta\phi^2\eta},$$

$$\frac{\eta}{S^{\frac{1}{3}} . \eta\phi\eta\phi^2\eta} = \frac{V . \xi\phi\xi}{S^{\frac{2}{3}} . \xi\phi\xi\phi^2\xi}.$$

From either of them we obtain the equation:

$$S\phi\xi\phi\eta = S^{\frac{1}{3}} . \xi\phi\xi\phi^2\xi S^{\frac{1}{3}} . \eta\phi\eta\phi^2\eta.$$

17. Show that in general any self-conjugate linear and vector function may be expressed in terms of two given ones, the expression involving terms of the second order.

Show also that we may write

$$\phi + z = a(\varpi + x)^2 + b(\varpi + x)(\omega + y) + c(\omega + y)^2,$$

where a, b, c, x, y, z are scalars, and ϖ, ω the given functions. What character of generality is necessary in ϖ and ω? How is the solution affected by non-self-conjugation in one or both?

18. Solve the equations:

$$(a) \quad V.\alpha\rho\beta = V.\alpha\gamma\beta,$$

$$(b) \quad \alpha\rho + \rho\beta = \gamma,$$

$$(c) \quad \rho + \alpha\rho\beta = \alpha\beta,$$

$$(d) \quad \alpha\rho\alpha^{-1} + \beta\rho\beta^{-1} = \gamma\rho\gamma^{-1},$$

$$(e) \quad \alpha\rho\beta\rho = \rho\alpha\rho\beta.$$

19. By throwing the cubic in ϕ into the form

$$(\phi^3 - m)^3 = (m_2\phi^2 - m_1\phi)^3$$

deduce the corresponding equation in ϕ^3, and so show how to extract the cube root of the linear vector function.

Do the same for the fourth and higher powers.

20. Show that, if ϕ be a linear vector function with three real roots, it may be expressed in an infinite number of ways as the product of two pure strains. (Tait.)

21. Find the directions which are most altered by a homogeneous strain. (Tait.)

22. If ϕ is the strain which converts the ellipsoid $S\rho\varpi\rho = -1$ into the ellipsoid $S\rho\psi\rho = -1$, ϕ must satisfy the equation

$$\phi'\psi\phi = \varpi.$$

Show that this is satisfied by

$$\phi = \psi^{-\frac{1}{2}}\chi\varpi^{\frac{1}{2}}$$

when χ is a *rotation* satisfying the equation $\chi'\chi = 1$. Interpret the result. (Joly.)

23. Show that $\theta\phi\theta^{-1}$ and ϕ have identical roots.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.