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CHAMBERS'S EDUCATIONAL COURSE.—EDITED BY W. AND R. CHAMBERS.

EXERCISES AND PROBLEMS

IN

ALGEBRA

WITH

ANSWERS AND HINTS TO THE SOLUTIONS

LONDON
W. AND R. CHAMBERS 47 PATERNOSTER ROW
AND HIGH STREET EDINBURGH
1855

Edinburgh:
Printed by W. and R. Chambers.

PREFACE.

The following collection of Exercises in Algebra is intended as a Supplement to the Algebra of Chambers's Educational Course, and is arranged in sections corresponding to those of that work. The Answers to all the Questions are given, and also such extensive hints for the solution of the more difficult Exercises, as will render the work convenient to Teachers, and useful to private Students. In arranging the Exercises, great care has been taken to place them as much as possible in the order of difficulty, so that the Student may be able to solve them by his own efforts. It will not be necessary to take all the sections in the exact order in which they are arranged; and Teachers may find it advantageous to direct the Pupil's attention to the section of Equations beginning on page 40, before he has gone through all the previous sections.

J. PRYDE, F.E.I.S.

CONTENTS.

EXERCISES IN ALGEBRA,

WITH THE ANSWERS.

NUMERICAL EVALUATION OF ALGEBRAIC EXPRESSIONS.

If $a = 2$, $b = 3$, $c = 4$, $d = 5$, find the numerical value of the following algebraic expressions:—

1. $a + b + c + 4d$, and $3a + 4b + 2c - d$, $=$ 29 and 21.
2. $2a + 3b - c$, and $5b + 6c - 4d$, . $=$ 9 " 19.
3. $a^2 + ab + b^2$, and $a^2 - ab + b^2$, . . $=$ 19 " 7.
4. $4a^2 + 4ab + b^2$, and $4a^2 - 4ab + b^2$, . $=$ 49 " 1.
5. $(d + b)(d - b)$, and $(d - a)d - a$, . . $=$ 16 " 13.
6. $a^2 + 2ac + c^2$, and $a^2 - 2ac + c^2$, . $=$ 36 " 4.
7. $(a + b + c)(a + b - c)(a - b + c)$, . $=$ 27.
8. $a^2b^2 + 2abc + c^2$, and $a^2b^2 - 2abc + c^2$, $=$ 100 " 4.
9. $abc + bc(d - b)$, and $abc - ab(d - c)$, $=$ 48 " 18.
10. $a^2bc + bc(a^2 - c)$, and $a^2bc + bc(a^2 + c)$, $=$ 48 " 144.

If $a = 4$, $b = 8$, $c = 3$, $d = 2$, and $x = 5$, find the numerical value of the following algebraic expressions:—

11. $3a + 5b + c + 2cdx - abd$, ... $=$ 51.
12. $4ac - 3dx + 5acd - a(b - c)$, ... $=$ 118.
13. $a^2(b - cd) + c^2(x - c) - d^3(x - a)$, . . . $=$ 42.
14. $(a + b)(a - d) + (x - c)^2 - a^{\frac{1}{2}}(b - c)$, . . $=$ 18.
15. $\dfrac{2a + 3c}{b - c} + \sqrt{abd} - \dfrac{5a - b - c}{x - d}$, ... $=$ $8\frac{2}{5}$.
16. $\{3b + c(x - d) + (a + c)(a - c)\}(b - a)^{\frac{1}{2}}$, . . $=$ 80.

17. $\{5(a+x)-3c(a-d)(c+d)\}\{a-(c-d)\}$, $= -135$.

18. $a^{\frac{1}{2}}b^{\frac{1}{2}}(c^2+d^2+a^2-x^2)+\sqrt{abd}$, . . $= 24$.

19. $\sqrt{5bdx}+\sqrt[3]{25abdx}-\dfrac{3a+b^{\frac{1}{2}}}{x+a^{\frac{1}{2}}}$, ... $= 38$.

20. $\{ab+cd-ax\}\{3a+cx+bx-c\}^{\frac{1}{2}}$, . . $= 72$.

21. $\dfrac{3a+b-c}{2a+2d}+\dfrac{3b+4c-2x}{b+c+d}-\dfrac{7a+4b-12c}{2c+3d}$, $= 1\frac{5}{12}$.

ADDITION.

CASE I.

Add together

1. $2a$, $3a$, $5a$, $4a$, and $7a$, ... $= 21a$.

2. $3ab$, $5ab$, ab, $6ab$, and $11ab$, ... $= 26ab$.

3. $4abc$, $3abc$, $5abc$, $7abc$, and $12abc$, . . . $= 31abc$.

4. $-2ac$, $-4ac$, $-7ac$, $-3ac$, and $-8ac$, . $= -24ac$.

5. $-3c(a-x)$, $-4c(a-x)$, $-7c(a-x)$, $-8c(a-x)$, and $-10c(a-x)$, ... $= -32c(a-x)$.

6. $16ac\sqrt{x}$, $15ac\sqrt{x}$, $12ac\sqrt{x}$, $10ac\sqrt{x}$, and $20ac\sqrt{x}$, $= 73ac\sqrt{x}$.

7. $2\sqrt{a-x}$, $5\sqrt{a-x}$, $7\sqrt{a-x}$, $10\sqrt{a-x}$, and $8\sqrt{a-x}$, $= 32\sqrt{a-x}$.

8. $3ac(a^2-x^2)^{\frac{1}{2}}$, $5ac(a^2-x^2)^{\frac{1}{2}}$, $4ac(a^2-x^2)^{\frac{1}{2}}$, $10ac(a^2-x^2)^{\frac{1}{2}}$, and $14ac(a^2-x^2)^{\frac{1}{2}}$, ... $= 36ac(a^2-x^2)^{\frac{1}{2}}$.

9. $4x+3y$, $7x+4y$, $5x+2y$, $3x+7y$, and $9x+10y$, $= 28x+26y$.

10. $10ab-2cd$, $5ab-4cd$, $8ab-7cd$, $3ab-10cd$, and $8ab-4cd$, ... $= 34ab-27cd$.

11. $5ac-d$, $3ac-7d$, $8ac-4d$, $3ac-6d$, and $8ac-12d$, $= 27ac-30d$.

12. $3a^2-ab+5b^2$, $4a^2-10ab+3b^2$, $7a^2-2ab+b^2$, and $3a^2-5ab+7b^2$, ... $= 17a^2-18ab+16b^2$.

13. $4a^2+3ax+7x^2$, $3a^2+7ax+4x^2$, $13a^2+22ax+x^2$, and $7a^2+3x^2$, ... $= 27a^2+32ax+15x^2$.

14. $3x^2 + 5xy + 7y^2$, $6x^2 + 5xy + 3y^2$, $2x^2 + 8xy + 9y^2$, and $x^2 + 6xy$, ... $= 12x^2 + 24xy + 19y^2$.

15. $4x^2 - 3xy + 2y^2$, $5x^2 - 7xy + 6y^2$, $7x^2 - 5xy + 9y^2$, and $x^2 + 7y^2$, ... $= 17x^2 - 15xy + 24y^2$.

CASE II.

Add together

1. $3a, 7a, -4a, -3a, +5a$, ... $= 8a$.

2. $4ac, 7ac, 5ac, -6ac, -2ac$, . . . $= 8ac$.

3. $7abx, 3abx, -5abx, -6abx, -2abx$, . . $= -3abx$.

4. $3x(a-c)$, $5x(a-c)$, $-3x(a-c)$, $-10x(a-c)$, $4x(a-c)$, $= -x(a-c)$.

5. $7a(x-y)^{\frac{1}{2}}$, $-5a(x-y)^{\frac{1}{2}}$, $8a(x-y)^{\frac{1}{2}}$, $6a(x-y)^{\frac{1}{2}}$, $-3a(x-y)^{\frac{1}{2}}$, ... $= 13a(x-y)^{\frac{1}{2}}$.

6. $3ac(a^2-x^2)^{\frac{1}{2}}$, $-5ac(a^2-x^2)^{\frac{1}{2}}$, $7ac(a^2-x^2)^{\frac{1}{2}}$, and $-2ac(a^2-x^2)^{\frac{1}{2}}$, ... $= 3ac(a^2-x^2)^{\frac{1}{2}}$.

7. $10ab - 4cd$, $3ab + 2cd$, $-6ab + 7cd$, and $12ab - 9cd$, $= 19ab - 4cd$.

8. $6ac - d$, $4ac + 3d$, $5ac - 7d$, $4ac + 2d$, and $-5ac + d$, $= 14ac - 2d$.

9. $a^2 + ab + b^2$, $3a^2 - 5ab + 4b^2$, $3a^2 + 12ab - 9b^2$, and $8a^2 - 7ab + 2b^2$, ... $= 15a^2 + ab - 2b^2$.

10. $3x^2 + 2x + 4$, $7x^2 + 4x - 9$, $8x^2 - 12x - 4$, and $3x^2 - 7x - 2$, ... $= 21x^2 - 13x - 11$.

11. $7a^2 - 4abx + 2b^2x^2$, $6a^2 + 3abx - 4b^2x^2$, and $-8a^2 - 7abx + 3b^2x^2$, ... $= 5a^2 - 8abx + b^2x^2$.

12. $a^3 - \frac{3}{4}a^2x + \frac{1}{3}ax^2 - \frac{5}{8}x^3$, and $a^3 + \frac{1}{2}a^2x + \frac{1}{4}ax^2 + \frac{2}{3}x^3$, $= 2a^3 - \frac{1}{4}a^2x + \frac{7}{12}ax^2 + \frac{1}{24}x^3$.

13. $x^2 - 11x + 7$, $5x^2 + x - 9$, $4x^2 + 3x + 5$, and $-4x^2 + 3x + 9$, ... $= 6x^2 - 4x + 12$.

14. $x^3 + (a-b)x^2 + 5x$, $14x^3 - 3(a-b)x^2 - 2x$, and $-5x^3 + 7(a-b)x^2 - x$, ... $= 10x^3 + 5(a-b)x^2 + 2x$.

15. $x^3 + 2x^2y - 4xy^2 + 5y^3$, $7x^3 - 12x^2y + 15xy^2 - 13y^3$, $-4x^3 + 5x^2y - 7xy^2 + 9y^3$, and $-2x^3 + 11x^2y - 12xy^2 + 3y^3$, $= 2x^3 + 6x^2y - 8xy^2 + 4y^3$.

16. $a^2(x^2 - 2ax)^{\frac{1}{2}} + 7x^3$, $3a^2(x^2 - 2ax)^{\frac{1}{2}} - 11x^3$, and $4a^2(x^2 - 2ax)^{\frac{1}{2}} - x^3$, . . . $= 8a^2(x^2 - 2ax)^{\frac{1}{2}} - 5x^3$.

CASE III.

Add together

1. $3ab, -7bc, 6ab, -4c, -2ab, +3bc, 7c$, and $2ab$,
$= 9ab - 4bc + 3c$.

2. $4ac, 5ad, 7bc, -2ad, 4bc, -2ac$, and $3ad$,
$= 2ac + 6ad + 11bc$.

3. $3a^2b - 2bd, 4bd - 3bc, -2a^2b$, and $3bd - bc$,
$= a^2b + 5bd - 4bc$.

4. $3a + x, 4a + 3b + 2c$, and $2a - 4b - c$,
$= 9a - b + c + x$.

5. $4x^2y + 5x, -3x^2y - 3x + y$, and $d - 2e$,
$= x^2y + 2x + y + d - 2e$.

6. $4a + 3b + 2c, 4a - 13b + 4y$, and $-3a + b + 3x$,
$= 5a - 9b + 2c + 4y + 3x$.

7. $3a^3 + 5ac - 2b^3, -2a^3 - 4ab - b^3$, and $5a^3 - 4ac + 7ab$,
$= 6a^3 + ac + 3ab - 3b^3$.

8. $a^2 + ax + x^2, 3a^2 - 4ax + 2x^2$, and $a^2 + x^2 + a + x$,
$= 5a^2 - 3ax + 4x^2 + a + x$.

9. $3a^2 + 4\sqrt{x}, 5a - 2\sqrt{x}, 5a^2 + 12$, and $b + 3\sqrt{x} + 7$,
$= 8a^2 + 5a + 5\sqrt{x} + 19 + b$.

10. $4a + b - 3(a - x), 3a + 7b - 3c, 3b + 7(a - x) + 9c - 8$, $4(a - x) - 2a + 7$, and $a + 2(a - x) + b + c - 3$,
$= 6a + 12b + 10(a - x) + 7c - 4$.

11. $ax^2 + bx - 3c, 4ax^2 - cy + 2d, 3bx + 2cy + 5c, 3cy + 7c - 5d$, and $4ax^2 + 3bx + cy + d - 4c$,
$= 9ax^2 + 7bx + 5c + 5cy - 2d$.

12. $3xy + 4(c - d) + 5(p + q), 3x + 2(c - d) - 4(p + q)$, $6xy - 7x + 7(c - d) + 6(p + q), 3y - 2x - 5(p + q)$, and $7xy + y + (p + q)$,
$= 16xy + 13(c - d) + 3(p + q) - 6x + 4y$.

13. $x + y + z, x + y - z, x - y + z$, and $-x + y + z$,
$= 2x + 2y + 2z$.

14. $x^2 + y^2 + z^2, x^2 + 2xy + y^2 + z^2$, and $-x^2 + y^2 - 2yz + z^2$,
$= x^2 + 2xy + 3y^2 - 2yz + 3z^2$.

15. $2ab + 3bc - 2a + b, 3ac + 2b - a$, and $4d + 6a - ab + bc$,
$= 3ac + ab + 4bc + 3a + 3b + 4d$.

SUBTRACTION.

1. From $2a + 3b - 5c$, take $a - 2b - 3c$, $= a + 5b - 2c$.

2. " $5a - 3b + 4c - d$, take $3a - 7b + 2c + 7d$, $= 2a + 4b + 2c - 8d$.

3. " $3a^2 - 2ab + b^2 - 3c^2$, take $a^2 - 5ab + 3b^2 - 2c^2$, $= 2a^2 + 3ab - 2b^2 - c^2$.

4. " $5m - 3n + p + 3$, take $m + 4n - 3p + 12$, $= 4m - 7n + 4p - 9$.

5. " $4m^2 - 6mn + n^2 + 7$, take $2m^2 + 4mn + 6n^2 + 4$, $= 2m^2 - 10mn - 5n^2 + 3$.

6. " $7x - 3a\sqrt{cx} - 5c + d$, take $3x - 7a\sqrt{cx} - 4c + 5d$, $= 4x + 4a\sqrt{cx} - c - 4d$.

7. " $7x^3 + 2x^2 - 5x + 4$, take $5x^3 + 6x^2 - 2x - 6$, $= 2x^3 - 4x^2 - 3x + 10$.

8. " $5x^2 - 10xy + 5y^2 - 7$, take $4x^2 - 8xy + 4y^2 - 9$, $= x^2 - 2xy + y^2 + 2$.

9. " $3x^2y - 3xy^2 + y^3 + 3d$, take $x^2y + 2xy^2 + 3y^3 + 4d$, $= 2x^2y - 5xy^2 - 2y^3 - d$.

10. " $12x - 5d + 4b + 3a$, take $2x - 3a + 8b + 7c$, $= 10x + 6a - 4b - 7c - 5d$.

11. " $3x^3 + 4x^2 - 5y^3 + 4y^2$, take $x^3 - 3x^2 - 8y^3 + 5y^2$, $= 2x^3 + 7x^2 + 3y^3 - y^2$.

12. " $7a^3 - 7a^2y + 4ay^3 - 4c^2$, take $a^3 - 2a^2y + 5ay^3$, $= 6a^3 - 5a^2y - ay^3 - 4c^2$.

13. " $ax^2 + 2bx + ac^2$, take $bx^2 - cx + 2bc^2$, $= (a - b)x^2 + (2b + c)x + (a - 2b)c^2$.

14. " $3x^2 + 4ax + 7a^2$, take $x^2 - 3bx + 5b^2$, $= 2x^2 + (4a + 3b)x + 7a^2 - 5b^2$.

15. " $3(x + y)^4 + 2a(x + y)^3 - 3a^2(x + y)^2 + 6a^3(x + y) + 3a^4$, take $2(x + y)^4 - 3a(x + y)^3 + 2a^2(x + y)^2 + a^3(x + y) - 2a^4$,
$= (x + y)^4 + 5a(x + y)^3 - 5a^2(x + y)^2 + 5a^3(x + y) + 5a^4$.

16. From $5a(x^2 + y^2) - b(x^2 - y^2) + 4c^2$, take $3a(x^2 + y^2) - 3b(x^2 - y^2)$, . . . $= 2a(x^2 + y^2) + 2b(x^2 - y^2) + 4c^2$.

17. From $5ac\sqrt{x^2 - y^2} + 3bc\sqrt{x^2 + y^2}$, take $3ac\sqrt{x^2 - y^2} - 4bc\sqrt{x^2 + y^2}$, ... $= 2ac\sqrt{x^2 - y^2} + 7\,bc\sqrt{x^2 + y^2}$.

18. From $3a(a^2 - 2ax)^{\frac{1}{2}} + 3c(x^2 + 2ax)^{\frac{1}{2}}$, take $a(a^2 - 2ax)^{\frac{1}{2}} - 5c(x^2 + 2ax)^{\frac{1}{2}}$, . . . $= 2a(a^2 - 2ax)^{\frac{1}{2}} + 8c(x^2 + 2ax)^{\frac{1}{2}}$.

Express the following without brackets:—

1. $2a + 3b - 4c + (3a - 2b + 3c) - (2a + 3b)$,
$= 3a - 2b - c$.

2. $5ab + (3ac - 4bc) - (2ab - 2ac + 3bc - e)$,
$= 3ab + 5ac - 7bc + e$.

3. $7x^3 + 3x^2a - (5xa^2 + 4a^3) - (4x^3 - 4x^2a + 3a^3)$,
$= 3x^3 + 7x^2a - 5xa^2 - 7a^3$.

4. $3ac - (ab + d) + \{4ac + d - (2ab - 4d)\}$,
$= 7ac - 3ab + 4d$.

5. $x^4 - (x^3 + 3x^2a + 3xa^2 + a^3) - (x^3 - 4a^3)$,
$= x^4 - 2x^3 - 3x^2a - 3xa^2 + 3a^3$.

6. $x^3 - (ay + y^3) - \{ax^2 - 3ay - (2y^3 - a^2)\}$,
$= x^3 + 2ay + y^2 - ax^2 - a^2$.

7. $x^4 + 4x^3 + 6x^2 + 4x + 1 - \{2x^3 - 2x^2 - (4x - 4)\}$,
$= x^4 + 2x^3 + 8x^2 + 8x - 3$.

8. $3a - (4b - 3c + 2d) - \{a - (2b - 4c + 3d)\} - 5a$,
$= -3a - 2b - c + d$.

9. $(5a^2 - 3ax + x^2) - \{4a^2 + 5ax - (3a^2 - 7ax + 5x^2)\} - 7x^2$,
$= 4a^2 - 15ax - x^2$.

10. $7a^3 - (5a^2x + 3ax^2 - 7x^3) - \{8a^3 - 4a^2x - (7ax^2 - 3x^3)\}$,
$= -a^3 - a^2x + 4ax^2 + 4x^3$.

11. $10x^4 - (4x^2y - 7xy^2 + 9y^4) - \{6x^3 + 5x^2y - (8xy^2 - 4y^4)\}$,
$= 4x^3 - 9x^2y + 15xy^2 - 13y^4$.

12. $3a^2 - 9ax + 4x^2 - (a^2 + 2ax - x^2) - (-7ax + 3x^2)$,
$= 2a^2 - 4ax + 2x^2$.

MULTIPLICATION.

CASES I. II. AND III.

1. Multiply $3ab$ by $2cx$, $5ac$ by $3dy$, and $4ad$ by $2cx$,
$= 6abcx$, $15acdy$, and $8acdx$.

2. " $4abx$ by $3ad$, $7bc$ by $3ad$, and $7ac$ by bx,
$= 12a^2bdx$, $21abcd$, and $7abcx$.

3. " $7ab$ by $-6a$, $4ac$ by $-2ac$, and $3bc$ by $-2cx$,
$= -42a^2b$, $-8a^2c^2$, and $-6bc^2x$.

4. " $-6ab$ by $2ac$, $-7ax$ by $5ac$, and $-3ax$ by $4ad$,
$= -12a^2bc$, $-35a^2cx$, and $-12a^2dx$.

5. " $-7a^2b$ by $3ab^2$, $-3a^2c$ by c^2x, and $3a^2x^2$ by $-2a^3$,
$= -21a^3b^3$, $-3a^2c^3x$, and $-6a^5x^2$.

6. " $5ax^2$ by $-3ac^2$, $-2pn$ by $-3cn$, and $4ax$ by $7cx^2$,
$= -15a^2c^2x^2$, $+6cn^2p$, and $28acx^3$.

7. " $4mn$ by $-7am^2$, $3acx$ by $4ab$, and $8c^2x^3$ by $4a^2c$,
$= -28am^3n$, $12a^2bcx$, and $32a^2c^3x^3$.

8. " $5abcx$ by $-7a^2cx^3$, and $-18x^2y$ by $-12ay$,
$= -35a^3bc^2x^4$, and $216ax^2y^2$.

9. " $-14a^2xy$ by $-7axy^2$, and $-24abd$ by $9abx^2$,
$= 98a^3x^2y^3$, and $-216a^2b^2dx^2$.

10. " $15a^2b^3y$ by $-6ax^3y$, and $-4ac^2x$ by $-7a^2bx^2$,
$= -90a^3b^3x^3y^2$, and $28a^3bc^2x^3$.

CASE IV.

1. Multiply $a + b$ by a, and $a + b$ by b,
$= a^2 + ab$, and $ab + b^2$.

2. " $2a + x$ by $3a^2$, and $2a + 4x$ by $3ax$,
$= 6a^3 + 3a^2x$, and $6a^2x + 12ax^2$.

3. " $4ac - 3ax$ by $7ax$, and $5ab + 4c$ by $-3ac$,
$= 28a^2cx - 21a^2x^2$, and $-15a^2bc - 12ac^2$.

4. " $3ab - 2cx$ by $4bc$, and $7a^2c - 4ac^2$ by $7acx$,
$= 12ab^2c - 8bc^2x$, and $49a^3c^2x - 28a^2c^3x$.

5. Multiply $5a^2b - 7acx$ by $2abc$, and $13a^2y + 4by^2$ by aby,
$= 10a^3b^2c - 14a^2bc^2x$, and $13a^3by^2 + 4ab^2y^3$.

6. " $13axy - 4x$ by $3a^2xy$, and $-11ay + 4ax$ by $7axy$,
$= 39a^3x^2y^2 - 12a^2x^2y$, and $-77a^2xy^2 + 28a^2x^2y$.

7. " $15a^2x - 5b^2y$ by $9aby$, and $2acx - 3ay$ by $-4ax$,
$= 135a^3bxy - 45ab^3y^2$, and $-8a^2cx^2 + 12a^2xy$.

8. " $3ac + 4ab + 7bc + 6abc$ by $3abc$,
$= 9a^2bc^2 + 12a^2b^2c + 21ab^2c^2 + 18a^2b^2c^2$.

9. " $14a^2b - 3a^2c - 2b^2c + 4b^2c^2$ by $-8abc^2$,
$= -112a^3b^2c^2 + 24a^3bc^3 + 16ab^3c^3 - 32ab^3c^4$.

10. " $13a + 14b - 15c - 16d + e$ by $5\ abc$,
$= 65a^2bc + 70ab^2c - 75abc^2 - 80abcd + 5abce$.

11. " $15ax - 14by + 13bx - 11ay$ by $-4abxy$,
$= -60a^2bx^2y + 56ab^2xy^2 - 52ab^2x^2y + 44a^2bxy^2$.

12. " $4a^2bc - 3ab^2d + 7bc^2d - 8bcd^2$ by $3abcd$,
$= 12a^3b^2c^2d - 9a^2b^3cd^2 + 21ab^2c^3d^2 - 24ab^2c^2d^3$.

13. " $11a^2x^3 - 4b^3x + 5a^2y^3 - 3b^3y$ by $-4a^2bx^2y$,
$= -44a^4bx^5y + 16a^2b^4x^3y - 20a^4bx^2y^4 + 12a^2b^4x^2y^2$.

14. " $-6a^2x^2y + 4ax^2y^2 - 7a^2xy^2$ by $-9a^2xy$,
$= 54a^4x^3y^2 - 36a^3x^3y^3 + 63a^4x^2y^3$.

15. " $5ab - 7bc + 8cd - 12ad$ by $12abcd$,
$= 60a^2b^2cd - 84ab^2c^2d + 96abc^2d^2 - 144a^2bcd^2$.

16. " $4x^2y^2 - 3x^2z^2 + 6y^2z^2 - 5xyz$ by $7xyz$,
$= 28x^3y^3z - 21x^3yz^3 + 42xy^3z^3 - 35x^2y^2z^2$.

CASE V.

1. Multiply $2a + 2b$ by $a + 2c$, $= 2a^2 + 2ab + 4ac + 4bc$.

2. " $3ax + 6ay + 7az$ by $4a + 3b$,
$= 12a^2x + 24a^2y + 28a^2z + 9abx + 18aby + 21abz$.

3. " $4a^2x^2 + 8axy + 4y^2$ by $2ax + 2y$,
$= 8a^3x^3 + 24a^2x^2y + 24axy^2 + 8y^3$.

4. " $2a^2x^2 - 4axz + 2z^2$ by $3ax - 3z$,
$= 6a^3x^3 - 18a^2x^2z + 18axz^2 - 6z^3$.

5. " $3a^2b^2 - 6abc + 3c^2$ by $5ab - 5c$,
$= 15a^3b^3 - 45a^2b^2c + 45abc^2 - 15c^3$.

6. Multiply $2a^2b^2 - 4abcd + 2cd$ by $3ab - 3cd$,
$= 6a^3b^3 - 18a^2b^2cd + 12abc^2d^2 + 6abcd - 6c^2d^2$.

7. " $5a^2x^2 + 10axy^2 + 5y^4$ by $7ax + 7y^2$,
$= 35a^3x^3 + 105a^2x^2y^2 + 105axy^4 + 35y^6$.

8. " $7a^2x^2 - 14abxy + 7b^2y^2$ by $ax - by$,
$= 7a^3x^3 - 21a^2x^2by + 21axb^2y^2 - 7b^3y^3$.

9. " $x^3 + 2x^2 + 2x + 1$ by $x^2 + 2x + 1$,
$= x^5 + 4x^4 + 7x^3 + 7x^2 + 4x + 1$.

10. " $3x^3 + 7x^2 - 5x + 1$ by $2x^2 - 3x + 2$,
$= 6x^5 + 5x^4 - 25x^3 + 31x^2 - 13x + 2$.

11. " $3x^2 - 2xy + 4y^2$ by $x^2 - 7xy + 3y^2$,
$= 3x^4 - 23x^3y + 27x^2y^2 - 34xy^3 + 12y^4$.

12. " $8x^3 - 4ax^2 + 7a^2x + 9a^3$ by $2x^2 + 3ax - 4a^2$,
$= 16x^5 + 16ax^4 - 30a^2x^3 + 55a^3x^2 - a^4x - 36a^5$.

13. " $x^3 + x^2y + xy^2 + y^3$ by $x^2 - 2xy + y^2$,
$= x^5 - x^4y - xy^4 + y^5$.

14. " $x^3 + x^2y + xy^2 + y^3$ by $x^2 + 2xy + y^2$,
$= x^5 + 3x^4y + 4x^3y^2 + 4x^2y^3 + 3xy^4 + y^5$.

15. " $x^3 + x^2y + xy^2 + y^3$ by $x^2 - xy + y^2$,
$= x^5 + x^3y^2 + x^2y^3 + y^5$.

16. " $x^3 + x^2y + xy^2 + y^3$ by $x^2 + xy + y^2$,
$= x^5 + 2x^4y + 3x^3y^2 + 3x^2y^3 + 2xy^4 + y^5$.

17. " $x^3 + ax^2 - bx - c$ by $x^2 - ax - b$,
$= x^5 - (a^2 + 2b)x^3 - cx^2 + (ac + b^2)x + bc$.

18. " $x^3 - ax^2 + bx - c$ by $x^2 - ax + b$,
$= x^5 - 2ax^4 + (a^2 + 2b)x^3 - (2ab + c)x^2 + (ac + b^2)x - bc$.

19. Multiply $x^3 - ax^2 + bx - c$ by $x^2 + ax + b$,
$= x^5 + (2b - a^2)x^3 - cx^2 + (b^2 - ac)x - bc$.

20. " $1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3$ by $1 + \frac{1}{2}x + \frac{1}{4}x^2$,
$= 1 + x + \frac{3}{4}x^2 + \frac{3}{8}x^3 + \frac{1}{8}x^4 + \frac{1}{32}x^5$.

21 " $1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3$ by $1 - \frac{1}{3}x + \frac{1}{9}x^2$,
$= 1 + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \frac{1}{243}x^5$.

22. " $1 + 2x + 4x^2 + 8x^3$ by $1 + 4x + 5x^2$,
$= 1 + 6x + 17x^2 + 34x^3 + 52x^4 + 40x^5$.

23. Multiply $9 + 7x + 5x^2 + 3x^3$ by $4 + 5x - 6x^2$,
$= 36 + 73x + x^2 - 5x^3 - 15x^4 - 18x^5$.

24. " $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$,
$= a^3 + b^3 + c^3 - 3abc$.

25. " $x + y + x^2 - y^2$ by $x - y + x^2 + y^2$,
$= x^2 + 2x^3 + x^4 - y^2 + 2y^3 - y^4$.

26. " $a + 2b + 3c$ by $a + b + c$,
$= a^2 + 2b^2 + 3c^2 + 3ab + 4ac + 5bc$.

Find the product of

27. $(a + x) \times (a + 2x) \times (a + 3x) \times (a + 4x)$,
$= a^4 + 10a^3x + 35a^2x^2 + 50ax^3 + 24x^4$.

28. $(2a + x) \times (3a + 2x) \times (4a + 3x) \times (5a + 4x)$,
$= 120a^4 + 326a^3x + 329a^2x^2 + 146ax^3 + 24x^4$.

29. $(a^2 + ax + x^2) \times (a^2 - ax + x^2) \times (a + x)$,
$= a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5$.

30. $(a^2 + ax + x^2) \times (a^2 - ax + x^2) \times (a - x)$,
$= a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5$.

31. $(a^m + x^m)(a^m - x^m)(a^n - x^n)(a^n + x^n)$,
$= a^{2(m+n)} - a^{2m}x^{2n} - a^{2n}x^{2m} + x^{2(m+n)}$.

32. $(a^2 + 1 + a^{-2})(a^2 - 1 + a^{-2})(a + a^{-1})$,
$= a^5 + a^3 + a + a^{-1} + a^{-3} + a^{-5}$.

33. $(x^4 + x^3 + x^2 + x + 1)(x - 1)(x + 1)(x + 2)$,
$= x^7 + 3x^6 + 2x^5 - x^2 - 3x - 2$.

34. $(1 + x^{-1} + x^{-2})(1 - x^{-1} + x^{-2})(1 - 2x^{-1} + x^{-2})$,
$= 1 - 2x^{-1} + 2x^{-2} - 2x^{-3} + 2x^{-4} - 2x^{-5} + x^{-6}$.

35. $(2x + 4y)(3x + 5y)(4x + 6y)(2x - 4y)(3x - 5y)$,
$= 144x^5 + 216x^4y - 976x^3y^2 - 1464x^2y^3 + 1600xy^4 + 2400y^5$.

36. $(a + b + c - d)(a + b - c + d)(a - b + c + d)(-a + b + c + d)$,

$$= \{(a + b) + (c - d)\}\{(a + b) - (c - d)\}\{(c + d) + (a - b)\}\{(c + d) - (a - b)\}$$

$$= \{(a + b)^2 - (c - d)^2\}\{(c + d)^2 - (a - b)^2\}$$

$$= (a + b)^2(c + d)^2 - (a^2 - b^2)^2 - (c^2 - d^2)^2 + (a - b)^2(c - d)^2$$

$$= - a^4 - b^4 - c^4 - d^4 + 2a^2b^2 + 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 2c^2d^2 + 8abcd.$$

DIVISION.

CASES I. II. AND III.

1. Divide $3abc$ by $2xy$, $6ac$ by $2bx$, and $4axy$ by $2bc$,

$$= \frac{3abc}{2xy}, \frac{3ac}{bx}, \text{ and } \frac{2axy}{bc}.$$

2. " $12acx$ by $7bd$, $8axy$ by $2bc$, and $3am$ by $4bn$,

$$= \frac{12acx}{7bd}, \frac{4axy}{bc}, \text{ and } \frac{3am}{4bn}.$$

3. " $4a^2c^3$ by $2ac^2$, $5b^4x^2$ by $3b^3x^3$, and $4a^5d^7$ by $3a^7d^2$,

$$= 2ac, \frac{5b}{3x}, \text{ and } \frac{4d^5}{3a^2}.$$

4. " $5a^4b^2c^3$ by $3ab^3c^2$, and $4x^3y^4z^5$ by $7xy^2z^3$,

$$= \frac{5a^3c}{3b}, \text{ and } \frac{4x^2y^2z^2}{3}.$$

5. " $7a^2bx^3$ by $5ac^2x$, and $6ax^2y^4$ by $3abc$,

$$= \frac{7abx^2}{5c^2}, \text{ and } \frac{2x^2y^4}{bc}.$$

6. " $3a^3b^2c$ by $6ax^2y$, and $8ax^3y$ by $3a^3xz$,

$$= \frac{a^2b^2c}{2x^2y}, \text{ and } \frac{8x^2y}{3a^2z}.$$

7. " $3abc$ by $12acx$, and $4ac$ by $-3bxy$,

$$= \frac{b}{4x}, \text{ and } -\frac{4ac}{3bxy}.$$

8. " $18ac^2x$ by $-6acx^2$, and $-12a^2cx^3$ by $-2ac^3$,

$$= -\frac{3c}{x}, \text{ and } \frac{6ax^3}{c^2}.$$

9. " $14a^4b^3x^5$ by $-3aby^2$, and $-15ac^5x^3$ by $5a^3cxy^2$,

$$= -\frac{14a^3b^2x^5}{3y^2}, \text{ and } -3\frac{c^4x}{a^2y^2}.$$

10. " $15ax$ by $12cy$, and $64a^3xz$ by $16a^2xy$,

$$= \frac{5ax}{4cy}, \text{ and } \frac{4az}{y}.$$

11. Divide $\frac{1}{2}a^2bx^3y$ by $-\frac{3}{5}abcx$, and $3cx^3$ by $-7ax^{-3}$,

$$= -\frac{5ax^2y}{6c}, \text{ and } -\frac{3cx^{n}}{7a}.$$

12. 〃 $4a^mb^{-3}$ by $-\frac{2}{3}ab$, and $3a^mb^n$ by $4a^pbc^n$,

$$= -6a^{m-1}b^{-4} \text{ or } \frac{6a^{m-1}}{b^4}, \text{ and } \frac{3a^{m-p}b^{n-1}}{4c^n}.$$

CASE IV.

1. Divide $a^2 + ac$ by a, and $4xy^2 - 3x^2y$ by $4xy$,
$= a + c$, and $y - \frac{3}{4}x$.

2. 〃 $a^4 - 3a^3x^2$ by $3a^2$, and $5a^2x^2 - 10ax$ by $5ax$,
$= \frac{1}{3}a^2 - ax^2$, and $ax - 2$.

3. 〃 $3ax - 54ay$ by $4ab$, and $a^2b - ab^2$ by $-2ab$,
$= \frac{3x}{4b} - \frac{27y}{2b}$, and $-\frac{1}{2}a + \frac{1}{2}b$.

4. 〃 $16a^2b^3x - 8a^3cx^2 - 24a^2dx$ by $4a^2x$,
$= 4b^3 - 2acx - 6d$.

5. 〃 $5x^3y^3 - 20a^2x^2y^2 + 3a^4xy$ by $-5xy$,
$= -x^2y^2 + 4a^2xy - \frac{3}{5}a^4$.

6. 〃 $12a^3x^2y - 18a^2x^3y - 16axy^3 + 24a^2x^2y^2$ by $2axy$,
$= 6a^2x - 9ax^2 - 8y^2 + 12axy$.

7. 〃 $3a^3 - 24a^2x + 21ax^2 - 3ax^3 + 3x^4$ by $3ax$,
$= a^2x^{-1} - 8a + 7x - x^2 + a^{-1}x^3$.

8. 〃 $6x^5 - 6x^4 + 5x^3 - 5x^2 + 4x - 4$ by $2x^3$,
$= 3x^2 - 3x + \frac{2}{5} - \frac{2x^{-1}}{5} + 2x^{-2} - 2x^{-3}$.

9. 〃 $5a^nx^m - 10a^{n+2}x^{m+2} - 15a^mx^n + 20a^mx^{n+1}$ by $-5a^2x$,
$= -a^{n-2}x^{m-1} + 2a^nx^{m+1} + 3a^{m-2}x^{n-1} - 4a^{m-2}x^n$.

10. 〃 $8a^nb^mc^2 - 4a^{n+2}b^{m+1}c^3 + 10a^3b^4c^4$ by $2a^nb^{m-1}c^2$,
$= 4b - 2a^2b^2c + 5a^{3-n}b^{5-m}c^2$.

CASE V.

1. Divide $a^2 + 4ab + 3b^2$ by $a + b$, . . $= a + 3b$.
2. 〃 $3a^2 + 14ax + 15x^2$ by $a + 3x$, . $= 3a + 5x$.
3. 〃 $6a^4 + a^2x - 15x^2$ by $2a^2 - 3x$, . . $= 3a^2 + 5x$.
4. 〃 $6x^2 + 2xy^2 - 20y^4$ by $2x + 4y^2$, . $= 3x - 5y^2$.

5. Divide $2x^4 - 9x^2z^2 - 5z^4$ by $2x^2 + z^2$, . . $= x^2 - 5z^2$.

6. " $3x^2 + 11\frac{1}{2}xc - 2c^2$ by $x + 4c$, . $= 3x - \frac{1}{2}c$.

7. " $x^3 + 6x^2y + 12xy^2 + 8y^3$ by $x + 2y$,
$= x^2 + 4xy + 4y^2$.

8. " $a^3x^3 - 6a^2x^2y + 12axy^2 - 8y^3$ by $ax - 2y$,
$= a^2x^2 - 4axy + 4y^2$.

9. " $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$ by $a + x$,
$= a^3 + 3a^2x + 3ax^2 + x^3$.

10. " $a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$ by $a - 2b$,
$= a^3 - 6a^2b + 12ab^2 - 8b^3$.

11. " $a^3c^3 + 9a^2c^2x + 26acx^2 + 24x^3$ by $ac + 4x$,
$= a^2c^2 + 5acx + 6x^2$.

12. " $a^3c^3 - a^2c^2x - 14acx^2 + 24x^3$ by $ac - 3x$,
$= a^2c^2 + 2acx - 8x^2$.

13. " $27a^3b^3 - 54a^2b^2c + 36abc^2 - 8c^3$ by $3ab - 2c$,
$= 9a^2b^2 - 12abc + 4c^2$.

14. " $8a^3b^3 - 36a^2b^2c + 54abc^2 - 27c^3$ by $2ab - 3c$,
$= 4a^2b^2 - 12abc + 9c^2$.

15. " $a^5x^5 + y^5$ by $ax + y$,
$= a^4x^4 - a^3x^3y + a^2x^2y^2 - axy^3 + y^4$.

16. " $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$,
$= a^2 + b^2 + c^2 - ab - ac - bc$.

17. " $a^2 + 2b^2 + 3c^2 + 3ab + 4ac + 5bc$ by $a + b + c$,
$= a + 2b + 3c$.

18. " $a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5$ by $a^2 - ax + x^2$,
$= a^3 + 2a^2x + 2ax^2 + x^3$.

19. " $a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5$ by $a^2 - ax + x^2$,
$= a^3 - x^3$.

20. " $a^5 + a^3 + a + a^{-1} + a^{-3} + a^{-5}$ by $a^2 - 1 + a^{-2}$,
$= a^3 + 2a + 2a^{-1} + a^{-3}$.

21. " $1 + 14x + 71x^2 + 154x^3 + 120x^4$ by $1 + 5x + 6x^2$,
$= 1 + 9x + 20x^2$.

22. " $1 + 18x + 117x^2 + 330x^3 + 303x^4$ by $1 + 7x + 10x^2$,
$= 1 + 11x + 30x^2 + \dfrac{10x^3 + 3x^4}{1 + 7x + 10x^2}$.

23. Divide $1 + x^4 + 2x^5$ by $1 + 2x + 4x^2$,

$$= 1 - 2x + 8x^3 - \frac{15x^4 + 30x^5}{1 + 2x + 4x^2}.$$

24. 〃 1 by $1 - 2x + x^2$ to five terms,

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \frac{6x^5 - 5x^6}{1 - 2x + x^2}.$$

25. 〃 1 by $1 + 2x + x^2$ to five terms,

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \frac{6x^5 + 5x^6}{1 + 2x + x^2}.$$

26. 〃 $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$ by $\frac{1}{2}x^2 - x + 3$,

$$= \frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9.$$

NOTE.—The above exercises, except 16 and 17, may be wrought by Synthetic Division.

GREATEST COMMON MEASURE, OR G.C.M.

CASE I.

1. Find the G.C.M. of $12a^2b^3cx^2$, and $24a^4b^2c^3x^5$, $= 12a^2b^2cx^2$.

2. 〃 〃 $63a^4b^2c^3x^4y^2$, and $42a^2b^3c^2xy^2$, $= 21a^2b^2c^2xy^2$.

3. 〃 〃 $144a^3b^4c^5d^2e^3$, and $108a^2bc^3e^2$, $= 36a^2bc^3e^2$.

4. 〃 〃 $12a^2bcx^2$, $20b^2c^2x^3$, and $32a^5b^2c^3x$, $= 4bcx$.

5. 〃 〃 $3ax^2y^2$, $12a^2x^3yz^2$, and $24a^3xy^4z^2$, $= 3axy$.

6. 〃 〃 $5ab^2c^3d^4$, $12a^2b^3c^2d$, and $30ab^4c^4d^3$, $= ab^2c^2d$.

7. 〃 〃 $7abcx$, $9a^2b^3c^2x^2$, and $21ab^2c^3x$, $= abcx$.

CASE II.

1. Find the G.C.M. of $a^2 + ax$, and $a^2 + 2ax + x^2$, $= a + x$.

2. 〃 〃 $a^2 - ax$, and $a^2 - 2ax + x^2$, $= a - x$.

3. 〃 〃 $a^2 + ab$, and $a^3 + b^3$, $= a + b$.

4. Find the G.C.M. of $a^2 - ab$, and $a^3 - b^3$, . $= a - b$.

5. " " $a^2 + 2ax + x^2$, and $a^2 - x^2$, $= a + x$.

6. " " $a^2 - 2ax + x^2$, and $a^3 - x^3$, $= a - x$.

7. " " $x^2 + x - 42$, and $x^2 + 10x + 21$, $= x + 7$.

8. " " $x^2 + 3x - 40$, and $x^2 + 8x - 65$, $= x - 5$.

9. " " $x^2 - 3x - 28$, and $x^2 + 5x + 4$, $= x + 4$.

10. " " $x^2 - 6x - 91$, and $x^2 - x - 156$, $= x - 13$.

11. " " $x^2 + 3x - 108$, and $x^2 + 5x - 126$, $= x - 9$.

12. " " $x^2 + 5x - 24$, and $x^2 + 23x + 120$, $= x + 8$.

NOTE.—The 7th to the 12th inclusive of the preceding exercises, may be very simply solved by resolving each of the expressions into factors, as shewn in Art. 73, CHAMBERS'S ALGEBRA; and then observing what factor is common to both expressions, that factor is the G.C.M. required.

13. Find the G.C.M. of $6x^2 + 51x + 99$, and $3x^2 + 57x + 144$, $= 3x + 9$.

14. " " $8x^2 + 58x + 77$, and $6x^2 + 65x + 176$, $= 2x + 11$.

15. " " $12x^2 - 15x - 63$, and $8x^2 + 74x + 105$, $= 4x + 7$.

16. " " $12x^2 - 108$, and $15x^2 - 36x - 27$, $= 3x - 9$.

17. " " $36x^2 + 96x - 36$, and $28x^2 + 100x + 48$, $= 4x + 12$.

18. " " $63x^2 + 114x - 45$, and $54x^2 - 63x + 15$, $= 9x - 3$.

19. " " $x^2 + 2x + 1$, and $x^3 + 1$, $= x + 1$.

20. " " $4x^2 + 12x + 9$, and $8x^3 + 27$, $= 2x + 3$.

21. Find the G.C.M. of $4x^2 - 25$, and $8x^3 + 60x^2 + 150x + 125$, $= 2x + 5$.

22. " " $9x^2 - 4$, and $36x^3 - 21x^2 - 20x + 12$, $= 3x - 2$.

23. " " $7x^2 - 4x - 3$, and $28x^3 - 2x^2 - 47x - 18$, $= 7x + 3$.

CASE III.

1. Find the G.C.M. of $x^2 - a^2$, $x^2 - 2ax + a^2$, and $4ax - 4a^2$, $= x - a$.

2. " " $x^2 - a^2$, $x^3 + a^3$, and $x^2 + 6ax + 5a^2$, $= x + a$.

3. " " $x^3 + a^3$, $x^2 - ax + a^2$, and $x^3 - 2ax^2 + 2a^2x - a^3$, ... $= x^2 - ax + a^2$.

4. Find the G.C.M. of $3a^2 + 6ab$, $7a^2 + 28ab + 28b^2$, and $5ax + 10bx$, ... $= a + 2b$.

5. Find the G.C.M. of $3ax + 4x^2$, $9a^2 - 16x^2$, and $27a^3 + 64x^3$, $= 3a + 4x$.

6. " " $7a^2 - 3ax$, $49a^4 - 9a^2x^2$, and $343a^6 - 27a^3x^3$, ... $= 7a^2 - 3ax$.

N.B.—Since each of the proposed quantities is divisible by the Greatest Common Measure, each of the exercises in the Greatest Common Measure will supply two or more exercises in Division, by dividing each of the given quantities by the G.C.M.

LEAST COMMON MULTIPLE.

CASE I.

1. Find the L.C.M. of $4a^2$, $12a^3$, and $15a^5$, . . $= 60a^5$.

2. " " $7a^2x$, $12a^3x^2y$, and $21a^4xy^2$, $= 84a^4x^2y^2$.

3. " " $5a^2b^2c^3$, $6a^3b^4c^3$, and $30a^4b^3c^2$, $= 30a^4b^4c^3$.

4. " " $3ab^3x^2y$, $7a^2b^2x^3y^4$, and $12a^3b^4x^2y^2$, $= 84a^3b^4x^3y^4$.

5. " " $9a^2x^2y^3$, $5a^3x^3y^4z$, and $15a^3x^2y^3z^4$, $= 45a^3x^3y^4z^4$.

6. Find the L.C.M. of $16m^2n^2p^3$, $12m^4n^3p^2$, and $24m^3n^2p^4$,
$= 48m^4n^3p^4$.

7. " " $15a^{\frac{1}{2}}b^2c^{\frac{1}{3}}$, $12a^{\frac{1}{2}}b^{\frac{1}{2}}c^2$, and $20a^3b^2c^{\frac{1}{2}}$,
$= 60a^3b^2c^2$.

8. " " $5x^{\frac{1}{3}}y^{\frac{1}{2}}z^3$, $9x^{\frac{2}{3}}y^2z^{\frac{1}{2}}$, and $36x^{\frac{3}{4}}y^{\frac{1}{3}}z^2$,
$= 180x^{\frac{3}{4}}y^2z^3$.

CASES II. AND III.

1. Find the L.C.M. of $3a^2 - 4ax$, and $6a^3 - 8a^2x$,
$= 6a^3 - 8a^2x$.

2. " " $8a^2 - 16ax$, and $a^2 - 4ax + 4x^2$,
$= 8a(a - 2x)^2$.

3. " " $3(a + b)$, $12(a - b)$, and $6(a^2 - b^2)$,
$= 12(a^2 - b^2)$.

4. " " $6(x^2y + xy^2)$, $9(x^3 - xy^2)$, and $4(y^3 + xy^2)$,
$= 36xy^2(x^2 - y^2)$.

5. " " $x^3 + 1$, $(x + 1)^2$, and $x^3 - 2x^2 + 2x - 1$,
$= (x - 1)(x + 1)^2(x^2 - x + 1)$.

6. " " $a^2 - ax + x^2$, $a^2 + ax + x^2$, $a^3 - x^3$, and $a^3 + x^3$, ... $= a^6 - x^6$.

7. Find the L.C.M. of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$,
$= x^4 + 6x^3 + 3x^2 - 26x - 24$.

8. " " $(a + b)^2$, $(a - b)^2$, $a^3 - b^3$, and $a^3 + b^3$,
$= (a^2 - b^2)^2(a^4 + a^2b^2 + b^4)$.

9. " " $a^4 - a^3x + a^2x^2 - ax^3 + x^4$, and $a^4 + a^3x + a^2x^2 + ax^3 + x^4$, . . $= a^8 + a^6x^2 + a^4x^4 + a^2x^6 + x^8$.

10. Find the L.C.M. of $x^2 - 1$, $x^2 - 9$, and $x^2 + 2x - 15$,
$= x^5 + 5x^4 - 10x^3 - 50x^2 + 9x + 45$.

N.B.—Since the *Least Common Multiple* is divisible without remainder by each of its component quantities, each of the exercises in the Least Common Multiple, with its answer, will supply two or more exercises in Division.

FRACTIONS.

REDUCTION OF FRACTIONS TO THEIR SIMPLEST FORM.

Reduce to their simplest form

1. $\frac{ab}{bx}$, $\frac{10ab^2c}{5ab}$, $\frac{14a^2bc^2}{7abcd}$, $\frac{-12acx^2}{4a^2c^2x}$, and $\frac{-32x^2yz}{-64xyz^2}$,
$$= \frac{a}{x}, 2bc, \frac{2ac}{d}, \frac{-3x}{ac}, \text{ and } \frac{x}{2z}.$$

2. $\frac{12x^2 - 2xy}{16x^2}$, $\frac{ax + x^2}{ab^2 + b^2x}$, $\frac{ac - c^2}{a^2 - c^2}$, and $\frac{x^2 - 2x + 1}{x^2 - 1}$,
$$= \frac{6x - y}{8x}, \frac{x}{b^2}, \frac{c}{a + c}, \text{ and } \frac{x - 1}{x + 1}.$$

3. $\frac{x^2 - y^2}{x^2 + 2xy + y^2}$, $\frac{a^2 + 2a - 3}{a^2 + 5a + 6}$, and $\frac{x^3 + 3x^2y + 3xy^2 + y^3}{x^2 + 3xy + 2y^2}$,
$$= \frac{x - y}{x + y}, \frac{a - 1}{a + 2}, \text{ and } \frac{(x + y)^2}{x + 2y}.$$

4. $\frac{6x^2 + 7x - 3}{6x^2 + 11x + 3}$, $\frac{x^2 - 2x - 3}{x^2 - 4x + 3}$, and $\frac{10x^2 + 7x + 1}{10x^2 + 3x - 1}$,
$$= \frac{3x - 1}{3x + 1}, \frac{x + 1}{x - 1}, \text{ and } \frac{5x + 1}{5x - 1}.$$

5. $\frac{ac + by + ay + bc}{af + 2bx + 2ax + bf}$, and $\frac{x^8 + x^6y^2 + x^2y + y^3}{x^4 - y^4}$,
$$= \frac{c + y}{f + 2x}, \text{ and } \frac{x^6 + y}{x^2 - y^2}.$$

6. $\frac{4a^2 - 12ax + 9x^2}{8a^3 - 27x^3}$, and $\frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc}$,
$$= \frac{2a - 3x}{4a^2 + 6ax + 9x^2}, \text{ and } \frac{a + b + c}{a - b - c}.$$

TO REDUCE A MIXED QUANTITY TO A FRACTIONAL FORM.

Reduce the following quantities to fractional forms :—

1. $2x - \frac{1}{3}$, $3x + \frac{2}{y}$, $4x - \frac{3x}{5a}$, and $3a + 4x - \frac{5}{2a}$,
$$= \frac{6x - 1}{3}, \frac{3xy + 2}{y}, \frac{20ax - 3x}{5a}, \text{ and } \frac{6a^2 + 8ax - 5}{2a}.$$

2. $2x + \frac{3-2x}{5}$, $5x + \frac{7x-4}{3x}$, and $3b - \frac{ab+b^2}{2a}$,

$$= \frac{8x+3}{5}, \frac{22x-4}{3x}, \text{ and } \frac{5ab-b^2}{2a}.$$

3. $1 + \frac{a-b}{a+b}$, $5c + \frac{a}{b+c}$, and $7x - \frac{2ax-x^2}{3a-x}$,

$$= \frac{2a}{a+b}, \frac{5bc+5c^2+a}{b+c}, \text{ and } \frac{19ax-6x^2}{3a-x}.$$

4. $a + x - \frac{2ax-x^2}{a+x}$, and $a + b - \frac{a^2-b^2}{a+2b}$,

$$= \frac{a^2+2x^2}{a+x}, \text{ and } \frac{3b(a+b)}{a+2b}.$$

5. $1 - x + x^2 - \frac{x^3}{1+x}$, and $1 + x + x^2 + \frac{x^3}{1-x}$,

$$= \frac{1}{1+x}, \text{ and } \frac{1}{1-x}.$$

6. $a^3 + a^2x + ax^2 + x^3 - \frac{a^4+x^4}{a-x}$, and $a^3 - a^2x + \frac{a^4+x^4}{a+x}$,

$$= -\frac{2x^4}{a-x}, \text{ and } \frac{2a^4-a^2x^2+x^4}{a+x}.$$

TO REDUCE IMPROPER FRACTIONS TO MIXED QUANTITIES.

Reduce to mixed quantities

1. $\frac{5ax+8x^2}{4x^2}$, $\frac{8a^2+7ax+3x^2}{a^2}$, and $\frac{9a^3+6a^2x+3ax}{3a^2}$,

$$= 2 + \frac{5a}{4x}, 8 + \frac{7ax+3x^2}{a^2}, \text{ and } 3a + 2x + \frac{x}{a}.$$

2. $\frac{3a^2+5ax}{a+2x}$, $\frac{8a^3+7a^2x}{a+x}$, and $\frac{4a^2-10ax+4x^3}{2a+x}$,

$$= 3a - \frac{ax}{a+2x}, 8a^2 - ax + x^2 - \frac{x^3}{a+x}, \text{ and } 4a - 3x + \frac{x^2}{2a+x}.$$

3. $\frac{2a^2+3ac+3c^2+a}{a+2c}$, and $\frac{6a^2x-5ax^2+4x^3}{2a-x}$,

$$= 2a - c + \frac{a+c^2}{a+2c}, \text{ and } 3ax - x^2 + \frac{3x^3}{2a-x}.$$

4. $\frac{1-2x+x^2}{1+x}$, and $\frac{a^8-x^{-12}-74a^3-x^2}{a^4+x^{-6}}$,

$$= 1 - 3x + 4x^2 - \frac{4x^3}{1+x}, \text{ and } a^4 - x^{-6} - \frac{74a^3+x^2}{a^4+x^{-6}}.$$

5. $\frac{1+3x^2+2x^4}{1-x^2}$, and $\frac{a^2-b^2-c^2-2bc}{a+b-c}$,

$$= 1 + 4x^2 + \frac{6x^4}{1-x^2}, \text{ and } a - b + c - \frac{4bc}{a+b-c}.$$

6. $\frac{3x^2+10x^3+13x^4+16x^5+15x^6+5}{x+2x^2}$,

$$= 3x + 4x^2 + 5x^3 + 6x^4 + \frac{3x^6+5}{x+2x^2}.$$

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR.

Reduce the following fractions to equivalent ones having a common denominator:—

1. $\frac{3a}{4b}, \frac{5a}{6c}$, and $\frac{2c}{3b}$, ... $= \frac{9ac}{12bc}, \frac{10ab}{12bc}$, and $\frac{8c^2}{12bc}$.

2. $\frac{7b}{8a}, \frac{11ab}{12c}$, and $\frac{5ac}{8b}$, . . $= \frac{56b^2c}{24abc}, \frac{22a^2b^2}{24abc}$, and $\frac{15a^2c^2}{24abc}$.

3. $\frac{3x}{4a}, \frac{7a}{9c}, \frac{4ax}{15b}$, and $\frac{5bx}{20ac}$,

$$= \frac{135bcx}{180abc}, \frac{140a^2b}{180abc}, \frac{48a^2cx}{180abc}, \text{ and } \frac{45b^2x}{180abc}.$$

4. $\frac{3a}{7x^2}, \frac{5b}{14y^2}, \frac{8ab}{21xy}$, and $\frac{5a^2b}{6x^2y}$,

$$= \frac{18ay^2}{42x^2y^2}, \frac{15bx^2}{42x^2y^2}, \frac{16abxy}{42x^2y^2}, \text{ and } \frac{35a^2by}{42x^2y^2}.$$

5. $\frac{2a}{mx}, \frac{5b}{my}, \frac{7c}{mz}$, and $\frac{14ab}{xy}$,

$$= \frac{2ayz}{mxyz}, \frac{5bxz}{mxyz}, \frac{7cxy}{mxyz}, \text{ and } \frac{14abmz}{mxyz}.$$

6. $\frac{a^2}{a+b}, \frac{ab}{a-b}$, and $\frac{3a^2-2ab}{a^2-b^2}$,

$$= \frac{a^2(a-b)}{a^2-b^2}, \frac{ab(a+b)}{a^2-b^2}, \text{ and } \frac{3a^2-2ab}{a^2-b^2}.$$

7. $\frac{ax}{a+x}$, $\frac{2a^2x^2}{a^2-ax+x^2}$, and $\frac{2a^2+x^2}{a^3+x^3}$,

$$=\frac{a^3x-a^2x^2+ax^3}{a^3+x^3}, \frac{2a^3x^2+2a^2x^3}{a^3+x^3}, \text{ and } \frac{2a^2+x^2}{a^3+x^3}.$$

8. $\frac{a+1}{a-1}$, $\frac{a-1}{a+1}$, $\frac{a^2+1}{a^2-1}$, and $\frac{a^2-1}{a^2+1}$,

$$=\frac{(a+1)^2(a^2+1)}{a^4-1}, \frac{(a-1)^2(a^2+1)}{a^4-1}, \frac{a^4+2a^2+1}{a^4-1}, \text{ and } \frac{a^4-2a^2+1}{a^4-1}.$$

9. $\frac{2x}{x+a}$, $\frac{x+a}{3a}$, $\frac{4a}{x-a}$, and $\frac{3a+x}{x^2-a^2}$

$$=\frac{6ax(x-a)}{3a(x^2-a^2)}, \frac{(x+a)^2(x-a)}{3a(x^2-a^2)}, \frac{12a^2(x+a)}{3a(x^2-a^2)}, \text{ and } \frac{9a^2+3ax}{3a(x^2-a^2)}.$$

10. $\frac{a^2+x^2}{(a+x)^2}$, $\frac{a^2-x^2}{(a-x)^2}$, $\frac{a^2+ax}{a^2-x^2}$, and $\frac{ax+x^2}{2ax}$,

$$=\frac{2ax(a^2+x^2)(a-x)^2}{2ax(a^2-x^2)^2}, \frac{2ax(a^2-x^2)(a+x)^2}{2ax(a^2-x^2)^2}, \frac{2ax(a^2+ax)(a^2-x^2)}{2ax(a^2-x^2)^2},$$

$$\text{and } \frac{(ax+x^2)(a^2-x^2)^2}{2ax(a^2-x^2)^2}.$$

ADDITION OF FRACTIONS.

1. $\frac{x}{3}+\frac{2x}{3}+\frac{5x}{3}+\frac{2}{3}$, and $\frac{x}{5a}+\frac{3x}{5a}+\frac{4x}{5a}+\frac{x-1}{5a}$,

$$=\frac{8x+2}{3}, \text{ and } \frac{9x-1}{5a}.$$

2. $\frac{a}{5}+\frac{a}{2}+\frac{3a}{10}+\frac{2a}{5}$, and $\frac{2a}{3x}+\frac{3a}{4x}+\frac{5a}{6x}+\frac{7a}{12x}$,

$$=\frac{14a}{10}, \text{ and } \frac{34a}{12x}.$$

3. $\frac{2a}{a+x}+\frac{3x}{a-x}+\frac{3x^2+a^2}{a^2-x^2}$, and $\frac{a+x}{2ax}+\frac{3a}{a-x}+\frac{7x}{a+x}$,

$$=\frac{3a^2+ax+6x^2}{a^2-x^2}, \text{ and } \frac{a^3(6x+1)+(20a^2-a)x^2+a^2x-(14a+1)x^3}{2ax(a^2-x^2)}.$$

4. $\frac{9x+7}{5}+\frac{6x+5}{4}+\frac{9x-8}{8}$, and $\frac{4x-5}{7}+\frac{2x+6}{9}+\frac{4x+8}{11}$,

$$=\frac{177x+66}{40}, \text{ and } \frac{802x+471}{693}.$$

5. $\frac{a}{a+b}+\frac{ab}{(a+b)^2}+\frac{a^3}{(a+b)^3}$, and $\frac{a^n}{(a+b)^{n+1}}+\frac{a^{n+1}}{(a+b)^{n+1}}$,

$=\frac{2a^3+3a^2b+2ab^2}{(a+b)^3}$, and $\frac{a^n(1+a)}{(a+b)^{n+1}}$.

6. $\frac{a}{(a-b)(x+a)}+\frac{b}{(a-b)(x+b)}$, and $\frac{a}{(a+b)(a-x)}+\frac{b}{(a+b)(b-x)}$,

$=\frac{(a+b)x+2ab}{(a-b)(x+a)(x+b)}$, and $\frac{2ab-(a+b)x}{(a+b)(a-x)(b-x)}$.

7. $\frac{1}{2a^2(a-x)}+\frac{1}{2a^2(a+x)}+\frac{1}{3a^3(a^2-x^2)}+\frac{1}{4a^3(a^2+x^2)}$,

$=\frac{12a^4+12a^2x^2+7a^2+x^2}{12a^3(a^4-x^4)}$.

8. $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}+\frac{x-1}{4(1+x^2)}$, $=\frac{1+x+x^2}{1-x-x^4+x^5}$.

SUBTRACTION OF FRACTIONS.

1. $\frac{14x}{8}-\frac{5x}{4}$, and $\frac{8a}{7}-\frac{3a}{5}$, . . . $=\frac{4x}{8}$ or $\frac{x}{2}$, and $\frac{19a}{35}$.

2. $\frac{3a}{4}-\frac{6a}{11}$, and $\frac{4a}{5}-\frac{3a}{7}$, ... $=\frac{9a}{44}$, and $\frac{13a}{35}$.

3. $\frac{2x}{5a}-\frac{3c}{4a}$, and $\frac{5y}{6c}-\frac{4c}{7y}$, . $=\frac{8x-15c}{20a}$, and $\frac{35y^2-24c^2}{42cy}$.

4. $\frac{9x+7}{4}-\frac{6x+4}{5}$, and $\frac{6x-5}{8}-\frac{4x-3}{7}$,

$=\frac{21x+19}{20}$, and $\frac{10x-11}{56}$.

5. $\frac{2x+y}{y}-\frac{y-2x}{x}$, and $\frac{x+1}{x-1}-\frac{x-1}{x+1}$,

$=\frac{2x^2+3xy-y^2}{xy}$, and $\frac{4x}{x^2-1}$.

6. $\left(5a-\frac{y-7}{4}\right)-\left(2a+\frac{y-3}{12}\right)$, and $\left(7a+\frac{2a}{b}\right)-\left(3a-\frac{a-3c}{b}\right)$,

$=3a-3-\frac{y}{3}$, and $4a+\frac{3a-3c}{b}$.

7. $\left(7x+\frac{x}{b}\right)-\left(3x-\frac{a-x}{b-x}\right)$, and $\frac{a}{(a+b)(x-a)}-\frac{b}{(a+b)(x-b)}$,

$$=4x+\frac{ab-x^2}{b(b-x)}, \text{ and } \frac{x(a-b)}{(a+b)(x-a)(x-b)}.$$

8. $\frac{x+y}{y}-\frac{2x}{x+y}$, and $\frac{2m+3n}{2m-3n}-\frac{2m-3n}{2m+3n}$,

$$=\frac{x^2+y^2}{y(x+y)}, \text{ and } \frac{24mn}{4m^2-9n^2}.$$

MULTIPLICATION OF FRACTIONS.

1. $\frac{a}{b}\times\frac{b}{c}$, $\frac{x}{5}\times\frac{4x}{9}$, $\frac{3}{2x}\times\frac{2}{3x}$, and $\frac{1}{3a}\times\frac{2}{7a}$,

$$=\frac{a}{c}, \frac{4x^2}{45}, \frac{1}{x^2}, \text{ and } \frac{2}{21a^2}.$$

2. $\frac{5x}{3}\times\frac{4}{9x}$, $\frac{8}{7x}\times-\frac{3x}{5}$, $\frac{ab}{cd}\times\frac{3d}{4b}$, and $-\frac{3a}{5}\times-\frac{3a}{5}$,

$$=\frac{20}{27}, -\frac{24}{35}, \frac{3a}{4c}, \text{ and } \frac{9a^2}{25}.$$

3. $\frac{a+x}{c+d}\times\frac{c-d}{a+x}$, and $\frac{a^2+ax+x^2}{a+x}\times\frac{a-x}{a^2-ax+x^2}$,

$$=\frac{c-d}{c+d}, \text{ and } \frac{a^3-x^3}{a^3+x^3}.$$

4. $\frac{a^2-x^2}{a}\times\frac{a^2+x^2}{ax}$, and $\frac{a^2x^2}{y^2}\times\frac{xy}{a(x+y)}\times\frac{x^2-y^2}{axy}$,

$$=\frac{a^4-x^4}{a^2x}, \text{ and } \frac{x^2(x-y)}{y^2}.$$

5. $\frac{a+x}{(m+n)^3}\times\frac{x^2-y^2}{12}\times\frac{(m+n)^2}{m-n}\times\frac{6(m^2-n^2)}{x+y}$,

$$=\tfrac{1}{2}(a+x)(x-y).$$

6. $\frac{a^3+x^3}{a^3-x^3}\times\frac{a^3-x^3}{a^2+x^2}\times\frac{a^4-x^4}{(a+x)^3}\times\frac{4ax}{a^2-ax+x^2}$,

$$=\frac{4ax(a-x)}{a^2+ax+x^2}.$$

7. $\left(\frac{a}{a+b}+\frac{b}{a-b}\right)\times\left(\frac{a}{a+c}-\frac{b}{b+c}\right)$,

$$=\frac{(a^2+b^2)c}{(a+b)(a+c)(b+c)}.$$

8. $\left(1 + x + x^2 + \frac{x^3}{1-x}\right) \times \left(1 - x + x^2 - x^3 + \frac{x^4}{1+x}\right),$
$$= \frac{1}{1-x^2}.$$

NOTE.—In the 8th exercise, reduce the factors to fractional forms, and then multiply; the first factor becomes $\frac{1}{1-x}$, and the second becomes $\frac{1}{1+x}$.

DIVISION OF FRACTIONS.

1. $\frac{a}{c} \div \frac{b}{c}, \frac{x}{5} \div \frac{2}{3x}, \frac{3x}{7} \div \frac{5b}{4x}$, and $\frac{5a}{b} \div \frac{7}{3b}$,
$$= \frac{a}{b}, \frac{3x^2}{10}, \frac{12x^2}{35b}, \text{ and } \frac{15a}{7}.$$

2. $\frac{4x}{9} \div \frac{4x^2}{45}, \frac{3ab}{c} \div \frac{4c}{a}, \frac{5b}{ac} \div \frac{2a}{3c}$, and $\frac{7x}{3a} \div \frac{3a}{5}$,
$$= \frac{5}{x}, \frac{3a^2b}{4c^2}, \frac{15b}{2a^2}, \text{ and } \frac{35x}{9a^2}.$$

3. $\frac{a+x}{c+d} \div \frac{a+x}{c-d}$, and $\frac{a^2+x^2}{3ax} \div \frac{a^2-x^2}{7a}$,
$$= \frac{c-d}{c+d}, \text{ and } \frac{7(a^2+x^2)}{3x(a^2-x^2)}.$$

4. $\frac{a^2-x^2}{4ax} \div \frac{a-x}{3x}$, and $\frac{a^3-x^3}{a^3+x^3} \div \frac{a-x}{a^2-ax+x^2}$,
$$= \frac{3(a+x)}{4a}, \text{ and } \frac{a^2+ax+x^2}{a+x}.$$

5. $\frac{a+x}{(m+n)^2} \div \frac{(m+n)}{a-x}$, and $\frac{9a^2+6ax+x^2}{2m+n} \div \frac{3a+x}{4m^2+4mn+n^2}$,
$$= \frac{a^2-x^2}{(m+n)^3}, \text{ and } (3a+x)(2m+n).$$

6. $\frac{x^2-x}{x-3} \div \frac{x^2-5x}{x-3}$, and $\frac{4a(a^2-x^2)}{3b(c^2-x^2)} \div \frac{a^2-ax}{bc+bx}$,
$$= \frac{x-1}{x-5}, \text{ and } \frac{4(a+x)}{3(c-x)}.$$

7. $\frac{3x^2}{a^3+x^3} \div \frac{x}{a+x}$, and $\frac{27(a^2-x^2)}{5} \div -\frac{9(a+x)}{5}$,
$$= \frac{3x}{a^2-ax+x^2}, \text{ and } -3(a-x).$$

8. $\frac{x^2+5x+4}{x^2+7x+12} \div \frac{x^2+2x+1}{x^2+3x+2}$, and $\frac{x^4-a^4}{(x-a)^2} \div \frac{x^2+ax}{x-a}$,

$$= \frac{x+2}{x+3}, \text{ and } \frac{x^2+a^2}{x}.$$

9. $\frac{x^2+7x+6}{x^2+12x+11} \div \frac{x^2-36}{x^2-121}$, and $\frac{x^2-3x-4}{x^2-7x-8} \div \frac{x^2-16}{x^2-64}$,

$$= \frac{x-11}{x-6}, \text{ and } \frac{x+8}{x+4}.$$

MISCELLANEOUS EXERCISES IN FRACTIONS.

1. $\frac{1}{4(1+a)} + \frac{1}{4(1-a)} + \frac{1}{2(1+a^2)}$, . . $= \frac{1}{1-a^4}$.

2. $\frac{1}{(x+2)(x+3)} - \frac{1}{(x+2)(x+3)(x+4)}$, $= \frac{1}{(x+2)(x+4)}$.

3. $\frac{1}{(x+1)(x+2)(x+3)} - \frac{1}{(x+1)(x+2)(x+3)(x+4)}$,

$$= \frac{1}{(x+1)(x+2)(x+4)}.$$

4. $\frac{1}{x-1} + \frac{1}{(x-1)(x-2)} - \frac{1}{(x-1)(x-2)(x-3)}$,

$$= \frac{x^2-4x+2}{(x-1)(x-2)(x-3)}.$$

5. $\frac{1}{x-1} - \frac{1}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)(x-3)}$,

$$= \frac{x^2-6x+10}{(x-1)(x-2)(x-3)}.$$

6. $\frac{1}{x-3} + \frac{1}{(x-3)(x-4)} - \frac{1}{(x-3)(x-4)(x-5)}$,

$$= \frac{x^2-8x+14}{(x-3)(x-4)(x-5)}.$$

7. $\frac{1}{1-x} - \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^3} - 1$, . . $= \frac{x^3}{(1-x)^3}$.

8. $\frac{b}{d} + \frac{ad-bc}{d(c+dx)} - \frac{a+bx}{c+dx} + \frac{a-b}{ab} + \frac{c-a}{ac} + \frac{b-c}{bc}$, $=0$.

NOTE.—Here the sum of the first three terms $= 0$, and the sum of the last three terms also $= 0$; hence the whole sum $= 0$.

9. $\frac{2x^2}{a^3+x^3} \div \frac{x}{a+x}$, and $\frac{6y^2}{5y-10} \div \frac{2y}{15y-30}$,
$$= \frac{2x}{a^2-ax+x^2}, \text{ and } 9y.$$

10. $\left(\frac{x^4-a^4}{(x-a)^2} \div \frac{x^2+ax}{x-a}\right) \times \frac{x^4+a^4}{(x+a)^2} \div \frac{x^2-ax}{x-a}$,
$$= \frac{(x^2+a^2)(x^4+a^4)}{x^2(x+a)^2}.$$

11. $\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}$,
$$= \frac{1}{abc}.$$

12. $\left(\frac{x}{x-y} + \frac{y}{x+y}\right)\left(\frac{x}{x-y} - \frac{y}{x+y}\right)\left(\frac{x+y}{x-y}\right)^2$,
$$= \frac{x^4+2x^3y+2xy^3-y^4}{(x-y)^4}.$$

13. $\frac{7ax}{11by} \times \frac{a^2-x^2}{c^2-x^2} \div \frac{a^2-ax}{bc+bx}$, . . . $= \frac{7x(a+x)}{11y(c-x)}$.

14. $\frac{x^2-9x+20}{x^2-6x} \times \frac{x^2-13x+42}{x^2-5x} \div \frac{x-7}{x(x+5)}$,
$$= \frac{x^2+x-20}{x}.$$

15. $\frac{x^2+8x+15}{x^2+x-12} \times \frac{x^2-x-20}{x^2+12x+35} \div \frac{x^2-2x-15}{x^2+11x+28}$,
$$= \frac{x+4}{x-3}.$$

16. $\frac{x^2-10x+21}{x^2+11x+30} \times \frac{x^2+12x+35}{x^2-11x+24} \div \frac{x^2+x-20}{x^2-2x-48}$,
$$= \frac{x^2-49}{x^2+x-20}.$$

17. $\left\{\frac{1}{x+2} + \frac{2}{x+3} + \frac{3}{x+4}\right\} \div \left\{\frac{1}{x+3} + \frac{1}{x+2} + \frac{1}{x+1}\right\}$,
$$= \frac{(x+1)(6x^2+34x+46)}{(x+4)(3x^2+12x+11)}.$$

18. $\left\{\frac{1}{x+5} + \frac{2}{x+4} - \frac{3}{x+3}\right\} \div \left\{\frac{1}{x+4} + \frac{1}{x+3} - \frac{2}{x+2}\right\}$,
$$= \frac{4x^2+26x+36}{3x^2+25x+50}.$$

INVOLUTION.

CASE I.

1. Find the square of $5a$, $3ax$, $\frac{7a}{2x}$, $\frac{4x}{3y}$, and $\frac{6a^2}{c}$,

$= 25a^2$, $9a^2x^2$, $\frac{49a^2}{4x^2}$, $\frac{16x^2}{9y^2}$, and $\frac{36a^4}{c^2}$.

2. " cube of $3a$, $7ab$, $5a^2x$, and $\frac{3x}{4a}$,

$= 27a^3$, $343a^3b^3$, $125a^6x^3$, and $\frac{27x^3}{64a^3}$.

3. " fourth power of $2a^{\frac{1}{2}}$, $3ac^{\frac{1}{2}}$, and $\frac{4a^{\frac{1}{2}}}{3x}$,

$= 16a^2$, $81a^4c^2$, and $\frac{256a^2}{81x}$.

4. " fifth " $2x$, $3ac$, and $\frac{4a^2}{5x}$,

$= 32x^5$, $243a^5c^5$, and $\frac{1024a^{10}}{3125x^5}$.

5. " sixth " $ax^{\frac{1}{2}}$, $a^{\frac{1}{2}}b^{\frac{1}{3}}c$, and $\frac{2}{x^{\frac{1}{2}}}$,

$= a^6x^3$, $a^3b^2c^6$, and $\frac{64}{x^3}$.

6. " seventh " $2ac^{\frac{1}{3}}$, $3abc^2$, and $\frac{3x}{2a}$,

$= 128a^7c^{\frac{7}{3}}$, $2187a^7b^7c^{14}$, and $\frac{2187x^7}{128a^7}$.

CASE II.

1. Find the square of $a + c$, $a + 2x$, and $ax + b$,
$= a^2 + 2ac + c^2$, $a^2 + 4ax + 4x^2$, and $a^2x^2 + 2abx + b^2$.

2. Find the square of $2a + c$, $3a + 2c$, and $x + 5y$,
$= 4a^2 + 4ac + c^2$, $9a^2 + 12ac + 4c^2$, and $x^2 + 10xy + 25y^2$.

3. Find the square of $a - c$, $a - 2b$, and $ac - b$,
$= a^2 - 2ac + c^2$, $a^2 - 4ab + 4b^2$, and $a^2c^2 - 2acb + b^2$.

4. Find the square of $2a - x$, $3a - 4c$, and $3x - 5y$,
$= 4a^2 - 4ax + x^2$, $9a^2 - 24ac + 16c^2$, and $9x^2 - 30xy + 25y^2$.

5. Find the cube of $m + n$, $a + 2x$, and $2a + 3b$,
$= m^3 + 3m^2n + 3mn^2 + n^3$, $a^3 + 6a^2x + 12ax^2 + 8x^3$, and $8a^3 + 36a^2b + 54ab^2 + 27b^3$.

6. Find the cube of $m - n$, $a - 2x$, and $3a - 2b$,
$= m^3 - 3m^2n + 3mn^2 - n^3$, $a^3 - 6a^2x + 12ax^2 - 8x^3$, and $27a^3 - 54a^2b + 36ab^2 - 8b^3$.

7. Find the cube of $a^2 + 2c^2$, $2x - y^2$, and $2a^2 - 3c^2$,
$= a^6 + 6a^4c^2 + 12a^2c^4 + 8c^6$, $8x^3 - 12x^2y^2 + 6xy^4 - y^6$, and $8a^6 - 36a^4c^2 + 54a^2c^4 - 27c^6$.

8. Find the square of $(a + b + c)$, and $(a + b - c)$,
$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$, and $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

9. Find the square of $(2a - b + c - d)$,
$= 4a^2 + b^2 + c^2 + d^2 - 4ab + 4ac - 4ad - 2bc + 2bd - 2cd$.

CASE III.

1. Find the square of $\frac{2a}{3b}$, $\frac{5ac}{4d}$, $\frac{4xy}{5cx}$, and $\frac{3x}{2y^2}$,

$$= \frac{4a^2}{9b^2}, \frac{25a^2c^2}{16d^2}, \frac{16x^2y^2}{25c^2x^2}, \text{ and } \frac{9x^2}{4y^4}.$$

2. " cube of $\frac{4a}{3c}$, $\frac{5ab}{6dx}$, $\frac{2a^2c}{5by^2}$, and $\frac{3ax}{4c^2y}$,

$$= \frac{64a^3}{27c^3}, \frac{125a^3b^3}{216d^3x^3}, \frac{8a^6c^3}{125b^3y^6}, \text{ and } \frac{27a^3x^3}{64c^6y^3}.$$

3. " square of $\frac{2a + c}{a + 3c}$, $\frac{x + 2y}{3x - y}$, and $\frac{2ac + x}{ac - 3x}$,

$$= \frac{4a^2 + 4ac + c^2}{a^2 + 6ac + 9c^2}, \frac{x^2 + 4xy + 4y^2}{9x^2 - 6xy + y^2}, \text{ and } \frac{4a^2c^2 + 4acx + x^2}{a^2c^2 - 6acx + 9x^2}.$$

4. Find the cube of $\frac{a + 2c}{3a + c}$, $\frac{2x - y}{x + 2y}$, and $\frac{2a + cx}{ac - 2x}$,

$$= \frac{a^3 + 6a^2c + 12ac^2 + 8c^3}{27a^3 + 27a^2c + 9ac^2 + c^3}, \frac{8x^3 - 12x^2y + 6xy^2 - y^3}{x^3 + 6x^2y + 12xy^2 + 8y^3},$$

$$\text{and } \frac{8a^3 + 12a^2cx + 6ac^2x^2 + c^3x^3}{a^3c^3 - 6a^2c^2x + 12acx^2 - 8x^3}.$$

5. Find the fourth power of $\frac{a-x}{a+x}$, and $\frac{2a+b}{a-2b}$,

$$=\frac{a^4-4a^3x+6a^2x^2-4ax^3+x^4}{a^4+4a^3x+6a^2x^2-4ax^3+x^4},$$

$$\text{and } \frac{16a^4+32a^3b+24a^2b^2+8ab^3+b^4}{a^4-8a^3b+24a^2b^2-32ab^3+16b^4}.$$

EVOLUTION.

CASE I.

1. Extract the square root of $16a^2$, $49b^2x^2$, and $81a^4c^2$, $= \pm 4a, \pm 7bx$, and $\pm 9a^2c$.

2. " cube " $8x^3y^3$, $-64a^6y^3$, and $125a^3x^6y^9$, $= 2xy, -4a^2y$, and $5ax^2y^3$.

3. " fourth " $16a^4x^8$, $256x^4y^{12}z^8$, and $6561a^4c^8$, $= \pm 2ax^2, \pm 4xy^3z^2$, and $\pm 9ac^2$.

4. " mth " $2^m \times 3^{2m}a^my^{2m}$, and $7^ma^{3m}x^{2m}$, $= (2\times 3^2ay^2 = 18ay^2)$, and $7a^3x^2$.

CASE II.

1. Extract the square root of $\frac{4x^2y^4}{9a^2c^2}$, $\frac{36a^2x^2}{49x^4y^2}$, and $\frac{144a^4}{81c^6}$,

$$= \pm\frac{2xy^2}{3ac}, \pm\frac{6ax}{7x^2y}, \text{ and } \pm\frac{12a^2}{9c^3}.$$

2. " cube " $\frac{8b^3x^6}{64a^6y^9}$, $\frac{a^3b^6}{27x^3y^3}$, and $-\frac{125a^9}{216c^3x^6}$,

$$= \frac{2bx^2}{4a^2y^3}, \frac{ab^2}{3xy}, \text{ and } -\frac{5a^3}{6cx^2}.$$

3. " fourth " $\frac{a^4b^8c^{12}}{16x^4y^8}$, and $\frac{625x^4y^4z^{12}}{1296a^4b^8c^{12}}$,

$$= \pm\frac{ab^2c^3}{2xy^2}, \text{ and } \pm\frac{5xyz^3}{6ab^2c^3}.$$

CASE III.

Extract the square root of the following quantities:—

1. $(9x^2 + 30xy + 25y^2)^{\frac{1}{2}}$, and $(16x^4 - 8x^2cy + c^2y^2)^{\frac{1}{2}}$,
$= 3x + 5y$, and $4x^2 - cy$.

2. $(a^2 - 2ab + b^2 + 2ac - 2bc + c^2)^{\frac{1}{2}}$, and $(4a^2 + 4ax + x^2 - 4ay - 2xy + y^2)^{\frac{1}{2}}$, . . . $= a - b + c$, and $2a + x - y$.

3. $(4a^4 - 4a^2b + b^2 + 8a^2c - 4bc + 4c^2 - 4a^2d^2 + 2bd^2 - 4cd^2 + d^4)^{\frac{1}{2}}$, ... $= 2a^2 - b + 2c - d^2$.

4. $\left(\frac{4a^2}{9} + \frac{4}{5}ax^2 + \frac{9x^4}{25}\right)^{\frac{1}{2}}$, and $\left(\frac{9a^4}{16} - \frac{15}{14}a^2x + \frac{25x^2}{49}\right)^{\frac{1}{2}}$,
$= \frac{2}{3}a + \frac{3}{5}x^2$, and $\frac{3}{4}a^2 - \frac{5}{7}x$.

5. $\left(\frac{4x^2}{9y^2} - 2 + \frac{9y^2}{4x^2}\right)^{\frac{1}{2}}$, and $\left(\frac{9a^2}{25x^2} + \frac{8a}{5c} + \frac{16x^2}{9c^2}\right)^{\frac{1}{2}}$,
$= \frac{2x}{3y} - \frac{3y}{2x}$, and $\frac{3a}{5x} + \frac{4x}{3c}$.

6. $\left(\frac{9x^2}{4a^2} - 6 + \frac{4a^2}{x^2} + \frac{5x^2}{a^2} - \frac{20}{3} + \frac{25x^2}{9a^2}\right)^{\frac{1}{2}}$, $= \frac{3x}{2a} - \frac{2a}{x} + \frac{5x}{3a}$.

CASE IV.

Extract the cube root of the following quantities:—

1. $(x^3 + 6x^2y + 12xy^2 + 8y^3)^{\frac{1}{3}}$, and $(8a^3 - 12a^2x + 6ax^2 - x^3)^{\frac{1}{3}}$,
$= x + 2y$, and $2a - x$.

2. $\left(\frac{27x^3}{y^3} + 9\frac{x}{y} + \frac{y}{x} + \frac{y^3}{27x^3}\right)^{\frac{1}{3}}$, and $\left(\frac{27x^3}{y^3} - \frac{9x^2}{y^2} + \frac{x}{y} - \frac{1}{27}\right)^{\frac{1}{3}}$,
$= \frac{3x}{y} + \frac{y}{3x}$, and $\frac{3x}{y} - \frac{1}{3}$.

3. $(a^6 + 9a^5x + 33a^4x^2 + 63a^3x^3 + 66a^2x^4 + 36ax^5 + 8x^6)^{\frac{1}{3}}$,
$= a^2 + 3ax + 2x^2$.

4. $(1 + 3x - 3x^2 - 11x^3 + 6x^4 + 12x^5 - 8x^6)^{\frac{1}{3}}$, $= 1 + x - 2x$.

5. $(x^3 - a)^{\frac{1}{3}}$, and $(x^3 + a)^{\frac{1}{3}}$,
$= x - \frac{1}{3}\frac{a}{x^2} - \frac{1}{9}\frac{a^2}{x^5} - \frac{5}{81}\frac{a^3}{x^8} - \frac{10a^4}{243x^{11}} -$ &c.,
and $x + \frac{1}{3}\frac{a}{x^2} - \frac{1}{9}\frac{a^2}{x^5} + \frac{5}{81}\frac{a^3}{x^8} - \frac{10}{243}\frac{a^4}{x^{11}} +$ &c.

NOTE.—These two expansions may be used for extracting the cube root of numbers that are nearly exact cubes—as, for instance, to find the cube root of $63 = (64 - 1) = (4^3 - 1)$; hence, in the first substitute 4 for x, and 1 for a, and the value of the cube root of 63 will be obtained. Or, to find the cube root of 1006 $= (1000 + 6) = 10^3 + 6$; hence, to find the cube root of 1006, in the second series, make $x = 10$, and $a = 6$; then find the value of the several terms, and the cube root of 1006 will be found.

IRRATIONAL QUANTITIES.

CASE I.

1. Reduce 7, $2a$, $5ac^2$, and $3a^2x$, to the form of the square root,
$= (49)^{\frac{1}{2}}, (4a^2)^{\frac{1}{2}}, (25a^2c^4)^{\frac{1}{2}}$, and $(9a^4x^2)^{\frac{1}{2}}$.

2. " 5, $3x^2$, $4a^2c$, and $2ay^2z$, to the form of the cube root,
$= (125)^{\frac{1}{3}}, (27x^6)^{\frac{1}{3}}, (64a^6c^3)^{\frac{1}{3}}$, and $(8a^3y^6z^3)^{\frac{1}{3}}$.

3. " 3, $2x$, $\frac{3x^2}{y}$, and $\frac{5ac^2}{2bx}$, to the form of the fourth root,
$= (81)^{\frac{1}{4}}, (16x^4)^{\frac{1}{4}}, \left(\frac{81x^8}{y^4}\right)^{\frac{1}{4}}$, and $\left(\frac{625a^4c^8}{16b^4x^4}\right)^{\frac{1}{4}}$.

4. " $2x$, $\frac{3ax}{b}$, $\frac{a^2c^2}{bx^3}$, and $\frac{ax^2y}{c^2z}$, to the form of the nth root,
$= (2^nx^n)^{\frac{1}{n}}, \left(\frac{3^na^nx^n}{b^n}\right)^{\frac{1}{n}}, \left(\frac{a^{2n}c^{2n}}{b^nx^{3n}}\right)^{\frac{1}{n}}$, and $\left(\frac{a^nx^{2n}y^n}{c^{2n}z^n}\right)^{\frac{1}{n}}$.

CASE II.

Reduce to their simplest form the following expressions:—

1. $\sqrt{12}$, $\sqrt{20a^3}$, $\sqrt{75a^2x^3}$, $\sqrt{128a^3x^5}$, and $\sqrt{200x^3y^4}$,
$= 2\sqrt{3}$, $2a\sqrt{5a}$, $5ax\sqrt{3x}$, $8ax^2\sqrt{2ax}$, and $10xy^2\sqrt{2x}$.

2. $\sqrt[3]{32a^4}$, $\sqrt[3]{81a^3x}$, $\sqrt[3]{500xy^5}$, and $\sqrt[3]{24a^4x^5}$,
$= 2a\sqrt[3]{4a}$, $3a\sqrt[3]{3x}$, $5y\sqrt[3]{4xy^2}$, and $2ax\sqrt[3]{3ax^2}$.

3. $\sqrt{x^3(a^4 - a^2y^2)}$, $\sqrt{(a^2x^4 - a^2x^2z^2)}$, and $\sqrt{a^6(a^5 + a^2x^3)}$,
$= ax\sqrt{(a^2x - xy^2)}$, $ax\sqrt{(x^2 - z^2)}$, and $a^4\sqrt{(a^3 + x^3)}$.

4. $\sqrt[3]{a^4(x + a)^4}$, $\sqrt[3]{a^3x^4(a^4 + a^3x)}$, and $\sqrt[3]{32a^3(a^2 - x^2)^2}$,
$= a(x + a)\sqrt[3]{a(x + a)}$, $a^2x\sqrt[3]{(ax + x^2)}$, and $2a\sqrt[3]{4(a^2 - x^2)}$.

CASE III.

1. Rationalise the denominators of $\frac{3}{\sqrt{3x^3}}$, $\frac{5x}{\sqrt{8x}}$, and $\frac{2a}{\sqrt{7a}}$,

$$=\frac{3\sqrt{3x}}{3x^2}=\frac{\sqrt{3x}}{x^2},\ \frac{5x\sqrt{2x}}{4x}=\frac{5\sqrt{2x}}{4},\ \text{and } \frac{2\sqrt{7a}}{7}.$$

2. " the denominators of $\frac{5}{\sqrt[3]{4x^2}}$, $\frac{7a}{\sqrt[3]{9a^2x}}$, and $\frac{6ax}{\sqrt[3]{25bc^2}}$,

$$=\frac{5\sqrt[3]{2x}}{2x},\ \frac{7a\sqrt[3]{3ax^2}}{3ax}=\frac{7\sqrt[3]{3ax^2}}{3x},\ \text{and } 6ax\frac{\sqrt[3]{5b^2c}}{5bc}.$$

3. " the denominators of $\frac{2x}{\sqrt{a^3(a-x)}}$, and $\frac{3bc}{\sqrt{a^5(a^2-x^2)^3}}$,

$$=\frac{2x\sqrt{a(a-x)}}{a^2(a-x)},\ \text{and } \frac{3bc\sqrt{a(a^2-x^2)}}{a^3(a^2-x^2)^2}.$$

4. " the denominators of $\frac{2}{\sqrt[m]{x^2}}$, $\frac{3a}{\sqrt[3]{x^{2n}}}$, and $\frac{4}{\sqrt[n]{a^2x^{2n+1}}}$,

$$=\frac{2\sqrt[m]{x^{(m-2)}}}{x},\ \frac{3a\sqrt[3]{x^n}}{x^n},\ \text{and } \frac{4\sqrt[n]{(a^{n-2}x^{n-1})}}{ax^3}.$$

CASE IV.

1. Reduce $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{6}$, and $\sqrt[6]{7}$, to similar radicals,

$$=(3^6)^{\frac{1}{12}},\ (5^4)^{\frac{1}{12}},\ (6^3)^{\frac{1}{12}},\ \text{and } (7^2)^{\frac{1}{12}};\ \text{or } 3^{\frac{6}{12}},\ 5^{\frac{4}{12}},\ 6^{\frac{3}{12}},\ \text{and } 7^{\frac{2}{12}};$$
$$\text{or } \sqrt[12]{729},\ \sqrt[12]{625},\ \sqrt[12]{216},\ \text{and } \sqrt[12]{49}.$$

2. " $a\sqrt{(x^2+y^2)}$, and $3b\sqrt[5]{(a^2-c^2)}$, to similar radicals,

$$=a(x^2+y^2)^{\frac{5}{10}},\ \text{and } 3b(a^2-c^2)^{\frac{2}{10}}.$$

3. " $\sqrt[n]{(a-b)^p}$, and $\sqrt[m]{(a^2+x^2)^r}$, to similar radicals,

$$=(a-b)^{\frac{mp}{mn}},\ \text{and } (a^2+x^2)^{\frac{nr}{mn}}.$$

4. " $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, $(ax)^{\frac{1}{4}}$, and $(a-x)^{\frac{1}{5}}$, to similar surds,

$$=a^{\frac{30}{60}},\ b^{\frac{20}{60}},\ (ax)^{\frac{15}{60}},\ \text{and } (a-x)^{\frac{12}{60}}.$$

CASE V.

ADDITION OF IRRATIONAL QUANTITIES.

1. Add together $\sqrt{12}$, $\sqrt{27}$, $\sqrt{48}$, and $\sqrt{108}$; also $\sqrt{8}$, $\sqrt{72}$, $\sqrt{128}$, and $\sqrt{32}$, ... $= 15\sqrt{3}$, and $20\sqrt{2}$.

2. Add together $\sqrt{54a^3}$, $\sqrt{24a^5}$, and $\sqrt{150a}$; also $\sqrt{3ax^2}$, $\sqrt{27a^3}$, and $\sqrt{48a^5x^4}$, $= (3a + 2a^2 + 5)\sqrt{6a}$, and $(x + 3a + 4a^2x^2)\sqrt{3a}$.

3. Add together $\sqrt[3]{56}$, $\sqrt[3]{189}$, and $\sqrt[3]{448}$; also $\sqrt[3]{24a^4}$, $\sqrt[3]{3a^4x^3}$, and $\sqrt[3]{81ax^6}$, ... $= 9\sqrt[3]{7}$, and $(2a + ax + 3x^2)\sqrt[3]{3a}$.

4. Add together $\sqrt{(9x + 27)}$, $3\sqrt{(4x + 12)}$, $5\sqrt{(25x + 75)}$, and $2\sqrt{(9a^2x + 27a^2)}$, ... $= (34 + 6a)\sqrt{(x + 3)}$.

5. Add together $\sqrt[3]{(16x^2 + 24)}$, $\sqrt[3]{(54x^2 + 81)}$, and $\sqrt[3]{(128a^3x^2 + 192a^3)}$, ... $= (5 + 4a)\sqrt[3]{(2x^2 + 3)}$.

6. Add together $\sqrt[5]{(32a^3+96x)}$, $\sqrt[5]{(a^8+3a^5x)}$, and $\sqrt[5]{(a^3x^5+3x^6)}$, $= (2 + a + x)\sqrt[5]{(a^3 + 3x)}$.

CASE VI.

SUBTRACTION OF IRRATIONAL QUANTITIES.

Find the difference of the following quantities :—

1. $(\sqrt{108} - \sqrt{27})$, $(\sqrt{320} - \sqrt{80})$, and $(\sqrt{448} - 2\sqrt{63})$, $= 3\sqrt{3}$, $4\sqrt{5}$, and $2\sqrt{7}$.

2. $(\sqrt{a^2x} - \sqrt{c^2x})$, $(\sqrt{64a^4x} - \sqrt{16a^2x^3})$, and $(3\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}})$, $= (a - c)\sqrt{x}$, $(8a^2 - 4ax)\sqrt{x}$, and $\frac{7}{6}\sqrt{3}$.

3. $(9\sqrt[3]{5x^4} - \sqrt[3]{135x^4})$, $(8\sqrt[3]{a^3b} - \sqrt[3]{a^6b})$, and $(\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{9}{32}})$, $= 6x\sqrt[3]{5x}$, $(8a - a^2)\sqrt[3]{b}$, and $\frac{1}{12}\sqrt[3]{18}$.

4. $(\sqrt{x^4y^2 - a^2x^2y^2} - \sqrt{x^2y^2 - a^2y^2})$, and $\left(\sqrt{\frac{a^2cd^2}{bf^2}} - \sqrt{\frac{a^2c}{b^3}}\right)$, $= y(x - 1)\sqrt{(x^2 - a^2)}$, and $\frac{abd - af}{b^2f}\sqrt{bc}$.

5. $(\sqrt[n]{a^{3n}c} - \sqrt[n]{a^{2n}c})$, and $(\sqrt[p]{a^{2p}c^{3p}b^5} - \sqrt[p]{a^pb^{2p+5}})$, $= a^2(a - 1)\sqrt[n]{c}$, and $a(ac^3 - b^2)\sqrt[p]{b^5}$.

6. $\left(\sqrt[3]{\frac{a^3b^2+a^3c}{b^2}} - \sqrt[3]{\frac{d^3b^2+d^3c}{b^2}}\right)$, and $\left(\sqrt[3]{\frac{a^7c-a^6b}{c^2}} - \sqrt[3]{\frac{a^4c-a^3b}{c^2}}\right)$, $= \frac{a - d}{b}\sqrt[3]{(b^3 + bc)}$, and $\frac{a(a - 1)}{c}\sqrt[3]{(ac^2 - bc)}$.

CASE VII.

MULTIPLICATION OF IRRATIONAL QUANTITIES.

1. Multiply $3\sqrt{15}$ by $\sqrt{6}$, $5\sqrt{3}$ by $4\sqrt{8}$, and $2\sqrt{27}$ by $3\sqrt{6}$,
$= 9\sqrt{10}$, $40\sqrt{6}$, and $54\sqrt{2}$.

2. 〃 $5\sqrt[3]{6}$ by $3\sqrt[3]{4}$, $\frac{1}{3}\sqrt[3]{15}$ by $5\sqrt[3]{18}$, and $\sqrt[3]{7a^2}$ by $\sqrt[3]{ac^2}$,
$= 30\sqrt[3]{3}$, $5\sqrt[3]{10}$, and $a\sqrt[3]{7c^2}$.

3. 〃 $\sqrt{9x} \times \sqrt{x}$, $\sqrt{ax} \times 6\sqrt{ac}$, and $\sqrt{5a} \times \sqrt{15ab}$,
$= 3x$, $6a\sqrt{ac}$, and $5a\sqrt{3b}$.

4. 〃 $6\sqrt[3]{a} \times \sqrt[3]{2c}$, $\sqrt[3]{a^2b} \times \sqrt[3]{7ac}$, and $\sqrt[3]{2c^2} \times 3\sqrt[3]{bc}$,
$= 6\sqrt[3]{2ac}$, $a\sqrt[3]{7bc}$, and $3c\sqrt[3]{2b}$.

5. 〃 $\frac{1}{2}\sqrt[3]{18ad^2} \times \frac{2}{3}\sqrt[3]{6dxy}$, and $\frac{2}{3}\sqrt[3]{7a} \times \frac{4}{7}\sqrt[3]{4a^2}$,
$= d\sqrt[3]{4axy}$, and $\frac{8a}{21}\sqrt[3]{28}$.

6. 〃 $x^{\frac{1}{4}} \times b^{\frac{1}{2}}$, $b^{\frac{1}{4}} \times (ax)^{\frac{1}{2}}$, and $a^{\frac{3}{2}} \times a^{\frac{1}{8}} \times - a^{\frac{1}{2}}$,
$= (b^2x)^{\frac{1}{4}}$, $(a^2bx^2)^{\frac{1}{4}}$, and $- a^{\frac{17}{8}}$, or $- a^2\sqrt[8]{a}$.

7. 〃 $(a^{\frac{1}{2}} + x^{\frac{1}{2}})(a^{\frac{1}{2}} - x^{\frac{1}{2}})$, and $(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}})$,
$= a - x$, and $a^{\frac{2}{3}} - b^{\frac{2}{3}}$.

8. 〃 $x^{\frac{1}{2}} \times x^{\frac{1}{3}} \times x^{\frac{1}{4}}$, and $(a^2\sqrt{a} + x\sqrt{x})(a^2\sqrt{a} - x\sqrt{x})$,
$= x^{\frac{13}{12}}$, and $a^5 - x^3$.

9. 〃 $(a + b)^{\frac{1}{2}} \times (a - b)^{\frac{1}{3}}$, and $(2 + \sqrt{x})(6 + \sqrt{x})$,
$= (a^5 + a^4b - 2a^3b^2 - 2a^2b^3 + ab^4 + b^5)^{\frac{1}{6}}$, and $12 + 8\sqrt{x} + x$.

10. Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$, . $= a - b$.

11. 〃 $(a^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}})(a^{\frac{1}{3}} - x^{\frac{1}{3}})$, and $(a^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}})$
$(a^{\frac{1}{3}} + x^{\frac{1}{3}})$, ... $= a - x$, and $a + x$.

12. Multiply $nax^5 + (n-1)a^2x^2 + (n-2)a^3x^{-1}$ by $a^3x^{\frac{1}{4}}(x - ax^{-2})$,
$= na^4x^{\frac{25}{12}} - a^5x^{\frac{13}{4}} - a^6x^{\frac{1}{4}} - (n - 2)a^7x^{-\frac{11}{4}}$.

CASE VIII.

DIVISION OF IRRATIONAL QUANTITIES.

1. Divide $\sqrt{336}$ by $\sqrt{12}$, $3\sqrt{300}$ by $\sqrt{5}$, and $\sqrt{54}$ by $5\sqrt{2}$,
$= 2\sqrt{7}$, $6\sqrt{15}$, and $\frac{3}{5}\sqrt{3}$.

2. " $\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$, and $\frac{3}{4}\sqrt[3]{\frac{3}{16}}$ by $\frac{3}{5}\sqrt[3]{\frac{2}{5}}$,
$= \frac{1}{8}\sqrt{3}$, and $\frac{5}{16}\sqrt[3]{30}$.

3. " $6\sqrt[3]{7a^7}$ by $3\sqrt[3]{2ac^2}$, and $2\sqrt[4]{19ax^2}$ by $\sqrt[4]{8a}$,
$= \frac{a}{c}\sqrt[3]{28c}$, and $\sqrt[4]{38x^2}$.

4. " $84a^2b^{\frac{1}{2}}c^2$ by $12a^{\frac{2}{3}}bc^{\frac{3}{4}}x$, and $6\sqrt{a^4b}$ by $3ab^{-\frac{1}{2}}$,
$= \frac{7a^{\frac{4}{3}}c^{\frac{5}{4}}}{b^{\frac{1}{2}}x}$, and $2ab$.

5. " $(a^2 - x^2)^{\frac{1}{2}}$ by $(a + x)^{\frac{1}{2}}$, and $(a^3 - x^3)^{\frac{1}{3}}$ by $(a - x)^{\frac{1}{3}}$,
$= (a - x)^{\frac{1}{2}}$, and $(a^2 + ax + x^2)^{\frac{1}{3}}$.

6. " $\sqrt{8} + \sqrt[3]{12} + \sqrt[4]{4}$ by $\sqrt{2}$, and $2\sqrt{32} + 3\sqrt{2} + 4$ by $\sqrt{8}$,
$= 2 + \sqrt[6]{18} + 1$, and $4 + \frac{3}{2} + \sqrt{2}$.

7. " $a^2 - a^{\frac{5}{3}}x^{\frac{1}{4}} - 2a^{\frac{1}{3}}x^{\frac{1}{4}} + 2x^{\frac{1}{2}}$ by $a^{\frac{1}{3}} - x^{\frac{1}{4}}$, . $= a^{\frac{5}{3}} - 2x^{\frac{1}{4}}$.

8. " $a^{\frac{3}{2}} + 2ab^{\frac{2}{3}} + 2a^{\frac{1}{2}}b^{\frac{4}{3}} + b^2$ by $a^{\frac{1}{2}} + b^{\frac{2}{3}}$,
$= a + a^{\frac{1}{2}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

9. " $a^{-1} - x^{-1}$ by $a^{-\frac{1}{3}} - x^{-\frac{1}{3}}$, and $a^{-6} - 64x^2$ by $a^{-1} + 2x^{\frac{1}{3}}$,
$= a^{-\frac{2}{3}} + a^{-\frac{1}{3}}x^{-\frac{1}{3}} + x^{-\frac{2}{3}}$, and $a^{-5} - 2a^{-4}x^{\frac{1}{3}} + 4a^{-3}x^{\frac{2}{3}}$
$- 8a^{-2}x + 16a^{-1}x^{\frac{4}{3}} - 32\sqrt[3]{x^5}$.

NOTE.—The above exercise may be reduced to the following form $\frac{x - a}{ax} \div \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{a^{\frac{1}{3}}x^{\frac{1}{3}}} = \frac{x - a}{ax} \times \frac{a^{\frac{1}{3}}x^{\frac{1}{3}}}{x^{\frac{1}{3}} - a^{\frac{1}{3}}} = \frac{1}{a^{\frac{2}{3}}x^{\frac{2}{3}}}\left(\frac{x - a}{x^{\frac{1}{3}} - a^{\frac{1}{3}}}\right)$. The latter factor can now be divided by using positive exponents; and then each term of the quotient, being divided by $a^{\frac{2}{3}}x^{\frac{2}{3}}$, will give the answer. In the same manner, the second part of exercise 9 may be reduced to the form $\frac{1 - 64a^3x^2}{a^3} \div \frac{1 + 2a^{\frac{1}{3}}x^{\frac{1}{3}}}{a^{\frac{1}{3}}} = \frac{1 - 64a^3x^2}{a^3} \times \frac{a^{\frac{1}{3}}}{1 + 2a^{\frac{1}{3}}x^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{a^5}}\left(\frac{1 - 64a^3x^2}{1 + 2a^{\frac{1}{3}}x^{\frac{1}{3}}}\right)$, the latter factor of which may again be divided by using positive exponents; and then the quotient, being divided by $\sqrt[3]{a^5}$, will give the answer. Both these exercises may also be very simply done by using Synthetic

Division. The coefficients of the quotient are found as follows:—

$$\text{1st,}\ \left\{\begin{array}{r|l} 1 & 1+0+0-1 = \text{coefficients of the dividend.} \\ +1 & \underline{\ \ +1+1+1} \\ & 1+1+1 \qquad\quad \text{"} \qquad \text{"} \qquad \text{quotient.} \end{array}\right.$$

$$\text{2d,}\ \left\{\begin{array}{r|l} 1 & 1+0+0+0+\ 0+\ 0-64=\text{coeff. of dividend.} \\ -2 & \underline{\ \ -2+4-8+16-32+64} \\ & 1-2+4-8+16-32 \quad = \quad \text{"} \quad \text{quotient.} \end{array}\right.$$

GENERAL EXERCISES.

1. Simplify $\dfrac{1}{2(1-x^{\frac{1}{2}})} + \dfrac{1}{2(1+x^{\frac{1}{2}})} + \dfrac{1}{1+x}, \quad . \quad = \dfrac{2}{1-x^2}.$

2. " $\left\{\dfrac{1}{a-(a^2-x^2)^{\frac{1}{2}}} - \dfrac{1}{a+(a^2-x^2)^{\frac{1}{2}}}\right\} \times \dfrac{x^2}{2(a^2-x^2)^{\frac{3}{2}}},$

$$= \frac{1}{a^2-x^2}.$$

3. " $\dfrac{1+x+(x^2-1)^{\frac{1}{2}}}{1+x-(x^2-1)^{\frac{1}{2}}} - \dfrac{1+x-(x^2-1)^{\frac{1}{2}}}{1+x+(x^2-1)^{\frac{1}{2}}},$

$$= 2(x^2-1)^{\frac{1}{2}}.$$

NOTE.—Here multiply both numerator and denominator of each fraction by its numerator, and then subtract; or, which comes to the same thing, reduce them to a common denominator.

4. Simplify $\dfrac{(x^2+1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}} + \dfrac{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}},$

$$= 2x^2.$$

5. " $\dfrac{x+a}{x+(xa^2)^{\frac{1}{3}}-(x^2a)^{\frac{1}{3}}}, \quad . \quad = \dfrac{x^{\frac{1}{3}}+a^{\frac{1}{3}}}{x^{\frac{1}{3}}}$, or $1+\left(\dfrac{a}{x}\right)^{\frac{1}{3}}.$

NOTE.—The denominator may be reduced to the form $x^{\frac{1}{3}}(x^{\frac{2}{3}} - x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}})$. The numerator is then divisible by the latter factor, and gives the quotient $x^{\frac{1}{3}} + a^{\frac{1}{3}}$.

6. Simplify $\dfrac{a^2+4a\sqrt{ax}+6ax+4x\sqrt{ax}+x^2}{\sqrt{a}+\sqrt{x}},$

$$= a^{\frac{3}{2}} + 3ax^{\frac{1}{2}} + 3a^{\frac{1}{2}}x + x^{\frac{3}{2}} = (\sqrt{a}+\sqrt{x})^3.$$

7. " $\dfrac{a^4+a^2x}{a^4-x^2}$, and $\dfrac{24a^5x^2-16a^4x^2-8a^3x^2}{32a^6x^2-16a^5x^2-24a^4x^2+8a^3x^2},$

$$= \frac{a^2}{a^2-x}, \text{ and } \frac{3a+1}{4a^2+2a-1}.$$

8. Simplify $\dfrac{1-\frac{1}{2}\{1-\frac{1}{3}(1-x)\}}{1-\frac{1}{3}\{1-\frac{1}{2}(1-x)\}}$, and $\dfrac{x+2}{2(x+1)} - \dfrac{x-2}{2(x-1)} - \dfrac{x}{x^2+1},$

$$= \frac{4-x}{5-x}, \text{ and } \frac{2x}{x^4-1}.$$

CASE IX.

INVOLUTION OF IRRATIONAL QUANTITIES.

1. Raise $3\sqrt{2}$, $4\sqrt{3}$, $7\sqrt{5}$, $5\sqrt{7}$, and $a\sqrt{3}$, to the second power,
$= 18, 48, 245, 175$, and $3a^2$.

2. " $2\sqrt{a}$, $3\sqrt{ax}$, $5\sqrt[3]{a^2x}$, and $7\sqrt[3]{(x^2-y^2)}$, to the third power,
$= 8a\sqrt{a}$, $27ax\sqrt{ax}$, $125a^2x$, and $343(x^2-y^2)$.

3. " $\frac{1}{6}\sqrt[3]{6a^3}$, $\frac{1}{2}\sqrt{4(a^2-x^2)}$, and $\frac{b\sqrt[3]{(c+d)}}{\sqrt{(a^2+d)}}$, to the fourth power,
$= \frac{a^4}{216}\sqrt[3]{6}$, $a^4 - 2a^2x^2 + x^4$, and $\frac{b^4(c+d)\sqrt[3]{(c+d)}}{a^4+2a^2d+d^2}$.

4. " $a^{\frac{8}{15}}b^{\frac{2}{3}}c^{\frac{4}{3}}$, $4x^{\frac{7}{15}}y^{\frac{2}{3}}z^{\frac{1}{3}}$, and $\frac{\frac{1}{2}a^{\frac{1}{3}}x^{\frac{2}{5}}}{\sqrt[3]{a^5c^2}}$, to the fifth power,
$= a^2b^3c^6\sqrt[3]{a^2bc^2}$, $1024x^2y^3z\sqrt[3]{xyz^2}$, and $\frac{a^{\frac{1}{3}}x^2c^{\frac{2}{3}}}{32a^6c^4x^{\frac{1}{3}}}$.

5. " $3x\sqrt[3]{(x+y)}$, and $(c+d)^{\frac{1}{8}}(a+b)^{\frac{1}{4}}$, to the fourth power,
$= 81x^4(x+y)\sqrt[3]{(x+y)}$, and $(c+d)^{\frac{1}{2}}(a+b)$.

6. " $\sqrt{x} - \sqrt{y}$, and $\sqrt[3]{a} - \sqrt[3]{b}$, to the third power,
$= (x+3y)\sqrt{x} - (y+3x)\sqrt{y}$, and $a - b - 3\sqrt[3]{ab}(\sqrt[3]{a} - \sqrt[3]{b})$,
or $= x^{\frac{3}{2}} - 3xy^{\frac{1}{2}} + 3x^{\frac{1}{2}}y - y^{\frac{3}{2}}$, and $a - 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}} - b$.

7. " $\sqrt{2} + \sqrt{5}$, $4 + \sqrt{3}$, and $3 - 2\sqrt{2}$, to the second power,
$= 7 + 2\sqrt{10}$, $19 + 8\sqrt{3}$, and $17 - 12\sqrt{2}$.

8. " $\frac{\sqrt{3}-1}{2}$, $\frac{2\sqrt{2}-1}{3}$, and $y + \frac{1}{3}\sqrt{x}$, to the third power,
$= \frac{3\sqrt{3}-5}{4}$, $\frac{22\sqrt{2}-25}{27}$, and $y^3 + y^2\sqrt{x} + \frac{1}{3}xy + \frac{1}{27}x\sqrt{x}$.

CASE X.

EVOLUTION OF IRRATIONAL QUANTITIES.

1. Extract the square root of $9\sqrt{3}$, 12^3, $49\sqrt{5}$, and $3^3 \times 2^3$,
$= 3\sqrt[4]{3}$, $24\sqrt{3}$, $7\sqrt[4]{5}$, and $6\sqrt{6}$.

2. " cube " $27\sqrt{5}$, $\frac{1}{64}\sqrt{2}$, 8×5^4, and $72\sqrt{3}$,
$= 3\sqrt[6]{5}$, $\frac{1}{4}\sqrt[6]{2}$, $10\sqrt[3]{5}$, and $2\sqrt[6]{243}$.

3. Extract the square root of $16a^3$, $a^2\sqrt{\frac{a}{x}}$, and $25a^4b^2c$,

$$= 4a\sqrt{a},\ \frac{a}{x}\sqrt[4]{ax^3},\ \text{and}\ 5a^2b\sqrt{c}.$$

4. " " " $4(a-x)^2(a+x)$, and $81(a+x)^3(a-x)$,

$$= 2(a-x)\sqrt{(a+x)},\ \text{and}\ 9(a+x)\sqrt{(a^2-x^2)}.$$

5. Extract the cube root of $\frac{1}{27}\sqrt{3a}$, $a^7b^{\frac{1}{2}}$, and $\frac{1}{2}x\sqrt{\frac{1}{2}x}$,

$$= \tfrac{1}{3}\sqrt[6]{3a},\ a^2\sqrt[6]{a^2b},\ \text{and}\ \tfrac{1}{2}\sqrt{2x}\ \text{or}\ \sqrt{\tfrac{1}{2}x}.$$

6. " " " $81a^4(a+x)^5$, and $\frac{24}{125}a^7(ax^3-x^4)$,

$$= 3a(a+x)\sqrt[3]{3a(a+x)^2},\ \text{and}\ \frac{2a^2x}{5}\sqrt[3]{3a(a-x)}.$$

7. " fourth root of $\frac{16}{81}x^{\frac{2}{3}}y^2$, $a^4b^{-5}c^{\frac{2}{3}}$, and $\frac{a^2b}{(c+d)^{\frac{4}{3}}}$,

$$= \tfrac{2}{3}x^{\frac{1}{6}}y^{\frac{1}{2}},\ \frac{a}{b^2}b^{\frac{3}{4}}c^{\frac{1}{6}},\ \text{and}\ \frac{a^{\frac{1}{2}}b^{\frac{1}{4}}}{(c+d)^{\frac{1}{3}}}.$$

Here $a^4b^{-5}c^{\frac{2}{3}} = \frac{a^4c^{\frac{2}{3}}}{b^5} = \frac{a^4}{b^8} \times b^3c^{\frac{2}{3}}$, the fourth root of which is evidently the answer given above.

8. Extract the fourth root of $\sqrt{a^4x^2}$, $32a^{\frac{4}{3}}b^{\frac{8}{3}}c$, and $48a^7(a-x)^{-6}$,

$$= a^{\frac{1}{2}}x^{\frac{1}{4}},\ 2a^{\frac{1}{3}}b^{\frac{2}{3}}\sqrt[4]{2c},\ \text{and}\ \frac{2a}{(a-x)^2}\sqrt[4]{3a^3(a-x)^2}.$$

9. " mth root of $a^{\frac{3}{2}}$, $\frac{a^m}{b^3}$, and $\frac{(a+x)^{pm}}{(a-x)^{m-1}}$,

$$= a^{\frac{3}{2m}},\ \frac{a}{b}\sqrt[m]{(b^{m-3})},\ \text{and}\ \frac{(a+x)^p}{a-x}(a-x)^{\frac{1}{m}}.$$

Here $\frac{(a+x)^{pm}}{(a-x)^{m-1}} \times \frac{a-x}{a-x} = \frac{(a+x)^{pm}(a-x)}{(a-x)^m}$, the mth root of which is that given above.

CASE XI.

TO EXTRACT THE SQUARE ROOT OF A BINOMIAL SURD.

1. Extract the square root of $3 + 2\sqrt{2}$, and $4 - 2\sqrt{3}$,

$$= \sqrt{2} + 1,\ \text{and}\ \sqrt{3} - 1.$$

2. " " " $9 + 4\sqrt{5}$, and $4 + 2\sqrt{3}$,

$$= \sqrt{5} + 2,\ \text{and}\ \sqrt{3} + 1.$$

3. Extract the square root of $11 + 4\sqrt{7}$, and $15 + 6\sqrt{6}$,
$= \sqrt{7} + 2$, and $\sqrt{6} + 3$.

4. " " " $8 + 2\sqrt{15}$, and $5 - 2\sqrt{6}$,
$= \sqrt{5} + \sqrt{3}$, and $\sqrt{3} - \sqrt{2}$.

5. " " " $24 - 6\sqrt{7}$, and $20 + 10\sqrt{3}$,
$= \sqrt{21} - \sqrt{3}$, and $\sqrt{15} + \sqrt{5}$;
or $= \sqrt{3}(\sqrt{7} - 1)$, and $\sqrt{5}(\sqrt{3} + 1)$.

6. " " " $4\frac{13}{16} - \frac{3}{2}\sqrt{10}$, and $14\frac{3}{16} + 3\frac{3}{4}\sqrt{6}$,
$= \frac{3}{4}\sqrt{5} - \sqrt{2}$, and $\frac{5}{2}\sqrt{2} + \frac{3}{4}\sqrt{3}$.

IMAGINARY QUANTITIES.

CASE I.

MULTIPLICATION OF QUADRATIC IMAGINARIES.

1. Multiply $2\sqrt{-5}$ by $7\sqrt{-2}$, and $4\sqrt{-3}$ by $3\sqrt{-2}$,
$= -14\sqrt{10}$, and $-12\sqrt{6}$.

2. " $5\sqrt{-2}$ by $-3\sqrt{-7}$, and $-4\sqrt{-11}$ by $3\sqrt{-5}$,
$= 15\sqrt{14}$, and $12\sqrt{55}$.

3. " $2 + 3\sqrt{-2}$ by $5 - 7\sqrt{-3}$,
$= 10 + 15\sqrt{-2} - 14\sqrt{-3} + 21\sqrt{6}$.

4. " $a + (n + 1)\sqrt{-1}$ by $a + (n - 1)\sqrt{-1}$,
$= a^2 + 2an\sqrt{-1} - n^2 + 1$.

5. " $a + b + a\sqrt{-c}$ by $a + b - a\sqrt{-c}$,
$= a^2 + 2ab + b^2 + a^2c$.

6. Square $a + b + a\sqrt{-c}$, and $a + b - a\sqrt{-c}$,
$= a^2 + 2ab + b^2 + 2a(a + b)\sqrt{-c} - a^2c$, and $a^2 + 2ab + b^2 - 2a(a + b)\sqrt{-c} - a^2c$.

7. Square $a + (m + n)\sqrt{-b}$, and $a - (m + n)\sqrt{-b}$,
$= a^2 + 2a(m + n)\sqrt{-b} - b(m + n)^2$, and $a^2 - 2a(m + n)\sqrt{-b} - b(m + n)^2$.

8. Find the product of $3a + 2\sqrt{-c}$, and $2b - 2c\sqrt{-d}$,
$= 6ab + 4b\sqrt{-c} - 6ac\sqrt{-d} + 4c\sqrt{cd}$.

CASE II.

DIVISION OF QUADRATIC IMAGINARY QUANTITIES.

1. Divide $6\sqrt{-12}$ by $3\sqrt{-3}$, and $5\sqrt{-15}$ by $7\sqrt{-\frac{3}{5}}$, $= 4$, and $\frac{25}{7}$.

2. " $13\sqrt{-15}$ by $7\sqrt{-5}$, and $12\sqrt{-7}$ by $8\sqrt{-3}$, $= \frac{13}{7}\sqrt{3}$, and $\frac{1}{2}\sqrt{21}$.

3. " $5 + 3\sqrt{-3}$ by $3 + 2\sqrt{-2}$,

$$= \tfrac{1}{17}\{15 + 9\sqrt{-3} - 10\sqrt{-2} + 6\sqrt{6}\}.$$

Here multiply both dividend and divisor by $3 - 2\sqrt{-2}$.

4. Divide $2a\sqrt{-c}$ by $3c\sqrt{-a}$, and $m\sqrt{-n}$ by $n\sqrt{-m}$,

$$= \frac{2}{3a}\sqrt{ac}, \text{ and } \frac{1}{n}\sqrt{mn}.$$

5. " $3a + 2x\sqrt{-3}$ by $a - 3x\sqrt{-2}$,

$$= \frac{3a^2 + 2ax\sqrt{-3} + 9ax\sqrt{-2} - 6x^2\sqrt{6}}{a^2 + 18x^2}.$$

Here multiply both dividend and divisor by $a + 3x\sqrt{-2}$.

6. Divide $5a + x\sqrt{-1}$, by $2a + x\sqrt{-1}$,

$$= \frac{10a^2 + x^2 - 3ax\sqrt{-1}}{4a^2 + x^2}.$$

SIMPLE EQUATIONS.

I. EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

1. $3x + 5 = 2x + 8$, ... $\therefore x = 3$.
2. $4x - 2 = 2x + 6$, ... $\therefore x = 4$.
3. $5x - 4 = 2x + 14$, ... $\therefore x = 6$.
4. $2x - 1 = 5x - 7$, ... $\therefore x = 2$.
5. $7x - 5 = 4x + 16$, ... $\therefore x = 7$.
6. $9x + 4 = 5x + 44$, ... $\therefore x = 10$.
7. $13x - 20 = 5x + 44$, ... $\therefore x = 8$.
8. $11x + 16 = 9x + 26$, ... $\therefore x = 5$.
9. $14x - 17 = 9x + 13$, ... $\therefore x = 6$.
10. $18x + 15 = 24x - 15$, ... $\therefore x = 5$.
11. $16x - 24 = 13x - 3$, ... $\therefore x = 7$.

12. $25x + 20 = 30x + 5$, ... $\therefore x = 3$.

13. $19x - 11 = 16x + 19$, ... $\therefore x = 10$.

14. $23x - 94 = 15x + 10$, ... $\therefore x = 13$.

15. $29x - 100 = 5x + 164$, ... $\therefore x = 11$.

16. $31x - 50 = 27x + 2$, ... $\therefore x = 13$.

17. $28x + 16 = 20x + 52$, ... $\therefore x = 4\frac{1}{2}$.

18. $20x + 11 = 13x + 17$, ... $\therefore x = \frac{6}{7}$.

19. $43x - 23 = 17x + 16$, ... $\therefore x = 1\frac{1}{2}$.

20. $59x - 104 = 42x + 17$, ... $\therefore x = 7\frac{4}{17}$.

21. $\frac{3x + 7}{2} = \frac{2x + 23}{3}$, ... $\therefore x = 5$.

22. $\frac{4}{5}x - 1 = \frac{11x - 61}{7}$, ... $\therefore x = 10$.

23. $\frac{5}{7}x + 3 = \frac{5x - 2}{6}$, ... $\therefore x = 28$.

24. $\frac{2}{3}x - 4 = \frac{1}{5}x + 3$, ... $\therefore x = 15$.

25. $\frac{x + 15}{3} + 1 = x - 14$, ... $\therefore x = 30$.

26. $\frac{2x - 8}{4} = 38 + \frac{x}{6}$, ... $\therefore x = 120$.

27. $\frac{3x}{4} + 12 = \frac{2x}{3} + 14$, ... $\therefore x = 24$.

28. $x + \frac{x}{2} + \frac{x}{5} + \frac{x}{6} = 44$, ... $\therefore x = 23\frac{4}{7}$.

29. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{75x}{12} - 700$, ... $\therefore x = 112$.

30. $3x - \frac{5x}{8} - 62 = \frac{5x}{3} - 13$, ... $\therefore x = 69\frac{3}{17}$.

31. $\frac{2x - 4}{3} - \frac{10x - 4}{18} = -1$, . . . $\therefore x = 1$.

32. $\frac{x+5}{7} - \frac{x-2}{5} = \frac{x+9}{11}$, ... $\therefore x = 2$.

33. $\frac{a}{x} + \frac{b}{x} + \frac{c}{x} = d$, ... $\therefore x = \frac{a+b+c}{d}$.

34. $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{7}{3}$, ... $\therefore x = \frac{11}{14}$.

35. $\frac{ax+bx+cx}{a+b} = x + d$, ... $\therefore x = \frac{d}{c}(a+b)$.

36. $\frac{a}{mx} + \frac{b}{nx} - \frac{c}{px} = r$, . . . $\therefore x = \frac{anp + bmp - cmn}{mnpr}$.

37. $\frac{c}{d+ex} = \frac{f}{a+bx}$, ... $\therefore x = \frac{df-ac}{bc-ef}$.

38. $\frac{c}{dx-e} = \frac{f}{ax-b}$, ... $\therefore x = \frac{bc-fe}{ac-df}$.

39. $\frac{5x-2}{2} - \frac{7x-7}{10} = \frac{66-5x}{10}$, ... $\therefore x = 3$.

40. $\frac{4x}{17} - \frac{252-5x}{3} = \frac{69-x}{2}$, ... $\therefore x = 49\frac{82}{245}$.

41. $x + \frac{14-x}{3} = \frac{21-x}{2}$, ... $\therefore x = 5$.

42. $\frac{x}{2} - \frac{5x+4}{3} = \frac{4x-9}{3}$, ... $\therefore x = \frac{2}{3}$.

43. $8(2x-2) + 3(3x-8) = 7(2x-4) + 32$, $\therefore x = 4$.

44. $\frac{2x+3}{4} + \frac{x+1}{6} - \frac{x}{16} = \frac{6x+2}{3} - \frac{4x}{3}$, . $\therefore x = 4$.

45. $\frac{3x-1}{2} + \frac{3-x}{4} - \frac{2x-2}{3} + \frac{x-5}{6} = \frac{16}{3}$, $\therefore x = 7$.

46. $\frac{3x-5}{4} - \frac{7x+3}{16} = \frac{8x+19}{8} - 8\frac{5}{8}$, . $\therefore x = 7$.

47. $\frac{11x+12}{25} + \frac{19x+3}{7} + \frac{17x-17}{21} = \frac{5x-25\frac{1}{3}}{4} + 28\frac{1}{7}$,

$\therefore x = 8$.

48. $\frac{5ab}{6} + 6cx - \frac{2cx}{3} = 2ab - \frac{4ac}{5} + \frac{3ac}{4}$, $\therefore x = \frac{70ab - 3ac}{320c}$.

49. $\frac{3bx - a^2}{a} + (a - b)x = \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4} + ab^2$,

$\therefore x = \frac{4ab^2 - 10a}{4a - 3b}$.

50. $ax + \frac{bx + 4a}{4} - \frac{6bx - 5a^2}{2a} = ab^2 + bx - \frac{3bx - a^2}{a}$,

$\therefore x = \frac{2a(2b^2 - 5)}{4a - 3b}$.

51. $\frac{7x + 5}{23} + \frac{9x - 1}{10} - \frac{x - 9}{5} + \frac{2x - 8}{15} = 23$, $\therefore x = 19$.

52. $\frac{7x + 9}{8} - \frac{9x - 13}{4} - \frac{3x + 1}{7} = \frac{9x - 249}{14}$, $\therefore x = 9$.

53. $\frac{3x + 3}{16} + \frac{7x + 11}{15} - \frac{7x + 1}{20} = 3$, . . . $\therefore x = 7$.

54. $\frac{2x + 1}{29} - \frac{402 - 3x}{12} = \frac{6x - 11}{2} - 221$, $\therefore x = 72$.

55. $10(x + \frac{1}{2}) - 6x\left(\frac{1}{x} - \frac{1}{3}\right) = 23$, . . . $\therefore x = 2$.

56. $\frac{1}{6}x + \frac{1}{4}(x - 8) + \frac{11}{2} = \frac{1}{2}(x + 5)$, . . $\therefore x = 12$.

57. $\frac{1}{14}(9x + 2) - \frac{1}{2}(15x - 18) = \frac{1}{7}(12x - 20)$, $\therefore x = 1\frac{2}{5}$.

58. $\frac{1}{4}(4 + \frac{3}{2}x) - \frac{1}{7}(2x - \frac{1}{3}) - 1\frac{3}{28} = 0$, . $\therefore x = \frac{2}{3}$.

59. $\frac{1}{6}(\frac{2}{3}x + 4) + \frac{2x - 15}{18} = \frac{x}{6}\left(\frac{6}{x} - 1\right)$, . $\therefore x = 3$.

60. $1\frac{2}{3} \times \left\{30 - \left(\frac{x}{8} + 26\right)\right\} = 1\frac{3}{4} \times \left\{2\frac{2}{3} + \frac{x}{6}\right\}$, $\therefore x = 4$.

WHEN THE UNKNOWN QUANTITY IS UNDER A RADICAL SIGN.

1. $\sqrt{(x + 4)} = 8$, ... $\therefore x = 60$.

2. $\sqrt{(3x + 4)} = 5$, ... $\therefore x = 7$.

3. $\sqrt{(2x + 15)} = 7$, ... $\therefore x = 17$.

4. $5 + \sqrt{(7x + 8)} = 13$, ... $\therefore x = 8$.

5. $7 + \sqrt{(4x + 5)} = 12$, ... $\therefore x = 5$.

6. $9 + 2\sqrt{(5x + 4)} = 23$, ... $\therefore x = 9$.

7. $3 + 5\sqrt{(x + 4)} = 28$, ... $\therefore x = 21$.

8. $3\sqrt[3]{(4x + 7)} = 9$, ... $\therefore x = 5$.

9. $7\sqrt{(\frac{2}{3}x - 6)} = 14$, ... $\therefore x = 15$.

10. $4\sqrt[3]{(5x - 9)} = 36$, ... $\therefore x = 147\frac{3}{5}$.

11. $\sqrt{(ax - bx)} = c$, ... $\therefore x = \frac{c^2}{a - b}$.

12. $a + x - \sqrt{(a^2 + x^2)} = b$, . $\therefore x = \frac{b}{2}\left\{2 + \frac{b}{a - b}\right\}$.

13. $\sqrt{(4a + x)} = 2\sqrt{(b + x)} - \sqrt{x}$, . $\therefore x = \frac{(a - b)^2}{2a - b}$.

14. $\sqrt{x} + \sqrt{a - \sqrt{(ax + x^2)}} = \sqrt{a}$, . . . $\therefore x = \frac{9a}{16}$.

15. $2\sqrt{(a + x)} + \sqrt{(a - x)} = \sqrt{a - x + \sqrt{(ax + x^2)}}$,

$$\therefore x = \frac{64a}{1025}.$$

Here square the equation as it stands; cancel from both sides $a - x$; then divide both sides by $\sqrt{(a + x)}$, and it will be reduced to the following $4\sqrt{(a + x)} = \sqrt{x} - 4\sqrt{(a - x)}$; which again square, and the result will reduce to $31x = -8\sqrt{(ax - x^2)}$, from which the answer can easily be found.

16. $\sqrt{(4a + x)} = 2\sqrt{(x - 2a)} - \sqrt{(a + x)}$, . . $\therefore x = \frac{17a}{8}$.

17. $\sqrt{(a + x)} + \sqrt{(a - x)} = b\sqrt[4]{(a^2 - x^2)}$,

$$\therefore x = \frac{ab}{b^2 - 2}\sqrt{(b^2 - 4)}.$$

Squaring, we have $a + x + a - x + 2\sqrt{(a^2 - x^2)} = b^2\sqrt{(a^2 - x^2)}$; hence, $(b^2 - 2)\sqrt{(a^2 - x^2)} = 2a$.

From which $x^2 = \frac{a^2(b^2 - 2)^2 - 4a^2}{(b^2 - 2)^2} = \frac{a^2b^2}{(b^2 - 2)^2}(b^2 - 4)$, from which the answer is evident.

18. $\sqrt{(1+x+x^2)}+\sqrt{(1-x+x^2)}=ax, \quad \therefore x=\frac{2}{a}\sqrt{\frac{(a^2-1)}{(a^2-4)}}.$

Squaring and collecting the terms, we obtain

$$2\sqrt{(1+x^2+x^4)}=(a^2-2)x^2-2.$$

Again, squaring, &c., $\quad a^2(a^2-4)x^2=4(a^2-1);$

hence, $$x^2=\frac{4(a^2-1)}{a^2(a^2-4)}; \quad \therefore x=\frac{2}{a}\left(\frac{a^2-1}{a^2-4}\right)^{\frac{1}{2}}.$$

19. $\frac{n+x+\sqrt{(2nx+x^2)}}{n+x-\sqrt{(2n\ \ +x^2)}}=m^2, \quad . \quad \therefore x=\frac{n}{2m}(m-1)^2.$

For m^2 write its equal $\frac{m^2}{1}$; then subtracting and adding the denominators and numerators on both sides, we obtain

$$\frac{\sqrt{(2nx+x^2)}}{n+x}=\frac{m^2-1}{m^2+1}.$$

Squaring, it becomes

$$\frac{2nx+x^2}{n^2+2nx+x^2}=\frac{(m^2-1)^2}{(m^2+1)^2},$$

or $$1-\frac{n^2}{(n+x)^2}=\frac{(m^2-1)^2}{(m^2+1)^2};$$

$$\therefore \frac{n^2}{(n+x)^2}=\frac{(m^2+1)^2-(m^2-1)^2}{(m^2+1)^2}=\frac{4m^2}{(m^2+1)^2}.$$

Inverting both sides, and extracting the square root, we have

$$\frac{x+n}{n}=\frac{m^2+1}{2m}, \text{ or } \frac{x}{n}=\frac{m^2-2m+1}{2m}=\frac{(m-1)^2}{2m};$$

$$\therefore x=\frac{n}{2m}(m-1)^2.$$

20. $\frac{n-\sqrt{(2nx-x^2)}}{n+\sqrt{(2nx-x^2)}}=m, \quad . \quad . \quad \therefore x=n\frac{(1-\sqrt{m})^2}{1+m}.$

After proceeding with this exercise in the same manner as the last, we will obtain

$$\frac{2nx-x^2}{n^2}=\frac{(m-1)^2}{(m+1)^2}.$$

Subtract both sides from 1, and reduce them to the form of a fraction; then $\frac{n^2-2nx+x^2}{n^2}=\frac{4m}{(m+1)^2}$, from which the answer can easily be obtained.

21. $\sqrt{\{a^2x^2 + b\sqrt{(abx + 4a^2x^2 + b\sqrt{2abx + 9a^2x^2})}\}} = ax + b,$

$$\therefore x = -\frac{b}{4a}.$$

Squaring both sides, cancelling the common term a^2x^2, and then dividing both sides by b, we have

$$\sqrt{(abx + 4a^2x^2 + b\sqrt{2ab + 9a^2x^2})} = 2ax + b.$$

Again, squaring both sides, cancelling the common term $4a^2x^2$, transposing abx, and then dividing both sides by b, it becomes

$$\sqrt{(2abx + 9a^2x^2)} = 3ax + b;$$

hence, $2abx + 9a^2x^2 = 9a^2x^2 + 6abx + b^2,$

$$\therefore 4abx = -b^2; \text{ whence, } x = -\frac{b}{4a}.$$

PROBLEMS PRODUCING SIMPLE EQUATIONS.

1. What number is that which being multiplied by 8, and 21 added to the product, the sum will be 93? . . . = 9.

2. Find the number to which 86 being added, the sum will be 5 times the number, and 14 more, ... = 18.

3. Divide 36 into two such parts that the greater shall exceed the less by 10, ... = 23 and 13.

4. What number is that which being increased by 6, and also multiplied by 6, the product shall be 4 times the sum? = 12.

5. What number is that which being increased by its third and fourth parts, the sum will be the excess of 62 above the number? $x + \frac{1}{3}x + \frac{1}{4}x = 62 - x$, ... = 24.

6. £1200 is to be divided between A, B, and C; A is to have £60 more than B, and B £30 more than C; required the share of each, . . . A = £450, B = £390, and C = £360.

7. A's age is double of B's, C's is triple of B's, and the sum of all their ages is 150; find the age of each,
A's age = 50, B's = 25, and C's = 75.

8. My watch and chain are together worth £35, and the watch is worth six times the chain; find the value of each,
Watch worth £30, and chain £5.

9. There is a person whose age is now just double that of his son; but twenty years ago, his age was three times that of his son; required the age of each, . The father's age = 80; the son's = 40.

10. At an election, 2143 persons voted, and the successful candidate had a majority of 193; what number voted for each?

For the successful, 1168; for the unsuccessful, 975.

11. From two towns, which are 187 miles apart, two travellers set out at the same time with the intention of meeting. One of them goes 8 miles, and the other 9 miles a day; in how many days will they meet? . . . . . . = 11 days.

12. A gentleman meeting 4 poor persons, distributed 5 shillings amongst them: to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first; what did he give to each? . = 6, 12, 18, and 24 pence respectively.

13. A gentleman bequeathed a legacy of £140 to three servants. A was to have twice as much as B, and B three times as much as C; what were their respective shares?

A's share = £84, B's = £42, and C's = £14.

14. Four merchants enter into a speculation, for which they subscribed £4755, of which B paid three times as much as A, C paid as much as A and B, and D paid as much as B and C; what did each pay?

Let x = the pounds that A paid, then $3x$ = the pounds that B paid, $4x$ = the pounds that C paid, and $7x$ = the pounds that D paid; $\therefore x + 3x + 4x + 7x = 4755$; hence, $x = 317$, $3x = 951$, $4x = 1268$, and $7x = 2219$.

15. A draper bought three pieces of cloth, which together measured 144 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second; what was the length of each?

The first = 30 yds., second = 45 yds., and third = 69 yds.

16. A person employed 4 workmen: to the first of which he gave 2 shillings more than to the second; to the second, 3 shillings more than to the third; and to the third, 4 shillings more than to the fourth. Their wages amounted to 32 shillings; what did each receive?

They received 12, 10, 7, and 3 shillings respectively.

17. A sum of money was to be divided among 6 poor persons: the second received 10d., the third 14d., the fourth 25d., the fifth 28d., and the sixth 33d. less than the first. Now, the whole sum distributed was 10d. more than the treble of what the first received; what did each receive?

They received 40, 30, 26, 15, 12, and 7 pence respectively, and the sum distributed was 10 shillings and 10 pence.

18. A farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93, and finds just twice as many remaining in the one as in the other; how many did each flock originally contain? . . = 147.

19. Bought 24 yards of cloth for £21, 8s. For part of it I paid 19 shillings a yard, and for the rest 17 shillings a yard; how many yards of each were bought?

Let $x =$ the number of yards at 19 shillings, then $24 - x$ = the yards at 17 shillings; hence, $19x =$ the price of the cloth at 19 shillings per yard, and $(24 - x) \times 17 =$ the price of the cloth at 17 shillings per yard, both being expressed in shillings; their sum will therefore be the whole price in shillings $= 428$; hence,

$$19x + 17(24 - x) = 428;$$

whence, $2x = 20$, and $x = 10$; $\therefore 24 - x = 14$.

Therefore there were 10 yards at 19 shillings, and 14 yards at 17 shillings.

20. Divide the number 197 into two such parts, that four times the greater may exceed five times the less by 50.

Let $x =$ the greater, and $197 - x =$ the less, then

$$4x - 5(197 - x) = 50; \therefore x = 115, \text{ and } 197 - x = 82.$$

21. What number is that whose third part exceeds its fourth part 16?

Let $12x =$ the number, then $4x =$ its third part, and $3x =$ its fourth part; hence, $4x - 3x = 16$, or $x = 16$;

$$\therefore \text{the number } 12x = 12 \times 16 = 192.$$

22. Divide the number 68 into two such parts, that 84 diminished by the greater, may be equal to three times 40 diminished by the less.

Let $x =$ the less, then $68 - x =$ the greater;

$\therefore 84 - (68 - x) = 3(40 - x)$; hence, $16 + x = 120 - 3x$, and $4x = 104$; $\therefore x = 26$, and $68 - x = 42$.

23. A courier, who travels 50 miles a day, had been despatched 5 days, when a second was sent to overtake him, in order to do which he must go 75 miles a day; in what time will the second overtake the first?

Let $x =$ the number of days the second courier travels, then $x + 5 =$ the number the first travels; $\therefore 75x =$ the miles the second travels, and $50(x + 5) =$ the miles the first travels.

But when the second overtakes the first, they will have travelled over the same distance;

$$\therefore 75x = 50(x + 5), \text{ and } \therefore x = 10 \text{ days}.$$

24. A gentleman bequeathed £210 to two servants: to one he left half as much as to the other; what did he leave to each?

Let $2x$ = what the first received, then x = what the second received;

$$\therefore 2x + x = 3x = 210;$$

hence, $x = 70$, and $2x = 140$.

25. A prize of £864 was divided between two persons, A and B, whose shares therein were in the proportion of 5 to 7; what was the share of each?

Let $5x$ = A's share, then $7x$ = B's share; and

$$\therefore 5x + 7x = 864; \text{ hence, } x = 72.$$

Therefore A's share $5x = 360$, and B's share $7x = 504$.

26. A sum of money is to be shared between A and B, in such a manner that as often as A gets £10, B shall get £7: now, A received £30 more than B; find the sum shared, and the share of each.

Let $10x$ = A's share, then $7x$ = B's share; hence, by the question, $10x - 7x = 3x = 30$; $\therefore x = 10$.

Therefore A's share $= 10x =$ £100, B's share $= 7x =$ £70, and the whole sum shared $= 17x =$ £170.

27. A sum of money was divided between two persons, A and B: A's share exceeded five-ninths of the whole by £50, and A's share was to B's as 5 to 3; what was the share of each?

Let $5x$ = A's share, then $3x$ = B's share, and the whole sum $= 8x$; hence, $5x = \frac{5}{9}8x + 50$,

or $x = \frac{8}{9}x + 10$; $\therefore x = 90$.

And A's share $= 5x =$ £450, B's share $= 3x =$ £270.

28. A's money is to B's as 9 is to 5, and A's money exceeds B's by £120; find the money of each.

If A's money be represented by $9x$, then B's will be $5x$, and their difference will be $4x$;

$$\therefore 4x = 120; \text{ hence, } x = £30.$$

Therefore A's $= 9x =$ £270, and B's $= 5x =$ £150.

29. A bankrupt owed to two creditors £560: the difference of the debts was to the less as 4 to 5; what were the debts?

Let $4x$ = the difference of the debts, then $5x$ = the less, and $9x$ = the greater;

$$\therefore 5x + 9x = 14x = 560; \text{ hence, } x = £40.$$

Therefore the debts are £200 and £360.

30. In a mixture of wine and brandy, half of the whole + 15 gallons was wine, and one-third of the whole − 3 gallons was brandy; how many gallons were there of each?

Let $6x$ = the number of gallons in all, then $3x + 15$ = the number of gallons of wine, and $2x - 3$ = the number of gallons of brandy;

$$\therefore 6x = 3x + 15 + 2x - 3;$$

hence, transposing, $x = 12$.

Therefore $3x + 15 = 51$ gallons of wine, and $2x - 3 = 21$ gallons of brandy.

31. From each of 16 equal coins an artist filed the worth of half-a-crown, and then offered them in payment for their original value; but being detected, the pieces were found to be really worth no more than 12 guineas in all; what was their original value?

Let x = the number of sixpences each was originally worth;

$\therefore x - 5$ = the sixpences each was worth after filing;

$\therefore 16(x - 5) = 504$, the sixpences in 12 guineas;

hence, $x = 36\frac{1}{2}$ = 18 shillings and 3 pence.

32. A and B make a joint-stock of £870, which, after a successful speculation, produced a clear gain of £174. Of this, A received £36 more than B; what did each person contribute to the stock?

Let x = the sum A contributed to the stock,

then $870 : x :: 174 :$ A's gain $= \frac{1}{5}x$;

$\therefore$ B's gain $= \frac{1}{5}x - 36$,

and $\frac{1}{5}x + \frac{1}{5}x - 36 = 174$.

From which x = A's stock = £525,

and B's stock = 870 − 525 = £345.

33. A has £800, and B £580; what sum must B give to A in order that A may be twice as rich as B?

Let x = the sum sought, then

$$800 + x = 2(580 - x), \text{ or } 3x = 360;$$

$$\therefore x = £120.$$

34. A has £600, and B £460; if A increase his capital by £4 per month, and B increase his by £1 per month, in how many months will A be twice as rich as B?

Let x = the months sought, then A will increase his capital by $4x$ pounds, and B will increase his by x pounds;

$$\therefore 600 + 4x = 2(460 + x).$$

From which x = 160 months, the time sought.

35. A gentleman gave in charity £46, a part of which he distributed in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now, a man and a woman had between them £8? what was given to a man, and what to a woman?

Let $x =$ the sum given to each woman, then $8 - x =$ the sum given to each man;

$$\therefore 7x + 5(8 - x) = 46 \text{ by the question.}$$

Therefore $x = £3$, a woman's share; and $8 - x = £5$, a man's share.

36. Divide the number 49 into two such parts, that the greater increased by 6 may be to the less diminished by 5 as 7 to 3.

Let $x =$ the less, then $49 - x =$ the greater; hence, the greater increased by $6 = 55 - x$, and the less diminished by $5 = x - 5$;

$$\therefore 55 - x : x - 5 :: 7 : 3;$$

hence, $7x - 35 = 165 - 3x$, or $10x = 200$.

Therefore $x = 20$, the less; and $49 - x = 29$, the greater.

37. A, B, and C make a joint-stock: A puts in £60 less than B, and £68 more than C; and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4; what did each put in?

Let $x =$ what A put in, then $x + 60 =$ what B put in, and $x - 68 =$ what C put in; then $2x + 60 : 2x - 8 :: 5 : 4$, by the question, from which $x = 140$.

Therefore they put in £140, £200, and £72 respectively.

38. The hold of a ship contained 442 gallons of water. This was emptied by two buckets, the greater of which, holding twice as much as the other, was emptied twice in three minutes, and the less three times in two minutes, and the whole time of emptying was 12 minutes; required the size of each.

Let $x =$ the gallons the less held, then $2x =$ the gallons the greater held; then, since the less was emptied three times in 2 minutes, it would be emptied 18 times in 12 minutes, $\therefore 18x =$ the gallons emptied by the less; and the greater being emptied twice in 3 minutes, would be emptied 8 times in 12 minutes, $\therefore 8 \times 2x = 16x =$ the gallons emptied by the greater; hence, by the question,

$$(18x + 16x) = 34x = 442; \therefore x = 13.$$

Hence, the less held 13 gallons, and the greater 26 gallons.

39. A besieged garrison had such a quantity of bread as would, if distributed to each man at 10 ounces a day, last 6 weeks; but

having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 8 weeks; required the number of men at first in the garrison.

Let $x =$ the number of men at first in the garrison, then $7 \times 6 \times 10x =$ the ounces of bread; also $x - 1200 =$ the men remaining after the sally, and $7 \times 8 \times 12(x - 1200) =$ the ounces of bread. But the quantity of bread was the same in both cases;

$$\therefore 96(x - 1200) = 60x,$$

or

$$8x - 9600 = 5x;$$

$$\therefore 3x = 9600, \text{ and } x = 3200.$$

40. Divide the number 198 into 5 such parts, that the first increased by 1, the second increased by 2, the third diminished by 3, the fourth multiplied by 4, and the fifth divided by 5, may be all equal.

Suppose that $x =$ the common result obtained by performing the operations mentioned in the question, then the first will evidently $= x - 1$, the second $x - 2$, the third $x + 3$, the fourth $\frac{x}{4}$, and the fifth $5x$;

$$\therefore x - 1 + x - 2 + x + 3 + \frac{x}{4} + 5x = 198,$$

or

$$8\tfrac{1}{4}x = 198; \therefore x = 24.$$

Therefore the numbers are 23, 22, 27, 6, and 120.

II. SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. $x + 3y = 9, \quad 3x + 2y = 13, \quad \therefore x = 3, y = 2.$

2. $3x + y = 24, \quad 4x + 5y = 65, \quad \therefore x = 5, y = 9.$

3. $x + y = 10, \quad 2x - 3y = 5, \quad \therefore x = 7, y = 3.$

4. $3x + 7y = 99, \quad 11x - 5y = 87, \quad \therefore x = 12, y = 9.$

5. $3x - 5y = 0, \quad 2x + 7y = 124, \quad \therefore x = 20, y = 12.$

6. $4x + 9y = 46, \quad 8x - 13y = 30, \quad \therefore x = 7, y = 2.$

7. $9x - 4y = 8, \quad 13x + 7y = 101, \quad \therefore x = 4, y = 7.$

8. $45x - 8y = 500, \quad 13x + 21y = 261, \quad \therefore x = 12, y = 5.$

9. $\frac{1}{2}x + \frac{1}{3}y = 18, \quad \frac{1}{3}x - \frac{1}{2}y = -1, \quad \therefore x = 24, y = 18.$

10. $\frac{1}{8}x + \frac{1}{9}y = 26, \quad \frac{1}{9}x + \frac{1}{8}y = 25, \quad \therefore x = 144, y = 72.$

11. $\dfrac{x+11}{10}+\dfrac{y-4}{6}=x-7,\ \dfrac{x+5}{7}-\dfrac{y-7}{3}=3y-x,$
$\therefore x=9,\ y=4.$

12. $\dfrac{x+2}{7}+\dfrac{y-x}{4}=2x-8,\ \dfrac{2y-3x}{3}+2y=3x+4,$
$\therefore x=5,\ y=9.$

13. $10-\dfrac{x}{2}=\dfrac{y}{3}+4,\ \dfrac{x-y}{2}+\dfrac{x}{4}=\dfrac{3y-x}{5}+1,$
$\therefore x=8,\ y=6.$

14. $\dfrac{3x+4y}{5}+\dfrac{x}{4}+2=12,\ \dfrac{6x-2y}{3}=14-\dfrac{y}{6},$
$\therefore x=8,\ y=4.$

15. $ax=by,\ x+y=c,$ . . $\therefore x=\dfrac{bc}{a+b},\ y=\dfrac{ac}{a+b}.$

16. $ax+by=m,\ a'x+b'y=m',$
$\therefore x=\dfrac{bm'-b'm}{ba'-b'a},\ y=\dfrac{am'-a'm}{ab'-a'b}.$

17. $\dfrac{x}{m}+\dfrac{y}{n}=a,\ \dfrac{x}{n}+\dfrac{y}{m}=b,$
$\therefore x=\dfrac{mn(bm-an)}{m^2-n^2}$, and $y=\dfrac{mn(am-bn)}{m^2-n^2}.$

18. $ax-by=a^2,\ bx-ay=b^2,$
$\therefore x=\dfrac{a^2+ab+b^2}{a+b}$, and $y=\dfrac{ab}{a+b}.$

19. $\dfrac{2x+y}{9}+\dfrac{7y+6x+11}{18}=\dfrac{19}{2}-\dfrac{5x-17}{6},\ \dfrac{3}{7}\{5x+3y+2\}$
$=\dfrac{1}{2}\{9y+6\},$ . . . . . . $\therefore x=7,\ y=4.$

20. $\left.\begin{array}{l} c(bx+ay)=axy\ \ (1), \\ c(ax-by)=bxy\ \ (2), \end{array}\right\}$ find x and y.

Divide each of the equations by cxy, then they become

$$\frac{b}{y}+\frac{a}{x}=\frac{a}{c}\ \dots\ (1),\ \text{and}\ \frac{a}{y}-\frac{b}{x}=\frac{b}{c}\ \dots\ (2).$$

Again, multiply (1) by a, and (2) by b, and they become

$$\frac{ab}{y}+\frac{a^2}{x}=\frac{a^2}{c}\ \dots\ (1),\ \text{and}\ \frac{ab}{y}-\frac{b^2}{x}=\frac{b^2}{c}\ \dots\ (2).$$

Subtracting, $\dfrac{a^2+b^2}{x}=\dfrac{a^2-b^2}{c};\ \therefore x=\dfrac{a^2+b^2}{a^2-b^2}\cdot c.$

In the same manner, by multiplying (1) by b, and (2) by a, and adding, we find that $y=\dfrac{a^2+b^2}{2ab}\cdot c.$

III. SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

1. $x + y + z = 12,\ x - y + z = 6,\ x - y - z = -2,$
$\therefore x = 5,\ y = 3,$ and $z = 4.$

2. $2x + 3y + 4z = 61,\ 3x + 2y + z = 54,\ 5x - 2y + 3z = 58,$
$\therefore x = 12,\ y = 7,$ and $z = 4.$

3. $4x - 3y + 2z = 28,\ 3x + 2y - 5z = 16,\ 2x + y - 3z = 10,$
$\therefore x = 10,\ y = 8,$ and $z = 6.$

4. $2x + 7y - 11z = 10,\ 5x - 10y + 3z = -15,\ -6x + 12y - z = 31,$
$\therefore x = 8,\ y = 7,$ and $z = 5.$

5. $x + y + z = 29,\ x + 2y + 3z = 62,\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 10,$
$\therefore x = 8,\ y = 9,$ and $z = 12.$

6. $x + 2y + 3z = 52,\ x + 3y + 4z = 70,\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 6,$
$\therefore x = 6,\ y = 8,$ and $z = 10.$

7. $$\frac{4x + 3y + z}{10} - \frac{y + 2z - x + 1}{15} = 5 + \frac{x - z - 5}{5},$$

$$\frac{9x + 5y - 2z}{12} - \frac{2x + y - 3z}{4} = \frac{7y + z + 3}{11} + \tfrac{1}{6},$$

$$\frac{5y + 3z}{4} - \frac{2x + 3y - z}{12} + 2z = y - 1 + \frac{3x + 2y + 7}{6}.$$

Here multiply the first equation by 30, the least common multiple of the denominators 5, 10, and 15; then

Multiply the second by 132, the least common multiple of the denominators 12, 4, 11, and 6; and lastly,

Multiply the third by 12, the least common multiple of the denominators 4, 12, and 6.

Having performed these multiplications, transpose the resulting equations, and they will be reduced to the following three:—

$$\left.\begin{aligned} 8x + 5y + 5z &= 122, \\ 33x - 62y + 65z &= 58, \\ \text{and}\quad 8x + 4y - 34z &= -2. \end{aligned}\right\} \text{From which} \left\{\begin{aligned} x &= 9, \\ y &= 7, \\ z &= 3. \end{aligned}\right.$$

8. $xy^2z^3 = a,\ x^2y^3z = b,\ x^3yz^2 = c.$

To find x, divide the cube of the third equation by the second, and the square of the third by the first, which gives

$$\frac{c^3}{b} = \frac{x^9y^3z^6}{x^2y^3z} = x^7z^5, \text{ and } \frac{c^2}{a} = \frac{x^6y^2z^4}{xy^2z^3} = x^5z.$$

Again, divide the fifth power of the latter result by the former, and we have

$$\frac{x^{25}z^5}{x^7z^5} = \frac{c^{10}}{a^5} \times \frac{b}{c^3};\ \therefore x^{18} = \frac{c^7b}{a^5};$$

hence,
$$x = \sqrt[18]{\frac{bc^7}{a^5}}.$$

To find y, divide the square of the second equation by the third, and then the cube of the second by the first, which gives

$$\frac{b^2}{c} = \frac{x^4y^6z^2}{x^3yz^2} = xy^5, \text{ and } \frac{b^3}{a} = \frac{x^6y^9z^3}{xy^2z^3} = x^5y^7.$$

Again, raising the first of these results to the fifth power, and dividing it by the second, gives

$$\frac{x^5y^{25}}{x^5y^7} = \frac{b^{10}}{c^5} \times \frac{a}{b^3};\ \therefore y^{18} = \frac{ab^7}{c^5};$$

hence,
$$y = \sqrt[18]{\frac{ab^7}{c^5}}.$$

To find z, divide the cube of the first by the third, and the square of the first by the second, which will give

$$\frac{a^3}{c} = y^5z^7, \text{ and } \frac{a^2}{b} = yz^5.$$

Again, divide the fifth power of the latter result by the former, and we have

$$\frac{y^5z^{25}}{y^5z^7} = \frac{a^{10}}{b^5} \times \frac{c}{a^3} = \frac{a^7c}{b^5},$$

$$\therefore z^{18} = \frac{a^7c}{b^5};\ \text{hence, } z = \sqrt[18]{\frac{a^7c}{b^5}}.$$

Therefore $x = \sqrt[18]{\frac{bc^7}{a^5}},\ y = \sqrt[18]{\frac{ab^7}{c^5}}$, and $z = \sqrt[18]{\frac{a^7c}{b^5}}$.

9. $xyz = 231, xyv = 420, yzv = 1540, xzv = 660.$

To solve this exercise, multiply the four equations together, which will give

$$x^3y^3z^3v^3 = 2^6 \times 3^3 \times 5^3 \times 7^3 \times 11^3.$$

Extracting the cube root,

$$xyzv = 4 \times 3 \times 5 \times 7 \times 11 = 4620,$$

which, being divided successively by the third, fourth, first, and second equations, gives

$$x = 3, y = 7, v = 20, \text{ and } z = 11.$$

QUESTIONS PRODUCING EQUATIONS WITH TWO OR MORE UNKNOWN QUANTITIES.

1. Find two numbers whose sum is 50, and whose difference is 16, ... = 33 and 17.

2. Find two numbers such that the greater is to the less as their sum is to 10, and as their difference is to 2, = 9 and 6.

3. Find a fraction such that if its numerator be increased by 3, the value will be $\frac{1}{2}$; and if its denominator be diminished by 3, the value will be $\frac{1}{5}$.

Here $\frac{x+3}{y} = \frac{1}{2}$, and $\frac{x}{y-3} = \frac{1}{5}$; $\therefore$ the fraction is $= \frac{1}{8}$.

4. There are two numbers such that if twice the less be added to the greater, the sum will be 18; and if three times the greater be diminished by the less, the remainder will be 19; find the numbers.

Here $x + 2y = 18$, and $3x - y = 19$; $\therefore x = 8$, and $y = 5$.

5. Find the fraction, to the numerator of which if one be added it shall $= \frac{1}{5}$; but if one be added to the denominator, it shall $= \frac{1}{10}$.

Here $\frac{x+1}{y} = \frac{1}{5}$, and $\frac{x}{y+1} = \frac{1}{10}$; $\therefore$ the fraction $= \frac{10}{99}$.

6. Two pieces of cloth, measuring together 57 yards, were sold for £36; the first was valued at 15s. per yard, the second at 10s.; what were the number of yards in each piece?

Here $x + y = 57$, and $15x + 10y = 720$ shillings;

$\therefore x = 30$, and $y = 27$.

7. The ages of two persons are such, that if to the sum of their ages 22 be added, the sum will be double the age of the elder; and

if 1 be taken from the difference of their ages, the remainder will be the age of the younger; what are their ages?

Let $x =$ the age of the elder, and $y =$ the age of the younger; then $x + y + 22 = 2x$, and $x - y - 1 = y$;

hence, $x = 43,\ y = 21.$

8. The sum of two numbers $= 12$, and the difference of their squares $= 72$; what are those numbers?

Here $x + y = 12$, and $x^2 - y^2 = 72$; divide the second equation by the first, and the quotient is $x - y = 6$;

$$\therefore x = 9, \text{ and } y = 3.$$

9. Find two numbers whose sum is to their difference as 3 is to 2, and whose difference is to their product as 1 is to 5.

Let $x =$ the greater, and $y =$ the less; then $x + y : x - y :: 3 : 2$, and $x - y : xy :: 1 : 5$; from which $x = 20$, and $y = 4$.

10. There are two kinds of gunpowder, one worth 1s. per lb., the other worth 1s. 3d. per lb.; how many pounds of each must be taken, so that a hundredweight of the mixture may be worth £6, 10s.?

Let $x =$ the number of pounds at 1s. per lb., and $y =$ the pounds at 1s. 3d.; then $x + y = 112$, and $x + 1\frac{1}{4}y = 130$;

$$\therefore x = 40, \text{ and } y = 72.$$

11. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it; how many will it hold of each?

Let $x =$ the number of crowns it will hold, and $y =$ the guineas;

then $x : 4 :: 1 :$ the space occupied by 4 crowns $= \dfrac{4}{x}$,

and $y : 5 :: 1 :$ " " " 5 guineas $= \dfrac{5}{y}$;

hence, $\dfrac{4}{x} + \dfrac{5}{y} = \frac{17}{63}$, or multiplying by 6, $\dfrac{24}{x} + \dfrac{30}{y} = \frac{102}{63}$,

and $\dfrac{19}{x} + \dfrac{6}{y} = 1$, " " 5, $\dfrac{95}{x} + \dfrac{30}{y} = \frac{315}{63}$.

From the latter equations, we easily find $x = 21$, and $y = 63$.

12. There is a number expressed by two figures, the sum of whose digits is $= 13$; and if 27 be subtracted from the number, the digits will be inverted; what is that number?

Let $x =$ the digit in the place of tens, and $y =$ the unit's digit; then $x + y = 13$, and $10x + y - 27 = 10y + x$;

$$\therefore \text{the number} = 85.$$

13. Divide £200 among three persons, A, B, and C, so that twice A's share + £80, three times B's share + £30, and four times C's share + £40, may be all equal to one another.

Let $x =$ A's share, $y =$ B's share, and $z =$ C's share; then $x + y + z = 200$, and $2x + 80 = 3y + 30 = 4z + 40$;

$$\therefore x = £80,\ y = £70, \text{ and } z = £50.$$

14. A's money, together with twice that of B and C, amounts to £1050; B's, together with thrice that of A and C, amounts to £1400; and C's, together with four times that of A and B, amounts to £1650; how much money had each?

The equations are $x + 2y + 2z = 1050$, $3x + y + 3z = 1400$, and $4x + 4y + z = 1650$;

$$\therefore x = 150,\ y = 200,\ z = 250.$$

15. Find three numbers such that the sum of the first and second shall $= a$, the sum of the first and third shall $= b$, and the sum of the second and third shall $= c$.

Let $x =$ the first, $y =$ the second, and $z =$ the third; then $x + y = a$, $x + z = b$, and $y + z = c$, and half their sum gives $x + y + z = \frac{1}{2}(a + b + c)$, from which, subtracting each of the original equations, we obtain $x = \frac{1}{2}(a + b - c)$, $y = \frac{1}{2}(a + c - b)$, and $z = \frac{1}{2}(b + c - a)$.

16. Find three quantities such that the product of the first and second, divided by the third, shall $= a$; the product of the second and third, divided by the first, shall $= b$; and the product of the third and first, divided by the second, shall $= c$.

Here $xyz^{-1} = a$ (1), $x^{-1}yz = b$ (2), and $xy^{-1}z = c$ (3);

(1) × (2), gives $y^2 = ab$; $\therefore y = \sqrt{ab}$;

(2) × (3), " $z^2 = bc$; $\therefore z = \sqrt{bc}$;

and (3) × (1), " $x^2 = ac$; $\therefore x = \sqrt{ac}$.

17. Find three numbers such that the product of the first and second, divided by their sum, shall $= \frac{12}{5}$; the product of the second and third, divided by their sum, shall $= \frac{18}{5}$; and the product of the first and third, divided by their sum, shall $= \frac{36}{13}$.

Here since the product of the first and second, divided by their sum, $= \frac{12}{5}$; their sum, divided by their product, $= \frac{5}{12} = \frac{15}{36}$; and performing the division on the first side, we obtain

$$\frac{1}{x} + \frac{1}{y} = \frac{15}{36}.$$

Similarly, $$\frac{1}{y} + \frac{1}{z} = \frac{10}{36},$$

and $$\frac{1}{x} + \frac{1}{z} = \frac{13}{36},$$

from which $x = 4$, $y = 6$, and $z = 9$.

18. A farmer mixes barley at 2s. 4d. a bushel, with rye at 3s. a bushel, and wheat at 4s. a bushel, so that the whole is 100 bushels, and worth 3s. 4d. a bushel. Had he added as much rye as before, and 10 bushels of wheat to the former quantity, the whole would have been worth exactly the same per bushel as before; how much of each kind of grain was there?

Let $x =$ the bushels of barley, $y =$ those of rye, and $z =$ those of wheat; then express the price of each in fourpences, and we have the following equations:—

$$x + y + z = 100,\ 7x + 9y + 12z = 1000,$$
$$9y + 12 \times 10 = 10(y + 10);$$

whence, $x = 28,\ y = 20,\ z = 52.$

QUADRATIC EQUATIONS.

1. $x^2 + 7 = 88,$ . . . . . . $\therefore x = \pm\ 9.$
2. $x^2 - 15 = 10,$ . . . . . . $\therefore x = \pm\ 5.$
3. $x^2 + 9 = 58,$ . . . . . . $\therefore x = \pm\ 7.$
4. $x^2 - 90 = 31,$ . . . . . . $\therefore x = \pm\ 11.$
5. $x^2 + 7x = 7(x + 3) + 4,$ . . . . $\therefore x = \pm\ 5.$
6. $x(x + 4) = 4x + 16,$ . . . . $\therefore x = \pm\ 4.$
7. $9x^2 + 4x = 4(x + \frac{1}{4}),$ . . . . . $\therefore x = \pm\ \frac{1}{3}.$
8. $4x^2 + 6x = 2(3x + 8),$ . . . . $\therefore x = \pm\ 2.$
9. $x^2 + 6x = 40,$ . . . . . $\therefore x = 4, \text{ or } - 10.$
10. $x^2 - 8x = 84,$ . . . . . $\therefore x = 14, \text{ or } - 6.$
11. $3x^2 - 12x = 96,$ . . . . . $\therefore x = 8, \text{ or } - 4.$
12. $5x^2 = 20x + 25,$ . . . . $\therefore x = 5, \text{ or } - 1.$
13. $7x^2 + 28x = 224,$ . . . . . $\therefore x = - 8, \text{ or } 4.$
14. $x^2 + 7x = 12\frac{3}{4},$ . . . . $\therefore x = 1\frac{1}{2}, \text{ or } - 8\frac{1}{2}.$
15. $x^2 - 8x + 12 = 0,$ . . . . $\therefore x = 2, \text{ or } 6.$
16. $x^2 - 20x + 96 = 0,$ . . . $\therefore x = 12, \text{ or } 8.$
17. $x^2 - 2x - 120 = 0,$ . . . . $\therefore x = 12, \text{ or } - 10.$
18. $x^2 + 3x - 54 = 0,$ . . . . $\therefore x = 9, \text{ or } - 6.$

19. $x^2 - 14x - 120 = 0$, . . . $\therefore x = 20$, or -6.

20. $x^2 - 8x = 180$, . . . $\therefore x = 18$, or -10.

21. $x(x-5) = 204$, . . . $\therefore x = 17$, or 12.

22. $x(x+3) = 180$, . . . $\therefore x = 12$, or -15.

23. $x(x+4) = 192$, . . . $\therefore x = 12$, or -16.

24. $x(x-5) = 500$, . . . $\therefore x = 25$, or -20.

25. $\dfrac{x(x+6)}{7} = 13$, ... $\therefore x = 7$, or -13.

26. $\dfrac{x(x-5)}{12} = 8\frac{2}{3}$, . . . $\therefore x = 13$, or -8.

27. $x^2 - 7x + \frac{13}{2} = \frac{1}{2}$, . . . $\therefore x = 6$, or 1.

28. $x^2 + x - 40 = 170$, . . . $\therefore x = 14$, or -15.

29. $6x + \dfrac{35 - 3x}{x} = 44$, ... $\therefore x = 7$, or $\frac{5}{6}$.

30. $x + \dfrac{24}{x-1} = 3x - 4$, . . $\therefore x = 5$, or -2.

31. $\dfrac{80}{x+4} + 1 = \dfrac{80}{x}$, ... $\therefore x = 16$, or -20.

32. $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = \frac{7}{3}$, . . $\therefore x = 3$, or $-\frac{4}{5}$.

33. $\dfrac{4x}{9} + \dfrac{x-5}{x+3} = \dfrac{4x+7}{19}$, ... $\therefore x = 3$, or $-8{\cdot}7$.

34. $\dfrac{3}{x+5} - \dfrac{4}{x-3} + \frac{1}{18} = 0$, . $\therefore x = 8 \pm \sqrt{601}$.

35. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \frac{13}{6}$, ... $\therefore x = 2$, or -3.

36. $\dfrac{x+1}{x-1} - \dfrac{x-1}{x+1} = 1$, . . $\therefore x = 2 \pm \sqrt{5}$.

37. $\dfrac{1}{x-1} - \dfrac{1}{x+3} = \frac{1}{35}$, . . . $\therefore x = 11$, or -13.

38. $\dfrac{12}{5-x} + \dfrac{8}{4-x} = \dfrac{32}{x+2}$, . $\therefore x = 2$, or $4\frac{6}{13}$.

39. $\frac{5x+4}{5x-4}+\frac{5x-4}{5x+4}=2\frac{1}{6}$, . . . $\therefore x=\pm 4$.

40. $\frac{18}{5-x}+\frac{12}{4-x}=\frac{21}{x+2}$, . . . $\therefore x=4$, or $-4\frac{1}{3}$.

41. $x+2-\frac{6}{x+2}=1$, ... $\therefore x=1$, or -4.

42. $x^2+x+1=\frac{42}{x^2+x}$, . . . $\therefore x=2$, or -3.

NOTE.—In 41, for $x+2$ write z, and the equations become, after clearing from fractions and transposing,

$$z^2-z=6, \text{ from which } z=3, \text{ or } -2;$$

$$\therefore x+2=3, \text{ or } -2, \text{ and } x=1, \text{ or } -4.$$

In 42, let $x^2+x=z$, and the equation becomes $z+1=\frac{42}{z}$; hence, $z^2+z=42$, from which $z=6$, therefore, also, $x^2+x=6$; $\therefore x=2$, or -3.

43. $\frac{(1+x)^3}{1+x^3}=\frac{16}{7}$, ... $\therefore x=3$, or $\frac{1}{3}$.

44. $\frac{(x-1)^3}{x^3-1}=\frac{16}{31}$, ... $\therefore x=5$, or $\frac{1}{5}$.

In 43, divide the numerator and denominator of the first side by $1+x$, and it becomes $\frac{1+2x+x^2}{1-x+x^2}=\frac{16}{7}$; then clearing from fractions and reducing, it can be solved in the common way.

Also, in 44, divide the numerator and denominator of the first side by $x-1$, and there results the equation $\frac{x^2-2x+1}{x^2+x+1}=\frac{16}{31}$, which being cleared from fractions can be solved as before.

45. $x^6-19x^3=216$, ... $\therefore x=3$, or -2.

46. $3x^6+42x^3=3321$, ... $\therefore x=3$, or $\sqrt[3]{-41}$.

In 45 and 46, observe that the given equations are quadratics, in which x^3 is the unknown quantity; hence, solve the equations for x^3, and then extract the cube root of each of the resulting values for the values of x.

47. $\sqrt{(x+12)} + \sqrt[4]{(x+12)} = 6$, . . $\therefore x = 4$, or 69.

Let the $\sqrt[4]{(x+12)} = \pm z$, then $\sqrt{(x+12)} = z^2$, and $x + 12 = z^4$, and the equation becomes $z^2 + z = 6$, from which the values of x given above can easily be found.

48. $x^2 + 11 + \sqrt{(x^2+11)} = 42$, $\therefore x = \pm 5$, or $\pm \sqrt{38}$.

Let $\sqrt{(x^2+11)} = \pm z$, then $x^2 + 11 = z^2$; hence the equation becomes $z^2 + z = 42$; whence, $z = 6$, or -7, and therefore $x^2 + 11 = 36$, or 49.

49. $x + 4 + \left(\frac{x+4}{x-4}\right)^{\frac{1}{2}} = \frac{12}{x-4}$, $\therefore x = \pm 5$, or $\pm 4\sqrt{2}$.

Multiplied by $(x-4)$, becomes $x^2 - 16 + (x^2 - 16)^{\frac{1}{2}} = 12$; and the 50, by subtracting 9 from both sides, and transposing, becomes

$$(x^2 - 9) - (x^2 - 9)^{\frac{1}{2}} = 12.$$

50. $x^2 = 21 + (x^2 - 9)^{\frac{1}{2}}$, . . $\therefore x = \pm 5$, or $\pm 3\sqrt{2}$.

QUADRATIC EQUATIONS WITH TWO UNKNOWN QUANTITIES.

1. $x + y = 12$, $x^2 + y^2 = 74$, $\therefore x = 7$ or 5, $y = 5$ or 7.

2. $x - y = 4$, $x^2 + y^2 = 106$,
$\therefore x = 9$ or -5, $y = 5$ or -9.

3. $x - 3y = 2$, $x^2 - 5y^2 = 76$,
$\therefore x = 11$ or -16, $y = 3$ or -6.

4. $3x + 2y = 19$, $9x^2 + 5y^2 = 245$,
$\therefore x = 2\frac{1}{27}$ or 5, $y = 6\frac{4}{9}$ or 2.

5. $x - y = 15$, $\frac{xy}{2} = y^2$, $\therefore x = 19\frac{3}{4}$ or $14\frac{1}{4}$, $y = 4\frac{3}{4}$ or $-\frac{3}{4}$.

6. $x + 2y = 7$, $x^2 + 3xy - y^2 = 23$,
$\therefore x = 3$ or $15\frac{2}{3}$, $y = 2$ or $-4\frac{1}{3}$.

7. $2x - 3y = 1$, $2x^2 + xy - 5y^2 = 20$,
$\therefore x = 5$ or $-9\frac{1}{4}$, $y = 3$ or $-6\frac{1}{2}$.

8. $x^2y^2 + 3xy = 88$, $x + y = 6$, $\therefore x = 4$ or 2, $y = 2$ or 4.

9. $x^2 + y^2 = 13$, $x + y = 5$, $\therefore x = 3$ or 2, $y = 2$ or 3.

10. $x + y = a$, $x^2 + y^2 = b$,
$\therefore x = \frac{1}{2}\{a \pm \sqrt{(2b - a^2)}\}$, $y = \frac{1}{2}\{a \mp \sqrt{(2b - a^2)}\}$.

11. $x^2 + xy = 66$, $xy - y^2 = 5$,
$\therefore x = \pm 6$ or $\pm \frac{11}{2}\sqrt{2}$, $y = \pm 5$ or $\pm \frac{1}{2}\sqrt{2}$.

12. $x^2 - xy - y^2 = 295$, $x^2 + y^2 = 370$,
$\therefore x = \pm 19$ or $\pm 7\sqrt{5}$, $y = \pm 3$ or $\pm 5\sqrt{5}$.

In the 11th and 12th exercises, for y put vx; then divide the first equation by the second, and x^2 will divide out of numerator and denominator; from the resulting equation find the value of v. Substitute this value of v in the equation which is least involved, and find the value of x, then $y = vx$ will give y.

In 11, $v = \frac{5}{6}$ or $\frac{1}{11}$; and in 12, $v = \frac{3}{19}$ or $\frac{5}{7}$.

13. $x^2 + y^2 = 34$, $x^2 - xy = 10$,
$\therefore x = \pm 5$ or $\pm \sqrt{2}$, $y = \pm 3$ or $\mp 4\sqrt{2}$.

14. $9x^2 = 4y^2$, $3xy + 2x + y = 485$,
$\therefore x = 10\frac{1}{9}$ or $-10\frac{8}{9}$, $y = 15\frac{1}{6}$ or $-16\frac{1}{3}$.

15. $\dfrac{x^2 + y}{x^2 - y} = \frac{5}{4}$, $4x + 5y = 17$,
$\therefore x = 3$ or $-10\frac{1}{5}$, $y = 1$ or $11\frac{14}{25}$.

16. $\dfrac{x + y}{x - y} = \frac{13}{5}$, $y^2 + x = 25$,
$\therefore x = 9$ or $-14\frac{1}{16}$, $y = 4$ or $-6\frac{1}{4}$.

17. $x^2 + y^2 = 58$, $xy = 21$, . . . $\therefore x = 7$, $y = 3$.

18. $x^2 - y^2 = 135$, $\dfrac{x}{y} = 4$, . . . $\therefore x = 12$, $y = 3$.

19. $x^4 + y^4 = 97$, $x + y = 5$, . $\therefore x = 3$ or 2, $y = 2$ or 3.

Let $x = u + v$, and $y = u - v$; then the second equation gives $u = \frac{5}{2}$, and the first becomes

$$2u^4 + 12u^2v^2 + 2v^4 = 97.$$

Again, substituting the value of u, and reducing their results,

$$(4v^2)^2 + 150(4v^2) = 151; \ \therefore v = \pm \tfrac{1}{2}.$$

20. $x^4 - y^4 = 369$, $x^2 - y^2 = 9$, . . $\therefore x = 5$, $y = 4$.

Dividing the first equation by the second,

$$x^2 + y^2 = 41.$$

21. $x^4 + y^4 = 1 + 2xy + 3x^2y^2$, $x^3 - y^3 = 3x^2y - 3xy^2 + 1$,

$$\therefore x = 2,\ y = 1.$$

Here, in the first, transpose $2x^2y^2$, and it becomes

$$x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2 = (xy + 1)^2;$$

and the second, by transposing, becomes

$$x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3 = 1;\ \therefore x - y = 1.$$

22. $(x^2 - xy + y^2)(x^2 + y^2) = 221$, $(x^2 - xy + y^2)(x^2 + xy + y^2)$ $= 273$, . . . $\therefore x = \pm 4, \pm 1, \pm 4\sqrt{-1}, \pm \sqrt{-1}$;

$$y = \pm 1, \pm 4, \pm \sqrt{-1}, \pm 4\sqrt{-1}.$$

Here the second equation, minus the first, gives

$$(x^2 - xy + y^2)xy = 52;$$

which, being subtracted from the first, we have

$$(x^2 - xy + y^2)^2 = 169;$$

whence, $$x^2 - xy + y^2 = \pm 13.$$

Dividing each of the given equations, and also their difference by this root, the quotients are

$$x^2 + y^2 = \pm 17,\ x^2 + xy + y^2 = \pm 21, \text{ and } xy = \pm 4,$$

from which the values given above can easily be found.

23. $x^3 - y^3 = 63$, $x - y = 3$, $\therefore x = 4$ or -1, $y = 1$ or -4.

The cube of the second minus the first, gives

$$3xy(x - y) = 36;$$

whence, $xy = 4$, then $(x - y)^2 + 4xy = (x + y)^2 = 25$;

$$\therefore x + y = \pm 5.$$

24. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4$, $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 28$, $\therefore x = 9$ or 1, $y = 1$ or 9.

The cube of the first minus the second, gives

$$3x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 36;$$

whence, $x^{\frac{1}{2}}y^{\frac{1}{2}} = 3$; then $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2 - 4x^{\frac{1}{2}}y^{\frac{1}{2}} = (x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 = 4$;

$$\therefore x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 2.$$

25. $x + 3(x + y)^{\frac{1}{2}} = 18 - y$, $x^2 - y^2 = 9$,

$$\therefore x = 5 \text{ or } 18\tfrac{1}{8},\ y = 4 \text{ or } 17\tfrac{7}{8}.$$

26. $x^2 + y^2 - (x + y) = 18$, $x + y = 19 - xy$,

$$\therefore x = 4 \text{ or } 3, \text{ or } -4 \pm \sqrt{-11};$$

$$y = 3 \text{ or } 4, \text{ or } -4 \mp \sqrt{-11}.$$

In 26, add twice the second equation to the first, then by transposition the result may easily be reduced to the form

$$(x+y)^2+(x+y)=56, \text{ from which } x+y=7, \text{ or } -8;$$

whence, $y=(7-x)$, or $-(8+x)$.

Substituting successively these values of y in the second equation, the values of x will be obtained.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. What is that number, from the square of which, if we take seven times the number, the remainder will be 44?

$$x^2-7x=44;\ \therefore x=11, \text{ or } -4.$$

2. Find a number such that if its third part increased by one be multiplied by its half, the product will be 30.

$$\left(\frac{x}{3}+1\right)\times\frac{x}{2}=30;\ \therefore x=12, \text{ or } -15.$$

3. The difference of two numbers is 13, and their product exceeds 7 times the less by 135; find the numbers.

$$x(x+13)=7x+135, \text{ or } x^2+6x=135;$$

$$\therefore x=9, \text{ or } -15.$$

The numbers are therefore 9 and 22.

4. In a court there are two square grass-plots, a side of one of which is 10 yards longer than the side of the other, and their areas are as 25 to 9; what is the lengths of their sides?

$$(x+10)^2:x^2::25:9, \text{ or } 4x^2-45x=225;$$

$$\therefore x=15, \text{ or } -3\tfrac{3}{4}.$$

5. A merchant sold a quantity of brandy for £39, and gained as much per cent. as the brandy cost him; what was the price of the brandy?

Let $x =$ the price of the brandy;

then $100:x::x:\frac{x^2}{100}=$ the gain $=$ £39 $-x$,

and $\therefore \frac{x^2}{100}=39-x$, or $x^2+100x=3900$;

whence, $x=$ £30.

6. It is required to find two numbers, the first of which may be

to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225.

Let $x =$ the first, then the second will be $\sqrt{16x}$; and by the second condition,

$$x^2 + 16x = 225;$$

whence, $x = 9$, and $\sqrt{16x} = 12$.

7. A draper bought some pieces of cloth for 180 crowns; had he for the same sum received three pieces more, each piece would have cost 3 crowns less; how many pieces did he buy, and what was the price of a piece?

Let $x =$ the number of pieces, then $\frac{180}{x} =$ the price of a piece in crowns;

$$\therefore \frac{180}{x} - 3 = \frac{180}{x+3}, \text{ or } x^2 + 3x = 180;$$

whence, $x = 12$, and $\frac{180}{x} = 15$ crowns.

8. A gentleman bought a horse for a certain number of pounds; and having sold it for £119, gained as much per cent. as the horse cost him; what was paid for the horse? See question 5 for the notation.

$$\therefore \frac{x^2}{100} = 119 - x, \text{ or } x^2 + 100x = 11900;$$

whence, $x =$ £70, the price paid.

9. The sum of two numbers is 16, and the quotient of the greater divided by the less, is to the quotient of the less divided by the greater as 25 is to 9; find the numbers.

Let $x =$ the less, then $16 - x =$ the greater; and

$$\therefore \frac{16-x}{x} : \frac{x}{16-x} :: 25 : 9; \text{ whence, } x^2 + 18x = 144;$$

hence, $x = 6$, and $16 - x = 10$.

10. Two flocks of sheep, one containing 5 sheep more than the other, were sold for £106, 5s. Each sheep cost as many shillings as there were sheep in the flock; required the number of sheep in each flock.

Let $x =$ the sheep in the least flock, then $x + 5 =$ the sheep in the other; also, $x^2 =$ the shillings the least flock cost, and $(x+5)^2 = x^2 + 10x + 25 =$ the shillings the other cost;

$$\therefore x^2 + x^2 + 10x + 25 = 2125;$$

hence, $x = 30$, and $x + 5 = 35$.

11. A tailor bought a piece of cloth for £73, 10s., from which he cut off 12 yards, and sold the remainder for £64, 15s., gaining

5s. per yard; how many yards were there in the piece, and what did he pay for a yard?

Let $x =$ the number of yards in the piece, then $\frac{1470}{x} =$ the shillings he paid for a yard; also, $\frac{1295}{x-12} =$ the shillings he received for a yard;

$$\therefore \frac{1470}{x} + 5 = \frac{1295}{x-12}, \text{ or } x^2 + 23x = 3528;$$

$$\therefore x = 49 \text{ yards, and the price} = 30s. \text{ per yard.}$$

12. What number is that which being added to its square root, the sum will be 552?

Let $x =$ the number sought, then $x + \sqrt{x} = 552$;

whence, $x = 529$.

13. A grazier bought as many sheep as cost him £50, out of which he reserved 15, and sold the remainder for £54, by which he gained 2s. a head on those sold; how many sheep did he buy, and what did he pay for each?

Let $x =$ the number of sheep bought, then $x - 15 =$ the number sold, and $\frac{60}{x} =$ the price of each in pounds;

$$\therefore (x - 15)\left(\frac{60}{x} + \tfrac{1}{10}\right) = 54;$$

whence, $x = 75$, the sheep bought,

and $\frac{60}{x} = \frac{60}{75} = £\frac{4}{5} = 16s.$

14. There is a field in the form of a rectangular parallelogram, whose length exceeds its breadth by 20 yards, and its area is 6300 yards; required its length and breadth.

Let $x =$ the breadth, and $x + 20 =$ the length;

then $x(x + 20) = 6300$;

$$\therefore x = 70, \text{ and } x + 20 = 90 \text{ yards.}$$

15. A person being asked his age, answered, if to half my age you add the square root of my age, and from the sum subtract 12, the remainder will be the third part of my age; required his age.

Let $x =$ the age sought; then

$$\frac{x}{2} + \sqrt{x} - 12 = \frac{x}{3}, \text{ or } x + 6\sqrt{x} = 72; \ \therefore x = 36.$$

16. What number is that, the sum of whose third and fourth parts is less by 2 than the square of its sixth part?

Here $\frac{x}{3} + \frac{x}{4} + 2 = \left(\frac{x}{6}\right)^2$; whence, $x^2 - 21x = 72$,

and $\therefore x = 24$.

17. The base of a right-angled triangle is less than the perpendicular by 3, and less than the hypotenuse by 6; required the sides.

Here $(x+6)^2 = x^2 + (x+3)^2$; whence the sides are 9, 12, and 15.

18. What two numbers are those whose difference is 3, and whose sum multiplied by the greater $= 405$?

Let $x =$ the less, then $x + 3 =$ the greater, and their sum $= 2x + 3$; whence, $(2x+3)(x+3) = 405$, or $2x^2 + 9x + 9 = 405$; and $\therefore x = 12$, and $x + 3 = 15$.

19. There are two square courts paved with stones, each a foot square. The side of one court exceeds that of the other by 10 feet, and the two pavements together required 3092 stones; find the length of the sides of the courts.

Let $x =$ the length of the less, then $x + 10 =$ that of the greater; and $x^2 + (x+10)^2 = 3092$; and

$$\therefore x = 34, \text{ and } x + 10 = 44.$$

20. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but on the enemy coming in sight, the front was increased by 845 men, and the whole was thus drawn up in 5 lines; find the number of men.

Let $x =$ the number in front at first, then $x + 5 =$ the depth, and the whole $= x(x+5)$.

Again, in the second position, the men in front $= x + 845$, and the men in the depth $= 5$, and the whole $= 5(x + 845)$; therefore $x(x+5) = 5(x+845)$; whence, $x = 65$.

Whence the whole men $= x(x+5) = 65 \times 70 = 4550$ men.

21. A farmer purchased a number of oxen for £112, and observed that if he had had one more for the same money, each of them would have cost him £2 less; required the number he purchased, and the price of each.

Let $x =$ the number of oxen purchased;

$$\therefore \frac{112}{x} = \text{the price of each in pounds.}$$

Also, by the second condition, $\dfrac{112}{x+1} = \dfrac{112}{x} - 2$;

$$\therefore 112x = 112x + 112 - 2x^2 - 2x, \text{ or } x^2 + x = 56.$$

Hence, $x = 7$, and $\dfrac{112}{x} =$ £16, the price of each.

22. A farmer purchased a number of oxen for £112, and observed that if he had purchased one fewer for the same money,

each of them would have cost him £2 more; required the number he purchased, and the price of each.

Let $x =$ the number of oxen purchased; then, reasoning as in the last question, we obtain

$$\frac{112}{x-1} = \frac{112}{x} + 2, \text{ or } x^2 - x = 56;$$

hence, $x = 8$, and $\frac{112}{x} =$ £14, the price of each.

23. A person buys 18 ells of cloth, part blue and part green; for each lot he gave 40s., and he pays for every yard of the blue cloth 1s. per ell more than for the green; how many ells of each kind were there?

Let $x =$ the ells of blue, then $18 - x =$ the ells of green;

$$\text{also, } \frac{40}{x} = \frac{40}{18-x} + 1, \text{ or } x^2 - 98x = -720.$$

Therefore $x = 8$, and $18 - x = 10$ ells.

24. Find a number, to the quadruple of which, if 80 be added, the sum will be to the square of the number as 2 to 9.

Here $4x + 80 : x^2 :: 2 : 9$, or $x^2 - 18x = 360$; and

$$\therefore x = 30.$$

25. The length of a rectangle exceeds its breadth by 12, and the sum of the squares of the length and breadth is 20880; what are the sides of the rectangle, and what is its area?

Let $x =$ the breadth, then $x + 12 =$ the length; and by the question,

$$x^2 + (x + 12)^2 = 20880, \text{ or } x^2 + 12x = 10368.$$

$$\therefore x = 96,\ x + 12 = 108, \text{ and } x(x + 12) = 10368 \text{ area.}$$

26. Two travellers set out to meet each other from two towns, A and B, which are 120 miles distant from each other; the first goes 6 miles a day, and the other 1 mile a day more than the number of days in which they meet; in how many days will they meet?

Let $x =$ the days sought, then $x + 1 =$ the miles the second walked per day;

$$\therefore 6x + x(x + 1) = 120, \text{ or } x^2 + 7x = 120;$$

$\therefore x = 8$, the number of days sought.

27. A had 40 yards of silk, and B 90, which they sold together

for £42. Now, A sold for £1 a third of a yard more than B did; how many yards did each sell for £1?

Let $x =$ the yards B sold for £1, then $x + \frac{1}{3} =$ what A sold for £1.

Also, $\frac{90}{x} =$ the pounds B received, and $\frac{40}{x + \frac{1}{3}} =$ what A received; but the sum received was £42.

$$\therefore \frac{90}{x} + \frac{40}{x + \frac{1}{3}} = 42, \text{ or } 21x^2 - 58x = 15; \text{ and}$$

$$\therefore x = 3.$$

Therefore B sold 3 yards, and A $3\frac{1}{3}$ for £1.

28. Two detachments of foot are ordered to a station distant 39 miles: they begin their march at the same time; but one party, by travelling one-fourth of a mile an hour more than the other, arrives one hour sooner: required the rates of marching.

Let $x =$ the slower rate, then $x + \frac{1}{4} =$ the quicker;

hence, $\frac{39}{x} = \frac{39}{x + \frac{1}{4}} + 1$, or $x^2 + \frac{1}{4}x = \frac{39}{4}$.

Therefore their rates of walking were 3 and $3\frac{1}{4}$ miles per hour.

29. Divide each of the numbers 21 and 30 into two parts, so that the first part of 21 may be three times as great as the first part of 30, and that the sum of the squares of the remaining parts may be 585.

Let $x =$ the first part of 30, then $3x =$ the first part of 21; hence, by the question,

$$(21 - 3x)^2 + (30 - x)^2 = 585, \text{ or } 5x^2 - 93x = -378.$$

Whence the parts are = 18 and 3, 6 and 24.

30. It is required to divide each of the numbers 11 and 17 into two parts, so that the product of the first parts of each may be 45, and the product of the second parts 48.

Let $x =$ the first part of 11, and $y =$ the first part of 17; then their second parts will be $11 - x$, and $17 - y$; hence, by the question,

$$xy = 45, \text{ and } (11 - x)(17 - y) = 48.$$

The difference of these equations is $187 - 11y - 17x = 3$, from which $y = \frac{184 - 17x}{11}$.

This value of y, being substituted in the first equation, gives

$$17x^2 - 184x = -495.$$

$\therefore x = 5$, and hence $y = 9$; and the numbers are 5, 6, and 9, 8.

31. There are two numbers whose product is 45, and the difference of their squares is to the square of their difference as 7 is to 2; what are the numbers?

Let $x =$ the greater, and $y =$ the less; then $x^2 - y^2 : (x - y)^2 :: 7 : 2$, and $xy = 45$; from which is found $x = 9$, and $y = 5$.

32. A and B engage in partnership with a capital of £100: A leaves his money in the partnership for 3 months, and B for 2 months, and each takes out £99 of capital and profit; determine the original contribution of each.

Let $x =$ A's capital, then $100 - x =$ B's; also, let $y =$ the rate of gain per pound per month; then, by the question,

$$x + 3xy = 100 - x + 2y(100 - x),$$

from which $$y = \frac{100 - 2x}{5x - 200}.$$

Again, by the question, $x + 3xy = 99$, in which, substituting the above value of y, and reducing, we obtain,

$$x^2 + 395x = 19800; \text{ and } \therefore x = 45.$$

Hence, A's capital = £45, and B's = £55.

33. There are two numbers such that if the first be increased by 2, and the second diminished by 3, their product will be 110: but if the first be diminished by 3, and the second increased by 2, their product will be 80; what are these two numbers?

Let $x =$ the first, and $y =$ the second; then, by the question,

$$(x + 2)(y - 3) = 110, \text{ and } (x - 3)(y + 2) = 80.$$

From the difference of these equations is found $y = x + 6$, which value, being substituted in either equation, gives

$$x^2 + 5x = 104; \text{ whence, } x = 8, \text{ and } y = 14.$$

34. Two retailers jointly invest £500 in business, to which each contributes a certain sum: the one let his money remain 5 months, the other only 2, and each received £450 capital and profit; how much did each advance at first?

Let $x =$ the sum contributed by the one whose money was in for 5 months, then £$500 - x =$ the sum the other contributed; also, let $y =$ the gain per pound per month; then $x + 5xy = 450$, and $500 - x + 2y(500 - x) = 450$;

whence, $$y = \frac{450 - x}{5x}, \text{ from the first equation;}$$

And substituting this value of y in the second equation, it becomes, after reduction,

$$x^2 + 550x = 150000; \text{ and } \therefore x = 200,$$

and $$500 - x = 300.$$

Hence the sums contributed were £200 and £300.

35. A regiment of foot was ordered to send 160 men on garrison duty, each company to furnish a like number; but before the detachment marched, three of the companies were sent on another service, when it was found that each company which remained was obliged to furnish 12 additional men, in order to make up the number 160; required the number of companies in the regiment.

Let $x =$ the number of companies in the regiment,

then $\frac{160}{x} =$ the men each company would require to send,

and $\frac{160}{x-3} =$ the men the remaining companies sent;

hence, $\frac{160}{x} + 12 = \frac{160}{x-3}$, or $x^2 - 3x = 40$;

$\therefore x = 8$, the number of companies.

36. A charitable person distributed a certain sum amongst some poor men and women, the number of the former were to the latter in the ratio of 4 to 5. Each man received one-third as many shillings as there were persons relieved, and each woman received twice as many shillings as there were women more than men. Now, the men received altogether 18 shillings more than the women; how many were there of each?

Let $4x =$ the number of men, and $5x =$ the women; then $3x =$ the shillings each man received, and $2x =$ the shillings each woman received; then, by the question, $4x \times 3x = 5x \times 2x + 18$, or $12x^2 = 10x^2 + 18$; and $\therefore x = 3$.

Hence there were 12 men, and 15 women.

37. There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the left-hand digit; but if the digits be reversed, and the number so formed be divided by a number greater by 1 than the sum of the digits, the quotient will be greater than the preceding quotient by 2; required the number.

Let $x =$ the digit in the place of tens, and $y =$ the digit in the place of units of the number sought; then $10x + y$ is the number sought, and $10y + x =$ the number when the digits are reversed; hence, by the question, we have

$$\frac{10x + y}{x + y} = x + 2, \text{ and } \frac{10y + x}{x + y + 1} = x + 4.$$

Clearing from fractions, and subtracting the first from the second, we find

$$7y - 12x = 4; \text{ and } \therefore y = \tfrac{1}{7}(12x + 4).$$

Substituting this value for y in the first equation, and reducing, we obtain

$$19x^2 - 40x = -4; \text{ and } \therefore x = 2, \text{ and } y = \tfrac{1}{7}(12x + 4) = 4.$$

Therefore the number sought is 24.

38. A cask, whose content is 30 gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water, and then the first cask filled with the mixture; and it appears, that if 10 gallons be now drawn off from the first into the second cask, there will be equal quantities of brandy in each; required the quantity of brandy first drawn off.

Let $x =$ the gallons first drawn off; then, after the second cask is filled with water, $\frac{x}{30} =$ the brandy in each gallon of the mixture; and since it would take x gallons of this mixture to fill the first cask, $\frac{x^2}{30} =$ the quantity of brandy put back.

Again, 10 gallons being taken out from 30 gallons, $\frac{2}{3}$ of the whole is left, one-half of which is brandy;

$$\therefore \tfrac{2}{3}\left(30 - x + \frac{x^2}{30}\right) = 15;$$

whence, $x^2 - 30x = -225$,

and $\therefore x = 15$, the brandy drawn off.

39. There is a number expressed by two digits, such that the product of the number by the sum of its digits is 1012; and if 63 be subtracted from the number, its digits will be inverted; what is the number?

Let $x =$ the digit in the place of tens, and y that in the place of units; then, by the first condition of the question,

$$(10x + y)(x + y) = 10x^2 + 11xy + y^2 = 1012;$$

and, by the second, $10x + y - 63 = 10y + x$, from which $y = x - 7$.

Substituting this value for y in the first equation, and reducing, it becomes

$$22x^2 - 91x = 963; \therefore x = 9.$$

Also, $y = x - 7 = 9 - 7 = 2$; hence the number is 92.

40. Find two numbers such that their product increased by their sum may be 79, and their sum taken from the sum of their squares may leave a remainder of 114.

Let x and y represent the two numbers sought; then, by the question,

$$xy + x + y = 79,$$

and $$x^2 + y^2 - (x + y) = 114;$$

adding twice the first to the second, gives

$$(x + y)^2 + (x + y) = 272; \therefore x + y = 16.$$

From this result, and the first equation, $x = 9$, and $y = 7$.

RATIOS AND PROPORTION.

1. Which is greater, $3:7$, or $5:13$? . . . The former.

2. Arrange the following ratios in their order of magnitude:—

$$m:n,\ mx:mx+c, \text{ and } mx:mx-c,$$

$$= mx:mx+c,\ m:n, \text{ and } mx:mx-c.$$

3. Find the ratio compounded of $a:m$, $m:n$, and $n:b$, $= a:b$.

4. Of the ratios $5:7$, $3:4$, and $4:9$, which is the greatest, and which is the least? $= 3:4$ is the greatest, and $4:9$ is the least.

5. What quantity must be added to each of the terms of the ratio $m:n$, that it may become the ratio of $a:b$? $= \frac{bm-an}{a-b}$.

6. Find a fourth proportional to ab, cd, and ax, . . $= \frac{cdx}{b}$.

7. What is the proportion deducible from the equation $ab = m^2 - n^2$? . . . . . $= a:m+n::m-n:b$.

8. What are the proportions deducible from the equation $y^2 = 4mx$? $= 4m:y::y:x,\ 2m:y::y:2x,\ m:y::y:4x$, &c.

9. Find a fourth proportional to 6, 4, 3, and to 5, 10, 6, $= 2$ and 12.

10. Find the number to which if 2 and 5 be successively added, the resulting sums are in the ratio of $5:11$, . . . $= \frac{1}{2}$.

11. If $a:b::c:d$, then $ma+nb:a::mc+nd:c$.

For $\frac{a}{b} = \frac{c}{d}$; $\therefore \frac{a}{nb} = \frac{c}{nd}$, and $\frac{ma}{nb} = \frac{mc}{nd}$;

hence, $$\frac{ma+nb}{nb} = \frac{mc+nd}{nd}.$$

And multiplying the two latter by the equals $\frac{nb}{a}$ and $\frac{nd}{c}$,

we have $$\frac{ma+nb}{a} = \frac{nc+nd}{c}.$$

Hence, $$ma+nb:a::mc+nd:c.$$

12. If four quantities of the same kind be proportional, prove that the sum of the greatest and least is greater than the sum of the other two.

Let the four proportional quantities be $a : b :: c : d$, of which a is the greatest, and consequently d the least; then, by conversion,

$$a : a - b :: c : c - d;$$

and, by alternation,

$$a : c :: a - b : c - d;$$

but, by hypothesis,

$$a > c; \therefore a - b > c - d.$$

Adding $b + d$ to each, we have $a + d > b + c$.

13. If $x : y :: a^4 : b^4$, and $a : b :: \sqrt[4]{(c+x)} : \sqrt[4]{(d+y)}$, shew that $cy = dx$.

Here raise the terms of the second proportion to the fourth power, then it is evident that $x : y :: c + x : d + y$; and taking the product of the extremes equal to the product of the means, and subtracting xy from both sides, there remains $cy = dx$.

14. If $a : b :: c : d$, shew that $a(c + d - a - b) = (a + b)(c - a)$.

Here $a + b : a :: c + d : c$; hence, $a(c + d) = c(a + b)$; from each of these equals take $a(a + b)$, and we have

$$a(c + d - a - b) = (a + b)(c - a).$$

15. If $a : b :: c : d$, shew that $a(a + b + c + d) = (a + c)(a + b)$.

Here proceed as in the last, only add $a(a + b)$.

16. Given $(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d)$, to prove that $a : b :: c : d$.

Here convert the two equal products into a proportion; then mixing the terms, and taking half of the results, and again mixing, and taking half the result, gives the proportion required.

17. If $a : b :: b : c$, then $a^2 - b^2 : a :: b^2 - c^2 : c$.

Here squaring the terms of the given proportion, we have then by division,

$$a^2 - b^2 : a^2 :: b^2 - c^2 : b^2; \text{ but } a^2 : b^2 :: a : c;$$

whence, by alternation and substitution, we have

$$a^2 - b^2 : a :: b^2 - c^2 : c.$$

18. A person in a railway-carriage observes that another train, running on a parallel line in the opposite direction, occupies

2 seconds in passing his train; but if the two trains had been proceeding in the same direction, they would have taken 30 seconds to pass each other; compare the speeds of the two trains.

Let $x : y =$ the ratio of the rates; then

$$x - y : x + y :: 2 : 30;$$

hence, $$x : y :: 32 : 28 :: 8 : 7.$$

19. If $a : b :: c : d$, then $a^2 + ab + b^2 : a^2 - ab + b^2 :: c^2 + cd + d^2 : c^2 - cd + d^2$.

For $a^3 : b^3 :: c^3 : d^3$, $\therefore a^3 - b^3 : a^3 + b^3 :: c^3 - d^3 : c^3 + d^3$;

and since $a : b :: c : d$, $\therefore a + b : a - b :: c + d : c - d$.

Compounding these latter proportions, and omitting the common factor $(a + b)(a - b)$ from the first and second terms, and the common factor $(c + d)(c - d)$ from the third and fourth, we obtain

$$a^2 + ab + b^2 : a^2 - ab + b^2 :: c^2 + cd + d^2 : c^2 - cd + d^2.$$

20. If $a : b :: c : d$, shew that $\dfrac{(a - b)(a - c)}{a} = (a+d) - (b+c)$; and that $\dfrac{(a-b)(a - c)}{abc} = \left(\dfrac{1}{a} + \dfrac{1}{d}\right) - \left(\dfrac{1}{b} + \dfrac{1}{c}\right)$.

For since $a : b :: c : d$, $\therefore a - b : b :: c - d : d$;

and $a : c :: b : d$, $\therefore a - c : c :: b - d : d$.

By compounding $(a - b)(a - c) : bc :: (c - d)(b - d) : d^2$; but $bc = ad$; hence, substituting and dividing d out of the second and fourth terms,

$$(a - b)(a - c) : a :: (c - d)(b - d) : d;$$

$$\therefore \frac{(a - b)(a - c)}{a} = \frac{(c - d)(b - d)}{d} = \left(\frac{c}{d} - 1\right)(b - d)$$

$$= \left(\frac{bc}{d} - b - c + d\right) = (a + d) - (b + c), \text{ since } a = \frac{bc}{d}.$$

Again, dividing both sides by bc, we obtain the second equation.

EQUIDIFFERENT PROGRESSION, OR ARITHMETICAL PROGRESSION.

N.B.—Let a = the first term of an Equidifferent Progression, d = the common difference, n = the number of terms, l = the last term, and s = the sum of the series; then $l = a + (n-1)d$, and $s = \{2a + (n-1)d\}\frac{n}{2}$; and from these two equations, when any three of the five elements are given, the other two can be determined. The contraction E.P. stands for Equidifferent Progression, and E. D. for Equidifferent.

1.	Find the	6th	term of the E.P.		2,	5,	8, &c.,	= 17.
2.	"	9th	"	"	3,	7,	11, &c.,	= 35.
3.	"	11th	"	"	10,	$11\frac{1}{2}$,	13, &c.,	= 25.
4.	"	15th	"	"	7,	12,	17, &c.,	= 77.
5.	"	18th	"	"	2,	7,	12, &c.,	= 87.
6.	"	24th	"	"	2,	5,	8, &c.,	= 71.
7.	"	31st	"	"	12,	$13\frac{1}{2}$,	15, &c.,	= 57.
8.	"	7th	"	"	24,	22,	20, &c.,	= 12.
9.	"	11th	"	"	5,	$4\frac{1}{2}$,	4, &c.,	= 0.
10.	"	17th	"	"	100,	96,	92, &c.,	= 36.
11.	"	23d	"	"	25,	23,	21, &c.,	= − 19.
12.	"	30th	"	"	25,	22,	19, &c.,	= − 62.
13.	"	41st	"	"	92,	89,	86, &c.,	= − 28.

14. Find the 81st term of the E.P. 7, 10, 13, &c., $= 247$.

15. " 90th " " 12, $11\frac{1}{3}$, $10\frac{2}{3}$, &c., $= -47\frac{1}{3}$.

16. " 365th " " 1, 3, 5, 7, &c., $= 729$.

17. " 181st " " 7, $7\frac{1}{3}$, $7\frac{2}{3}$, &c., $= 67$.

18. " 53d " " 24, $22\frac{2}{3}$, $21\frac{1}{3}$, &c., $= -45\frac{1}{3}$.

19. Sum the E.P. $3 + 5 + 7 +$ &c., to 10 terms, $= 120$.

20. " " $1 + 5 + 9 +$ &c., to 13 " , $= 325$.

21. " " $5 + 8 + 11 +$ &c., to 19 " , $= 608$.

22. " " $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} +$ &c., to 15 " , $= 112\frac{1}{2}$.

23. " " $\frac{1}{4} + \frac{3}{8} + \frac{1}{2} +$ &c., to 12 " , $= 11\frac{1}{4}$.

24. " " $2\frac{1}{2} + 2\frac{5}{6} + 3\frac{1}{6} +$ &c., to 60 " , $= 740$.

25. " " $40 + 37 + 34 +$ &c., to 30 " , $= -105$.

26. " " $-3 - 1 + 1 +$ &c., to 20 " , $= 320$.

27. " " $-17 - 12 - 7 -$ &c., to 11 " , $= 88$.

28. " " $2\frac{1}{2} + 3 + 3\frac{1}{2} +$ &c., to n terms, $= \frac{n(n+9)}{4}$.

29. " " $1 + 2 + 3 +$ &c., to n terms, $= \frac{n(n+1)}{2}$.

30. " " $1 + \frac{n-1}{n} + \frac{n-2}{n} +$ &c., to n terms, $= \frac{n+1}{2}$.

31. " " $(a-x)^2 + (a^2+x^2) + (a+x)^2 +$ &c., to n terms, ... $= n\{a^2 + x^2 + (n-3)ax\}$.

32. Sum the E.P. $\frac{a+2b}{a+b} + \frac{3a+b}{a+b} + \frac{5a}{a+b} +$ &c., to n terms, $= \frac{2na - (n-5)b}{a+b} \times \frac{n}{2}$.

33. Insert 2 E.D. means between 5 and 17, $= 9$ and 13.

34. " 3 " " 20 " 48,
$= 27, 34,$ and 41.

35. " 4 " " 100 " 40,
$= 88, 76, 64,$ and 52.

36 " 7 " " $-1\frac{1}{4}$ " $4\frac{3}{4}$,
$= -\frac{1}{2}, +\frac{1}{4} + 1 + 1\frac{3}{4} + 2\frac{1}{2} + 3\frac{1}{4}$, and $+ 4$.

37. Find the common difference by which 12 E.D. means may be inserted between 7 and 72, ... $= \frac{72 - 7}{13} = 5$.

38. Find the common difference by which n E.D. means may be inserted between a and b, ... $= \frac{b - a}{n + 1}$.

39. Insert n E.D. means between 1 and 37, so that the 5th may be to the $(n - 1)^{\text{th}}$ as 1 is to 3; find n and the 5th term.

Here $d = \frac{37 - 1}{n + 1} = \frac{36}{n + 1}$; the 5th mean $= 1 + \frac{5 \times 36}{n + 1}$; and the $(n - 1)^{\text{th}}$ mean $= 1 + \frac{(n - 1) \times 36}{n + 1}$.

Therefore, by the question,

$$1 + \frac{180}{n + 1} : 1 + \frac{(n - 1) \times 36}{n + 1} :: 1 : 3,$$

or $\quad n + 181 : 37n - 35 :: 1 : 3$; and $\therefore n = 17$.

Hence, $d = \frac{36}{n + 1} = \frac{36}{18} = 2$, and the 5th mean $= 1 + 5 \times 2 = 11$.

40. The first term of an E.P. is 5, the number of terms 30, and their sum 1455; find the common difference, and the last term.

Since $s = \{2a + (n - 1)d\} \times \frac{n}{2}$, $\therefore 1455 = \{10 + 29d\} \times 15$; and hence, $d = 3$, and $l = a + (n - 1)d = 92$.

41. The sum of an E.P. is 72, the first term is 17, and the common difference is $- 2$; find the number of terms, and explain the double answer, ... $= 6$, or 12.

NOTE.—The reason of the double answer is, that the terms after the 6th up to the 12th inclusive, are $-5, -3, -1, +1, +3$, and $+5$; the sum of which is evidently 0. Therefore the sum of 6 terms is equal to the sum of 12 terms.

42. How far does a person travel in gathering up 120 stones placed in a straight line, at intervals of 4 feet from each other, supposing that he fetches each stone singly and deposits it in a basket which is in the same straight line produced, and 20 yards distant from the nearest stone, and that he starts from the basket?

Here since all the distances are travelled twice, the first term is 40 yards, and the common difference is $2\frac{2}{3}$ yards; also the number of terms is 120;

$\therefore s = \{2 \times 40 + 119 \times 2\frac{2}{3}\}\frac{120}{2} = 23840$ yards $= 13\frac{6}{11}$ miles.

EQUIRATIONAL OR GEOMETRICAL PROGRESSION.

N.B.—Let $a =$ the first term of an Equirational Progression, $r =$ the common ratio, $n =$ the number of terms, $l =$ the last term, and $s =$ the sum of the terms; then $l = ar^{n-1}$, and $s = \dfrac{a(r^n - 1)}{r - 1}$, when $r > 1$; and $s = \dfrac{a(1 - r^n)}{1 - r}$, when $r \angle 1$; and $s = \dfrac{a}{1 - r}$, when n is infinite. From these equations, when any three of the five quantities a, r, n, l, and s are given, the other two can be found. The contraction E. R. stands for Equirational, and E. R. P. for Equirational Progression.

1.	Find the	7th	term of the	E. R. P.	3, 6, 12, &c.,	$= 192$.
2.	"	5th	"	"	5, 15, 45, &c.,	$= 405$.
3.	"	6th	"	"	$2\frac{1}{4}$, $1\frac{1}{2}$, 1, &c.,	$= \frac{8}{27}$.
4.	"	8th	"	"	$\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, &c.,	$= 8$.
5.	"	7th	"	"	2, $-$ 6, 18, $-$ &c.,	$= 1458$.
6.	"	9th	"	"	-3, 6, $-$ 12, &c.,	$= -\ 768$.
7.	"	10th	"	"	27, 9, 3, &c.,	$= \frac{1}{729}$.
8.	"	12th	"	"	$1\frac{2}{5}$, $3\frac{1}{2}$, $8\frac{3}{4}$, &c.,	$= 33378\frac{1231}{2048}$.
9.	"	11th	"	"	64, $-$ 32, 16, $-$ &c.,	$= \frac{1}{16}$.

10. Sum $1 + 3 + 9 + 27 +$ &c., to 8 terms, $s = 3280$.

11. " $2 + 4 + 8 + 16 +$ &c., to 10 " , $s = 2046$.

12. " $96 + 48 + 24 + 12 +$ &c., to 9 " , $s = 191\frac{5}{8}$.

13. " $3 + 6 + 12 + 24 +$ &c., to 8 " , $s = 765$.

14. " $\frac{1}{4} + \frac{3}{4} + \frac{9}{4} + \frac{27}{4} +$ &c., to 7 " , $s = 273\frac{1}{4}$.

15. " $\frac{1}{2} + 1 + 2 + 4 +$ &c., to n " , $s = 2^{n-1} - \frac{1}{2}$.

16. " $\frac{1}{2} + 1 + 2 + 4 +$ &c., to 20 " , $s = 524287\frac{1}{2}$.

17. " $12, - 6, + 3 - \frac{3}{2} +$ &c., to 12 " , $s = 7\frac{511}{512}$.

18. " $\frac{2}{3} + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} +$ &c., to n " ,

$$s = \tfrac{8}{3} \cdot \frac{4^n - 3^n}{4^n}.$$

19. " $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} +$ &c., to n " , $s = \dfrac{3^n - 2^n}{3 \times 2^{n-2}}$.

20. " $\frac{3}{2} - \dfrac{3}{2^3} + \dfrac{3}{2^5} - \dfrac{3}{2^7} +$ &c., to n " ,

$$s = \tfrac{6}{5} \cdot \frac{2^{2n} + 1}{2^{2n}}.$$

21. " $4 + 2 + 1 + \frac{1}{2} +$ &c., to infinity, . $s = 8$.

22. " $\frac{3}{2} - \dfrac{3}{2^3} + \dfrac{3}{2^5} - \dfrac{3}{2^7} +$ &c., " , . $s = \frac{6}{5}$.

23. " $4 + 3 + \frac{9}{4} + \frac{27}{16} +$ &c., " , . $s = 16$.

24. " $\cdot2 + \cdot02 + \cdot002 +$ &c., " , . $s = \frac{2}{9}$.

25. " $\cdot27 + \cdot0027 + \cdot000027 +$ &c., " , . $s = \frac{27}{99}$.

26. " $1 + 2x + 4x^2 + 8x^3 +$ &c., " , x being less than $\frac{1}{2}$, ... $s = \dfrac{1}{1 - 2x}$.

27. Sum $a^n + ba^{n-1} + b^2a^{n-2} +$ &c., to infinity, a being greater than b, ... $s = \dfrac{a^{n+1}}{a - b}$.

28. Sum $\frac{m}{n} - \frac{(m-n)x}{n} + \frac{(m-n)x^2}{n^2} - \frac{(m-n)x^3}{n^3} + \text{\&c.}$, to infinity, n being greater than x.

In this exercise, the first term does not form a part of the E.R.P., therefore find the sum of the series after the first term, and subtract it from the first term, and the result will be the sum required;

$$\therefore s = \frac{m}{n} - \frac{\frac{(m-n)x}{n}}{1+\frac{x}{n}} = \frac{m}{n} - \frac{(m-n)x}{n+x}$$

$$= \frac{mn + mx - mnx + n^2x}{n(n+x)}.$$

29. Sum $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} + \text{\&c.}$, to infinity, a being greater than b, ... $s = \frac{a^2}{a-b}$.

30. Insert an E.R. mean between 3 and 48, . . . $= 12$.

Note.—Any number of E.R. means may be inserted between two numbers, a and b, as follows: let $n =$ the number of means to be inserted, and $r =$ the common ratio; then

$$r = \sqrt[n+1]{\frac{b}{a}}, \text{ then the means are } ar,\ ar^2,\ ar^3,\ \text{\&c.}$$

31. Insert 2 E.R. means between 5 and 135, $= 15$ and 45.

32. " 2 " " 7 " 448, $= 28$ and 112.

33. " 3 " " 3 " 48, $= 6$, 12, and 24.

34. " 3 " " 9 " 2304, $= 36$, 144, and 576.

35. " 4 " " 11 " 352, $= 22$, 44, 88, and 176.

36. " 4 " " 5 " 1215, $= 15$, 45, 135, and 405.

37. " 5 " " 7 " 448, $= 14$, 28, 56, 112, and 224.

38. In an equirational progression, consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the difference of the odd and even terms; required a proof.

Let the progression be $a + ar + ar^2 + \&c.$, then the sum of the squares of the terms will form the progression

$$a^2 + a^2r^2 + a^2r^4 + \&c.,$$

in which the first term is a^2, and the common ratio r^2;

$$\therefore s = \frac{a(r^{2n} - 1)}{r^2 - 1}.$$

Also, let s' = the sum of the given series, then $s' = \frac{a(r^n - 1)}{r - 1}$;

and s'' = the difference of the odd and even terms, which will evidently be the sum of the E.R.P. $a - ar + ar^2 - ar^3 + \&c.$, in which r is negative;

$$\therefore s'' = \frac{a(-r^n - 1)}{-r - 1} = \frac{a(r^n + 1)}{r + 1},$$

in which r^n is negative, because by the question n is odd.

We have now to prove that $s = s's''$, which is manifest, since

$$\frac{a(r^n - 1)}{r - 1} \times \frac{a(r^n + 1)}{r + 1} = \frac{a^2(r^{2n} - 1)}{r^2 - 1}.$$

39. If there be n equirational progressions, each having the same ratio r, whose first terms are a, $2a$, $3a$, &c., na, and sums s_1, s_2, s_3, &c., s_n, where each is summed to n terms; prove that $s_1 + s_2 + s_3 + \&c. + s_n = \frac{n(n + 1)}{2}\left(\frac{r^n - 1}{r - 1}\right)a.$

N.B.—Since the common ratio in each of the progressions is r, the sum of any one of them will evidently be the first term $\times \left(\frac{r^n - 1}{r - 1}\right)$; hence, the sum of all the equirational progressions will be

$$(a + 2a + 3a + 4a + \&c. + na)\left(\frac{r^n - 1}{r - 1}\right).$$

But the coefficients a, $2a$, $3a$, &c., na is evidently an equidifferent progression, whose first term is a, and common difference also a, and the number of terms is n; hence their sum is $\frac{n(n + 1)}{2}a$;

and $\therefore s_1 + s_2 + s_3 + \&c. + s_n = \frac{n(n + 1)}{2}\left(\frac{r^n - 1}{r - 1}\right)a.$

40. The sum of an equirational progression carried to infinity

is 6, and the sum of its first and second terms is $4\frac{1}{2}$; what is the series?

Let $x =$ the first term, and $r =$ the common ratio; then the sum to infinity $= \dfrac{x}{1-r} = 6$, and the sum of the first and second terms is $x + rx = 4\frac{1}{2}$.

Divide the latter equation by the former, and we obtain

$$1 - r^2 = \tfrac{3}{4};\ \therefore r^2 = \tfrac{1}{4}, \text{ and } r = \pm\tfrac{1}{2};$$

hence, $x = 3$ or 9, and the series is $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \text{\&c.}$,

or $\qquad 9 - \frac{9}{2} + \frac{9}{4} - \frac{9}{8} + \frac{9}{16} - \text{\&c.}$

HARMONIC PROGRESSION.

1. Continue to two terms each way the harmonic series 3, 4, 6,
$= 2, 2\frac{2}{5}, 3, 4, 6, 12, \infty$.

2. " " " " harmonic series $4, 5, 6\frac{2}{3}$,
$= 2\frac{6}{7}, 3\frac{1}{3}, 4, 5, 6\frac{2}{3}, 10, 20$.

3. Insert 2 harmonic means between 5 and 10, $= 6$ and $7\frac{1}{2}$.

4. " 3 " " " 2 " 8,
$= 2\frac{6}{13}, 3\frac{1}{5}, 4\frac{4}{7}$.

5. " 4 " " " 6 " 36,
$= 7\frac{1}{5}, 9, 12, 18$.

6. " 5 " " " 6 " 42,
$= 7, 8\frac{2}{5}, 10\frac{1}{2}, 14, 21$.

7. " 8 " " " 7 " 70,
$= 7\frac{7}{9}, 8\frac{3}{4}, 10, 11\frac{2}{3}, 14, 17\frac{1}{2}, 23\frac{1}{3}, 35$.

8. The sum of three numbers in harmonic progression is 11, the excess of the third above the second is 3, and the sum of their squares is 49; find the numbers, . . . $= 2, 3$, and 6.

9. Insert 6 harmonic means between 1 and 2,
$= \frac{14}{13}, \frac{7}{6}, \frac{14}{11}, \frac{7}{5}, \frac{14}{9}$, and $\frac{7}{4}$.

10. " 7 " " " 1 and 2,
$= \frac{16}{15}, \frac{8}{7}, \frac{16}{13}, \frac{4}{3}, \frac{16}{11}, \frac{8}{5}$, and $\frac{16}{9}$.

11. The sum of three numbers in harmonic progression is 11, and the sum of their squares is 49; find the numbers.

Let x = the less extreme, and y = the greater; then, since the sum of the three numbers = 11, the mean may be represented by $11 - y - x$.

Again, by the properties of harmonic progressionals, the mean is

$$\frac{2xy}{x+y}; \text{ and } \therefore \frac{2xy}{y+x} = 11 - (y+x);$$

multiplying by $y + x$, and transposing,

$$11(y+x) - y^2 - x^2 = 4xy \quad \ldots \quad (a).$$

Also, $x^2 + y^2 + (11 - y - x)^2 = 49$ by the question,

or $x^2 + y^2 + 121 + x^2 + y^2 - 22(y + x) + 2xy = 49.$

Transposing, and dividing by 2,

$$y^2 + x^2 + xy - 11(y+x) = -36 \quad \ldots \quad (b);$$

$$(a) + (b) \text{ gives } yx = 4yx - 36,$$

$$\therefore yx = 12 \quad \ldots \quad \ldots \quad (c);$$

$$(b) + (c) \text{ gives } (y+x)^2 - 11(y+x) = -24,$$

$$\therefore y + x = 8 \quad \ldots \quad \ldots \quad (d).$$

From (c) and (d) the values of x and y are easily found to be 2 and 6, and $11 - 2 - 6 = 3$.

Therefore the numbers are = 2, 3, and 6.

12. The sum of three numbers in harmonic progression is 37, and the sum of their squares is 469; find the numbers,
= 10, 12, and 15.

13. The sum of three numbers in harmonic progression is 23, and the sum of their squares is 259; find the numbers,
= 3, 5, and 15.

PROBLEMS IN EQUIDIFFERENT, EQUIRATIONAL, AND HARMONIC PROGRESSION.

1. Find three numbers in equidifferent progression such that their sum may be 21, and the third with double the first may be 19, ... = 5, 7, and 9.

2. Find three numbers in equidifferent progression such that their sum may be 15, and the third with double of the second may be 18, ... = 2, 5, and 8.

3. Find three numbers in equidifferent progression such that their sum may be 27, and that twice the third together with the second may be 35, ... = 5, 9, and 13.

4. Find three numbers in equidifferent progression such that their sum may be 24, and that twice the first with twice the second may be 40, ... = 12, 8, and 4.

5. A person bought 5 books, whose price in shillings were in equidifferent progression. The price of the second was 21s., and that of the dearest 27s.; what did he give for each, and how much for the whole?

NOTE.—This question admits of two solutions, for the dearest book might either be the first or the fifth: if the dearest was the first, then the common difference was three; but if the dearest was the fifth, then the common difference was two.

∴ { 27, 21, 15, 9, and 3, sum, £3, 15s.;
or 19, 21, 23, 25, " 27, " , £5, 15s.

6. Find the series in equidifferent progression, consisting of $(n + 7)$ terms, of which the sum of the first n terms is 40, the sum of the next 4 is 86, and that of the last 3 is 96.

From the terms of the question, the three following equations are easily obtained:—

$$\{2a + (n - 1)d\}\frac{n}{2} = 40, \text{ or } 2a + (n - 1)d = \frac{80}{n} \quad (a);$$

$$\{2a + (2n + 3)d\} \times 2 = 86, \text{ or } 2a + (2n + 3)d = 43 \,(b);$$

$$\{2a + (2n + 10)d\} \times \tfrac{3}{2} = 96, \text{ or } 2a + (2n + 10)d = 64 \,(c);$$

$(c) - (b)$ gives $7d = 21$,

and $\therefore d = 3$;

$$(b) - (a) \text{ gives } (n + 4)d = 43 - \frac{80}{n};$$

or substituting the value of d, and reducing, we have

$$3n^2 - 31n = -80,$$

and $\therefore n = 5$.

The value of a is now easily found from either of the equations to be 2;

∴ series = 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, and 35.

7. There are three numbers in equidifferent progression whose sum is 18, and the sum of the first and second is to the sum of the second and third as five to seven; required the numbers,

= 4, 6, and 8.

8. The sum of four numbers in equidifferent progression is 26, and four times the product of the first and last is 88; what are the numbers? . . . . . . = 2, 5, 8, and 11.

9. Divide £35 among three persons, so that their shares may be in equirational progression, and that the last may have £15 more than the first.

Let x = the share of the first, and r = the common ratio; then the equations are

$$x + rx + r^2x = 35\ (a),\ \text{and}\ r^2x - x = 15\ (b);$$

then $(b) \div (a)$ gives $\dfrac{r^2 - 1}{r^2 + r + 1} = \frac{3}{7}$;

whence, $4r^2 - 3r = 10$, $\therefore r = 2$;

and from (a), $x + 2x + 4x = 35$, $\therefore x = 5$,

= £5, £10, and £20.

10. There are four numbers in equirational progression: the sum of the first and second is 28, and the sum of the third and fourth is 252; what are the numbers? . = 7, 21, 63, and 189.

11. There are four numbers in equirational progression, and the sum of the first and fourth is to the sum of the second and third as 7 is to 3; also, the sum of the first, second, and third is 65; find the numbers, ... = 5, 15, 45, and 135.

12. The sum of three numbers in equirational progression is 62, and the sum of the first and second is to the third as 6 is to 25; what are the numbers? . . . . . . = 2, 10, and 50.

13. A traveller set out from a place, and went 1 mile the first day, but increased his speed afterwards by 2 miles every day. After he had gone 3 days, a second set out from the same place and travelled 12 miles the first day, 13 the second, and so on. In how many days will the first be overtaken by the second? And if they both continue their travelling at the same rate, in what time will the first overtake the second, and how far will they have travelled between the places of meeting?

Let n = the days the second travelled, then $n + 3$ = the days the first travelled;

hence, $\{2 + (n + 2) \times 2\}\dfrac{n + 3}{2} = \{24 + n - 1\}\dfrac{n}{2}$ ... (a);

whence, $n^2 - 11n = -18$,

and $\therefore n = 2$ or 9;

and the difference of the values of either side of equation (a), for $n = 9$ and $n = 2$ is the distance between the places of meeting, . . = 2 and 9 days, and 119 miles.

14. The equidifferent mean between two numbers exceeds the equirational mean by 13, and the equirational exceeds the harmonic mean by 12; what are the numbers?

Let $x =$ the greater number, and $y =$ the less; then, by the question,

$$\frac{x+y}{2} = \sqrt{xy} + 13 \quad \ldots \quad \ldots \quad (a),$$

and
$$\frac{2xy}{x+y} + 12 = \sqrt{xy} \quad \ldots \quad \ldots \quad (b).$$

Multiplying the corresponding sides of (a) and (b) gives

$$xy + 6(x+y) = xy + 13\sqrt{xy}, \text{ or } 6(x+y) = 13\sqrt{xy} \ldots (c).$$

Squaring, $36x^2 + 72xy + 36y^2 = 169xy$;

subtract $144xy$, $\therefore 36x^2 - 72xy + 36y^2 = 25xy$,

$$6(x-y) = 5\sqrt{xy} \quad \ldots \quad \ldots \quad (d).$$

Multiplying (c) by 5, and (d) by 13, and subtracting the results, we find $y = \frac{4}{9}x$; which value, being substituted in (a) or (b), gives the answer,

$$x = 234, \text{ and } y = 104.$$

15. The equidifferent mean between two numbers exceeds the equirational mean by 153, and the equirational mean exceeds the harmonic by 72; what are the numbers? . . = 34 and 544.

16. The equidifferent mean between two numbers exceeds the equirational mean by 459, and the equirational mean exceeds the harmonic by 216; what are the numbers? . = 102 and 1632.

17. £700 was divided among four people, whose shares were in equirational progression; the difference between the greatest and least shares was to the difference between the mean shares as 37 to 12; what were their several shares?

= £108, £144, £192, and £256.

18. The sum of four numbers in equirational progression is 200, and the sum of the first and second is to the sum of the third and fourth as 2 is to 18; what are the numbers?

= 5, 15, 45, and 135.

19. A debt can be discharged in a year of 52 weeks by paying 3 shillings the first week, 5 the second, 7 the third, and so on; required the amount of the debt, and of the last payment,

= £140, 8s., and £5, 5s.

20. Determine the relations that must exist between a, b, and c;

p, q, and r; so that a, b, and c may be the p^{th}, q^{th}, and r^{th} terms of an equidifferent progression.

Let $f =$ the first term, and $d =$ the common difference; then

$$f + (p - 1)d = a \ (1), f + (q - 1)d = b \quad \ldots \quad (2),$$

and $$f + (r - 1)d = c \quad \ldots \quad \ldots \quad (3).$$

Multiplying the first by $(q - r)$, the second by $(r - p)$, and the third by $(p - q)$; and then adding the three results (CHAMBERS's *Algebra*, Art. 272), the terms on the first side become zero, and the second

$(q - r)a + (r - p)b + (p - q)c$, which therefore $= 0$;

and dividing by abc, we have

$$\frac{q - r}{bc} + \frac{r - p}{ac} + \frac{p - q}{ab} = 0.$$

21. There are four numbers in equidifferent progression such that if 1 be added to the first and second terms, 3 to the third, and 9 to the fourth, the sums will then be in equirational progression; required the numbers.

Let the numbers be represented by a, $a + d$, $a + 2d$, and $a + 3d$; then we have

$$a + 1 : a + d + 1 :: a + d + 1 : a + 2d + 3 \quad \ldots (a),$$

and $a + d + 1 : a + 2d + 3 :: a + 2d + 3 : a + 3d + 9 \ldots (b)$.

From (a), $d^2 = 2a + 2$; and from (b), $d^2 = 4a$; whence, $a = 1$, and $d = 2$; and the numbers are 1, 3, 5, and 7.

22. There are two series—one equidifferent, the other equirational—each consisting of three terms. The 1st, 2d, and 3d terms of the equidifferent series are severally contained twice, thrice, and six times in the corresponding terms of the equirational series; and the sum of the first and third terms of the equidifferent series is 20; what are the series?

Equidifferent = 5, 10, 15; equirational = 10, 30, 90.

23. There are four numbers in equidifferent progression such that the product of the extremes is 3700, and the product of the means is 3828; find the numbers.

Assume the series to be $x - 3y$, $x - y$, $x + y$, and $x + 3y$; then $x^2 - 9y^2 = 3700$, and $x^2 - y^2 = 3828$, from which $y = 4$, and $x = 62$.

Hence the numbers are 50, 58, 66, and 74.

24. In an equidifferent progression consisting of ten terms, the

excess of the last four terms above the first four is 48, and the sum of the second and fifth terms is 20; find the series,

= 5, 7, 9, 11, 13, 15, 17, 19, 21, and 23.

25. In an equirational progression consisting of seven terms, the sum of the first three terms is to the sum of the last three as 13 is to 1053, and the sum of the first and third terms is 20; find the series, . . . = 2, 6, 18, 54, 162, 486, and 1458.

PROPERTIES OF NUMBERS.

1. What is the least number by which 12 can be multiplied, so that the product may be an exact cube?

Since $12 = 4 \times 3 = 2^2 \times 3$, and no number composed of different factors can be an exact cube, except each of the different factors be cubes; 2 and 3 must both be made cubes, therefore $2^3 \times 3^3 \div 2^2 \times 3 = 2 \times 3^2 = 2 \times 9 = 18$, is the multiplier required.

2. By what number must 54 be multiplied, that the product may be a complete fourth power?

Here $54 = 2 \times 27 = 2 \times 3^3$; thence the multiplier sought is $2^4 \times 3^4 \div 2 \times 3^3 = 2^3 \times 3 = 24$.

3. If n be any whole number, either odd or even, prove that $n(n + 1)(n + 2)$, and $n(n - 1)(n - 2)$, are each divisible by 6 without a remainder.

Since n or $n \pm 1$ is divisible by 2 without a remainder, and n or $n \pm 1$, or $n \pm 2$ is divisible by 3 without a remainder, there is at least one factor divisible by 2, and one of the factors is divisible by 3; therefore the whole is divisible by $2 \times 3 = 6$.

4. If n be any whole number, either odd or even, prove that $n^2(n \pm 1)^2(n \pm 2)^2(n \pm 3)^2$ is divisible by 576 without a remainder.

In the same manner, as was shewn in Ex. 3, one of the simple factors is divisible by 2, another by 3, and another by 4; and as they are all squares in the given expression, their product will be divisible by $2^2 \times 3^2 \times 4^2 = 576$.

5. If m be any whole number, odd or even, then $m^3 + 5m$ is divisible by 6 without a remainder; required a proof.

Since every whole number is either of the form $3n$ or $3n \pm 1$, $m^3 + 5m$ is either of the form $27n^3 + 15n = 3n(9n^2 + 5)$, in which, if n be even, $3n$ is divisible by 6; but if n be odd, $9n^2 + 5$ is even, and therefore divisible by 2, and $3n$ is divisible by 3, and therefore the whole is divisible by 6; or $3m^3 + 5m$ is of the form $(3n \pm 1)^3 + 5(3n \pm 1) = 27n^3 \pm 27n^2 + 9n \pm 1 + 15n \pm 5 = 27n^3 \pm 27n^2 + 24n \pm 6 = 3n\{n(9n \pm 9) + 8\} \pm 6$, which is evidently divisible by 6, if $3n\{n(9n \pm 9) + 8\}$ be divisible by 6. Now, if n be even, $3n$ is divisible by 6; but if n be odd, then $n(9n \pm 9) + 8 = 9n(n \pm 1) + 8$; where, if n be odd, $n \pm 1$ is therefore even; and $n(9n \pm 9) + 8$ is divisible by 2, and therefore the whole by 6.

6. Decompose 277200 into its prime factors, and find the multiplier that will make it a perfect cube.

Its prime factors are 11, 7, 3^2, 5^2, and 2^4; hence the factor by which it must be multiplied, that the product may be a perfect cube, is

$$11^2 \times 7^2 \times 3 \times 5 \times 2^2 = 355740.$$

7. If n be any whole number, either n^2 or $n^2 \pm 1$ is divisible by 5 without a remainder; required a proof.

Since every square number ends in one of the six digits, 0, 1, 4, 5, 6, or 9, and that every number that has its right-hand digit either 5 or 0 is divisible by 5, the truth of the theorem is obvious.

8. Shew that the difference of the squares of any two odd numbers is divisible by 8.

Since every odd number is of the form of $2n + 1$, let the one be $2n + 1$, and the other $2n' + 1$; we have

$$(2n + 1)^2 - (2n' + 1)^2 = 2(n + n' + 1) \times 2(n - n')$$
$$= 4(n + n' + 1)(n - n');$$

now $n - n'$ is even, and is therefore divisible by 2, and hence the whole is divisible by 8.

9. If two numbers differ by 1, prove that the difference of their squares is equal to the sum of the numbers.

Let the less $= n$, then the greater $= n + 1$, and the difference of their squares is

$$(n + 1)^2 - n^2 = (n + 1 + n)(n + 1 - n)$$
$$= 2n + 1 = \text{their sum}.$$

10. If two numbers differ by any number a, prove that the difference of their squares is equal the sum of the two numbers multiplied by a.

Let $n =$ the less, then $n + a =$ the greater, and the difference of their squares is

$$(n + a)^2 - n^2 = (n + a + n)(n + a - n) = (2n + a)a.$$

11. If the sum of two fractions = 1, prove that their difference is equal to the difference of their squares.

Let the fractions be represented by $\frac{x}{y}$ and $\frac{y-x}{y}$, then their sum is evidently = 1, and their difference $\frac{2x-y}{y}$, also the difference of their squares is

$$\left(\frac{x}{y}\right)^2 - \left(\frac{y-x}{y}\right)^2 = \frac{x+y-x}{y} \times \frac{2x-y}{y} = \frac{2x-y}{y}.$$

12. Prove that the product of two odd numbers is odd; and that the product of two even numbers, or of an even and an odd number, is even.

Let the two odd numbers be represented by $2n+1$ and $2n'+1$, their product is

$$4nn' + 2(n+n') + 1 = 2(nn' + n + n') + 1,$$

which is evidently odd; again, let the two even numbers be represented by $2n$ and $2n'$, their product $4nn'$ is evidently even; lastly, if one be even, and the other odd, they may be represented by $2n$ and $2n'+1$, and their product is $2n(2n'+1)$, an even number.

13. The product of the sum of two squares, by the sum of other two squares, is the sum of two squares; required a proof.

Let the first two squares be represented by a^2 and b^2, and the second by m^2 and n^2; then

$$\begin{aligned}(a^2+b^2)(m^2+n^2) &= a^2m^2 + b^2m^2 + a^2n^2 + b^2n^2 \\ &= a^2m^2 + b^2n^2 + a^2n^2 + b^2m^2 \\ &= a^2m^2 + 2ambn + b^2n^2 + a^2n^2 - 2ambn + b^2m^2 \\ &= (am+bn)^2 + (an-bm)^2.\end{aligned}$$

14. Prove that twice the sum of two squares is equal to the sum of two squares.

Let a^2 and b^2 be any two squares; then

$$\begin{aligned}2(a^2+b^2) &= a^2 + 2ab + b^2 + a^2 - 2ab + b^2 \\ &= (a+b)^2 + (a-b)^2.\end{aligned}$$

15. Prove that three times the sum of three squares is equal to the sum of four squares.

Let the three squares be represented by a^2, b^2, and c^2; then

$$\begin{aligned}3(a^2+b^2+c^2) &= (a+b+c)^2 + (a-b)^2 \\ &\quad + (a-c)^2 + (b-c)^2.\end{aligned}$$

For $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$,

$$(a-b)^2 = a^2 + b^2 - 2ab,$$
$$(a-c)^2 = a^2 + c^2 - 2ac,$$
$$(b-c)^2 = b^2 + c^2 - 2bc;$$

and adding, we have

$$(a+b+c)^2 + (a-b)^2 + (a-c)^2 + (b-c)^2$$
$$= 3(a^2 + b^2 + c^2).$$

16. Prove that four times the sum of four squares is equal to the sum of seven squares.

Let the squares be represented by a^2, b^2, c^2, and d^2; then

$$4(a^2 + b^2 + c^2 + d^2) = (a+b+c+d)^2 + (a-b)^2$$
$$+ (a-c)^2 + (a-d)^2$$
$$+ (b-c)^2 + (b-d)^2 + (c-d)^2.$$

This may now be proved by expanding and adding, as in the last exercise.

17. Prove that five times the sum of five squares is equal to the sum of eleven squares.

For, as in the last two exercises, it may easily be shewn that

$$5(a^2 + b^2 + c^2 + d^2 + e^2) = (a+b+c+d+e)^2$$
$$+ (a-b)^2 + (a-c)^2$$
$$+ (a-d)^2 + (a-e)^2 + (b-c)^2 + (b-d)^2$$
$$+ (b-e)^2 + (c-d)^2 + (c-e)^2$$
$$+ (d-e)^2 = \text{the sum of eleven squares.}$$

NOTE.—Combining the above three exercises with the principles of combinations, it may easily be shewn that n times the sum of n squares is equal to the sum of $\frac{n(n-1)}{1.2} + 1$ squares; so that 12 times the sum of 12 squares may easily be shewn to be equal to $\frac{12 \times 11}{1 \times 2} + 1 = 67$ squares.

18. In what scale of notation is the number 187 equal to 247 in the common denary scale?

Let $r =$ the base of the scale, then we have the following equation:—

$$r^2 + 8r + 7 = 247;$$

hence, $r = 12$, the duodenary.

19. In what scale of notation is the number 473 equal to 189 in the common denary scale?

Here $4r^2 + 7r + 3 = 189;$

hence, $r = 6$, the senary scale.

20. In what scale of notation is the number 10404 equal to 729 in the common denary scale?

Here $r^4 + 4r^2 + 4 = 729;$

hence, $r = 5$, the quinary scale.

21. In what scale of notation is the number 50407 equal to 20743 in the common denary scale?

Here $5r^4 + 4r^2 + 7 = 20743;$

hence, $r = 8$, the octary scale.

PERMUTATIONS AND COMBINATIONS.

1. How many days can 6 persons be placed in different positions about a table at dinner?

Here the number of days $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ days.

2. How many permutations can be formed from the letters of the word Edinburgh?

Here the permutations sought are

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880.$$

3. The number of combinations of $(n + 1)$ things, taken $(n - 1)$ together, is 78; shew that the number of permutations of n things, taken four together, is 11880.

The combinations of $(n + 1)$ things, taken $(n - 1)$ together, is the same as the number of combinations of $(n + 1)$ things, taken two together;

hence, $\frac{(n + 1)n}{1 . 2} = 78$, or $n^2 + n = 156;$

$$\therefore n^2 + n + \tfrac{1}{4} = \tfrac{625}{4},$$

or $n + \frac{1}{2} = \frac{25}{2}; \therefore n = 12.$

Hence, $n(n - 1)(n - 2)(n - 3) = 12 \times 11 \times 10 \times 9$
$= 11880.$

4. The number of permutations of $(n+1)$ things, taken two together, is 240; shew that the number of combinations of n things, taken five together, is 3003.

The permutations of $(n+1)$ things, taken two and two together, is $(n+1)n$; and therefore $(n+1)n = 240$.

Hence, $n^2 + n + \frac{1}{4} = \frac{961}{4}$; $\therefore n = 15$.

The combinations, taken five together, are

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5},$$

or $$\frac{15 \times 14 \times 13 \times 12 \times 11}{1 \times 2 \times 3 \times 4 \times 5} = 3003.$$

5. The number of combinations of $(m+n)$ things, taken two together, is 78; and the number of permutations of $(m-n)$ things, taken two together, is 6; find the number of permutations and of combinations of m things, taken n together.

The combinations of $(m+n)$, two together, is

$$\frac{(m+n)(m+n-1)}{1.2} = 78;$$

and hence, $(m+n)^2 - (m+n) + \frac{1}{4} = \frac{625}{4}$;

$$\therefore m + n = 13 \quad \ldots \quad (a).$$

Again, the permutations of $(m-n)$ things, taken two together, is

$$(m-n)(m-n-1) = (m-n)^2 - (m-n) = 6,$$

or $$(m-n)^2 - (m-n) + \tfrac{1}{4} = \tfrac{25}{4};$$

$$\therefore m - n = 3 \quad \ldots \quad (b).$$

From $(a) + (b)$, $2m = 16$, and $\therefore m = 8$;

and from $(a) - (b)$, $2n = 10$, and $\therefore n = 5$.

Hence the permutations are $8 \times 7 \times 6 \times 5 \times 4 = 6720$,

and the combinations are $\frac{8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5} = 56$.

6. The number of permutations of n things, taken five together, $= 7$ times the number, taken four together; find n.

Here $$n(n-1)(n-2)(n-3)(n-4)$$
$$= 7n(n-1)(n-2)(n-3);$$
$$\therefore n - 4 = 7,$$

by dividing both sides by $n(n-1)(n-2)(n-3)$,

and $\therefore n = 11$.

7. Of the permutations formed with the nine letters of the word Edinburgh, taken all together, how many begin with Ed?—how many with Edi?—how many with Edin?

The number of permutations beginning with Ed is evidently the number of permutations formed with the other seven letters, and is therefore

$$\therefore 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$$

2d, Those beginning with Edi are the number of permutations of the remaining six letters, and is therefore

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

3d, Those beginning with Edin is the number of permutations of the remaining five letters, and is therefore

$$5 \times 4 \times 3 \times 2 \times 1 = 120.$$

8. How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence?

Since there are 6 different coins, and they may be taken in any number to form a sum, we have as follows:—

The combinations, 1 at a time . . . . . . = 6.

" " , 2 " $= \frac{6 \times 5}{1 . 2}$, ... = 15.

" " , 3 " $= \frac{6 \times 5 \times 4}{1 . 2 . 3}$, . . = 20.

" " , 4 " $= \frac{6 \times 5 \times 4 \times 3}{1 . 2 . 3 . 4}$, . = 15.

" " , 5 " $= \frac{6 \times 5 \times 4 \times 3 \times 2}{1 . 2 . 3 . 4 . 5}$, . = 6.

" " , 6 " $= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 . 2 . 3 . 4 . 5 . 6}$, = 1.

Therefore the total number of different sums that can be paid with these six coins $= 6 + 15 + 20 + 15 + 6 + 1 = 63$.

9. Of the different combinations of the nine letters of the word Edinburgh, taken five together, in how many will the letter *e* occur, and in how many will it not occur?

The number of combinations in which *e* occurs, will evidently be the number of combinations of the remaining eight letters, taken four together, which is $\frac{8 \times 7 \times 6 \times 5}{1 . 2 . 3 . 4} = 70$.

And the number of combinations in which the letter *e* does not occur, will be the number of combinations of the remaining eight letters, taken five together, which is

$$\frac{8 \times 7 \times 6 \times 5 \times 4}{1 . 2 . 3 . 4 . 5} = 56.$$

10. Out of 19 consonants and 6 vowels, how many different words can be formed, having 2 consonants and 1 vowel in each?

The number of different permutations that can be formed of 19 consonants, taken 2 together, is 19×18.

And since the vowel may either be placed before the 2 consonants, between them, or after them, $19 \times 18 \times 3$ is the number of words that can be formed with 1 vowel; but as there are 6 vowels, and the same number of words can be formed with each, the whole number of words will be

$$19 \times 18 \times 3 \times 6 = 6156.$$

METHOD OF UNDETERMINED COEFFICIENTS.

1. Develop $\frac{1}{3 + x}$ into a series.

Assume $\frac{1}{3 + x} = A + Bx + Cx^2 + Dx^3 + \&c.;$

and multiply both sides by $3 + x$, which gives

$$1 = 3A + (3B + A)x + (3C + B)x^2 + (3D + C)x^3 + \&c.$$

Then equating the coefficients of the like powers of x on both sides, we have

$$3A = 1,\ 3B + A = 0,\ 3C + B = 0,\ 3D + C = 0;$$

from which $A = \frac{1}{3}$, $B = -\frac{1}{9}$, $C = \frac{1}{27}$, and $D = -\frac{1}{81}$;

hence the series is $= \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3 + \&c.$

2. Convert $\frac{1}{1 + 2x + x^2}$ into a series.

Assume, as in the last example, the series to be

$$A + Bx + Cx^2 + Dx^3 + \&c.;$$

then multiply both sides by $1 + 2x + x^2$, and equate the coefficients of the like powers of x on both sides, observing that those on the first side are all zero, and we have the following equations:—

$$A = 1,\ B + 2A = 0,\ C + 2B + A = 0,$$

and $$D + 2C + B = 0;$$

whence, $A = 1$, $B = -2$, $C = 3$, and $D = -4$;

$$\therefore \frac{1}{1 + 2x + x^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \&c.$$

3. Develop $\sqrt{(a^2 - x^2)}$ into a series.

Here, if we assume the series to be

$$A + Bx + Cx^2 + Dx^3 + \&c.,$$

it will be found that the coefficients of the odd powers of x are all zero; therefore let

$$\sqrt{(a^2 - x^2)} = a + Bx^2 + Cx^4 + Dx^6 + \&c.;$$

and square both sides, which will give, by equating the coefficients,

$$A^2 = a^2,\ 2AB = -1,\ 2AC + B^2 = 0,$$
$$2AD + 2BC = 0;$$

$$\therefore A = a,\ B = -\frac{1}{2a},\ C = -\frac{1}{8a^3},$$

and $$D = -\frac{1}{16a^5}.$$

Hence, $$\sqrt{(a^2 - x^2)} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \&c.$$

4. Find three fractions whose sum is $\frac{1}{x^4 - a^4}$.

Since $x^4 - a^4 = (x^2 + a^2)(x + a)(x - a)$, assume the fractions to be

$$\frac{A}{x^2 + a^2} + \frac{B}{x + a} + \frac{C}{x - a} = \frac{1}{x^4 - a^4};$$

hence, $$A(x^2 - a^2) + B(x^2 + a^2)(x - a)$$
$$+ C(x^2 + a^2)(x + a) = 1;$$

$$\therefore A = -\frac{1}{2a^2},\ B = -\frac{1}{4a^3},\ C = \frac{1}{4a^3}.$$

Hence the fractions are

$$-\frac{1}{2a^2(a^2 + x^2)} - \frac{1}{4a^3(x + a)} + \frac{1}{4a^3(x - a)}.$$

5. Find three fractions whose sum is $\frac{x - 1}{x^3 + 8x^2 + 21x + 18}$.

Since $x^3 + 8x^2 + 21x + 18 = (x + 3)^2(x + 2)$, assume the fractions to be

$$\frac{A}{x + 2} + \frac{B}{x + 3} + \frac{C}{(x+3)^2} = \frac{x - 1}{x^3 + 8x^2 + 21x + 18};$$

hence, $$A(x + 3)^2 + B(x + 2)(x + 3) + C(x + 2)$$
$$= x - 1.$$

Assume $x = -3$, $\therefore C = 4$;

assume $x = -2$, $\therefore A = -3$;

substituting these values of A and C, we find $B = 3$;

$$\therefore \frac{x-1}{x^3+8x^2+21x+18} = -\frac{3}{x+2} + \frac{3}{x+3} + \frac{4}{(x+3)^2}.$$

6. Find three fractions whose sum is $\frac{3x^2-7x+6}{(x-1)^3}$.

Let $\frac{3x^2-7x+6}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)}$,

and let $z = x - 1$;

then $\frac{3x^2-7x+6}{(x-1)^3} = \frac{3z^2-z+2}{z^3} = \frac{3}{z} - \frac{1}{z^2} + \frac{2}{z^3}$

$$= \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3};$$

$$\therefore A = 3,\ B = -1, \text{ and } C = 2;$$

and hence, $\frac{3x^2-7x+6}{(x-1)^3} = \frac{3}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}$.

7. Find three fractions whose sum is $\frac{x^2+px+q}{(x-a)(x-b)(x-c)}$.

Assume the three fractions to be $\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$;

then since the relation sought is independent of the value of x, we may assume any value we please for x in the equation

$$x^2+px+q = A(x-b)(x-c)$$
$$+ B(x-a)(x-c) + C(x-a)(x-b),$$

without destroying the equality.

Now, if $x = a$, it reduces to

$$a^2+pa+q = A(a-b)(a-c);$$

$$\therefore A = \frac{a^2+pa+q}{(a-b)(a-c)}.$$

Similarly, by assuming $x = b$, and $x = c$, we find

$$B = \frac{b^2+pb+q}{(b-a)(b-c)}, \text{ and } C = \frac{c^2+pc+q}{(c-a)(c-b)};$$

and therefore

$$\frac{x^2+px+q}{(x-a)(x-b)(x-c)} = \frac{a^2+pa+q}{(a-b)(a-c)(x-a)}$$
$$+ \frac{b^2+pb+q}{(b-a)(b-c)(x-b)} + \frac{c^2+pc+q}{(c-a)(c-b)(x-c)}.$$

8. Find three fractions whose sum is $\dfrac{3x+5}{(x-2)(x-3)(x-4)}$.

Proceeding exactly as in the last exercise, and assuming $x=2$, $x=3$, and $x=4$ in succession, we find

$$A=\tfrac{11}{2},\ B=-14,\ \text{and } C=\tfrac{17}{2};$$

$$\therefore \frac{3x+5}{(x-2)(x-3)(x-4)}=\frac{11}{2(x-2)}-\frac{14}{x-3}+\frac{17}{2(x-4)}.$$

9. Find four fractions whose sum is $\dfrac{x^2+3x+4}{(x-2)(x-4)(x-6)(x-8)}$.

Assuming the fractions to be

$$\frac{A}{x-2}+\frac{B}{x-4}+\frac{C}{x-6}+\frac{D}{x-8},$$

we have $\quad x^2+3x+4$

$$=A(x-4)(x-6)(x-8)+B(x-2)(x-6)(x-8)$$
$$+C(x-2)(x-4)(x-8)+D(x-2)(x-4)(x-6),$$

in which let $\quad x=2$, then $A=-\tfrac{7}{24}$;

$x=4$, " $B=+2$;

$x=6$, " $C=-\tfrac{29}{8}$;

$x=8$, " $D=+\tfrac{23}{12}$;

$$\therefore \frac{x^2+3x+4}{(x-2)(x-4)(x-6)(x-8)}$$

$$=-\frac{7}{24(x-2)}+\frac{2}{x-4}-\frac{29}{8(x-6)}+\frac{23}{12(x-8)}.$$

10. Find the sum of the series $1.2+2.3+3.4+$ &c., to n terms.

Here the general term is $n(n+1)$, and such series can all be summed as follows:—

Assume the series $=An^3+Bn^2+Cn$, in which the highest power of n is one greater than the number of factors in each term of the series; then if for n we put $n+1$, this will be carrying the series one term further, and the difference of the two series will plainly just be the additional term; thus

$$An^3+Bn^2+Cn$$
$$=1.2+2.3+3.4+\dots n(n+1) \quad \dots \quad (a);$$

hence, $\quad A(n+1)^3+B(n+1)^2+C(n+1)$

$$=1.2+2.3+3.4+\dots n(n+1)$$
$$+(n+1)(n+2) \quad \dots \quad \dots \quad (b);$$

$$\therefore 3An^2+(3A+2B)n+(A+B+C)$$
$$=n^2+3n+2 \quad \dots \quad (b)-(a).$$

Equating the coefficients of the like powers of n on both sides, gives

$$3A = 1,\ 3A + 2B = 3, \text{ and } A + B + C = 2;$$

whence, $A = \frac{1}{3}$, $B = 1$, and $C = \frac{2}{3}$;

and the series is therefore

$$1.2 + 2.3 + 3.4 \ldots n(n + 1)$$
$$= \tfrac{1}{3}n^3 + n^2 + \tfrac{2}{3}n = \frac{n}{3}(n + 1)(n + 2).$$

11. Find the sum of the series $1^2 + 2^2 + 3^2 + 4^2 +$ &c., to 12 terms.

Here the general term is n^2; and proceeding, as in last exercise, we find the sum of n terms

$$= \frac{n(n + 1)(2n + 1)}{6};\ \therefore \text{ the sum of } 12 = 650.$$

12. Find the sum of the series $3.5 + 6.8 + 9.11 + 12.14$ + &c., to 9 terms.

Here the general term is $3n(3n + 2)$; and proceeding, as in the 10th exercise,

$$3An^2 + (3A + 2B)n + A + B + C$$
$$= 9n^2 + 24n + 15;$$

$$\therefore A = 3,\ B = \tfrac{15}{2}, \text{ and } C = \tfrac{9}{2};$$

hence the sum of n terms is $= \frac{3n}{2}(n + 1)(2n + 3)$,

which, for $n = 9$, becomes $\frac{27}{2} \times 10 \times 21 = 2835$.

13. Find the sum of n terms of the series $1.2^2 + 2.3^2 + 3.4^2 +$ &c.

Let $$An^4 + Bn^3 + Cn^2 + Dn$$
$$= 1.2^2 + 2.3^2 + \ldots n(n + 1)^2;$$

then $$A(n + 1)^4 + B(n + 1)^3 + C(n + 1)^2 + D(n + 1)$$
$$= 1.2^2 + 2.3^2 + \ldots n(n + 1)^2 + (n + 1)(n + 2)^2.$$

Subtracting the first equation from the second, we have

$$4An^3 + (6A + 3B)n^2 + (4A + 3B + 2C)n$$
$$+ A + B + C + D = n^3 + 5n^2 + 8n + 4;$$

hence, $4A = 1$, $6A + 3B = 5$, $4A + 3B + 2C = 8$,

and $$A + B + C + D = 4;$$

$$\therefore A = \tfrac{1}{4},\ B = \tfrac{7}{6},\ C = \tfrac{7}{4}, \text{ and } D = \tfrac{5}{6}.$$

The sum is therefore $\frac{n^4}{4} + \frac{7n^3}{6} + \frac{7n^2}{4} + \frac{5n}{6}$
$= \frac{n}{12}(3n + 5)(n + 2)(n + 1)$.

14. Find the sum of n terms of the series $1^3 + 2^3 + 3^3 + \&c., n^3$.

Assuming the series, as in the 13th exercise, we find, after subtracting,

$$4An^3 + (6A + 3B)n^2 + (4A + 3B + 2C)n$$
$$+ A + B + C + D = n^3 + 3n^2 + 3n + 1;$$

hence, $4A = 1$, $6A + 3B = 3$, $4A + 3B + 2C = 3$,

and $A + B + C + D = 1$;

$$\therefore A = \tfrac{1}{4}, B = \tfrac{1}{2}, C = \tfrac{1}{4}, \text{ and } D = 0.$$

The sum is therefore $\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \frac{n^2(n + 1)^2}{4}$ = the square of the sum of the numbers.

15. Sum the series $1 . 2 . 3 + 2 . 3 . 4 + 3 . 4 . 5 + \&c.$, to n terms.

Here the general or n^{th} term is $n(n + 1)(n + 2)$; and assuming the same series, and operating, as in the 13th exercise, we find

$$A = \tfrac{1}{4}, B = \tfrac{6}{4}, C = \tfrac{11}{4}, \text{ and } D = \tfrac{6}{4};$$

$$\therefore 1 . 2 . 3 + 2 . 3 . 4 + \ldots n(n + 1)(n + 2)$$
$$= \frac{n^4}{4} + \frac{6n^3}{4} + \frac{11n^2}{4} + \frac{6n}{4}$$
$$= \frac{n}{4}(n + 1)(n + 2)(n + 3).$$

16. Sum the series $1 . 3 . 5 + 2 . 4 . 6 + 3 . 5 . 7 + \&c.$, to n terms, and find the sum when $n = 20$.

Here the general or n^{th} term is $n(n + 2)(n + 4)$; and assuming the same, as in the 13th exercise, we find

$$A = \tfrac{1}{4}, B = \tfrac{10}{4}, C = \tfrac{29}{4}, \text{ and } D = \tfrac{20}{4};$$

$$\therefore 1 . 3 . 5 + 2 . 4 . 6 + \ldots n(n + 2)(n + 4)$$
$$= \frac{n^4}{4} + \frac{10n^3}{4} + \frac{29n^2}{4} + \frac{20n}{4}$$
$$= \frac{n}{4}(n + 1)(n + 4)(n + 5).$$

And making $n = 20$, in the above result, we have the sum of 20 terms

$$= 5 \times 21 \times 24 \times 25 = 63000.$$

17. Find the sum of 16 terms of the series whose general term is $(2n+1)(3n-1)$; and find the general expression for the sum of n terms.

Since each term of this series contains only two factors, assume the same series as in exercise 10, and it will be found that

$$A = 2,\ B = \tfrac{7}{2},\ C = \tfrac{1}{2};$$

and therefore the sum of n terms

$$= 2n^3 + \tfrac{7}{2}n^2 + \tfrac{1}{2}n = \frac{n}{2}(4n+3)(n+1) - n.$$

For $n = 16$, the sum is $\frac{16}{2} \times 67 \times 17 - 16 = 9096$.

BINOMIAL THEOREM.

Expand by the Binomial Theorem

1. $(a + x)^2$, ... $= a^2 + 2ax + x^2$.
2. $(a - x)^2$, ... $= a^2 - 2ax + x^2$.
3. $(1 + 2x)^2$, ... $= 1 + 4x + 4x^2$.
4. $(1 - x)^3$, ... $= 1 - 3x + 3x^2 - x^3$.
5. $(1 + 2x)^3$, ... $= 1 + 6x + 12x^2 + 8x^3$.
6. $(a + 3x)^3$, ... $= a^3 + 9a^2x + 27ax^2 + 27x^3$.
7. $(2a - 3x)^3$, $= (2a)^3 - 3(2a)^2(3x) + 3(2a)(3x)^2 - (3x)^3$
   $= 8a^3 - 36a^2x + 54ax^2 - 27x^3$.
8. $(1 + x)^4$, ... $= 1 + 4x + 6x^2 + 4x^3 + x^4$.
9. $(1 - 2x)^4$, . . . $= 1 - 8x + 24x^2 - 32x^3 + 16x^4$.
10. $(1 + 3x)^4$, . . . $= 1 + 12x + 54x^2 + 108x^3 + 81x^4$.
11. $(a + x)^5$, $= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$.
12. $(1 + 2x)^5$, . $= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$.
13. $(1 - 3x)^5$, $= 1 - 15x + 90x^2 - 270x^3 + 405x^4 - 243x^5$.
14. $(a + \frac{1}{2}x)^5$, $= a^5 + \frac{5}{2}a^4x + \frac{5}{2}a^3x^2 + \frac{5}{4}a^2x^3 + \frac{5}{16}ax^4 + \frac{1}{32}x^5$.

15. $(2a - 3x)^5$,

$$= (2a)^5 - 5(2a)^4(3x) + 10(2a)^3(3x)^2 - 10(2a)^2(3x)^3 + 5(2a)(3x)^4 - (3x)^5$$
$$= 32a^5 - 240a^4x + 720a^3x^2 - 1080a^2x^4 + 810ax^4 - 243x^5.$$

16. $(1 + 3x)^6$,

$$= 1 + 18x + 135x^2 + 540x^3 + 1215x^4 + 1458x^5 + 729x^6.$$

17. $(2a - x)^6$,

$$= 64a^6 - 192a^5x + 240a^4x^2 - 160a^3x^3 + 60a^2x^4 - 12ax^5 + x^6.$$

18. $(1 - 4x)^6$,

$$= 1 - 24x + 240x^2 - 1280x^3 + 3840x^4 - 6144x^5 + 4096x^6.$$

19. $(2x + 3y)^5$,

$$= (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

20. $(3x - 2y)^5$,

$$= (3x)^5 - 5(3x)^4(2y) + 10(3x)^3(2y)^2 - 10(3x)^2(2y)^3 + 5(3x)(2y)^4 - (2y)^5$$
$$= 240x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5.$$

21. $(1 + x)^{-2}$,

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \frac{7x^6 + 6x^7}{(1 + x)^2}.$$

22. $(1 - x)^{-2}$,

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \frac{7x^6 - 6x^7}{(1 - x)^2}.$$

23. $(1 + 2x)^{-3}$,

$$= 1 - 6x + 24x^2 - 80x^3 + 240x^4 - 672x^5 + \&c.$$

24. $\left(1 + \frac{1}{x}\right)^{-3}$,

$$= \left(\frac{x}{1 + x}\right)^3 = x^3 - 3x^4 + 6x^5 - 10x^6 + 15x^7 - 21x^8 + \&c.$$
$$= \left(\frac{x}{x + 1}\right)^3 = 1 - \frac{3}{x} + \frac{6}{x^2} - \frac{10}{x^3} + \frac{15}{x^4} - \frac{21}{x^5} + \&c.$$

The former expansion will give a converging series, if x is very small; and the latter, if x is very great.

25. $(1 - \frac{1}{3}x)^{-3}$,
$$= 1 + x + \tfrac{2}{3}x^2 + \tfrac{10}{27}x^3 + \tfrac{5}{27}x^4 + \tfrac{7}{81}x^5 + \&c.$$

26. $(1 + x)^{\frac{1}{2}}$, . $= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \&c.$

27. $(1 - x)^{\frac{1}{2}}$, . $= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \&c.$

28. $(a^2 + x^2)^{\frac{1}{2}}$,
$$= a\{1 + \tfrac{1}{2}\frac{x^2}{a^2} - \tfrac{1}{8}\frac{x^4}{a^4} + \tfrac{1}{16}\frac{x^6}{a^6} - \tfrac{5}{128}\frac{x^8}{a^8} + \&c.\},$$
$$\text{or } a + \tfrac{1}{2}\frac{x^2}{a} - \tfrac{1}{8}\frac{x^4}{a^3} + \tfrac{1}{16}\frac{x^6}{a^5} - \tfrac{5}{128}\frac{x^8}{a^7} + \&c.$$

29. $(a^2 - x^2)^{\frac{1}{2}}$,
$$= a\{1 - \tfrac{1}{2}\frac{x^2}{a^2} - \tfrac{1}{8}\frac{x^4}{a^4} - \tfrac{1}{16}\frac{x^6}{a^6} - \tfrac{5}{128}\frac{x^8}{a^8} - \&c.\},$$
$$\text{or } a - \tfrac{1}{2}\frac{x^2}{a} - \tfrac{1}{8}\frac{x^4}{a^3} - \tfrac{1}{16}\frac{x^6}{a^5} - \tfrac{5}{128}\frac{x^8}{a^7} - \&c.$$

30. $(a + x)^{\frac{1}{3}}$,
$$= a^{\frac{1}{3}}\{1 + \tfrac{1}{3}\frac{x}{a} - \tfrac{1}{9}\frac{x^2}{a^2} + \tfrac{5}{81}\frac{x^3}{a^3} - \tfrac{10}{243}\frac{x^4}{a^4} + \&c.\}.$$

31. $(a - x)^{\frac{1}{3}}$,
$$= a^{\frac{1}{3}}\{1 - \tfrac{1}{3}\frac{x}{a} - \tfrac{1}{9}\frac{x^2}{a^2} - \tfrac{5}{81}\frac{x^3}{a^3} - \tfrac{10}{243}\frac{x^4}{a^4} - \&c.\}.$$

32. $(1 - x)^{-\frac{1}{2}}$,
$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 + \ldots$$
$$+ \frac{1 . 3 . 5 . \&c. (2r - 1)}{2 . 4 . 6 . \&c. (2r)}x^r + \&c.$$

33. $(a^2 - x^2)^{-\frac{1}{3}}$,
$$= a^{-\frac{2}{3}} + \tfrac{1}{3}a^{-\frac{8}{3}}x^2 + \tfrac{2}{9}a^{-\frac{14}{3}}x^4 + \tfrac{14}{81}a^{-\frac{20}{3}}x^6 + \ldots$$
$$+ \frac{1 . 4 . 7 . \&c. (3r - 2)}{3 . 6 . 9 . \&c. (3r)}a^{-(2r+\frac{2}{3})}x^{2r} + \&c.$$

34. $(1 - x^2)^{-\frac{3}{2}}$,
$$= 1 + \tfrac{3}{2}x^2 + \tfrac{15}{8}x^4 + \tfrac{35}{16}x^6 + \tfrac{315}{128}x^8 + \ldots$$
$$+ \frac{3 . 5 . 7 . \&c. (2r + 1)}{2 . 4 . 6 . \&c. (2r)}x^{2r} + \&c.$$

35. $(1 - 2x + x^2)^3$,
$$= (1 - x)^6 = 1 - 6x + 15x^2 - 20x^3 + 15x^4$$
$$- 6x^5 + x^6.$$

36. $(1 + x + x^2)^5$,

$$= \{(1 + x) + x^2\}^5 = (1 + x)^5 + 5(1 + x)^4x^2 + 10(1 + x)^3x^4 + 10(1 + x)^2x^6 + (1 + x)x^8 + x^{10}.$$

Again, expanding each of the terms within parentheses, and collecting the same powers of x into one term, the expansion becomes

$$1 + 5x + 15x^2 + 30x^3 + 45x^4 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + x^{10}.$$

37. $(1 - x + x^2)^5$,

$$= \{(1 - x) + x^2\}^5 = (1 - x)^5 + 5(1 - x)^4x^2 + 10(1 - x)^3x^4 + 10(1 - x)^2x^6 + 5(1 - x)x^8 + x^{10}$$

$$= 1 - 5x + 15x^2 - 30x^3 + 45x^4 - 51x^5 + 45x^6 - 30x^7 + 15x^8 - 5x^9 + x^{10}.$$

38. Find an approximate value to the cube root of 31 by the Binomial Theorem.

$$(31)^{\frac{1}{3}} = (27 + 4)^{\frac{1}{3}} = 3(1 + \tfrac{4}{27})^{\frac{1}{3}}.$$

Hence, in 30th exercise, for a substitute 27, and for x substitute 4, and then find the numerical value of each of the terms, and the result will be the value sought, which is

3·1414; or, more nearly, 3·14138.

39. Find an approximate value to the $\sqrt[7]{120}$.

Since $1^7 = 1$, and $2^7 = 128$; $\therefore \sqrt[7]{120}$ is less than 2;

but $\sqrt[7]{120} = \sqrt[7]{(128 - 8)} = 2\sqrt[7]{(1 - \frac{8}{128})} = 2(1 - \frac{1}{16})^{\frac{1}{7}}$,

and $$2\left(1 - \frac{1}{16}\right)^{\frac{1}{7}} = 2\left(1 - \frac{1}{2^4}\right)^{\frac{1}{7}}$$

$$= 2\left(1 - \tfrac{1}{7} \cdot \frac{1}{2^4} - \tfrac{1}{7} \cdot \tfrac{3}{7} \cdot \left(\frac{1}{2^4}\right)^2 - \tfrac{1}{7} \cdot \tfrac{3}{7} \cdot \frac{13}{3 \times 7} \cdot \left(\frac{1}{2^4}\right)^3 - \text{\&c.}\right.$$

$$= 2\left(1 - \frac{1}{112} - \frac{3}{12544} - \frac{39}{4214784} - \text{\&c.}\right)$$

$$\frac{1}{112} = \cdot0089286$$

$$\frac{3}{12544} = \cdot0002392$$

$$\frac{39}{4214784} = \cdot0000090$$

Sum	= ·0091768
Subtract from	1·
Remainder,	·9908232
	2

$$\therefore \sqrt[7]{120} = 1\cdot9816464$$

40. Find the term which involves $a^{13}b^7$ in the expansion of $(a^3 + ab)^9$.

Here the power of a in the first term of the expansion is the 27th, and its powers will diminish by 2 in each successive term, $(27 - 13) \div 2 = 14 \div 2 = 7$, the number of term before that sought; it is therefore the 8th, which is also evident from the power of b.

Therefore the term is $\frac{9.8.7.6.5.4.3}{1.2.3.4.5.6.7}a^{13}b^7 = 36a^{13}b^7$.

41. Find the 5th term in the expansion of $(a + b)^8$,

$$= \frac{8 \times 7 \times 6 \times 5}{1.2.3.4}a^4b^4 = 70a^4b^4.$$

42. " 6th " expansion of $(a - x)^n$,

$$= -\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}.a^{n-5}x^5.$$

43. " 8th term in the expansion $\left(1 - \frac{1}{x}\right)^{12}$,

$$= -\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6}{1.2.3.4.5.6.7}.\frac{1}{x^7} = -792.\frac{1}{x^7}.$$

44. " 5th term in the expansion of $(a^2 - x^2)^{\frac{3}{2}}$,

$(a^2 - x^2)^{\frac{3}{2}} = a^3\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}}$; and hence the 5th term

$$= a^3 \times \frac{\frac{3}{2} \times \frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{1.2.3.4}\left(-\frac{x^2}{a^2}\right)^4 = \frac{3}{128}\frac{x^8}{a^5}.$$

45. Find the $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term in the expansion of $(a + x)^n$.

Here n must be even, and the term sought is the middle term, which is

$$\frac{n(n-1)(n-2).\&c.(\frac{1}{2}n + 1)}{1.2.3.4.\&c.\frac{1}{2}n}a^{\frac{n}{2}}x^{\frac{n}{2}}.$$

SERIES.

1. Find the first term of the second order of differences of the series $1.2 + 2.3 + 3.4 + \&c.$,

$$d_2 = a - nb + \frac{n(n-1)}{2}c = 2 - 2.6 + 12 = 2.$$

2. Find the first term of the third order of differences of the series $1.2 + 2.3 + 3.4 + 4.5 + \&c.$,

$$d_3 = -a + nb - \frac{n(n-1)}{1.2}c + \frac{n(n-1)(n-2)}{1.2.3}d$$
$$= -2 + 3.6 - 3.12 + 20 = 0.$$

3. Find the 7th term of the series $2 + 6 + 12 + 20 + \&c.$, in terms of its several orders of differences.

The general formula for finding any term of a series in terms of its differences is, the n^{th} term $=$

$$a + (n-1)d_1 + \frac{(n-1)(n-2)}{1.2}d_2$$
$$+ \frac{(n-1)(n-2)(n-3)}{1.2.3}d_3 + \&c.,$$

where $a =$ the first term of the given series, $n =$ the number of the term sought; and d_1, d_2, d_3, &c., the first term of the first, second, third, &c., orders of differences;

$$\therefore \text{ the term sought} = 2 + 6 \times 4 + \frac{6 \times 5}{1.2} \times 2 = 56.$$

4. Find the first term of the third order of differences in the series $1.2.3 + 2.3.4 + 3.4.5 + 4.5.6 + \&c.$, . . $= 6$.

5. Find the 12th term of the series whose first term is 6, first difference $d_1 = 18$, second difference $d_2 = 18$, third difference $d_3 = 6$, and fourth difference $= 0$;

$$12\text{th term} = 6 + 11 \times 18 + \frac{11 \times 10}{1.2} \times 18 + \frac{11 \times 10 \times 9}{1.2.3} \times 6 = 2184;$$

or, from the first term and the differences the series itself may be found, which is that given in the 4th exercise; hence the term sought is evidently

$$12 \times 13 \times 14 = 2184.$$

6. Find the sum of 18 terms of the series 3 + 11 + 31 + 69 + 131 + &c.

Here $a = 3$, $d_1 = 8$, $d_2 = 12$, $d_3 = 6$, $d_4 = 0$, and $n = 18$;

$$\therefore s = 18 \times 3 + \frac{18 \times 17}{1.2} \times 8 + \frac{18 \times 17 \times 16}{1.2.3} \times 12 + \frac{18 \times 17 \times 16 \times 15}{1.2.3.4} \times 6 = 29430.$$

The general term of this series is $\{n^3 + (n + 1)\}$.

7. Find the sum of 20 terms of the series 4 + 10 + 18 + 28 + 40 + &c.

Here $a = 4$, $d_1 = 6$, $d_2 = 2$, $d_3 = 0$, and $n = 20$;

$$\therefore s = 20 \times 4 + \frac{20 \times 19}{1.2} \times 6 + \frac{20 \times 19 \times 18}{1.2.3} \times 2 = 3500.$$

The general term of this series is $n^2 + 3n$.

8. Find the sum of 30 terms of the series 3 + 7 + 13 + 21 + 31 + &c.

Here $a = 3$, $d_1 = 4$, $d_2 = 2$, $d_3 = 0$, and $n = 30$;

$$\therefore s = 30 \times 3 + \frac{30 \times 29}{1.2} \times 4 + \frac{30 \times 29 \times 28}{1.2.3} \times 2 = 9950.$$

The general term of this series is $n^2 + n + 1$.

9. Find the 100th term of the series 2 + 7 + 14 + 23 + 34 + &c.

Here $a = 2$, $d_1 = 5$, $d_2 = 2$, $d_3 = 0$, and $n = 100$;

$$\therefore \text{ the 100th term} = a + (n-1)d_1 + \frac{(n-1)(n-2)}{2}d_2$$

$$= 2 + 99 \times 5 + \frac{99 \times 98}{2} \times 2 = 10199.$$

The general term of this series is $n^2 + 2n - 1$.

10. Find the 40th term of the series 2 + 11 + 32 + 71 + 134 + &c.

Here $a = 2$, $d_1 = 9$, $d_2 = 12$, $d_3 = 6$, $d_4 = 0$, and $n = 40$;

$$\therefore \text{ the 40th term} = a + (n-1)d_1 + \frac{(n-1)(n-2)}{1.2}d_2 + \frac{(n-1)(n-2)(n-3)}{1.2.3}d_3$$

$$= 2 + 39 \times 9 + \frac{39 \times 38}{1.2} \times 12 + \frac{39 \times 38 \times 37}{1.2.3} \times 6$$

$$= 2 + 351 + 8892 + 54834 = 64079.$$

11. Find the sum of 30 terms of the series $2 + 11 + 32 + 71 + 134 +$ &c.

Here $a = 2$, $d_1 = 9$, $d_2 = 12$, $d_3 = 6$, $d_4 = 0$, and $n = 30$;

$$\therefore s = na + \frac{n(n-1)}{1 \cdot 2}d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}d_2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}d_3 = 217125.$$

The general term is $n^3 + 2n - 1$.

12. Find the series of which $a = 3$, $d_1 = 6$, $d_2 = 2$, and $d_3 = 0$; and find its n^{th} term, and the sum of 12 terms.

The series is $3 + 9 + 17 + 27 + 39 +$ &c.

Its n^{th} term is $n^2 + 3n - 1$.

The sum of 12 terms is $s = 872$.

13. Find the series of which $a = 2$, $d_1 = 7$, $d_2 = 12$, $d_3 = 6$, and $d_4 = 0$; and find its general term, and the sum of 20 terms.

The series is $2 + 9 + 28 + 65 + 126 +$ &c.

Its general term is $n^3 + 1$.

The sum of 20 terms is $s = 44120$.

14. Find the series of which $a = 5$, $d_1 = 6$, $d_2 = 2$, and $d_3 = 0$; find also the general term, and the sum of n terms.

The series is $5 + 11 + 19 + 29 + 41 +$ &c.

Its general term is $n^2 + 3n + 1$.

The sum of n terms is

$$s = 5n + \frac{n(n-1)}{1 \cdot 2} \times 6 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \times 2$$

$$= \frac{30n}{6} + \frac{18n^2 - 18n}{6} + \frac{2n^3 - 6n^2 + 4n}{6}$$

$$= \frac{2n^3 + 12n^2 + 16n}{6} = \frac{n}{3}(n+2)(n+4).$$

15. The general term of a series $n^4 - n^3 + 1$; find the series, the first term of the several orders of differences, and the sum of 20 terms.

The series is $1 + 9 + 55 + 193 + 501 +$ &c.

The differences are $d_1 = 8$, $d_2 = 38$, $d_3 = 54$, $d_4 = 24$, $d_5 = 0$.

The sum of 20 terms $= 678586$.

16. Find, by the method of subtraction, the sum of the series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\&c. \ldots \frac{1}{(2n-1)(2n+1)}$ to n terms; and also to infinity.

Let $$\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\frac{1}{2n-1}=s \quad \ldots \quad (a),$$

then $$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ldots\frac{1}{2n+1}=s+\frac{1}{2n+1}-1 \,(b);$$

$$\therefore (a)-(b)\ \frac{2}{1.3}+\frac{2}{3.5}+\frac{2}{5.7}+\ldots\frac{2}{(2n-1)(2n+1)}=\frac{2n}{2n+1};$$

hence, $$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$

Therefore the sum of n terms is $\frac{n}{2n+1}$; and if n become infinite,

$$\frac{n}{2n+1}=\frac{1}{2+\frac{1}{n}}=\frac{1}{2}.$$

17. Find, by the method of subtraction, the sum of the series $\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots\frac{1}{(4n-1)(4n+3)}$, to n terms; and also to infinity.

The sum to n terms $=\frac{n}{3(4n+3)}$; and to infinity $=\frac{1}{12}$.

18. Find the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots\frac{1}{n(n+3)}$, by the method of subtraction to n terms, and to infinity.

Let $$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n}=s,$$

then $$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots\frac{1}{n+3}$$
$$=s+\frac{3n^2+12n+11}{(n+1)(n+2)(n+3)}-\frac{11}{6};$$

subtracting, $$\frac{3}{1.4}+\frac{3}{2.5}+\frac{3}{3.6}+\ldots\frac{3}{n(n+3)}$$
$$=\frac{11}{6}-\frac{3n^2+12n+11}{(n+1)(n+2)(n+3)}.$$

Hence, $$\frac{1}{1\,.\,4}+\frac{1}{2\,.\,5}+\frac{1}{3\,.\,6}+\ldots\frac{1}{n(n+3)}$$
$$=\tfrac{11}{18}-\frac{3n^2+12n+11}{3(n+1)(n+2)(n+3)},$$
which is the sum to n terms.

If n be infinite, this sum reduces simply to $\tfrac{11}{18}$.

20. Find the sum of the series $\frac{1}{1\,.\,2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}+\frac{1}{3\,.\,4\,.\,5}$ + &c., to n terms; and also to infinity.

$$n \text{ terms} = \frac{n(n+3)}{4(n+1)(n+2)}; \text{ to infinity} = \tfrac{1}{4}.$$

21. Find the sum of the series $\frac{1}{1\,.\,3\,.\,5}+\frac{2}{3\,.\,5\,.\,7}+\frac{3}{5\,.\,7\,.\,9}$ + &c., to n terms; and also to infinity.

$$n \text{ terms} = \frac{n(n+1)}{2(2n+1)(2n+3)}; \text{ to infinity} = \tfrac{1}{8}.$$

22. Find the sum of the series $\frac{5}{1\,.\,2\,.\,3\,.\,4}+\frac{7}{2\,.\,3\,.\,4\,.\,5}+\frac{9}{3\,.\,4\,.\,5\,.\,7}$ + &c., to n terms; and also to infinity.

$$n \text{ terms} = \frac{n(n+4)}{3(n+1)(n+3)}; \text{ to infinity} = \tfrac{1}{3}.$$

23. Find the sum of the series $\frac{7}{1\,.\,3\,.\,4\,.\,6}+\frac{9}{2\,.\,4\,.\,5\,.\,7}+\frac{11}{3\,.\,5\,.\,6\,.\,8}$ + &c., to n terms; and also to infinity.

The general term of this series is
$$\tfrac{1}{3}\,.\,\frac{(n+3)(n+5)-n(n+2)}{n(n+2)(n+3)(n+5)}.$$

The sum of n terms $= \dfrac{n(7n^3+84n^2+303n+306)}{40(n+1)(n+2)(n+4)(n+5)}$,

and the sum to infinity $= \tfrac{7}{40}$.

Let $\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+$ &c., to n terms $= s_1$,

" $\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+$ &c., " " $= s_2$,

" $\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+$ &c., " " $= s_3$,

and $\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{9}+$ &c., " " $= s_4$,

then the series may be formed, and its sum found to n terms, by performing the following operation :—

$$\tfrac{1}{3}\{\tfrac{1}{2}(s_1-s_2)-\tfrac{1}{2}(s_3-s_4)\}.$$

24. Find the sum of the series $\frac{2}{1.3.5.7}+\frac{3}{3.5.7.9}+\frac{4}{5.7.9.11}+$ &c., to n terms, and to infinity.

The sum of n terms $=\frac{n(n+3)}{10(2n+1)(2n+5)}$,

and the sum to infinity $=\frac{1}{40}$.

HIGHER EQUATIONS HAVING NUMERICAL COEFFICIENTS.

1. Form the equation whose roots are 3, 7, and -9, $= x^3 - x^2 - 69x + 189 = 0$.

2. " " " " are 10, 6, and -1, $= x^3 - 15x^2 + 44x - 60 = 0$.

3. " " " " are 5, -6, and -4, $= x^3 + 5x^2 - 26x - 120 = 0$.

4. " " " " are 2, 4, and 8, $= x^3 - 14x^2 + 56x - 64 = 0$.

5. " " " " are 2, -3, 5, and -6, $= x^4 + 2x^3 - 35x^2 - 36x + 180 = 0$.

6. " " whose roots are 2, -2, 5, and 1, $= x^4 - 6x^3 + 5x^2 + 24x - 20 = 0$.

7. " " whose roots are 3, 4, 5, and 6, $= x^4 - 18x^3 + 119x^2 - 342x + 360$.

8. " " whose roots are 12, -1, -3, and 4, $= x^4 - 12x^3 - 13x^2 + 144x + 144 = 0$.

9. " " whose roots are 1, -1, 2, 3, and 5, $= x^5 - 10x^4 + 30x^3 - 20x^2 - 31x + 30 = 0$.

10. " " whose roots are 3, 7, -4, -6, and 1, $= x^5 - x^4 - 55x^3 + 25x^2 + 534x - 504 = 0$.

11. " " whose roots are 2, 3, $2+\sqrt{-3}$, and $2-\sqrt{-3}$, . . . $= x^4 - 9x^3 + 33x^2 - 59x + 42 = 0$.

12. Form the equation whose roots are $3+\sqrt{-2}$, $3-\sqrt{-2}$, 5, and 7, . . . $= x^4 - 18x^3 + 118x^2 - 342x + 385 = 0$.

13. Change the equation $x^3 - 6x^2 + 21x - 30 = 0$, into another wanting the second term, . . $= y^3 - 9y - 4 = 0$.

14. Change the equation $x^3 + 12x^2 - 72x + 36 = 0$, into another wanting the second term, . $= y^3 - 120y + 452 = 0$.

15. Change the equation $x^4 - 12x^3 - 13x^2 + 144x + 204 = 0$, into another wanting the second term,
$= y^4 - 67y^2 - 150y + 276 = 0.$

16. Change the equation $x^5 - 10x^4 + 30x^3 - 20x^2 - 31x + 30 = 0$, into another wanting the second term, $= y^5 - 10y^3 + 9y = 0.$

17. Change the equation $x^3 + 6x^2 - 21x + 30 = 0$, into another wanting the second term, . . . $= y^3 - 33y + 88 = 0.$

18. Given the equation $x^3 - 5x^2 - 2x + 24 = 0$, to find the equation whose roots are the double of the roots of the given equation; and also the equation whose roots shall be half the roots of the given equation,

$$= \begin{cases} y^3 - 10y^2 - 8y + 192 = 0, \\ \text{and } 8y^3 - 20y^2 - 4y + 24 = 0. \end{cases}$$

19. Given the equation $x^4 + x^3 - 32x^2 + 12x + 144 = 0$; write the equations whose roots are respectively double, and one-half the roots of the given equation,

$$= \begin{cases} y^4 + 2y^3 - 128y^2 + 96y + 2304 = 0, \\ \text{and } 16y^4 + 8y^3 - 128y^2 + 24y + 144 = 0. \end{cases}$$

20. Find the equation whose roots are three times as great as the roots of the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$,
$= y^4 - 12y^3 - 63y^2 + 594y + 1944 = 0.$

21. Find the equation whose roots are one-third part of the roots of the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$,
$= 81y^4 - 108y^3 - 63y^2 + 66y + 24 = 0.$

22. Change the equation $x^4 - 6x^3 + x^2 + 24x - 20 = 0$, into another whose roots shall be the reciprocals of its roots,
$= 20y^4 - 24y^3 - y^2 + 6y - 1 = 0.$

23. Change the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$, into another whose roots shall be the reciprocals of its roots,
$= 24y^4 + 22y^3 - 7y^2 - 4y + 1 = 0.$

24. Find the equation whose roots are greater than the roots of the equation $x^3 - x^2 - 69x + 189 = 0$, by 2,
$= y^3 - 7y^2 - 53y + 315 = 0.$

25. Find the equation whose roots are less than the roots of the equation $x^4 + 2x^3 - 35x^2 - 36x + 180 = 0$, by 1,
$= y^4 + 6y^3 - 23y^2 - 96y + 112 = 0.$

26. Find the equation whose roots are less by 2 than the roots of the equation $x^4 - 18x^3 + 119x^2 - 342x + 360 = 0$,
$= y^4 - 10y^3 + 35y^2 - 50y + 24 = 0.$

27. Find the equation whose roots are less by 5 than the roots of the equation $x^4 - 12x^3 - 13x^2 + 144x + 144 = 0$,
$= y^4 + 8y^3 - 43y^2 - 386y - 336 = 0.$

28. Diminish the roots of $x^4 - 6x^3 + x^2 + 24x - 20 = 0$, by a quantity greater than the greatest positive root,
$= y^4 + 18y^3 + 109y^2 + 252y + 160 = 0.$

Find the roots of the following nine equations by the method of divisors :—

29. $x^3 - 4x^2 - 17x + 60 = 0$, $= 3$, 5, and -4.

30. $x^3 - x^2 - 44x + 84 = 0$, $= 6$, -7, 〃 2.

31. $x^3 - 10x^2 + 31x - 30 = 0$, $= 5$, 3, 〃 2.

32. $x^3 - 9x^2 - 10x + 168 = 0$, $= 7$, 6, 〃 -4.

33. $x^3 - 20x^2 + 111x - 180 = 0$, $= 3$, 5, 〃 12.

34. $x^3 - 18x^2 + 15x + 34 = 0$, $= 2$, 17, 〃 -1.

35. $x^3 - 11x^2 - 17x + 195 = 0$, $= 7$, 9, 〃 -5.

36. $x^3 - 79x + 210 = 0$, . . $= 3$, 7, 〃 -10.

37. $x^3 + 15x^2 + 2x - 312 = 0$, $= 4$, -6, 〃 -13.

38. The equation $x^4 + 7x^3 + 9x^2 - 27x - 54 = 0$, has three equal roots; find all its roots.

The three equal roots are each $= -3$, and the other root $= +2$.

39. The equation $x^4 - 4x^3 - 13x^2 + 28x + 60 = 0$, has two equal roots; find all its roots.

The two equal roots are each $= -2$, and the other roots are $= 3$ and 5.

40. The equation $x^5 - 19x^4 + 114x^3 - 238x^2 + 205x - 63 = 0$, has three equal roots; find all its roots.

The three equal roots are each $= 1$, and the other roots are $= 7$ and 9.

41. The equation $x^5 - 12x^4 + 39x^3 + 44x^2 - 432x + 576 = 0$, has three equal roots; find all its roots.

The three equal roots are each $= 4$, and the other roots are $= 3$ and -3.

42. Find a root of the equation $x^3 - 7x^2 - 45x + 96 = 0$,
$= 1{\cdot}7694192$.

43. Prove by Sturm's Theorem that the equation $x^3 - 7x^2 - 45x + 96 = 0$, given in last exercise, has other two real roots, one between 10 and 11, and the other between -5 and -6; and find these roots, ... $= 10{\cdot}4316089$, and $-5{\cdot}2010281$.

44. Prove that the equation $x^4 + 2x^3 - 51x^2 + 44x + 252 = 0$, has two positive and two negative roots; analyse it by Sturm's Theorem, and find its four roots.

The roots are $3{\cdot}8284272$, $4{\cdot}3245554$,
$-1{\cdot}8284272$, and $-8{\cdot}3245554$.

45. Find one positive and one negative root of the equation $x^4 + 3x^3 + 7x^2 + 28x - 100 = 0$; and prove by Budan's or Sturm's Theorems that the other two roots are imaginary.

$x = 1{\cdot}7897628$, and $-4{\cdot}2305476$.

46. Prove by Budan's Theorem that the roots of the equation $x^4 + 2x^3 - 39x^2 + 80x - 45 = 0$, are all real; and find the four roots.

The roots are 1·1097722, 1·381966,
3·618034, and $-$ 8·1097722.

47. Prove by Budan's Theorem that the equation $x^4 + 6x^3 - 4x^2 + 7x - 9 = 0$, has two imaginary roots, and find its two real roots.

The real roots are ·950065, and $-$ 6·7833039.

48. Find the four roots of the equation $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$.

The roots are 32·060291, 12·756442, ·350987, and 34·832280.

49. Find the four roots of the equation $x^4 + 312x^3 + 23337x^2 - 14874x + 2360 = 0$.

The roots are ·316664473, $-$ 126·316664473, ·316665178, and $-$ 186·316665178.

50. Find the negative root of the equation $x^5 - 32x^3 + 72x^2 - 185x + 360 = 0$, which lies between 6 and 7, and prove that the equation has two imaginary roots.

The root sought is $= 6{\cdot}88855$ nearly.

CONTINUED FRACTIONS.

1. Reduce $\frac{339}{945}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 2, 1, 3, 1, 2, 2, 3.$

$$\text{Convergents} = \frac{1}{2}, \frac{1}{3}, \frac{4}{11}, \frac{5}{14}, \frac{14}{39}, \frac{33}{92}, \text{ and } \frac{113}{315}.$$

2. Reduce $\frac{743}{611}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 1, 4, 1, 1, 1, 2, 3, 1, 3.$

$$\text{Convergents} = \frac{5}{4}, \frac{6}{5}, \frac{11}{9}, \frac{17}{14}, \frac{45}{37}, \frac{152}{125}, \frac{197}{162}, \text{ and } \frac{743}{611}.$$

3. Reduce $\frac{351}{965}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 2, 1, 2, 1, 87.$

Convergents $= \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}$, and $\frac{351}{965}$.

4. Reduce $\frac{251}{764}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 3, 22, 1, 4, 2.$

Convergents $= \frac{1}{3}, \frac{22}{67}, \frac{23}{70}, \frac{114}{347}$, and $\frac{251}{764}$.

5. Reduce $\frac{1769}{5537}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 3, 7, 1, 2, 4, 5, 1, 2.$

Convergents $= \frac{1}{3}, \frac{7}{22}, \frac{8}{25}, \frac{23}{72}, \frac{100}{313}, \frac{523}{1637}, \frac{623}{1950}$, and $\frac{1769}{5537}$.

6. Reduce $\frac{907}{18564}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 20, 2, 7, 5, 2, 1, 3.$

Convergents $= \frac{1}{20}, \frac{2}{41}, \frac{15}{307}, \frac{77}{1576}, \frac{169}{3459}, \frac{246}{5035}$, and $\frac{907}{18564}$.

7. Reduce $\frac{587}{1943}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 3, 3, 4, 2, 3, 1, 1, 2.$

Convergents $= \frac{1}{3}, \frac{3}{10}, \frac{13}{43}, \frac{29}{96}, \frac{100}{331}, \frac{129}{427}, \frac{229}{758}$, and $\frac{587}{1943}$.

8. Reduce $\frac{1947}{3359}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 3.$

Convergents $= \frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12}, \frac{11}{19}, \frac{40}{69}, \frac{91}{157}, \frac{131}{226}, \frac{222}{383}, \frac{575}{992}$,

and $\frac{1947}{3359}$.

9. Reduce $\frac{13957}{59476}$ to a continued fraction, and find the approximate convergent fractions.

Quotients $= 4, 3, 1, 4, 1, 2, 1, 11, 2, 6.$

$$\text{Convergents} = \frac{1}{4}, \frac{3}{13}, \frac{4}{17}, \frac{19}{81}, \frac{23}{98}, \frac{65}{277}, \frac{88}{375}, \frac{1033}{4402}, \frac{2154}{9179},$$

and $\frac{13957}{59476}$.

10. Find the successive converging fractions of the moon's sidereal period, which is 27·3216612 days.

Quotients of $\frac{10000000}{3216612} = 3, 9, 5, 2, 1, 1, 1, 16, 1, 11, 1, 8.$

$$\text{Convergents} = \frac{1}{3}, \frac{9}{28}, \frac{46}{143}, \frac{101}{314}, \frac{147}{457}, \frac{248}{771}, \frac{395}{1228}, \frac{6568}{36419}, \text{\&c.}$$

and adding 27 days to each, give the approximate lunar period in days

$$= \frac{82}{3}, \frac{765}{28}, \frac{3907}{143}, \frac{8579}{314}, \frac{12486}{457}, \text{\&c.};$$

that is, the moon revolves 3 times in 82 days nearly; 28 times in 765 days, more nearly; 143 times in 3907, still more nearly; and so on. The third and fourth fractions differ by $\frac{1}{44902}$ of a day, and hence each of them differs from the true period by a fraction of a day less than $\frac{1}{44902}$.

11. Find the successive converging fractions which express the approximate ratio of the years of Mercury and the earth, the year of Mercury being 87·969255 days, and that of the earth 365·256384 days.

Quotients $= 4, 6, 1, 1, 2, 1, 5, 131, 1, 12, 1, 5, 10.$

$$\text{Convergents} = \frac{1}{4}, \frac{6}{25}, \frac{7}{29}, \frac{13}{54}, \frac{33}{137}, \frac{46}{191}, \frac{263}{1092}, \frac{34499}{143243}, \frac{34762}{144335}, \text{\&c.}$$

12. Find the successive converging fractions which express the approximate ratio of the years of Venus and the earth, the year of Venus being 224·700817 days, and that of the earth 365·256384 days.

Quotients $= 1, 1, 1, 1, 2, 29, 2, 27, 1, 1, 2, 2, 4, 7, 8, 1, 3.$

$$\text{Convergents} = \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{8}{13}, \frac{235}{382}, \frac{478}{777}, \frac{13141}{21361}, \frac{13619}{22138}, \frac{26760}{43499}, \text{\&c.}$$

13. Find the converging fractions of the $\sqrt{3}$.

The continued fraction $= 2 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \&c.}}}$

Convergents $= \frac{7}{4}, \frac{26}{15}, \frac{97}{56}, \frac{362}{209}, \frac{1351}{780}$, &c.

14. Find the converging fractions to the $\sqrt{5}$.

The continued fraction $= 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4} + \&c.}}$

Convergents $= \frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \frac{682}{305}, \frac{2889}{1292}$, &c.

15. Find the converging fractions to the $\sqrt{19}$.

The continued fraction $= 4 + \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{8+}\ \frac{1}{2+}$, &c.

Convergents $= \frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}, \frac{1421}{326}, \frac{3012}{691}$, &c.

16. Find the converging fractions to the $\sqrt{31}$.

The continued fraction $= 5 + \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{5+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{10+}$, &c.

Convergents $= \frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \frac{206}{37}, \frac{657}{118}, \frac{863}{155}, \frac{1520}{273}, \frac{16063}{2885}$, &c.

17. Find the converging fractions to the $\sqrt{45}$.

The continued fraction $= 6 + \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{12+}$, &c.

Convergents $= \frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}, \frac{161}{24}, \frac{20496}{305}$, &c.

18. Find the converging fractions to the $\sqrt{28}$.

The continued fraction $= 5 + \frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{10+}$, &c.

Convergents $= \frac{5}{1}, \frac{16}{3}, \frac{37}{7}, \frac{127}{24}, \frac{1307}{247}$, &c.

INDETERMINATE EQUATIONS.

1. Divide 142 into two such parts that one part may be divisible by 9, and the other by 14.

Here $$9x + 14y = 142;$$

$$\therefore x = 8, \text{ and } y = 5;$$

$$\therefore \text{the parts required are } 9x = 72, \text{ and } 14y = 70.$$

2. Find the least integral values of x and y that fulfil the conditions of the equation $19x - 117y = 11$, . $x = 56, y = 9$.

3. Find all the integral values of x and y that the equation $13x + 14y = 200$ admits of, ... $x = 10, y = 5$.

4. Find all the integral values of x and y that fulfil the conditions of the equation $7x - 13y = 6$, . . $x = 12, y = 6$.

5. Find a number between 100 and 200, which, when divided by 12, leaves a remainder of 10; and when divided by 15, leaves a remainder of 4.

Equation $12x + 10 = 15y + 4$; number $= 154$.

6. A boy has between 100 and 400 marbles, and finds that when he counts them by 13's, there are 9 over; and when he counts them by 17's, there are 14 over: how many had he?

Equation $13x + 9 = 17y + 14$; number $= 269$.

7. Find the integral values of x and y in the equation $9x + 13y = 2000$;

$$\left.\begin{array}{l} x = 215, 202, 189, 176, 163, 150, 137, 124, \&c. \\ y = \quad 5, \ \ 14, \ \ 23, \ \ 32, \ \ 41, \ \ 50, \ \ 59, \ \ 68, \&c. \end{array}\right\} \text{17 solutions.}$$

8. Find the integral values of x and y, and the number of solutions in the equation $11x + 13y = 290$;

$$\left.\begin{array}{l} x = 24 \text{ or } 11, \\ y = \ 2 \text{ or } 13. \end{array}\right\} \text{Only two solutions.}$$

9. Find two integers such that if the first be multiplied by 17,

and the second by 26, the first product shall be greater than the second by 7.

The equation is $17x = 26y + 7$; and the numbers sought are

$$\left.\begin{array}{l} x = 5, 31, 57, \\ y = 3, 20, 37, \end{array}\right\} \text{ \&c., number of solutions infinite.}$$

10. Divide 1591 into two such parts, that one may be divisible by 23, and the other by 34.

Here $23x + 34y = 1591$;

$$\therefore x = 47 \text{ or } 13, \; y = 15 \text{ or } 38.$$

The parts sought are 1081 and 510, or 299 and 1292.

11. Find two fractions having 7 and 11 for denominators, whose sum is equal to $\frac{43}{77}$.

Here $\frac{x}{7} + \frac{y}{11} = \frac{43}{77}$, or $11x + 7y = 43$;

$$\therefore x = 2, \; y = 3,$$

and the fractions are $\frac{2}{7}$, and $\frac{3}{11}$.

12. Find three fractions having prime numbers for their denominators, and whose sum shall be $\frac{222}{385}$, . . $= \frac{1}{5}, \frac{2}{7}, \frac{3}{11}$.

13. A person buys 124 head of cattle—namely, pigs, sheep, and lambs—for £400. The pigs cost £4, 10s. a head; the sheep, £3, 3s. 4d.; and the lambs, £1, 5s.; how many did he buy of each kind?

Here $$\begin{array}{r} 4\tfrac{1}{2}x + 3\tfrac{1}{6}y + 1\tfrac{1}{4}z = 400, \\ x + y + z = 124; \end{array} \quad \therefore \left\{\begin{array}{l} x = 17, 40, 63, \\ y = 99, 60, 21, \\ z = 8, 24, 40. \end{array}\right.$$

14. Find three integers such that if the first be multiplied by 5, the second by 13, and the third by 18, the sum of the products shall be 997; but if the first be multiplied by 11, the second by 20, and the third by 37, the sum of the products shall be 1866.

Here $$\begin{array}{r} 5x + 13y + 18z = 997, \\ 11x + 20y + 37z = 1866; \end{array} \quad \therefore \left\{\begin{array}{l} x = 16, \\ y = 29, \\ z = 30. \end{array}\right.$$

15. Find the number of integral solutions of the equation $3x + 7y + 17z = 100$.

Since z cannot be less than one, and must be an integer, take $z = 1, 2, 3$, &c., and we have the following equations in x and y.

$3x + 7y = 83$, in which $x = 23, 16, 9, 2$, $y = 2, 5, 8, 11$;

$3x + 7y = 66$, " $x = 15, 8, 1$, $y = 3, 6, 9$;

$3x + 7y = 49$, " $x = 14, 7$, $y = 1, 4$;

$3x + 7y = 32$, " $x = 6$, $y = 2$;

$3x + 7y = 15$, no solution:

$\therefore$ the total number of solutions is

$$= 4 + 3 + 2 + 1 = 10.$$

16. Divide 30 into three such parts, that if the first be multiplied by 7, the second by 19, and the third by 38, the sum of the products shall be 745.

Here
$$7x + 19y + 38z = 745,$$
$$x + y + z = 30;$$
$$\therefore x = 6, y = 11, \text{ and } z = 13.$$

THE END.

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