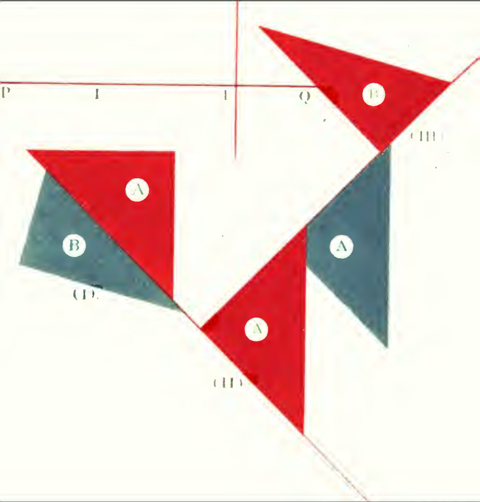


The young geometrician; or, Practical geometry without ...

Oliver Byrne

THE YOUNG GEOMETRICIAN.

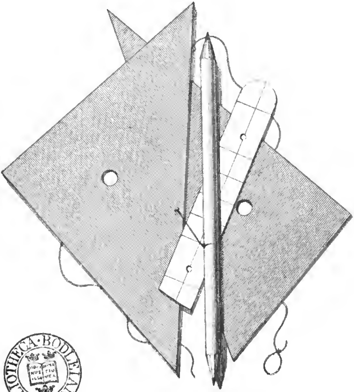
THE YOUNG GEOMETRICIAN;

OR,

Practical Geometry without Compasses.

BY OLIVER BYRNE,

FORMERLY PROFESSOR OF MATHEMATICS AT THE COLLEGE OF CIVIL ENGINEERS; AUTHOR OF "THE ELEMENTS OF EUCLID BY COLOURS," AND MANY OTHER WORKS; INVENTOR OF DUAL ARITHMETIC, A NEW ART; AND THE CALCULUS OF FORM, A NEW SCIENCE, ETC. ETC.



LONDON:

CHAPMAN & HALL, 193, PICCADILLY.

1865.

PREFACE.

A SLIGHT inspection is sufficient to show that this original work, although small, may be of great value to the young Practical Geometrician, those intended for the Engineering profession, or practical Mechanical Draftsmen.

In the geometrical constructions, instead of scale, compasses, and protractor, two set squares or triangular rulers, and a slip of brass, tin, or iron are employed. One of those triangular rulers is half a square, the other half an equilateral triangle; in operating with them they are laid flat on the paper, and as the case may be, any one of the three sides of the one made to coincide with any one of the three sides of the other: the rulers are thus kept in close contact, but one of them is made to slide, while the other, by gentle pressure, is kept fixed until the required lines are drawn; both rulers are then shifted to another position. Throughout the work each ruler is represented by a small similar triangle, to be moved when coloured red, but to remain stationary when coloured blue.

In my Euclid the demonstrations are carried on by the use of coloured symbols, signs, and diagrams which renders the reasoning more precise, and the attainment more expeditious; but in this work, colour is made to indicate motion and also rest; for example, the red ruler has to be moved, while the blue ruler remains at rest. Compasses damage the surfaces upon which they are employed; besides, a required opening is obtained or retained by them with uncertainty, there being no guide to the motion of opening or of closing, and most mechanical motions are irregular when the guide principle is not applied; the triangular parallel rulers, or set squares, remedy those defects. The slip of brass, steel, or tin employed is not more than $\frac{1}{16}$th or $\frac{1}{18}$th of an inch thick, and answers several purposes, namely, straight white lines may be drawn with one of its corners, which is smooth and rounded, so as not to cut the paper, but indent it, which in many cases dispenses with the use of the black lead pencil and india-rubber. This strip of metal also assists in drawing ellipses, logarithmic spirals, and circles. A few permanent lines drawn on one of its sides serve to bring it into any known or required position. This slip or strip of metal is likewise used to lay off and measure lines and angles. With respect to the extent of the work, it may be observed that but few problems are introduced, compared with the number that can be submitted to similar treatment. The methods employed to lay off and measure angles, and to describe ellipses and logarithmic spirals, are new. I have also solved the celebrated problems of the duplication of the cube, and the trisection of an angle; my constructions will be found more accurate, and as simple, as the construction of a square on a given straight line, by compasses and ruler. These problems have been discussed since and before the time of Hippocrates, Plato, and Menechme, more than five hundred years before the Christian Era; at present, the solutions of these problems by the help of ruler and compasses only are considered impossible. The problem of the duplication of the cube is called the Delian problem, and is said to have been proposed by an oracle. It may be necessary to add that other set squares or triangular rulers may be purchased cheaply at mathematical instrument makers, and employed with advantage, as well as those used and described in this work.

OLIVER BYRNE.

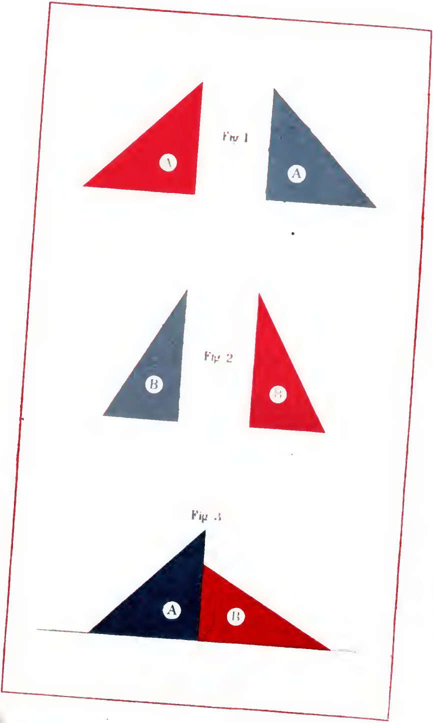


Fig 1
A
A
Fig 2
B
B
Fig 3
A
B

THE YOUNG GEOMETRICIAN.

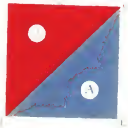
DEFINITIONS AND DIRECTIONS.

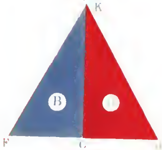
1. A, A, is a triangular ruler, coloured red when it is supposed to slide, and blue when supposed stationary; this flat ruler forms half a square. Fig. 1.

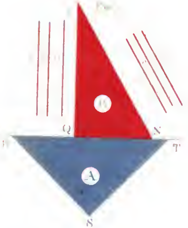
2. B, B, is another triangular ruler, and as in the former case, coloured red when moveable, and blue when stationary; this flat triangular ruler is half an equilateral triangle. Fig. 2.

These triangular rulers are made of thin pear-wood, or hard gutta-percha, and sold by most mathematical instrument makers. A, B, represent rulers of all sizes of their respective shapes; in trade these rulers are termed set squares.

By looking along the edges it is easily observed if these rulers be correct or not, so far as the straightness of the edges is concerned; the right angles may be tested by placing one on the other, to see if they coincide, and stand together on a straight line, as shown in Fig. 3.


A

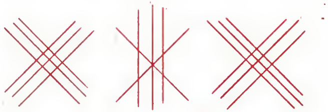

K
B
F

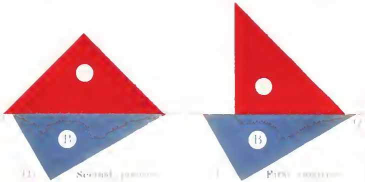

Q
N
T
A
S

3. If one side of A be supposed red, and the other blue, a square may be formed as in Fig. 4. A straight line may be laid off the length of either the diagonal C D, or the side D E = E C. The angle C D E = E C D = 45°; C E or E D, is about five inches long, but may be obtained of greater or less length.

4. If one side of B be supposed blue, and the other red, an equilateral triangle may be formed as in Fig. 5. A straight line may be laid off the length of K H, H G, or G K. The angle K F G = 60°; the angle F K G = G K H = 30°. K H is double the length of H G. The length of K H is generally five or six inches long, but may be had of greater or less length.

5. Fig. 6 shows the positions of the rulers in drawing the parallel lines *a*, *a*, *a*, and *b*, *b*, *b*. A being stationary and B moved, or slipped along the common base line R T, *a*, *a*, *a*, ... if produced, will meet *b*, *b*, *b*, ... at an angle of 30°. The faces N Q and T R must be brought together without blow or shock. If A is found to shift during the operation, it must again be properly placed. Other parallel lines may be drawn, making angles of 60° and 30°, with one another, by sliding the faces P N, Q P, upon R T kept stationary.



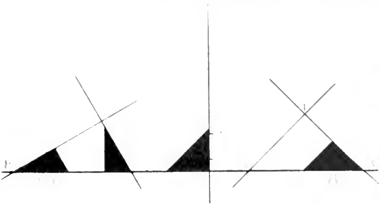

B
B
Second
First


d
c

6. The first position of the rulers, determines the parallel lines *a*, *a*, *a*, ... perpendicular to P Q, Fig. 7; and the parallel lines *b*, *b*, *b*, ... making angles of 45° with P Q. The second position determines the parallel lines *b*, *b*, *b*, ... and *c*, *c*, *c*, ... making right angles with *b*, *b*, *b*, ... All these lines may be drawn by allowing B to retain its position throughout, and only changing A. It must not be forgotten that the triangular ruler coloured red is always considered moveable, and the blue one stationary. Lines drawn as in this case make right angles with one another and the fixed line P Q, or cut one another at angles of 45°.

7. C D E and F G H represent the two triangular rulers employed, Fig. 8. The small blue triangle standing on a line alone shows that line to be the exact length of one of the sides of one of the triangular rulers; the side represented is similar to the small side in contact, thus H T = the side H G, *t h* and *g h* being the small sides corresponding to the side G H of the ruler F G H.

The exact length of the line C E is pointed out by the little blue triangle *c d e*, and the exact length of D E is shown in a similar manner by the position of the small blue triangle *c d e*.



The triangular ruler when small and coloured red also points out that the line upon which it is placed is one drawn between two known or given points, or a line found by producing a known line.

8. The small red triangle Fig. 10 placed on the line C D E F indicates that C F is obtained by either joining the points C F, or by producing D E, E F, or C D.

Both the set squares are coloured red when both have to be moved while being operated with (see Probs. 39 and 40).

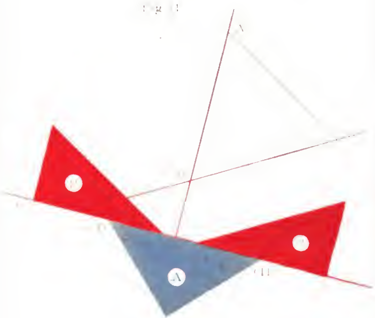
9. When the triangular ruler is coloured red and represented without its stationary blue companion, it is supposed to be moved alone from some fixed position or in the direction of a given line, as P O, Fig. 9.

(I) The line T N is drawn perpendicular to P O, by first drawing *g h* a portion of T N and then producing *g h* both ways, by moving the ruler in the direction of *g h*. In a similar way one of the rulers may be employed to draw the line *e f* making an angle of 45° with P O (II). (III), *a c* makes an angle of 60° with P O. (IV) *b c* makes an angle of 30° with P O; all these lines are drawn by moving a single ruler. In each of the following constructions the first position of the rule or rulers will be marked (I); the second, (II); the third, (III); and so on consecutively.

10. A great variety of positions may be given to the set squares, from which the same line or lines may be drawn; but from the nature of particular geometrical constructions, one position may be found more convenient than another. However, suitable positions are instantly found, since when straight lines are given or drawn, corresponding positions are soon determined, and *vice versâ;* (5) (6).

PART I.

PROBLEMS SOLVED BY DRAWING STRAIGHT LINES ONLY, WITHOUT THE USE OF GRADUATED LINES OR COMPASSES.

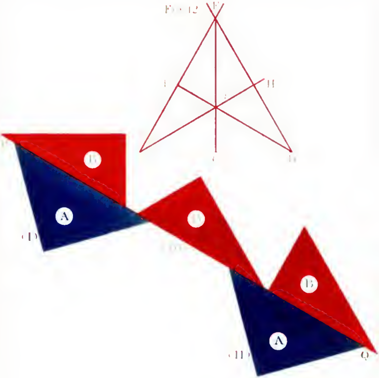


PROBLEM I.

To describe an equilateral triangle upon a given finite straight line A B.

A suitable position for D E is taken, Deffs. 5, 6, 10, and C A drawn from (I); from (II) C B is drawn and the equilateral triangle A B C is described. Fig. 1.

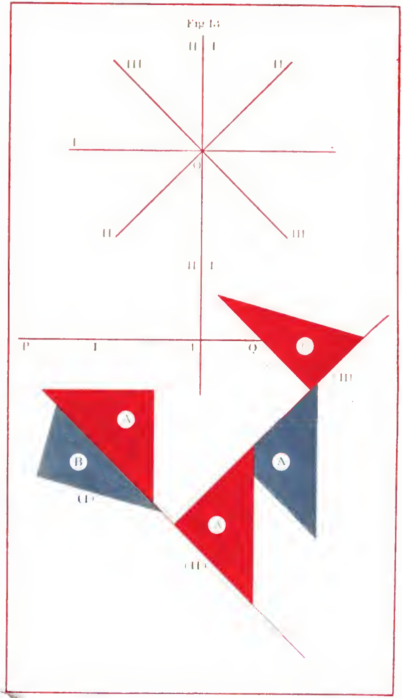
A B and AC are parallel lines of the first position (I). B C a parallel line of (II).



PROBLEM II.

To describe an equilateral triangle C D E on a given straight line C D, and draw perpendiculars from the angular points to the opposite sides. Fig. 12.

(I) gives the base line P Q, and determines E G, after C E and E D are drawn by (II); (III) gives C H and the perpendicular F D, found by joining the points O, D. When the position of the blue ruler A is found, it may not be necessary to shift it during the construction; the red ruler as usual being considered the only one moved. C D, E G, parallels of (I); C H, D E, parallels of (III); E C a parallel of (II); F D is parallel to P Q.

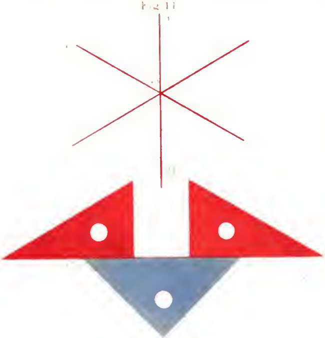

B
A
A
A

Problem III.

Through a given point O to draw two straight lines, one perpendicular and the other parallel to a given straight line P Q; and to divide each of the four right angles thus formed into two equal angles. Fig. 13.

The corresponding numbers (I) and I, (II) and II, (III) and III, show how the required lines are drawn through O. Although five triangular rulers are represented, but two are employed; coloured blue when stationary, and red when considered moveable, one marked A, and the other B; these remarks will not be again repeated.

I, O, I, are parallel lines of the first position (I). II, O, II, parallels of the second position, marked (II). III, O, III, parallels of the third position, marked (III). See Definitions 5 and 6.

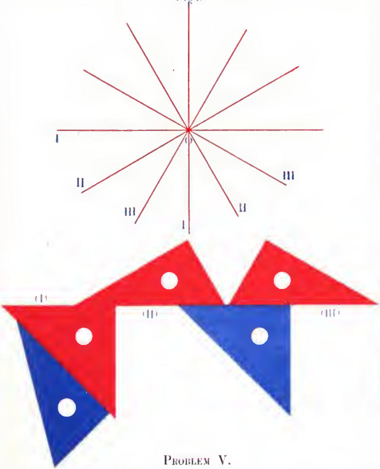


Problem IV.

Through a given point O, Fig. 14, to draw three straight lines forming six equal angles.

How the lines are drawn is apparent by inspecting the figure. One of the lines is marked both I and II, because it may be drawn from either of the positions (I) (II).

Fig. 15.

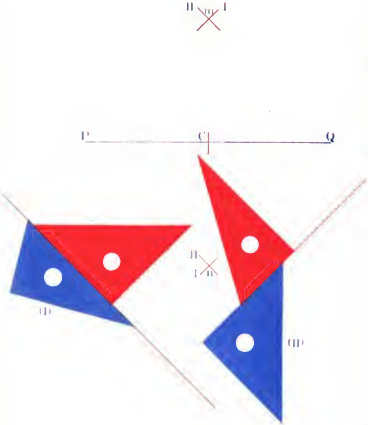


Problem V.

From a given point O to draw twelve straight lines so as to lay off twelve equal angles, each equal 30°. Fig. 15.

Compare the numerals on the figure with the positions corresponding, and the construction is readily understood. In future, when comparisons from positions to the numerals on the figure have only to be made, the problem will be merely announced. See Definitions 5, 6, 10, and Problems I., II., III., IV., with respect to parallel lines.

Fig. 16.

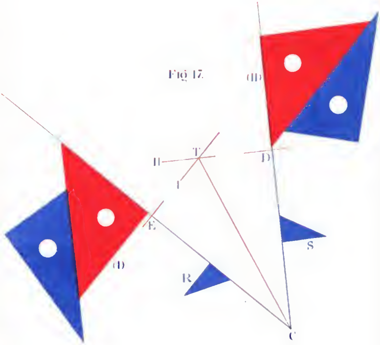


Problem VI.

To bisect a given finite straight line, P Q. Fig. 16.

The points *m* and *n* being found from (I) and (II), the line joining *m* and *n* divides the line P Q into two equal parts in the point C. P Q becomes the diagonal of a square of which P, *n*, Q, *m*, are the angular points. In a similar manner a given line may be divided into four, eight, sixteen, &c., equal parts.

Fig. 17.

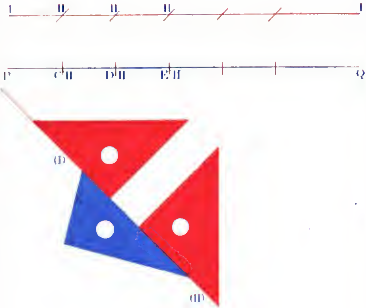


Problem VII.

To bisect a given rectilinear angle, E C D. Fig. 17.

The positions of the small blue triangles R, S, show, as before observed, that the lines, C E, C D, are taken the exact lengths of the shortest side of the triangular ruler B. T is found as indicated, and the line T C bisects the given angle E C D. In this way an angle may be divided into two, four, eight, &c., equal angles.

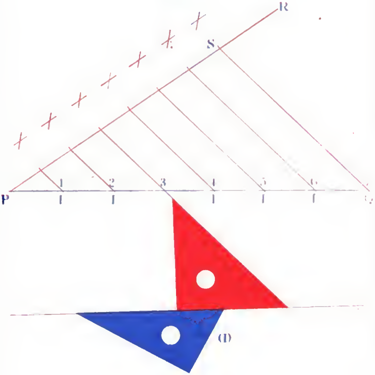
Fig. 18.



Problem VIII.

On the straight line P Q to lay off any number of equal parts P C, C D, D E, ... each equal to P C. Fig. 18.

Fig 19

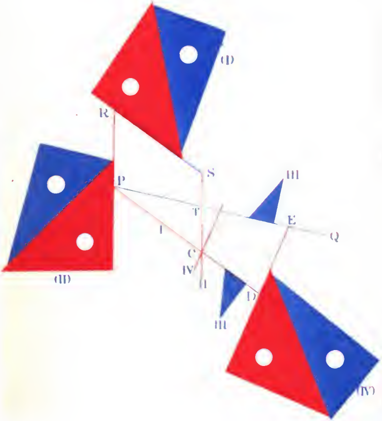


PROBLEM IX.

To divide a given line P Q into any proposed number of equal parts, say seven. Fig. 19.

Draw any line P R, making any angle with P Q; in the line P R, beginning at P, set off seven equal parts of any length by Prob. VIII.; join S and Q, then the parallel lines, given by (1), divide the line P Q into seven equal parts.

Fig 20



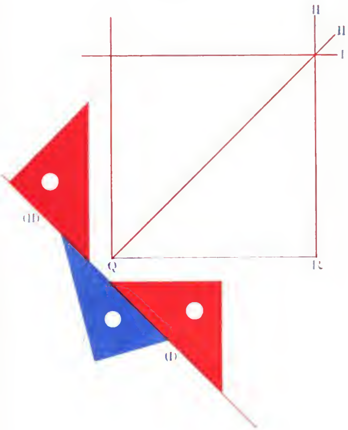
PROBLEM X.

From the greater of two straight lines P Q, to cut off a part P F equal to the lesser R S, anyhow posited. Fig. 20.

Join the points P and R, then the positions of the rulers show how the required lines are to be drawn.

R S = P C = P F, and P D = P E.

Fig. 21



PROBLEM XI.

On a given line Q R, to construct a square. Fig. 21.

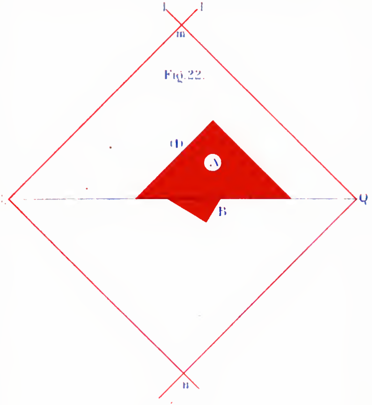
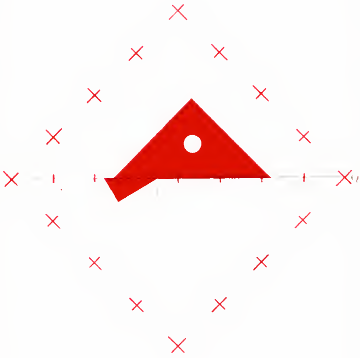


Fig. 22.

Problem XII.

To construct a square on a given diagonal Q P. Fig. 22.

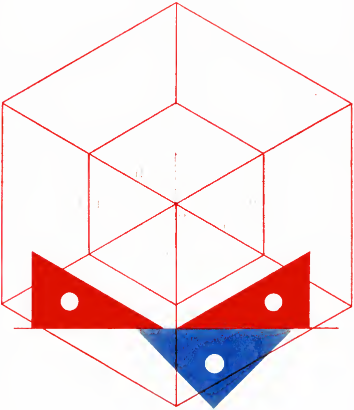
The triangular ruler B is moved along the diagonal, below, as in the figure, to determine *m*; and above in a similar manner to find *n*. The employment of the small red triangle, as agreed upon before, represents this motion along a line. This process will be found very convenient in dividing any given straight line, as P Q into 2, 4, 8, 16 . . . . equal parts as the points *m*, *m*, *m*, ... and *n*, *n*, *n*, ... are only required. The next problem illustrates this remark.



PROBLEM XIII.

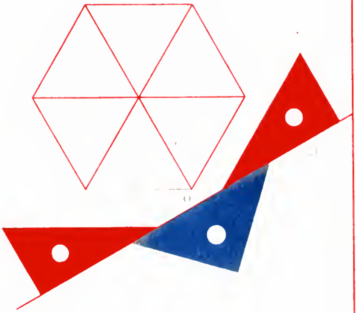
To divide a given straight line P Q into eight equal parts. Fig. 23.

(I) indicates that both the triangular rulers move on the line P Q, and on either side of it.



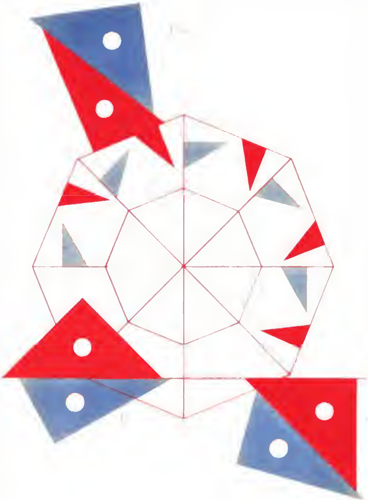
PROBLEM XIV.

To describe a regular hexagon, so that the distance from the centre to each of the angles may be a given length, O P. Fig. 24.



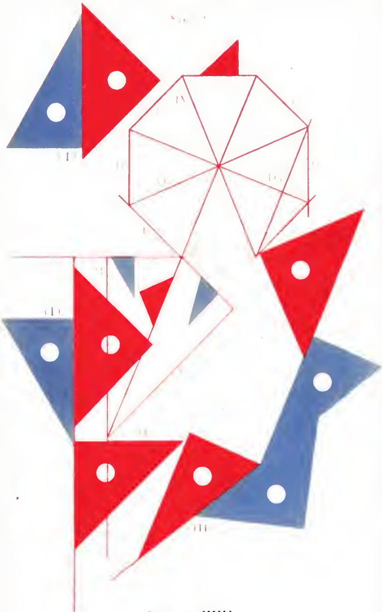
Problem XV.

On a given line, P Q to construct a regular hexagon. Fig. 25.



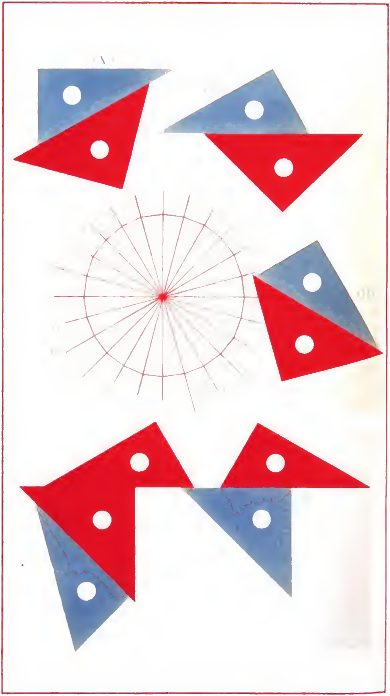
PROBLEM XVI.

To describe a regular octagon, so that the distance from the centre to each of the angular points may be of a given length, O P. Fig. 26.



Problem XVII.

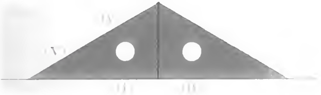
On a given line P Q to describe a regular octagon. Fig. 27.



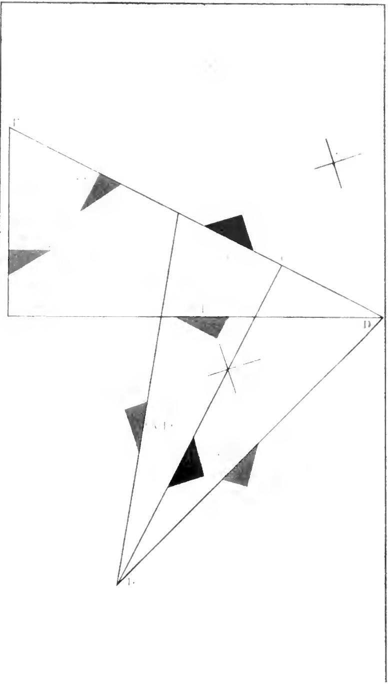
Problem XVIII.

Round a given point as centre to describe a regular polygon of twenty-four sides, Fig. 28.

Regular polygons of 12, 8, 6, 4, and 3 equal sides may at the same time be constructed round the given point.



The bases are on a triangle of the form, Fig. 29, and may stand in any position round the given point. All the bases may be at the same side of the base, P Q.


D

Problem XIX.

To construct a triangle F G D H (Fig. 30), so that each of the angles at the base F D will be double the angle H.

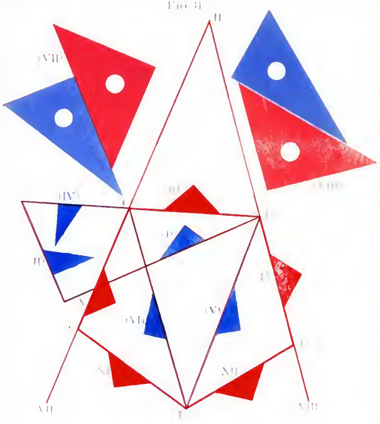
Explanation of the Figure.

It is indicated that C D is taken equal the longest side of the lesser triangular ruler, and that F H and H D are equal and each equal C D.

C E = E F = the length of the shortest side of the lesser triangular, and hence C E is equal half the length of C D. It is also indicated that E D is found by joining the points E and D, and G H by joining the points r and s. Let the degrees in the angle F H D be called x; then the degrees in H F G $= 2x$, and H D G $= 2x$ also :

$$\therefore 5x = 180^\circ \therefore x = 36^\circ = \text{F H D}$$
$$\text{and } 2x = 72^\circ = \text{H F G} = 72^\circ.$$
$$\text{F H G} = \text{G H D} = 18^\circ = \text{half of } x.$$

This problem being understood, a pentagon is easily constructed.

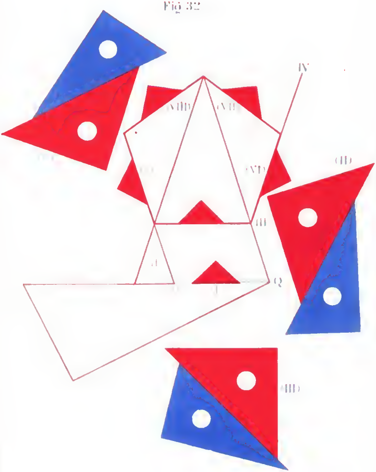


PROBLEM XX.

To construct a regular pentagon. Fig. 31.

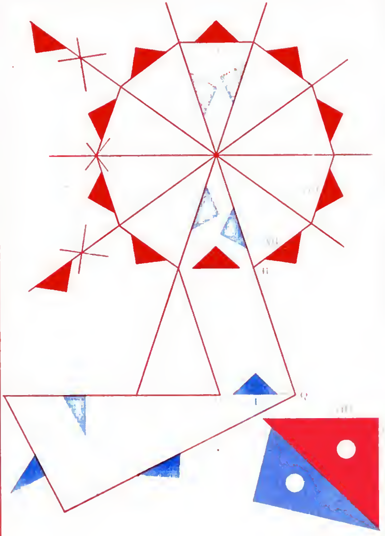
Although the construction indicates how made, yet it may be necessary to state, the triangle C E D, or its equal C H D, is, by the last problem, so constructed as to have each of the angles at the base double the vertical angle; that is angle E C D = E D C = twice C E D. By following the numbers of the position (I), (II), (III),... the construction is readily understood. (VII) shows that G C is parallel to E D, and (VIII) that D H is drawn parallel to C E.

Fig. 32



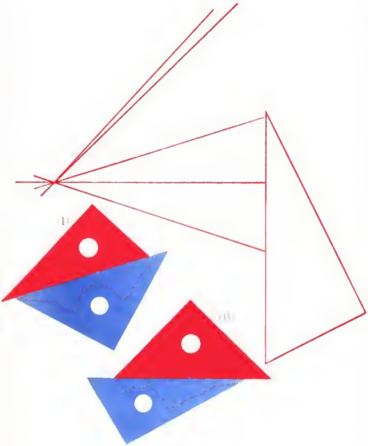
Problem XXI.

To construct a regular pentagon each of whose equal sides is equal to a given length, P Q. Fig. 32.



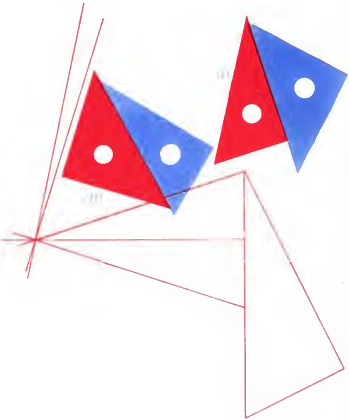
PROBLEM XXII.

To describe a regular dodecagon, each of whose equal sides shall be of a given length, P Q. Fig. 33.



PROBLEM XXIII.

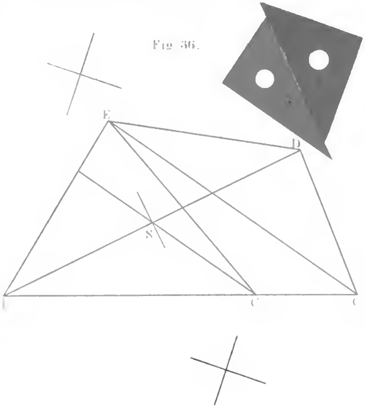
To lay off an angle of three degrees. Fig. 34.



PROBLEM XXIV.

To lay off an angle of six degrees. Fig. 35.

In a similar manner angles of 9°, 12°, 15°, . . . may be laid off.

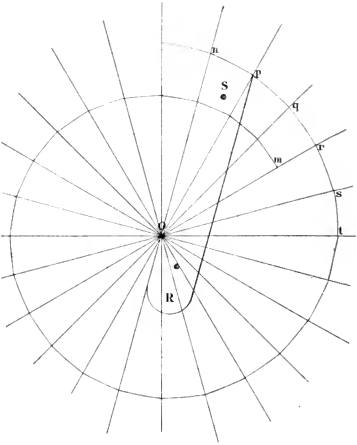


Problem XXV.

To bisect a trapezium C D E F, by a line E G, drawn from one of its angles E. Fig. 36.

The diagonal D F, opposite the angle E, is bisected in S.

Fig 37.



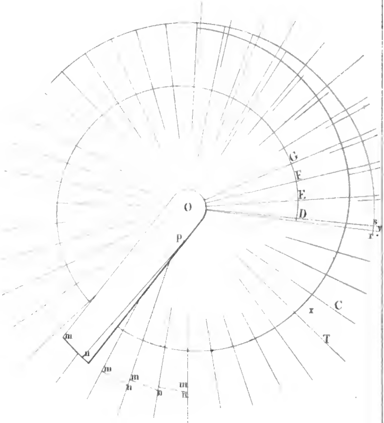
Problem XXVI.

To describe a logarithmic spiral, without the use of compasses, about a given point, O. Fig. 37.

The logarithmic spiral given in Fig. 37 is formed by drawing twenty-four equal angles from the given point O, by Problem XVIII., and applying the thin slip of brass or steel to touch the radii vector O*n*, O*p*, O*q*, ..., or the radii vector produced. As the spiral is being drawn the point *n* is moved round from *n* to *p*, from *p* to *q*, &c. R S represents the thin slip of brass or steel, which may also be employed to draw white lines to avoid pencil marks.

It is not necessary that the equal angles be an aliquot part of a circle; and the points *m* and *n* may be selected at pleasure; for example, see the next problem.

Fig.38



Problem XXVII.

To describe a logarithmic spiral round the centre, origin, or pole O (Fig. 38), without dividing the circle into equal parts.

In the figure $m\,n$ taken at random measures of the equal angles $m\,O\,n, \ldots$ the measurement is continued round and round as often as necessary. During this operation the point O is kept stationary by a needle or common brass pin. When the angles are laid off, the slip of brass is removed and applied to lay off the spiral, commencing at any point D, in the first radius vector O D, and in succession, bringing the points n and P to coincide with E and the line E O; F and the line F O; G and the line G O. An immense number of spirals may be formed in this way m, n, P, being taken anyhow. It may be observed that when the angle $P\,n\,m$ forms a series of isosceles triangles $O\,m\,n$, the curve traced will be a circle.

Let x = the degrees in each of the equal angles C O T, and y = angle $r\,O\,s$. $5\,y = x$, very nearly; but

$$y + 39\,x = 360^\circ, \therefore \frac{x}{5} + 39\,x = 360,$$

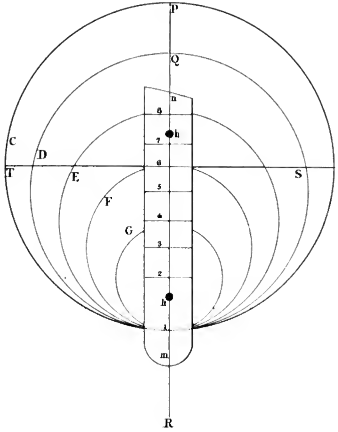
$$x + 195\,x = 1800\cdot$$

$$196\,x = 1800\cdot$$

$$x = \frac{1800}{196} = 9\tfrac{9}{49} \text{ degrees nearly.}$$

$$= \text{angle C O T.}$$

Fig 39.



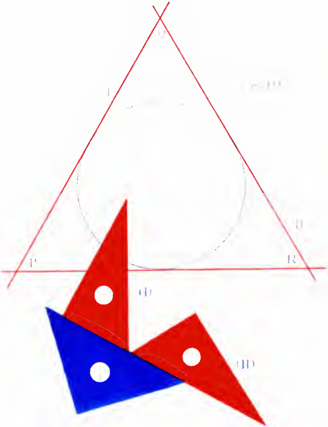
Problem XXVIII.

To describe a circle without compasses. Fig. 39.

In the brass slip there are holes *h h*, by which the line *m n* may be placed over any given line P R, and by the cross lines on the slip, over any perpendicular line S T at the same time, so that the small pin-hole 6, 5, 4, ... may be made to fall over any point O, where R P meets S T at right angles. Let 1 be a small hole capable of admitting the point of a pencil; then, if the point 6 be held fast by the point of pin or needle, the circle C will be described by moving the pencil point round in contact with the paper. If the point 5 be made fast, the circle D will be described; if 4, E, and so on.

PART II.

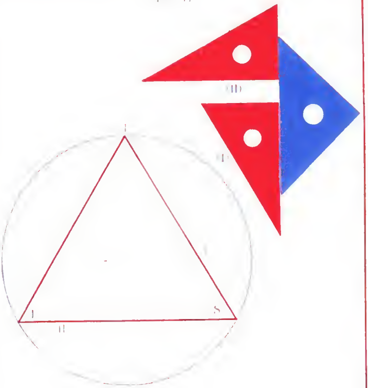
THE CIRCLE AND REGULAR POLYGONS.



PROBLEM XXIX.

About a given circle to describe an equilateral triangle P Q R. Fig. 40.

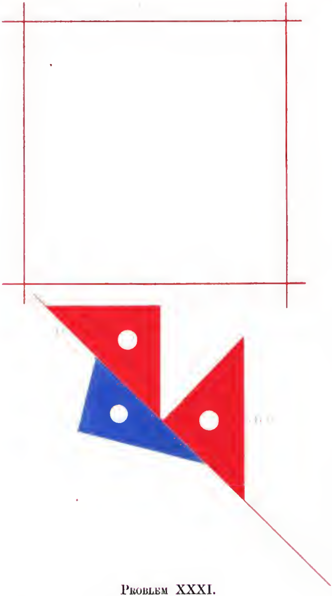
The line P R is drawn to touch the given circle at any point; their position (I) determines P Q, and (II), Q R.



PROBLEM XXX.

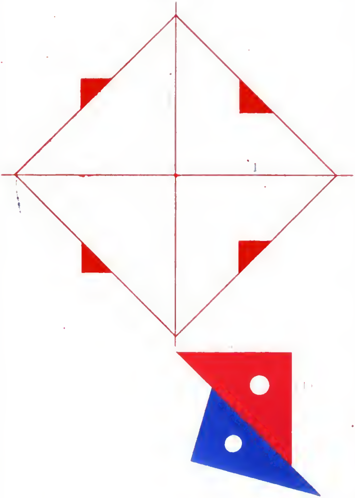
In a given circle to inscribe an equilateral triangle R S T. Fig. 41.

Draw the diameter, then (I) and (II) determine the points R and S.



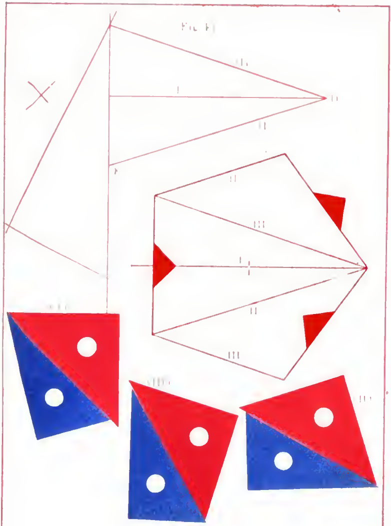
PROBLEM XXXI.

To describe a square about a given circle. Fig. 42.



PROBLEM XXXII.

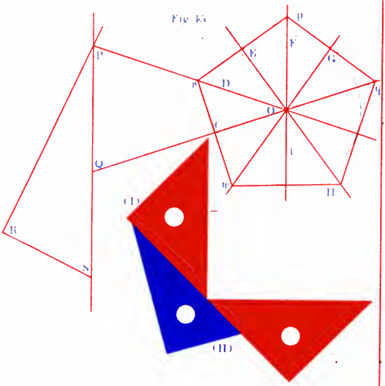
In a given circle to inscribe a square. Fig. 43.



PROBLEM XXXIII.

To inscribe a regular pentagon in a given circle. Fig. 44.

The triangle C D E is so constructed that each of the angles C, E, is double the angle D.



PROBLEM XXXIV.

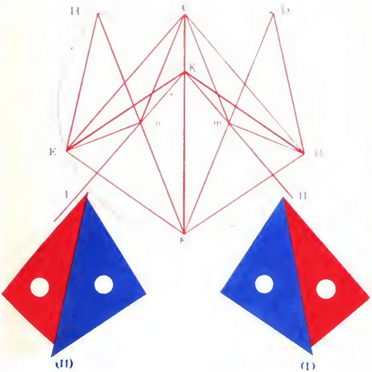
About a given circle to describe a regular pentagon. Fig. 45.

The triangle P O Q being constructed, so that angle P=twice angle O=Q, the given circle is readily divided into ten equal parts D O C, E O D, F O E, ... then the pentagon *m*, *n*, *p*, . . . circumscribing the given circle, may be drawn by drawing perpendiculars to the radii, through the points C, E, G, ...

P R=2 R S, R S=S Q.

P R=P O=O Q.

Fig 46



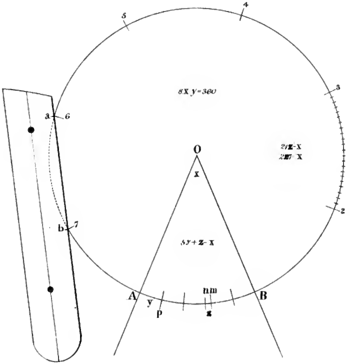
Problem XXXV.

To divide the area of any circular segment D G F E H into four equal areas by three straight lines E K, G K, F C. Fig. 46.

Divide the arc into four equal parts in the points E, F, G, draw the chords D G, G F, F E, and D F, F H. Drop perpendiculars E n, G m, F C, draw C E, E n, C n, and also C G, m C, m G; then draw m K parallel to C G and n K parallel to E C, join E and K, and G and K, and the four areas.

C K G D, C K E H, F K G, E K F are equal to one another.

Fig 47.



Problem XXXVI.

To find the number of degrees, minutes, &c., in any given angle A O B, without the use of compasses or protractor. Fig. 47.

The chord of the arc A B may be marked off on the edge of the slip of brass by slight dots of ink a, b, and applied round the circle, A B=B, 2,=2, 3,=3, 4,=4, 5, =5, 6,=6, 7,=7, p.

$\therefore$ Putting x=the degrees, &c.
in the arc A B, and y=A p, the arc in excess of an even number of applications,

$$\therefore 8x-y=360^\circ. \quad (\text{I}).$$

In a similar manner A $p=y$, is found to be contained five times in A B, and $m\,n$ over;

Putting $m\,n=z$,

$$\therefore 5y+z=x. \quad (\text{II}).$$

Similarly,

Find how often z is contained in A B, or in 2, 3, which is equal A B.

It may in the example before us be taken for 21 or 22 times, it matters not which is selected, as the sequel will show.

$$\therefore 21z=x \quad Or\ 22z=x \quad (\text{II}).$$

To find x from (I), (II), (III), taking $21z=x$

$$z=\frac{x}{21}, \text{ from (III)};$$

which when substituted in (II), gives

$$5y+\frac{x}{21}=x \qquad \therefore y=\frac{4x}{21}$$

$$\therefore 8x-\frac{4x}{21}=360, \text{ from (I)}.$$

$\therefore$ 41 x=1890 (46° 6′ very nearly.

```
164
———
 250
 246
 ———
   4
  60
 ———
 240
 246
 ———
```

Again let the three equation be

$8x - y = 360$, (I).

$5y + z = x$, (II).

$22z = x$, (III).

$$z = \frac{x}{22} \text{ from (III).}$$

$$\text{then } 5y + \frac{x}{22} = x \text{ from (II).}$$

$$\therefore y + \frac{21x}{110},$$

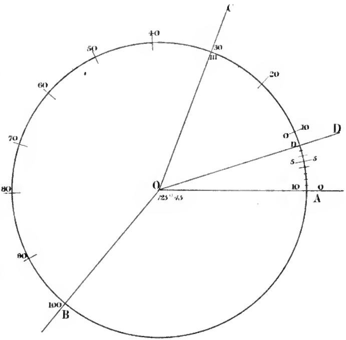
$$\text{and } 8x - \frac{21x}{110} = 360, \text{ from (I).}$$

859 x=39600 (46° 6′ very nearly.

```
3436
————
 5240
 5154
 ————
   86
   60
 ————
 5160
 5154
 ————
```

Hence the angle A O B=46° 6′, and it makes no practical difference whether we take 21 $z=x$ or 22 $z=x$, a fact worth notice.

Fig 18.



Problem XXXVII.

To lay off an angle of any given number of degrees, &c., without compasses or protractor. Give an example with 75° 40′.

Draw the radius O A, Fig. 48, and lay off ten small arcs of any length, so that each may be less than one-tenth of the radius O A. Then set off ten of these small divisions ten times and draw the radius O B.

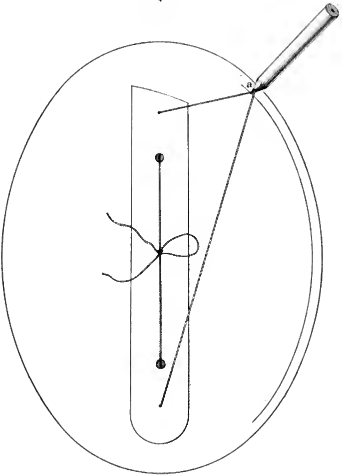
Measure the angle A O B by the last problem; in the given case A O B is found = 125° 45′.

°	′
360	0
125	45
234	15

234° 15′ : 100 :: 75° 40′ : 32·3.

Take from m to $n = 32{\cdot}3$, then the angle D O C= 75° 40′. By these methods angles may be measured and laid off more accurately than by an expensive protractor. See the Author's work, entitled, "The Practical Model Calculator."

Fig. 49.

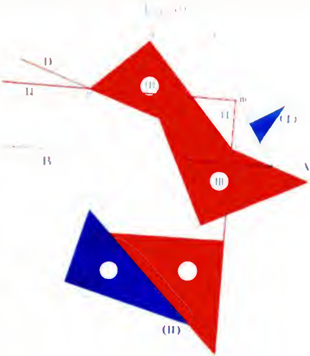


Problem XXXVIII.

TO DESCRIBE AN ELLIPSIS.

With the sharp point of a penknife make a very fine hole in a black-lead pencil; the lead must be good, as the operation is a very delicate one. A piece of soft lead metal may be employed for the same purpose. The small hole *a*, Fig. 49, is to be through the lead, not the wood; the nearer the point the better. Then a silk thread is passed through this hole, and through any two of the small holes in the slip of brass, and secured in the manner shown in the figure. The thread or fine cord may be lengthened or shortened at pleasure, and the Ellipsis described in the way shown in the figure.

Circles of considerable extent and of given radii may be described in a similar way, by allowing both ends of the thread to pass through the same hole in the strip of brass.



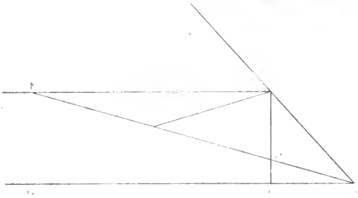
Problem XXXIX.

To divide a given angle B A C into three equal angles, Fig. 50.

The line A *m* is made=*p q*, the least side of the lesser triangular ruler; by (II) *p m* is drawn parallel, and *m n* perpendicular to A B.

Then both rulers are kept in motion, and at the same time in close contact, as represented in the figure, until *p* falls on the line *p m*, and *n* on the line *m n*; *r n* A passing through the angular point A.

Then the angle D A B is one third of the angle C A B. This is a problem not capable of solution by the straight line and circle.



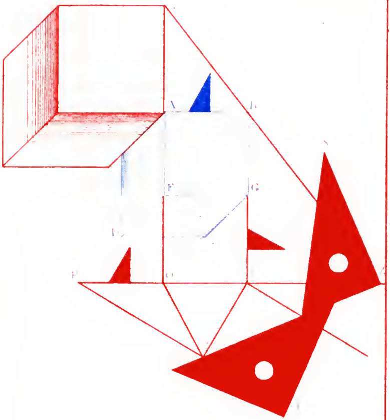
The truth of the solution of Problem XXXIX. may be thus established, Fig. 51.

Suppose F B parallel and B C perpendicular to A G, and further, suppose F D = twice A B; bisect F D in E, and draw E B, because F B D is a right angle—

F E = E B = E D = A B

Then the angle B F E = F B E = E A C, on account of the parallels.

Again B E A = twice B F E, but B E A = B A E,
$\therefore$ B A E = twice B F E = twice D A C,
and $\therefore$ the angle D A C is one-third of the angle B A C.



PROBLEM XL.

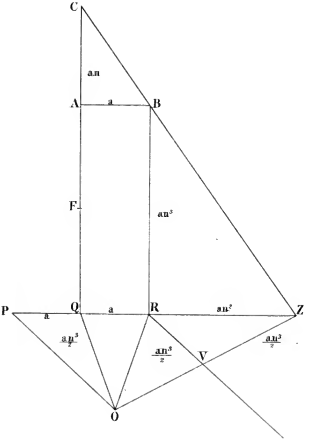
Let A B be the side of a given cube B D. It is required to find A C, the side of another cube C E, so that the solid contents of the cube C E are double the solid contents of the cube B D, Fig. 52.

This is the famous problem of the duplication of the cube, and cannot be solved by the help of ruler and compasses only; it is sometimes called the Delian problem. Before the time of Plato, A.C. 390, and since, mathematicians have in vain attempted to solve this and the last problem geometrically, that is, by the ruler and compasses only.

Let A B = B G = G R = R Q = Q P = Q O = O R = V Z. The length of the shortest side of the lesser set square; a line of any other given length may be applied. Draw O P and V R parallel to it; then apply the set squares in close contact, the edge O V of O V T passing through the point O, while the points V and Z of Z S V fall exactly on the lines R V, R Z. Then draw the line Z B C, cutting F A produced in C; then the cube on A C is double the cube on A B.

However desirable it may be to employ compasses, yet no construction in which compasses might be employed can be capable of a higher degree of accuracy than the one given in the text.

Fig 53.



Demonstration, Fig. 53.

Let $AB = a$, $AC = an$, $BR = an^3$, $RZ = an^2$.

a, an, an^2, an^3, are the four terms a and an^3, the extremes, and an, an^2, the means.

$$PQ = a,$$

$$AF = FQ = QO = OR = VZ = \frac{an^3}{2}.$$

$$OZ^2 = OR^2 + ZR^2 + RQ \times RZ$$

$$= \left(\frac{an^3}{2}\right)^2 + a^2 n^4 + a^2 n^2 = \left(\frac{an^3}{2} + an\right)^2$$

$$\therefore OZ = \frac{an^3}{2} + an.$$

Now, if RV be parallel to PO,

$$PZ : RZ :: OZ : VZ, \text{ that is}$$

$$2a + an^2 : an^2 :: \frac{an^3}{2} + an : \frac{an^3}{2}$$

$$\text{for } \frac{an^3}{2}(2a + an^2) = an^2\left(\frac{an^3}{2} + an\right)$$

$$\text{or } \frac{n}{2}(2a + an^2) = \left(\frac{an^3}{2} + an\right)$$

$$\text{or } \frac{n}{2}(2 + n^2) = \left(\frac{n^3}{2} + n\right).$$

Then it is also evident that

$$ZR : RB :: BA : AC = an.$$

Then, if $a = 1$ and $an^3 = 2$, it is evident that in the case of duplication, if AB be taken $= 1$, then $BR = 2$, and $AC = an = \sqrt[3]{2}$, $a^3 = 1$, $a^3 n^3 = 2$, in this particular case.

If BR be taken $= 3$ times AB, then the cube of $AC = 3$ times the cube of AB, and so on.

THE END.

Cox and Wyman, Printers, Great Queen Street, London, W.C.