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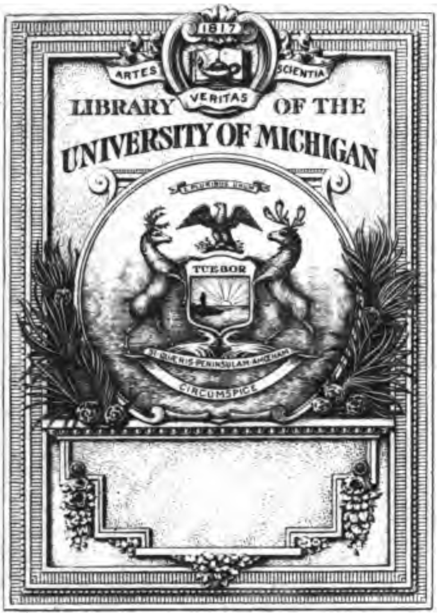
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Samuel H. Wright, A. M., Ph. D.

A

KEY

TO

BLAND'S

ALGEBRAICAL PROBLEMS.

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A

KEY

TO

BLAND'S, Miles, 1786-1868

ALGEBRAICAL PROBLEMS;

CONTAINING THE

SOLUTIONS

OF THE

EQUATIONS AND PROBLEMS

IN THE

PRAXIS CONTAINED IN SECTION XI.

At pages 293 – 383
of the Book of Problems

LONDON:
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AVE MARIA LANE.

1836.

A KEY.

1. *Simple Equations involving one unknown Quantity.*

1. (17) By transposition, $19x + 4x = 59 - 13$,

$$\text{or } 23x = 46,$$

$$\text{and } \therefore \text{ (18. Cor. 2.) } x = \frac{46}{23} = 2.$$

2. (18. Cor. 1.) multiplying every term by 3,

$$9x + 12 - x = 138 - 6x,$$

$$\therefore \text{ (17) by transposition, } 9x + 6x - x = 138 - 12,$$

$$\text{or } 14x = 126,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{126}{14} = 9.$$

3. (18. Cor. 2.) dividing every term by x,

$$x + 15 = 35 - 3x,$$

$$\therefore \text{ (17) by transposition, } x + 3x = 35 - 15,$$

$$\text{or } 4x = 20,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{20}{4} = 5.$$

4. Here 12 is the least common multiple of 6, 4, 3, 2; therefore (18. Cor. 1.) multiplying both sides of the equation by 12,

$$2x - 3x + 120 = 4x - 6x + 132,$$

$\therefore$ (17) by transposition, $2x + 6x - 3x - 4x = 132 - 120$,

$$\text{or } x = 12.$$

5. (18. Cor. 1.) multiplying by 15 the least common multiple of 3 and 5,

$$3x + 3 + 45 = 10x - 15,$$

$\therefore$ (17) by transposition, $3 + 45 + 15 = 10x - 3x$,

$$\text{or } 63 = 7x,$$

$$\therefore \text{(18. Cor. 2.) } \frac{63}{7} = 9 = x.$$

6. (18. Cor. 1.) multiplying by 21 the least common multiple of 3 and 7,

$$49x + 14 + 105x = 588 + 15x - 18,$$

$\therefore$ (17) by transposition, $49x + 105x - 15x = 588 - 18 - 14$,

$$\text{or } 139x = 556,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{556}{139} = 4.$$

7. (18. Cor. 1.) multiplying every term by 5,

$$3x + 4 + 10x = 22 - x + 80,$$

$\therefore$ by transposition, $3x + 10x + x = 22 + 80 - 4$,

$$\text{or } 14x = 98,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{98}{14} = 7.$$

8. Here 12 is the least common multiple of 2, 4, 6; (18. Cor. 1.) multiplying therefore both sides of the equation by 12,

$$42 - 6x + 48 = 9x - 33 + 16x + 30,$$

$\therefore$ (17) by transposition,

$$42 + 48 + 33 - 30 = 9x + 16x + 6x,$$

$$\text{or } 93 = 31x,$$

$$\therefore \text{(18. Cor. 2.) } \frac{93}{31} = 3 = x.$$

9. Since 18 contains 9, 3, and 2, a certain number of times exactly, it will be the least common multiple of 18, 9, 3, 2; and ∴ (18. Cor. 1.) multiplying both sides of the equation by 18,

$$2x - 5 + 114 - 6x = 20x - 14 - 45,$$

$$\therefore \text{(17) by transposition, } 114 + 14 + 45 - 5 = 20x + 6x - 2x,$$

$$\text{or } 168 = 24x,$$

$$\therefore \text{(18. Cor. 2.) } \frac{168}{24} = 7 = x.$$

10. (18. Cor. 1.) multiplying both sides of the equation by 12, the product of 3 and 4,

$$12x - 8x - 4 = 3x + 9,$$

$$\therefore \text{(17) by transposition, } 12x - 8x - 3x = 9 + 4,$$

$$\text{or } x = 13.$$

11. (18. Cor. 1.) multiplying both sides of the equation by 24, the product of 3 and 8.

$$9x + 15 - 168 - 8x = 936 - 120x,$$

$$\therefore \text{(17) by transposition, } 9x + 120x - 8x = 936 + 168 - 15,$$

$$\text{or } 121x = 1089.$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{1089}{121} = 9.$$

12. (18. Cor. 1.) multiplying both sides of the equation by $4 \times 5 = 20$,

$$80x - 76 - 8x = 300 - 35x - 55,$$

$$\therefore \text{(17) by transposition, } 80x + 35x - 8x = 300 + 76 - 55,$$

$$\text{or } 107x = 321,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{321}{107} = 3.$$

13. (18. Cor. 1.) multiplying both sides of the equation by 36, the least common multiple of 3, 4, and 9,

$$252 - 36x - 16x - 24 = 216 - 45x - 9,$$

$\therefore$ (17) by transposition,

$$252 + 9 - 216 - 24 = 36x + 16x - 45x,$$

$$\text{or } 21 = 7x,$$

$$\therefore \text{(18. Cor. 2.) } \frac{21}{7} = 3 = x.$$

14. Since 16 is a multiple of 8 and 4, it is the least common multiple of 16, 8, and 4; $\therefore$ (18. Cor. 1.) multiplying both sides of the equation by 16,

$$122 + 12x - 4 - 7x - 3 = 16x + 38,$$

$\therefore$ (17) by transposition,

$$122 - 4 - 3 - 38 = 16x + 7x - 12x,$$

$$\text{or } 77 = 11x,$$

$$\therefore \text{(18. Cor. 2.) } \frac{77}{11} = 7 = x.$$

15. (18. Cor. 1.) multiplying both sides of the equation by $2 \times 3 \times 11 = 66$,

$$36x + 48 - 165x - 99 = 594 - 88x - 99x - 297,$$

$\therefore$ (17) by transposition,

$$36x + 88x + 99x - 165x = 594 + 99 - 297 - 48,$$

$$\text{or } 58x = 348,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{348}{58} = 6.$$

16. Since 12 is a multiple of 6, 4, 3, it is the least common multiple of 12, 6, 4, 3; $\therefore$ (18. Cor. 1.) multiplying both sides of the equation by 12,

$$12x + 81 - 27x - 10x - 4 = 61 - 8x - 20 - 29 - 4x,$$

$\therefore$ (17) by transposition,

$$81 + 20 + 29 - 4 - 61 = 27x + 10x - 12x - 8x - 4x,$$

$$\text{or } 65 = 13x,$$

$$\therefore \text{(18. Cor. 2.) } \frac{65}{13} = 5 = x.$$

17. (18. Cor. 1.) multiplying both sides of the equation by $2 \times 11 \times 13 = 286$,

$$182x - 208 + 330x + 176 = 858x - 4433 + 143x,$$

$\therefore$ (17) by transposition,

$$176 + 4433 - 208 = 858x + 143x - 182x - 330x,$$

$$\text{or } 4401 = 489x,$$

$$\therefore \text{(18. Cor. 2.)} \frac{4401}{489} = 9 = x.$$

18. (18. Cor. 1.) multiplying by 10 the least common multiple of the denominators,

$$25x - 5 - 7x + 2 = 66 - 5x,$$

$$\therefore \text{(17) by transposition, } 25x + 5x - 7x = 66 + 5 - 2,$$

$$\text{or } 23x = 69,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{69}{23} = 3.$$

19. (18. Cor. 1.) multiplying by 36 the least common multiple of the denominators,

$$27x - 27 - 36x + 48 = 192 - 108 - 16x,$$

$\therefore$ (17) by transposition,

$$27x + 16x - 36x = 192 + 27 - 108 - 48,$$

$$\text{or } 7x = 63,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{63}{7} = 9.$$

20. (18. Cor. 1.) multiplying by $2 \times 3 \times 17 = 102$,

$$24x - 204 - 8772 + 170x = 3519 - 51x,$$

$\therefore$ (17) by transposition,

$$24x + 170x + 51x = 3519 + 204 + 8772,$$

$$\text{or } 245x = 12495,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{12495}{245} = 51.$$

21. (18. Cor. 1.) multiplying by $5 \times 7 \times 13 = 455$,

$$910x - 140x + 70 = 182x + 1001 - 455 + 520x,$$

∴ (17) by transposition,

$$910x - 140x - 182x - 520x = 1001 - 455 - 70,$$

$$\text{or } 68x = 476,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{476}{68} = 7.$$

22. (18. Cor. 1.) multiplying by 29 × 12 the least common multiple of the denominators,

$$24x + 12 - 11658 + 87x = 3132 - 81954 + 1044x,$$

∴ by transposition,

$$12 + 81954 - 11658 - 3132 = 1044x - 24x - 87x,$$

$$\text{or } 67176 = 933x,$$

$$\therefore \text{ (18. Cor. 2.) } \frac{67176}{933} = 72 = x.$$

23. (18. Cor. 1.) multiplying by 56 the least common multiple of the denominators,

$$49x + 63 - 24x - 8 = 126x - 182 - 996 + 36x,$$

∴ by transposition,

$$63 + 182 + 996 - 8 = 126x + 36x + 24x - 49x,$$

$$\text{or } 1233 = 137x,$$

$$\therefore \text{ (18. Cor. 2.) } \frac{1233}{137} = 9 = x.$$

24. (18. Cor. 1.) multiplying by 72 the least common multiple of the denominators,

$$360 - 54x - 450 - 136 + 48x = 146 + 72x - 81x - 360,$$

∴ (17) by transposition,

$$48x + 81x - 54x - 72x = 146 + 450 + 136 - 360 - 360,$$

$$\text{or } 3x = 12,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{12}{3} = 4.$$

25. (18. Cor. 1.) multiplying by 72 the least common multiple of the denominators,

$42x - 258 + 972 - 60 - 48x = 18432 - 24x + 96 - 45x - 261 - 864x$,

$\therefore$ (17) by transposition, $42x + 24x + 45x + 864x - 48x = 18432 + 96 + 258 + 60 - 261 - 972$,

$$\text{or } 927x = 17613,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{17613}{927} = 19.$$

26. (18. Cor. 1.) multiplying by 720 the least common multiple of the denominators,

$2880x + 72 - 135x + 585 - 960 - 560x = 5040x - 23760 - 648 - 360x - 990x + 1530$,

$\therefore$ (17) by transposition,

$72 + 585 + 23760 + 648 - 960 - 1530 = 5040x + 135x + 560x - 360x - 990x - 2880x$,

$$\text{or } 22575 = 1505x,$$

$$\therefore \text{ (18. Cor. 2.) } \frac{22575}{1505} = 15 = x.$$

27. (18. Cor. 1.) multiplying by $3 \times 7 \times 11 \times 16 = 3696$,

$38192 + 4928x - 1386x - 21714 - 693x + 4389 = 176484 + 5376 - 336x - 2640x - 10560$,

$\therefore$ (17) by transposition,

$4928x + 336x + 2640x - 1386x - 693x = 176484 + 5376 + 21714 - 10560 - 38192 - 4389$,

$$\text{or } 8849x = 150433,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{150433}{8849} = 17.$$

28. Multiplying both sides of the equation by x,

$$3a + x - 5x = 6.$$

$$\therefore \text{ (17) by transposition, } 3a - 6 = 4x.$$

$$\text{and (18. Cor. 2.) } \frac{3a - 6}{4} = x.$$

29. (17) by transposition,

$$6cx - \frac{2}{3}cx = 2ab - \frac{5}{6}ab + \frac{3}{4}ac - \frac{4}{5}ac,$$

$$\text{or } \frac{16cx}{3} = \frac{7ab}{6} - \frac{ac}{20},$$

$$= \frac{(70b - 3c).a}{60},$$

$$\therefore (18.)\ x = \frac{(70b - 3c).a}{320c}.$$

30. (18.) multiplying by 21,

$$\frac{231x - 273}{25} + 57x + 9 - \frac{105x - 532}{4} = 591 - 17x - 4,$$

(17) by transposition,

$$\frac{231x - 273}{25} - \frac{105x - 532}{4} = 578 - 74x,$$

and multiplying by 100,

$$924x - 1092 - 2625x + 13300 = 57800 - 7400x,$$

$$\therefore \text{ by transposition, } 5699x = 45592,$$

$$\text{and (18. Cor. 2.) } x = \frac{45592}{5699} = 8.$$

31. (18. Cor. 1.) multiplying by 36, the least common multiple of the denominators,

$$9x - 24 - 16x + 28 + 36x = 96 - 6x - 24 + 72,$$

$$\therefore \text{ by transposition, } 35x = 140,$$

$$\text{and (18. Cor. 2.) } x = \frac{140}{35} = 4.$$

32. (18. Cor. 1.) multiplying by 156, the least common multiple of the denominators,

$$78x - 78 - 75 - 24 + 72x = 156x - 20x + 10 - 3x,$$

$$\therefore \text{ by transposition, } 17x = 187,$$

$$\text{and (18. Cor. 2.) } x = \frac{187}{17} = 11.$$

33. (17) by transposition,

$$\left(\frac{a^2}{bc} + b - \frac{e}{f} - d - b\right) . x = \frac{d^2}{a} - b,$$

$$\text{or } \frac{a^2f - bce - bcd}{bcf} . x = \frac{d^2 - ab}{a},$$

$$\therefore (18)\ x = \frac{(d^2 - ab) . bcf}{a^2f - abce - abcd}.$$

34. (17) by transposition, $\left(\frac{a}{b} + \frac{c}{d} + \frac{e}{f}\right) . x = g + h,$

$$\text{or } \frac{adf + bcf + bde}{bdf} . x = g + h,$$

$$\therefore (18)\ x = \frac{(g + h) . bdf}{adf + bcf + bde}.$$

35. (17) by transposition, $\left(\frac{a^2}{b - c} - b\right) . x = (d - a) . c,$

$$(18)\ (a^2 - b^2 + bc) . x = c . (b - c) . (d - a),$$

$$\text{and } \therefore (18. \text{ Cor. } 2.)\ x = \frac{c . (b - c) . (d - a)}{a^2 - b^2 + bc}.$$

36. (18) multiplying by 12,

$$8x - \frac{3 - \frac{3}{2}x}{x} = 6x - 6 + 2x + 7,$$

$$\therefore \text{ by transposition, } \frac{3}{2} - \frac{3}{x} = 1,$$

$$\text{and } \frac{1}{2} = \frac{3}{x},$$

$$\therefore (18. \text{ Cor. } 2.)\ x = 6.$$

37. Multiplying both sides of the equation by 36,

$$9x + 20 = \frac{144x - 432}{5x - 4} + 9x,$$

$$\therefore \text{ (17. Cor. 3.) } 20 = \frac{144x - 432}{5x - 4},$$

$$\text{and } 5 = \frac{36x - 108}{5x - 4},$$

$$\therefore \text{ (18. Cor. 1.) } 25x - 20 = 36x - 108,$$

$$\text{(17) by transposition, } 88 = 11x,$$

$$\therefore \text{ (18. Cor. 2.) } \frac{88}{11} = 8 = x.$$

38. Multiplying both sides of the equation by 25,

$$20x + 36 + \frac{125x + 500}{9x - 16} = 20x + 86,$$

$$\therefore \text{ (17. Cor. 3.) } \frac{125x + 500}{9x - 16} = 50,$$

$$\text{and (18. Cor. 2.) } \frac{5x + 20}{9x - 16} = 2,$$

$$\therefore \text{ (18. Cor. 1.) } 5x + 20 = 18x - 32,$$

$$\text{(17) by transposition, } 20 + 32 = 18x - 5x,$$

$$\text{or } 52 = 13x,$$

$$\therefore \text{ (18. Cor. 2.) } \frac{52}{13} = 4 = x.$$

39. Multiplying both sides of the equation by 18,

$$10x + 17 - \frac{216x + 36}{13x - 16} = 10x - 8,$$

$$\therefore \text{ (17. Cor. 3.) } 25 = \frac{216x + 36}{13x - 16},$$

$$\text{and (18. Cor. 1.) } 325x - 400 = 216x + 36,$$

$$\text{(17) by transposition, } 325x - 216x = 36 + 400,$$

$$\text{or } 109x = 436,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{436}{109} = 4,$$

40. Multiplying both sides of the equation by 28,

$$18x - 19 + \frac{154x + 294}{3x + 7} = 18x + 30,$$

$$\therefore \text{(17. Cor. 3.)} \ \frac{154x + 294}{3x + 7} = 49,$$

$$\text{and (18. Cor. 2.)} \ \frac{22x + 42}{3x + 7} = 7,$$

$$\text{(18. Cor. 1.)} \ 22x + 42 = 21x + 49,$$

$$\text{(17) by transposition,} \ 22x - 21x = 49 - 42,$$

$$\text{or } x = 7.$$

41. Multiplying both sides of the equation by 36,

$$8x + 34 - \frac{468x - 72}{17x - 32} + 12x = 21x - x - 16,$$

$$\therefore \text{(17. Cor. 3.)} \ 50 = \frac{468x - 72}{17x - 32},$$

$$\text{and (18. Cor. 1.)} \ 850x - 1600 = 468x - 72,$$

$$\therefore \text{ by transposition, } 382x = 1528,$$

$$\text{and (18. Cor. 2.)} \ x = \frac{1528}{382} = 4.$$

42. Multiplying both sides of the equation by 84,

$$21x + 18 - \frac{168x + 360}{23x - 6} + 21x = 44x - 2x + 6,$$

$$\therefore \text{(17. Cor. 3.)} \ 12 = \frac{168x + 360}{23x - 6},$$

$$\text{and (18. Cor. 2.)} \ 1 = \frac{14x + 30}{23x - 6},$$

$$\therefore \text{(18. Cor. 1.)} \ 23x - 6 = 14x + 30,$$

$$\text{by transposition, } 9x = 36,$$

$$\therefore \text{(18. Cor. 2.)} \ x = \frac{36}{9} = 4.$$

43. Multiplying both sides of the equation by 105,

$$42 - 35x - \frac{15 \cdot (7 - 2x^2)}{2 \cdot (x - 1)} = 5 + 15x - 35x + \frac{77}{2} + 1,$$

$$\therefore \text{(17. Cor. 3.)} - \frac{15}{2} \cdot \frac{7 - 2x^2}{x - 1} = 15x + \frac{5}{2},$$

$$\text{and (18. Cor. 2.)} - \frac{21 - 6x^2}{x - 1} = 6x + 1,$$

$$\text{whence (18. Cor. 1.) } 6x^2 - 21 = 6x^2 - 5x - 1,$$

$$\text{by transposition, } 5x = 20,$$

$$\therefore \text{(18. Cor. 2.) } x = \frac{20}{5} = 4.$$

44. Multiplying both sides of the equation by $\frac{bx}{a}$,

$$b^2 + x^2 = bcx + x^2,$$

$$\therefore \text{(17. Cor. 3.) } b^2 = bcx,$$

$$\text{and (18. Cor. 2.) } \frac{b}{c} = x.$$

45. (18. Cor. 2.) $\frac{c}{a + bx} = \frac{d}{e + fx}$,

$$\text{(18. Cor. 1.) } ce + cfx = ad + bdx,$$

$$\therefore \text{(17) by transposition, } cfx - bdx = ad - ce,$$

$$\text{or } (cf - bd) \cdot x = ad - ce,$$

$$\therefore \text{(18. Cor. 8.) } x = \frac{ad - ce}{cf - bd}.$$

46. (18.) $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = kx$,

and $\therefore$ (18. Cor. 2.) $\frac{adfh + bcfh + bdeh + bdfg}{bdfhk} = x$.

47. By multiplication, $ab + (a + b) \cdot x + x^2 - ab - ac = \frac{a^2 c}{b} + x^2$,

$$\therefore \text{(17. Cor. 3.)}\ (a+b) \, . \, x = \frac{a^2 c}{b} + a c,$$

$$= (a+b) \, . \, \frac{ac}{b},$$

$$\text{and (18. Cor. 3.)}\ x = \frac{ac}{b}.$$

48. (21) Since the product of the extremes is equal to the product of the means,

$$10 + x = 8x - 18,$$

$$\therefore \text{(17) by transposition, } 10 + 18 = 8x - x,$$

$$\text{or } 28 = 7x,$$

$$\therefore \text{(18. Cor. 2.)}\ \frac{28}{7} = 4 = x.$$

49. (21) $4 \, . \, \dfrac{17 - 4x}{4} = 5 \, . \left(\dfrac{15 + 2x}{3} - 2x\right),$

$$\text{or } 17 - 4x = \frac{75 + 10x}{3} - 10x,$$

$$\therefore \text{(17) by transposition, } 17 + 6x = \frac{75 + 10x}{3},$$

$$\text{(18. Cor. 1.) } 51 + 18x = 75 + 10x,$$

$$\text{(17) by transposition, } 18x - 10x = 75 - 51,$$

$$\text{or } 8x = 24,$$

$$\therefore \text{(18. Cor. 2.)}\ x = \frac{24}{8} = 3.$$

50. (21) $16x + 5 = \dfrac{4x + 14}{9x + 31} \, . \, (36x + 10),$

$$\text{(18. Cor. 1.) } 144x^2 + 541x + 155 = 144x^2 + 544x + 140,$$

$$\therefore \text{(17. Cor. 3.) } 15 = 3x,$$

$$\text{and (18. Cor. 2.)}\ \frac{15}{3} = 5 = x.$$

51. (21) $\dfrac{12x^2 - 67x - 57}{6x - 43} = 2x + 19,$

$$(18.\ \text{Cor. } 1.)\ 12x^2 - 67x - 57 = 12x^2 + 28x - 817,$$
$$(17.\ \text{Cor. } 3.)\ 95x = 760,$$
$$\therefore\ (18.\ \text{Cor. } 2.)\ x = \frac{760}{95} = 8.$$

52. Multiplying both sides of the equation by $2x + 3$,

$$10x^2 + 15x + \frac{14x^2 + 39x + 27}{4x + 3} = 18x + 27 + 10x^2 - 18,$$
$$\therefore\ (17.\ \text{Cor. } 3.)\ \frac{14x^2 + 39x + 27}{4x + 3} = 3x + 9,$$
$$(18.\ \text{Cor. } 1.)\ 14x^2 + 39x + 27 = 12x^2 + 45x + 27,$$
$$\text{and } (17.\ \text{Cor. } 3.)\ 2x^2 = 6x,$$
$$\therefore\ (18.\ \text{Cor. } 2.)\ x = \frac{6}{2} = 3,$$

53. (17) by transposition, $\sqrt[3]{10x + 35} = 4 + 1 = 5$,
$\therefore$ cubing both sides of the equation,

$$10x + 35 = 125,$$
$$\therefore\ (17)\ \text{by transposition, } 10x = 125 - 35 = 90,$$
$$\text{and } (18.\ \text{Cor. } 2.)\ x = \frac{90}{10} = 9.$$

54. (17) by transposition, $\sqrt[5]{9x - 4} = 8 - 6 = 2$,
$\therefore$ (19) raising both sides of the equation to the fifth power,

$$9x - 4 = 32,$$
$$(17)\ \text{by transposition, } 9x = 32 + 4 = 36,$$
$$\text{and } (18.\ \text{Cor. } 2.)\ x = \frac{36}{9} = 4.$$

55. (19) squaring both sides of the equation,

$$x + 16 = 4 + 4\sqrt{x} + x,$$
$$\therefore\ (17.\ \text{Cor. } 3.)\ 12 = 4\sqrt{x},$$
$$\text{and } (18.\ \text{Cor. } 2.)\ 3 = \sqrt{x},$$
$$\therefore\ (19)\ 9 = x.$$

56. (19) squaring both sides of the equation,

$$x - 32 = 256 - 32\sqrt{x} + x,$$

$$\therefore \text{ (17. Cor. 3.) } 288 = 32\sqrt{x},$$

$$\text{(18. Cor. 2.) } \frac{288}{32} = 9 = \sqrt{x},$$

$$\therefore \text{ (19) } 81 = x.$$

57. (19) squaring both sides of the equation,

$$4x + 21 = 4x + 4\sqrt{x} + 1,$$

$$\therefore \text{ (17. Cor. 3.) } 20 = 4\sqrt{x},$$

$$\text{(18. Cor. 2.) } \frac{20}{4} = 5 = \sqrt{x},$$

$$\therefore \text{ (19) } 25 = x.$$

58. (19) cubing both sides of the equation,

$$a^3 . (bx - c) = d^3 . (ex + fx - g),$$

$\therefore$ (17) by transposition,

$$a^3 bx - d^3 . (e + f) . x = a^3 c - d^3 g,$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{a^3 c - d^3 g}{a^3 b - d^3 . (e + f)}.$$

59. (19) raising each side of the equation to the twelfth power,

$$(a^2 + c)^4 = \frac{(a^2 + c)^5}{d^3 . (x + b)^3},$$

$$\text{(18. Cor. 3.) } a^2 + c = \frac{1}{d^3 . (x + b)^3},$$

$$\therefore \text{ (18.) } (x + b)^3 = \frac{1}{d^3 . (a^2 + c)},$$

$$\therefore \text{ (19) } x + b = \frac{1}{d . \sqrt[3]{a^2 + c}},$$

and (17) by transposition, $x = \dfrac{1}{d\sqrt[3]{a^2 + c}} - b.$

60. (19) raising both sides of the equation to the $(2m)$th power,

$$a^2 + 2ax + x^2 = x^2 + 5ax + b^2,$$

$$\therefore \text{ (17. Cor. 3.) } a^2 - b^2 = 3ax,$$

$$\text{and (18. Cor. 2.) } \frac{a^2 - b^2}{3a} = x.$$

61. (17) by transposition, $b \cdot \sqrt[m]{x + d} = c - a$.
(19) raising each side of the equation to the mth power,

$$b^m \cdot (x + d) = (c - a)^m,$$

$$\therefore \text{ (18. Cor. 2.) } x + d = \left(\frac{c - a}{b}\right)^m,$$

$$\text{(17) by transposition, } x = \left(\frac{c - a}{b}\right)^m - d.$$

62. (18. Cor. 1.) $3x + 116\sqrt{x} - 160 = 3x + 21\sqrt{x} + 30,$

$$\therefore \text{ (17. Cor. 3.) } 95\sqrt{x} = 190,$$

$$\text{(18. Cor. 2.) } \sqrt{x} = \frac{190}{95} = 2,$$

$$\therefore \text{ (19) } x = 4.$$

63. (18. Cor. 1.) $b\sqrt{x} + b\sqrt{b} = a\sqrt{x} - a\sqrt{b},$

$$\therefore \text{ (17) by transposition, } (a + b) \cdot \sqrt{b} = (a - b) \cdot \sqrt{x},$$

$$\text{and (18. Cor. 2.) } \frac{a + b}{a - b} \cdot \sqrt{b} = \sqrt{x},$$

$$\therefore \text{ (19) } \left(\frac{a + b}{a - b}\right)^2 \cdot b = x.$$

64. (18. Cor. 1.) $24x - 2\sqrt{6x} - 12 = 24x - \sqrt{6x} - 18,$

$$\therefore \text{ (17. Cor. 3.) } 6 = \sqrt{6x},$$

$$\text{(18. Cor. 2.) } \sqrt{6} = \sqrt{x},$$

$$\therefore \text{ (19) } 6 = x.$$

65. Since $5x - 9 = (\sqrt{5x} + 3).(\sqrt{5x} - 3)$,

$$\therefore \frac{5x - 9}{\sqrt{5x} + 3} = \sqrt{5x} - 3,$$

$$\text{whence } \sqrt{5x} - 3 - 1 = \frac{\sqrt{5x} - 3}{2},$$

$$\text{or (17. Cor. 3.) } \frac{\sqrt{5x} - 3}{2} = 1,$$

$$\text{(18. Cor. 1.) } \sqrt{5x} - 3 = 2,$$

$$\therefore \text{ (17) by transposition, } \sqrt{5x} = 2 + 3 = 5,$$

$$\text{whence (19) } x = 5.$$

66. (19) squaring both sides of the equation,

$$1 + x\sqrt{x^2 + 12} = 1 + 2x + x^2,$$

$$\therefore \text{ (17. Cor. 3.) } x\sqrt{x^2 + 12} = 2x + x^2,$$

$$\text{(18. Cor. 2.) } \sqrt{x^2 + 12} = 2 + x,$$

$$\text{(19) } x^2 + 12 = 4 + 4x + x^2,$$

$$\therefore \text{ (17. Cor. 3.) } 8 = 4x,$$

$$\text{and (18. Cor. 2.) } \frac{8}{4} = 2 = x.$$

67. (17) by transposition,

$$\frac{ax}{b}\sqrt{c^2x^2 + d^2} = ex - \frac{acx^2}{b},$$

$$\text{(18. Cor. 2.) } \frac{a}{b}\sqrt{c^2x^2 + d^2} = e - \frac{acx}{b},$$

(19) squaring both sides,

$$\frac{a^2}{b^2}.(c^2x^2 + d^2) = e^2 - \frac{2acex}{b} + \frac{a^2c^2x^2}{b^2},$$

$$\therefore \text{ (17. Cor. 3.) } \frac{2ace}{b}.x = e^2 - \frac{a^2d^2}{b^2},$$

$$\therefore x = \frac{b^2e^2 - a^2d^2}{2abce}.$$

68. (18. Cor. 1.) $\sqrt{x^2 - 9x} + x - 9 = 36$,

$\therefore$ by transposition, $\sqrt{x^2 - 9x} = 45 - x$,

(19) squaring both sides, $x^2 - 9x = (45)^2 - 90x + x^2$,

$\therefore$ (17. Cor. 3.) $81x = (45)^2$,

and (18. Cor. 2.) $x = \frac{(45)^2}{81} = 25$.

SECTION II.

Simple Equations involving two unknown Quantities.

1. Multiplying the first equation by 3,

$$3x + 45y = 159,$$
$$\text{but } 3x + \quad y = \ 27;$$

$\therefore$ by subtraction, $44y = 132$,

whence $y = 3$;

and $3x = 27 - 3 = 24$,

$\therefore x = 8$.

2. Multiplying the first equation by 2,

$$8x + 18y = 102,$$
$$\text{but } 8x - 13y = \quad 9;$$

$\therefore$ by subtraction, $31y = 93$,

whence $y = 3$;

and $4x = 51 - 9y = 51 - 27 = 24$,

$\therefore x = 6$.

3. (18. Cor. 1.) clearing the equations of fractions, by multiplying each by 12,

$$2x + 3y = 72,$$
$$\text{and } 3x + 2y = 68;$$

and as the coefficients in this case are not aliquot parts, multiplying the first by 3, and the second by 2;

$$6x + 9y = 216,$$
$$6x + 4y = 136;$$

$\therefore$ by subtraction, $5y = 80,$

$$\therefore y = 16;$$

$$\therefore \frac{x}{6} = 6 - \frac{y}{4} = 2,$$

and $x = 12.$

4. Multiplying each equation by 8,

$$x + 64y = 1552,$$
and $64x + y = 1048;$

$\therefore$ by addition, $65x + 65y = 2600,$

and $x + y = 40,$

but $x + 64y = 1552,$

$\therefore$ by subtraction, $63y = 1512,$

$$\therefore y = 24;$$

and $x = 40 - y = 40 - 24 = 16.$

5. Multiplying the first equation by 5,

$$3x - 1 + 15y - 20 = 75,$$

$\therefore$ by transposition, $3x + 15y = 96,$

and $\therefore x + 5y = 32.$

also from the second equation, $3y - 5 + 12x = 94,$

$\therefore$ by transposition, $3y + 12x = 99,$

and $\therefore y + 4x = 33,$

but $20y + 4x = 128,$

$\therefore$ by subtraction, $19y = 95,$

$$\therefore y = 5.$$

and $x = 32 - 5y = 32 - 25 = 7.$

6. Multiplying the first equation by 5, and the second by 3,

$$45x + 8y = 350,$$
$$\text{and } 21y - 13x = 132.$$

Multiply the former of these by 21, and the latter by 8,

$$\text{and } 168y + 945x = 7350,$$
$$\text{also } 168y - 104x = 1056,$$

$\therefore$ by subtraction, $\quad 1049x = 6294,$

$$\therefore x = 6;$$
$$\text{and } 21y = 78 + 132 = 210,$$
$$\therefore y = 10.$$

7. Clearing the first equation of fractions,

$$28 + 4x - 10x + 5y = 60y - 100,$$
$$\therefore \text{ by transposition, } 55y + 6x = 128.$$

Clearing the second equation of fractions,

$$15y - 21 + 4x - 3 = 108 - 30x,$$
$$\therefore \text{ by transposition, } 34x + 15y = 132.$$

Multiplying the former by 3, and the latter by 11,

$$18x + 165y = 384,$$
$$374x + 165y = 1452,$$

$\therefore$ by subtraction, $\quad 356x = 1068,$

$$\therefore x = 3.$$
$$\text{and } 15y = 132 - 34x = 132 - 102 = 30,$$
$$\therefore y = 2.$$

8. Clearing the first equation of fractions,

$$14x + 14 - 6y - 8x = 98 - 9y - 33,$$
$$\therefore \text{ by transposition, } 6x + 3y = 51,$$
$$\text{and } 2x + y = 17.$$

Clearing the second equation of fractions,

$$4y - 12 - 10x + 8y = 4x - 11y + 19,$$
$$\therefore \text{ by transposition, } 23y - 14x = 31,$$
$$\text{but } 7y + 14x = 119,$$

$\therefore$ by addition, $30y = 150,$

$$\text{and } y = 5.$$
$$\therefore\ 2x = 17 - 5 = 12,$$
$$\text{and } x = 6.$$

9. From the first equation,

$$64x + 60 - 4x = 32y + 80 + 7x + 11,$$
$$\therefore \text{ by transposition, } 53x - 32y = 31.$$

From the second equation,

$$45y - 6x - 3y = 30x + 10y + 20,$$
$$\therefore \text{ by transposition, } 32y - 36x = 20,$$
$$\text{but } 32y - 53x = -31,$$
$$\therefore \text{ by subtraction, } 17x = 51,$$
$$\text{and } x = 3.$$
$$\text{whence } 8y = 9x + 5 = 32,$$
$$\therefore\ y = 4.$$

10. Clearing the first equation of fractions,

$$51x - 9x - 15y + 867 = 255y + 68x + 119,$$
$$\therefore \text{ by transposition, } 270y + 26x = 748,$$
$$\text{and } 135y + 13x = 374.$$

Also from the second equation,

$$132 - 36y - \frac{90x - 126}{11} = 3x + 3 - 8y - 5,$$
$$\therefore \text{ by transposition, } 134 - 28y - \frac{90x - 126}{11} = 3x,$$
$$\therefore\ 1474 - 308y - 90x + 126 = 33x,$$
$$\text{and } 308y + 123x = 1600.$$

Multiplying this equation by 13, and the former by 123,

$$4004y + 1599x = 20800,$$
$$16605y + 1599x = 46002,$$
$$\therefore \text{ by subtraction, } 12601y = 25202,$$
$$\text{and } y = 2.$$
$$\therefore\ 13x = 374 - 135y = 374 - 270 = 104,$$
$$\text{and } x = 8.$$

11. From the first equation multiplied by 6,

$$7x - 21 + 6y - 2x = 24 + 9x - 57,$$

$$\therefore \text{ by transposition, } 4x - 6y = 12,$$

$$\text{and } 2x - 3y = 6.$$

From the second equation multiplied by 16,

$$16x + 8y - 18x + 14 = 12y + 36 - 4x - 5y,$$

$$\therefore \text{ by transposition, } 2x + y = 22,$$

$$\text{but } 2x - 3y = 6,$$

$$\therefore \text{ by subtraction, } 4y = 16,$$

$$\text{and } y = 4.$$

$$\text{whence } 2x = 22 - y = 22 - 4 = 18,$$

$$\therefore x = 9.$$

12. From the first equation, $3a^2 + ax = b^2 + by$,

but from the second, $ax = c - 2by$,

$$\therefore \text{ by substitution, } 3a^2 + c - 2by = b^2 + by,$$

$$\text{and by transposition, } 3a^2 + c - b^2 = 3by,$$

$$\therefore y = \frac{3a^2 + c - b^2}{3b}.$$

$$\therefore ax = c - \frac{6a^2 + 2c - 2b^2}{3} = \frac{c + 2b^2 - 6a^2}{3},$$

$$\therefore x = \frac{c + 2b^2 - 6a^2}{3a}.$$

13. Multiplying the first equation by $2 \times 3 \times 5 \times 11 = 330$,

$$210x + 180 + 440y - 990 = 990x - 2145 + 165x - 198y + 66x,$$

$$\therefore \text{ by transposition, } 1011x - 638y = 1335.$$

Also from the second, $9x + 12 = 10y - 15$,

$$\therefore \text{ by transposition, } 9x - 10y = -27;$$

multiplying this equation by 337, and the former by 3,

$$3033x - 3370y = -9099,$$

$$\text{and } 3033x - 1914y = 4005,$$

$$\therefore \text{ by subtraction, } 1456y = 13104,$$

$$\text{and } \therefore\ y = 9.$$
$$\text{whence } 9x = 10y - 27 = 90 - 27 = 63,$$
$$\therefore\ x = 7.$$

14. From the first equation multiplied by 6,

$$15x + 39 - 8y + 3x + 5 = 54 + 14x - 6y + 2,$$
$$\therefore \text{ by transposition, } 4x - 2y = 12,$$
$$\text{and } \quad 2x - \ y = \ 6.$$

But from the second, $7x + 49 = 3y - 8 + 16x$,

$$\therefore \text{ by transposition, } 9x + 3y = 57,$$
$$\text{and } \quad 3x + \ y = 19,$$
$$\text{but } \quad 2x - \ y = \ 6,$$
$$\therefore \text{ by addition, } \quad 5x = 25,$$
$$\text{and } x = 5.$$
$$\therefore\ y = 2x - 6 = 10 - 6 = 4.$$

15. From the first equation (Alg. 181).

$$x + y : 3x :: 4 : 3,$$
$$\therefore \text{ (Alg. 186.) } \quad x + y : x :: 4 : 1,$$
$$\text{and (Alg. 180.) } \quad y : x :: 3 : 1,$$
$$\text{whence } y = 3x.$$

Multiplying the second equation by 60,

$$22y - 24x - 315 + 45y = 40 + 2x - 5,$$
$$\therefore \text{ by transposition, } 67y - 26x = 350,$$

in which, if the value of y be substituted from the first equation,

$$(201x - 26x =)\ 175x = 350,$$
$$\therefore \text{ (18. Cor. 2.) } x = \frac{350}{175} = 2,$$
$$\text{and } y = 3x = 6.$$

16. From the first equation multiplied by 30,

$$9x + 12y + 9 - 4x - 14 + 2y = 150 + 6y - 48,$$
$$\therefore \text{ by transposition, } 5x + 8y = 107.$$

Clearing the second equation of fractions,

$$99y + 55x - 88 - 33x - 33y = 84x + 72,$$

$$\therefore \text{ by transposition, } 66y - 62x = 160.$$

Multiplying this equation by 4, and the former by 33,

$$264y - 248x = 640,$$

$$\text{and } 264y + 165x = 3531,$$

$$\therefore \text{ by subtraction, } 413x = 2891,$$

$$\text{and } x = 7.$$

$$\therefore 8y = 107 - 5x = 107 - 35 = 72,$$

$$\text{and } y = 9.$$

17. Multiplying the first equation by 12,

$156x + 4y - 17 + x - 45 + 9x = 48y + 44 - 24x - 14y - 56,$

$$\therefore \text{ by transposition, } 190x - 30y = 50,$$

$$\text{or } 19x - 3y = 5.$$

Clearing the second equation of fractions,

$270x + 540 - 288 - 120y + 144x = 225x - 45y - 75 - 56x - 8y + 80,$

$$\therefore \text{ by transposition, } 245x - 67y = -247.$$

Multiplying this equation by 3, and the former by 67,

$$735x - 201y = -741,$$

$$\text{and } 1273x - 201y = 335,$$

$$\text{by subtraction, } 538x = 1076,$$

$$\text{and } x = 2.$$

$$\text{whence } 3y = 19x - 5 = 38 - 5 = 33,$$

$$\therefore y = 11.$$

18. Clearing the equations of fractions, and transposing,

$$3 . (a^2 - b^2) . x + 5 . (a^2 - b^2) . y = (8a - 2b) . ab,$$

$$\text{and } 3 . (a^2 - b^2) . x + (a + b + c) . 3by = 3 . (a + 2b) . ab + \frac{3acb^2}{a + b},$$

$\therefore$ by subtraction,

$$(5a^2 - 8b^2 - 3ab - 3bc) . y = (5a - 8b) . ab - \frac{3acb^2}{a + b},$$

$$= \frac{(5a^2 - 8b^2 - 3ab - 3bc) \cdot ab}{a + b},$$

$$\text{whence } y = \frac{ab}{a + b}.$$

$$\text{and } 3x = \frac{(8a - 2b) \cdot ab}{a^2 - b^2} - \frac{(5a - 5b) \cdot ab}{a^2 - b^2} = \frac{3 \cdot (a + b) \cdot ab}{a^2 - b^2},$$

$$\therefore \; x = \frac{ab}{a - b}.$$

19. Multiplying the first equation by 18,

$$4x + 2y + 7y + 6x + 11 = 171 - 15x + 51,$$

$$\therefore \text{ by transposition, } 25x + 9y = 211.$$

Also from the second, $30x + 18y + 12 = 63y + 42$,

$$\therefore \text{ by transposition, } 30x - 45y = 30,$$

$$\text{and } \quad 6x - 9y = 6;$$

$$\text{but } \quad 25x + 9y = 211,$$

$$\therefore \text{ by addition, } \quad 31x = 217,$$

$$\text{and } x = 7.$$

$$\text{whence } 3y = 2x - 2 = 14 - 2 = 12,$$

$$\text{and } y = 4.$$

20. Multiplying the first equation by 12,

$$12x - 20y - 2x + 8y + 9 = 6y + 4 + 3,$$

$$\therefore \text{ by transposition, } 10x - 18y = -2,$$

$$\text{or } \quad 5x - 9y = -1.$$

From the second, $3x + \frac{21y}{4} + 28 = 80x - \frac{5y}{2} - 480$,

$$\therefore \text{ by transposition, } 77x - \frac{31y}{4} = 508,$$

$$\text{and } 308x - 31y = 2032,$$

Multiplying this equation by 9, and the former by 31,

$$2772x - 279y = 18288,$$

$$\text{and } \quad 155x - 279y = -31,$$

$$\therefore \text{ by subtraction, } 2617x = 18319,$$

$$\therefore\ x = 7.$$

$$\text{whence } 9y = 5x + 1 = 35 + 1 = 36,$$

$$\text{and } y = 4.$$

21. From the first equation,

$$xy + 7x + 5y + 35 = xy - 9x + y - 9 + 112,$$

$$\therefore \text{ by transposition, } 16x + 4y = 68;$$

$$\text{or } 8x + 2y = 34;$$

$$\text{but from the second, } 8x - 12y = -36,$$

$$\therefore \text{ by subtraction, } 14y = 70,$$

$$\therefore\ y = 5;$$

$$\text{and } 2x = 3y - 9 = 15 - 9 = 6,$$

$$\therefore\ x = 3.$$

22. Multiplying the first equation by 4,

$$6x + 9 + \frac{6x + 10y}{2x - 3} = 13 + 6x + 8,$$

$$\therefore \text{ (17. Cor. 3.) } \frac{6x + 10y}{2x - 3} = 12.$$

$$\text{and } \therefore\ 3x + 5y = 12x - 18,$$

$$\therefore \text{ by transposition, } 9x - 5y = 18.$$

Multiplying the second equation by 10,

$$8y + 7 + \frac{30x - 15y}{y - 4} = 40 + 8y - 18,$$

$$\therefore \text{ (17. Cor. 3.) } \frac{30x - 15y}{y - 4} = 15,$$

$$\text{or } \frac{2x - y}{y - 4} = 1;$$

$$\therefore\ 2x - y = y - 4,$$

$$\therefore \text{ by transposition, } 2x - 2y = -4,$$

$$\text{or } x - y = -2,$$

$$\text{which being multiplied by 5, } 5x - 5y = -10,$$

$$\text{but } 9x - 5y = 18,$$

$$\therefore \text{ by subtraction, } 4x = 28,$$

$$\text{and } x = 7.$$
$$\therefore\ y = x + 2 = 9.$$

23. Multiplying the first equation by 3,

$$12x - 103 - \frac{4y + 13x}{9 - 2y} = 12x + 8,$$
$$\therefore\ (17.\ \text{Cor. } 3.)\ \frac{4y + 13x}{9 - 2y} = -111,$$
$$\text{and } 4y + 13x = 222y - 999,$$
$$\therefore\ \text{by transposition, } 218y - 13x = 999.$$

Multiplying the second equation by 6,

$$18x + \frac{63 - 12y}{2x - 5} = 18x + 13 - 12\tfrac{2}{3},$$
$$\therefore\ (17.\ \text{Cor. } 3.)\ \frac{63 - 12y}{2x - 5} = \frac{1}{3},$$
$$\text{and } 189 - 36y = 2x - 5,$$
$$\text{by transposition, } 2x + 36y = 194,$$
$$\text{and } x + 18y = 97.$$

Multiplying this equation by 13,

$$13x + 234y = 1261,$$
$$\text{but } -13x + 218y = 999,$$
$$\therefore\ \text{by addition, } 452y = 2260,$$
$$\text{and } y = 5.$$
$$\text{whence } x = 97 - 18y = 97 - 90 = 7.$$

24. From the first equation, (18. Cor. 1.)

$$128x^2 - 18y^2 + 24x + 15y - 2 = 128x^2 - 18y^2 + 217,$$
$$\therefore\ (17.\ \text{Cor. } 3.)\ 24x + 15y = 219,$$
$$\text{or } 8x + 5y = 73.$$

From the second equation, by division,

$$5 - \frac{50}{2x + 2y + 3} = 5 - \frac{54}{3x + 2y - 1},$$

$$\therefore \frac{50}{2x + 2y + 3} = \frac{54}{3x + 2y - 1},$$

$$\text{or } \frac{25}{2x + 2y + 3} = \frac{27}{3x + 2y - 1},$$

$$\therefore 75x + 50y - 25 = 54x + 54y + 81,$$

by transposition, $21x - 4y = 106$;

multiplying this equation by 5, and the former by 4,

$$105x - 20y = 530,$$

$$\text{and } 32x + 20y = 292,$$

$\therefore$ by addition, $137x = 822$,

and $x = 6$.

$$\therefore 5y = 73 - 8x = 73 - 48 = 25,$$

$$\therefore y = 5.$$

25. From the first equation multiplied by $4x - 2$,

$$16x^2 + 12xy - 8x - 6y + \frac{96x + 22xy - 48 - 11y}{2x + 1} =$$

$$16x^2 + 12xy - 8x + 5y + 28,$$

$$\therefore \text{(17. Cor. 3.) } \frac{96x + 22xy - 48 - 11y}{2x + 1} = 11y + 28,$$

and $96x + 22xy - 48 - 11y = 22xy + 56x + 11y + 28$,

by transposition, $40x - 22y = 76$,

and $20x - 11y = 38$.

From the second equation multiplied by $4x + 6y + 3$,

$$8x^2 - 18y^2 + 22x + 15y + 12 = 8x^2 - 18y^2 + 108,$$

$$\therefore \text{(17. Cor. 3.) } 22x + 15y = 96;$$

multiplying this equation by 10, and the former by 11,

$$220x + 150y = 960,$$

$$\text{and } 220x - 121y = 418,$$

$\therefore$ by subtraction, $271y = 542$,

and $y = 2$.

$$\therefore 20x = 38 + 11y = 38 + 22 = 60,$$

and $x = 3$.

26. From the first equation,

$$6x^2 - 24y^2 + 11x + 14y + 3 = 6x^2 + 130 - 24y^2,$$

$$\therefore \text{(17. Cor. 3.)}\ 11x + 14y = 127.$$

Multiplying the second by $3y - 4$,

$$9xy - 12x - \frac{453y - 48xy - 604 + 64x}{4y - 1} = 9xy - 110,$$

$$\therefore \text{(17. Cor. 3.)}\ \frac{453y - 48xy - 604 + 64x}{4y - 1} = 110 - 12x,$$

whence

$$453y - 48xy - 604 + 64x = 440y - 48xy - 110 + 12x,$$

$$\text{and (17. Cor. 3.)}\ 13y + 52x = 494,$$

$$\text{or}\quad y + 4x = 38.$$

Multiplying this equation by 14,

$$56x + 14y = 532,$$

$$\text{but}\quad 11x + 14y = 127,$$

$$\therefore \text{by subtraction,}\ 45x = 405,$$

$$\text{and } x = 9.$$

$$\therefore y = 38 - 4x = 38 - 36 = 2.$$

27. Multiplying the first equation by $56y$,

$$8x - 48 + \frac{(28x + 49) \cdot y}{3} - \frac{28xy - 4y^2}{3} = \frac{76y + 4y^2}{3} - \frac{11x}{3} - 6,$$

$$\therefore \text{by transposition,}\ \frac{11x}{3} + 8x - 9y = 42,$$

$$\text{and (18. Cor. 1.)}\ 35x - 27y = 126.$$

Dividing the first and third terms of the second equation by 3, and the second and fourth by 2,

$$4x - 5y + \frac{13}{12} : 5y - 4x + \frac{43}{3} :: 31 - 3x : 3x - \frac{7}{5},$$

$$\therefore \text{(Alg. 179.)}\ \frac{185}{12} : 5y - 4x + \frac{43}{3} :: \frac{148}{5} : 3x - \frac{7}{5},$$

$$:: 148 : 15x - 7,$$

and dividing the first and third terms by 37,

$$\frac{5}{12} : 5y - 4x + \frac{43}{3} :: 4 : 15x - 7,$$

$$\text{whence } \frac{75x - 35}{12} = 20y - 16x + \frac{172}{3},$$

$$\text{and } \therefore\ 75x - 35 = 240y - 192x + 688,$$

$$\text{by transposition, } 267x - 240y = 723,$$

$$\text{or } 89x - 80y = 241.$$

Multiplying this equation by 27, and the former by 80,

$$2403x - 27 \times 80y = 6507,$$

$$2800x - 80 \times 27y = 10080,$$

$$\therefore \text{ by subtraction, } 397x = 3573,$$

$$\text{and } x = 9.$$

$$\text{whence } 9 \times 35 - 126 = 27y,$$

$$\text{or } 35 - 14 = 3y,$$

$$\text{whence } y = 7.$$

28. Multiplying the first equation by 80, the least common multiple of 5, 8, 10, 16,

$$28x + 96y - 6y - 12 + 3x - 2 = 400 - 5x,$$

$$\therefore \text{ by transposition, } 36x + 90y = 414,$$

$$\text{or } 2x + 5y = 23.$$

From the second, (Wood's Alg. 184.)

$$9x + 4y + 15 : 3x - 2y + 1 :: \frac{21}{2} : \frac{7}{6} :: 9 : 1,$$

$$\text{whence } 9x + 4y + 15 = 27x - 18y + 9,$$

$$\therefore \text{ by transposition, } 18x - 22y = 6,$$

$$\text{but from the first, } 18x + 45y = 207,$$

$$\therefore \text{ by subtraction, } 67y = 201,$$

$$\therefore\ y = 3.$$

$$\text{whence } 2x = 23 - 5y = 8,$$

$$\text{and } x = 4.$$

29. Multiplying the first equation by 210,

$$280x - 140y + 210 - 540 + 30x - 150y = \frac{105x}{2} - 42y - 30 - 1617,$$

$$\therefore \text{ by transposition, } 310x - 248y = \frac{105x}{2} - 1317,$$

$$\text{whence } 515x - 496y = -2634.$$

From the second, (Alg. 182.)

$$30 : 4x - 2y :: \frac{5}{6} : \frac{2x}{3} - \frac{y}{2} + \frac{2}{3},$$

$$\text{and } 3 : 2x - y :: \frac{1}{6} : \frac{2x}{3} - \frac{y}{2} + \frac{2}{3},$$

$$:: 1 : 4x - 3y + 4,$$

$$\text{whence } 12x - 9y + 12 = 2x - y,$$

$$\therefore \text{ by transposition, } 10x - 8y = -12,$$

which being multiplied by 62,

$$620x - 496y = -744,$$

$$\text{but } 515x - 496y = -2634,$$

$$\therefore \text{ by subtraction, } 105x = 1890,$$

$$\text{and } x = 18.$$

$$\therefore 4y = 5x + 6 = 96.$$

$$\therefore y = 24.$$

30. Multiplying the first equation by $33x$,

$$132x + 36xy - 18y - 6 = 33xy + 3xy - 31 + 110x + 143,$$

$$\therefore \text{ (17. Cor. 3.) } 22x - 18y = 118,$$

$$\text{and } 11x - 9y = 59.$$

Multiplying the second equation by $6y + 27$,

$$4xy + 18x - \frac{18xy + 81x - 30y - 135}{y + 7} = 4xy + \frac{170}{3},$$

$$\therefore \text{ (17. Cor. 3.) } 18x - \frac{170}{3} = \frac{18xy + 81x - 30y - 135}{y + 7},$$

and

$$\therefore\ 18xy + 126x - \frac{170y + 1190}{3} = 18xy + 81x - 30y - 135,$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 45x - \frac{170y + 1190}{3} + 30y = -135,$$

$$\text{or } 9x - \frac{34y + 238}{3} + 6y = -27,$$

$$\therefore\ 27x - 34y - 238 + 18y = -81,$$

by transposition, $27x - 16y = 157$,

multiplying this equation by 9, and the former by 16,

$$243x - 144y = 1413,$$

$$\text{and } 176x - 144y = 944,$$

$$\therefore \text{ by subtraction, } 67x = 469,$$

whence $x = 7$.

$$\therefore\ 9y = 11x - 59 = 77 - 59 = 18,$$

and $y = 2$.

31. From the second equation, $\sqrt{y - x} = \frac{3}{2} . \sqrt{20 - x}$,

which being substituted in the first equation,

$$\sqrt{y} - \frac{3}{2}\sqrt{20 - x} = \sqrt{20 - x},$$

by transposition, $\sqrt{y} = \frac{5}{2}\sqrt{20 - x}$,

$$\therefore\ (19)\ y = \frac{25}{4} . (20 - x) = 125 - \frac{25x}{4};$$

This value of y being substituted in the first equation,

$$125 - \frac{29x}{4} : 20 - x :: 9 : 4,$$

$$\therefore\ 500 - 29x = 180 - 9x,$$

$$\text{and } 320 = 20x,$$

$$\therefore\ x = 16,$$

$$\text{and } y = 125 - 100 = 25.$$

SECTION III.

Pure Quadratics and others which may be solved without completing the Square.

1. By transposition, $2x^2 = 32$,

$$\therefore \ x^2 = 16,$$
$$\text{and } x = \pm 4.$$

2. (Alg. 180.) $x : y :: 2 : 1$,

$$\therefore \ x = 2y.$$

Which being substituted in the second equation,

$$2y^2 = 18,$$
$$\therefore \ y^2 = 9,$$
$$\text{and } y = \pm 3.$$
$$\therefore \ x = 2y = \pm 6.$$

3. (Alg. 179.) $x : y :: 9 : 5$,

$$\therefore (21) \ x = \frac{9y}{5}.$$

Substituting this value in the second equation,

$$\frac{81y^2}{25} + 4y^2 = 181,$$
$$\therefore \ 181y^2 = 181 \times 25,$$
$$\text{and } y^2 = 25,$$
$$\therefore \text{ extracting the square root, } y = \pm 5;$$
$$\text{whence } x = \frac{9y}{5} = \pm 9.$$

4. (Alg. 182 and 184.) $x : y :: a + b : a - b$,

$$\therefore \ x = \frac{a+b}{a-b} \cdot y;$$

and ∴ from the second equation,

$$\frac{a+b}{a-b} \cdot y^2 = c^2,$$

$$\text{or } y^2 = c^2 \cdot \frac{a-b}{a+b},$$

∴ extracting the square root, $y = \pm c\sqrt{\frac{a-b}{a+b}}$,

$$\therefore\ x = \pm c\sqrt{\frac{a+b}{a-b}},$$

5. (Alg. 182.) $2x^2 : 2y^2 :: 25 : 9$,

and (Alg. 184) $x^2 : y^2 :: 25 : 9$,

(Alg. 188.) $x : y :: 5 : 3$,

$$\therefore (21)\ x = \frac{5y}{3}.$$

Substituting this for x in the second equation,

$$\frac{5y^3}{3} = 45,$$

whence $y^3 = 27$,

extracting the cube root, $y = 3$.

$$\therefore\ x = \frac{5y}{3} = 5.$$

6. By subtraction, $x^2 - 2xy + y^2 = 36$,

∴ extracting the square root, $x - y = \pm 6$,

now, $xy - y^2 = (x - y) \cdot y = \pm 6y$,

$\therefore\ \pm 6y = 18$,

and $y = \pm 3$.

$\therefore\ x = y \pm 6 = \pm 9$.

7. Dividing the two first terms of the proportion by $x + y$, (Alg. 184.)

$$1 : x - y :: 1 : 4,$$

$$\therefore\ x - y = 4.$$

$$\text{whence } x^2 - 2xy + y^2 = 16,$$
$$\text{but } \quad 4xy = 84,$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 100,$$

$$\text{and extracting the square root, } x + y = \pm 10,$$
$$\text{but } x - y = 4,$$

$$\therefore \text{ by addition, } 2x = 14 \text{ or } -6,$$
$$\text{and } x = 7 \text{ or } -3;$$
$$\text{and by subtraction, } 2y = 6 \text{ or } -14,$$
$$\therefore y = 3 \text{ or } -7.$$

8. (Alg. 177.) $y : x :: n - m : n$,

$$\therefore (21)\ y = \frac{n - m}{n} . x;$$

substituting this in the first equation,

$$ax^2 + b . \frac{n - m}{n} . x^2 = c^3,$$

$$\text{or } (na + nb - mb) . x^2 = nc . c^2,$$

$$\therefore x^2 = c^2 . \frac{nc}{na + nb - mb},$$

$$\text{and } x = \pm c . \sqrt{\frac{nc}{na + nb - mb}}.$$

$$\therefore y = \pm \frac{n - m}{n} . c . \sqrt{\frac{nc}{na + nb - mb}}.$$

6. (Alg. 182 and 184.) $x^3 : y^3 :: 343 : 216$,

$$\therefore \text{ (Alg. 188.) } x : y :: 7 : 6,$$

$$\text{and } y = \frac{6x}{7}.$$

This being substituted in the second equation,

$$\frac{6x^3}{7} = 294,$$

$$\text{and } \therefore x^3 = 343,$$

$$\text{whence } x = 7;$$
$$\text{and } \therefore y = \frac{6x}{7} = 6.$$

10. (Alg. 184.) Dividing the two first terms by $x - y$.

$$x : y :: 3 : 7,$$
$$\therefore x = \frac{3y}{7};$$

substituting this for x in the second equation,

$$\frac{3y^3}{7} = 147,$$
$$\text{or } y^3 = 343,$$
$$\therefore \text{extracting the cube root, } y = 7;$$
$$\text{and } x = \frac{3y}{7} = 3.$$

11. (Alg. 182.) $2\sqrt{x} : 2\sqrt{y} :: 5 : 3,$

(Alg. 184.) $\sqrt{x} : \sqrt{y} :: 5 : 3,$

(Alg. 188.) $x : y :: 25 : 9,$

$$\text{and } x = \frac{25y}{9};$$

substituting this in the second equation,

$$\frac{25y}{9} - y = 16,$$
$$\text{or } 16y = 16 \times 9,$$
$$\therefore y = 9;$$
$$\text{and } x = \frac{25y}{9} = 25.$$

12. By addition, $2\sqrt[4]{x} = 10,$

$$\therefore \sqrt[4]{x} = 5,$$
$$\text{and } x = 625.$$

By subtracting the equations, $2\sqrt[4]{y} = 4$,

$\therefore \sqrt[4]{y} = 2$,

and $y = 16$.

13. (Alg. 184.) Dividing the two first terms by $\sqrt{x} - \sqrt{y}$,

$\sqrt{x} + \sqrt{y} :: 1 :: 8 : 1$,

$\therefore \sqrt{x} + \sqrt{y} = 8$.

and squaring this, $x + 2\sqrt{xy} + y = 64$,

but $4\sqrt{xy} = 60$,

$\therefore$ by subtraction, $x - 2\sqrt{xy} + y = 4$.

$\therefore$ extracting the square root, $\sqrt{x} - \sqrt{y} = \pm 2$,

but $\sqrt{x} + \sqrt{y} = 8$,

$\therefore$ by addition, $2\sqrt{x} = 10$ or 6,

$\sqrt{x} = 5$ or 3,

and $x = 25$ or 9.

By subtraction, $2\sqrt{y} = 6$ or 10,

$\sqrt{y} = 3$ or 5,

$y = 9$ or 25.

14. (Alg. 184.) Dividing the two first terms by $x - y$,

$x^2 + xy + y^2 : xy :: 7 : 2$,

$\therefore$ (Alg. 179.) $x^2 + 2xy + y^2 : xy :: 9 : 2$,

or $36 : xy :: 9 : 2$,

$\therefore xy = 8$.

Now, $x^2 + 2xy + y^2 = 36$,

and $4xy = 32$,

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 4$,

extracting the square root, $x - y = \pm 2$;

but $x + y = 6$,

$\therefore$ by addition, $2x = 8$ or 4,

and $x = 4$ or 2.

$$\text{By subtraction, } 2y = 4 \text{ or } 8,$$
$$\text{and } y = 2 \text{ or } 4.$$

15. From the first equation, $x + y = \frac{xy}{2}$,

$$\text{which from the second is} = 9.$$
$$\therefore\ x^2 + 2xy + y^2 = 81,$$
$$\text{but}\quad 4xy = 72,$$
$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 9,$$
$$\text{and extracting the square root, } x - y = \pm 3;$$
$$\text{but } x + y = 9,$$
$$\therefore \text{ by addition, } 2x = 12 \text{ or } 6,$$
$$\text{and } x = 6 \text{ or } 3.$$
$$\text{by subtraction, } 2y = 6 \text{ or } 12,$$
$$\text{and } y = 3 \text{ or } 6.$$

16. Since $x^4 - y^4 = (x^2 - y^2) \cdot (x^2 + y^2) = 9 \cdot (x^2 + y^2)$.

$$\therefore\ 9 \cdot (x^2 + y^2) = 369,$$
$$\text{and } x^2 + y^2 = 41;$$
$$\text{but } x^2 - y^2 = 9,$$
$$\therefore \text{ by addition, } 2x^2 = 50,$$
$$x^2 = 25,$$
$$\text{and } x = \pm 5.$$
$$\text{by subtraction, } 2y^2 = 32,$$
$$y^2 = 16,$$
$$\text{and } y = \pm 4.$$

17. Clearing the second equation of fractions, and multiplying by 3.

$$3x^2y - 3xy^2 = 48,$$
$$\text{but } x^3 - y^3 = 56,$$
$$\therefore \text{ by subtraction, } x^3 - 3x^2y + 3xy^2 - y^3 = 8,$$
$$\text{extracting the cube root, } x - y = 2,$$

$$\text{whence also } xy = 8.$$

$$\begin{aligned} \therefore\ x^2 - 2xy + y^2 &= 4, \\ \text{and}\quad 4xy &= 32, \\ \therefore \text{ by addition, } x^2 + 2xy + y^2 &= 36, \\ \text{and}\quad x + y &= \pm 6; \\ \text{but}\quad x - y &= 2, \\ \therefore \text{ by addition, } 2x &= 8 \text{ or } -4, \\ \text{and}\quad x &= 4 \text{ or } -2. \\ \text{by subtraction, } 2y &= 4 \text{ or } -8, \\ \text{and}\quad y &= 2 \text{ or } -4. \end{aligned}$$

18. Clearing the equation of fractions,

$$1 + \sqrt{1-x^2} - 1 + \sqrt{1-x^2} = \sqrt{3},$$

$$\text{or, } 2\sqrt{1-x^2} = \sqrt{3},$$

$$\therefore\ 4 - 4x^2 = 3,$$

$$\text{by transposition, } 4x^2 = 1,$$

$$\text{extracting the square root, } 2x = \pm 1,$$

$$\text{and } \therefore\ x = \pm \frac{1}{2}.$$

19. Squaring the second equation,

$$\begin{aligned} x^2y + 2xy^{\frac{3}{2}} + y^2 &= 196, \\ \text{but from the first, } 2x^2y \qquad + 2y^2 &= 232, \\ \therefore \text{ by subtraction, } x^2y - 2xy^{\frac{3}{2}} + y^2 &= 36, \\ \text{extracting the square root, } xy^{\frac{1}{2}} - y &= \pm 6; \\ \text{but } xy^{\frac{1}{2}} + y &= 14, \\ \therefore \text{ by subtraction, } 2y &= 8 \text{ or } 20, \\ \text{and}\quad y &= 4 \text{ or } 10. \\ \text{by addition, } 2xy^{\frac{1}{2}} &= 20 \text{ or } 8, \\ \therefore\ xy^{\frac{1}{2}} &= 10 \text{ or } 4. \end{aligned}$$

$$\left.\begin{aligned} \text{and } \therefore\ 2x &= 10, \\ \text{and}\quad x &= 5. \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \sqrt{10}\,.\,x &= 4, \\ \text{and } x &= 2\sqrt{\frac{2}{5}}. \end{aligned}\right.$$

20. Cubing the first equation,

$$x + 3\sqrt[3]{x^2 y} + 3\sqrt[3]{x y^2} + y = 216,$$

$$\text{but } x \qquad\qquad + y = 72,$$

$$\therefore \text{ by subtraction, } \quad 3\sqrt[3]{x^2 y} + 3\sqrt[3]{x y^2} = 144,$$

$$\text{or} \quad 3\,(\sqrt[3]{x} + \sqrt[3]{y})\,.\,\sqrt[3]{x y} = 144,$$

$$\text{or} \quad 3\,.\,6\,.\,\sqrt[3]{x y} = 144,$$

$$\therefore \sqrt[3]{x y} = 8,$$

$$\text{and } x y = 512.$$

Squaring the second equation, $x^2 + 2xy + y^2 = 5184$,

$$\text{but} \quad 4xy \qquad = 2048,$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 3136,$$

extracting the square root, $x - y = \pm 56$,

$$\text{but } x + y = 72,$$

$$\therefore \text{ by addition, } \quad 2x = 128 \text{ or } 16,$$

$$\text{and} \quad x = 64 \text{ or } 8,$$

$$\text{by subtraction, } \quad 2y = 16 \text{ or } 128,$$

$$\text{and} \quad y = 8 \text{ or } 64.$$

21. From the first equation by transposition,

$$4x^2 - \frac{x^2}{y} = 10y - \frac{5}{2},$$

$$\text{or } x^2\,.\,\frac{4y-1}{y} = \frac{5}{2}\,.\,(4y - 1,$$

$$\therefore x^2 = \frac{5y}{2}.$$

Substituting this value in the second equation,

$$\frac{5y}{2} + 3y = 55,$$

$$\therefore 11y = 110,$$

$$\text{and } y = 10.$$

$$\text{whence } x^2 = \frac{5y}{2} = 25,$$

$$\text{and } x = \pm 5.$$

22. Clearing the equation of fractions,

$$x - \sqrt{2 - x^2} + x + \sqrt{2 - x^2} = a\,x\,.\,(2x^2 - 2),$$

$$\text{or } 2x = a\,x\,.\,(2x^2 - 2),$$

$$\therefore\ 1 = a\,.\,(x^2 - 1),$$

$$\text{and } a\,x^2 = a + 1,$$

$$\therefore\ x = \pm\sqrt{\frac{a+1}{a}}.$$

23. $$\frac{\sqrt{a^2 + x^2} - x}{x} = \frac{1}{b},$$

$$\text{or } \sqrt{\frac{a^2}{x^2} + 1} - 1 = \frac{1}{b},$$

transposing, and squaring each side,

$$\frac{a^2}{x^2} + 1 = 1 + \frac{2}{b} + \frac{1}{b^2},$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ \frac{a^2}{x^2} = \frac{2}{b} + \frac{1}{b^2} = \frac{1 + 2b}{b^2},$$

$$\therefore\ \frac{a^2\,b^2}{1 + 2b} = x^2,$$

$$\text{and } x = \pm\frac{a\,b}{\sqrt{1 + 2b}}.$$

24. From the first equation,

$$4y - x + y - x + 2\sqrt{(y - x)\,.\,(4y - x)} = 4\,.\,(2y - x),$$

$$\text{or } 2\sqrt{(y - x)\,.\,(4y - x)} = 3y - 2x,$$

and squaring each side,

$$4\,.\,(y - x)\,.\,(4y - x) = 9y^2 - 12\,x\,y + 4x^2,$$

$$\text{or } 4\,.\,(4y^2 - 5\,x\,y + x^2) = 9y^2 - 12\,x\,y + 4x^2,$$

$$\therefore\ 7y^2 = 8xy,$$
$$\text{and } 7y = 8x.$$

From the second, $\sqrt{y^2 - 9x} : \frac{3}{4}\sqrt{x^2 - 6y} :: 1 : \frac{3}{4}$,

$$\therefore\ y^2 - 9x = x^2 - 6y,$$

and by substitution, $\frac{64x^2}{49} - 9x = x^2 - \frac{48x}{7}$,

$$\text{whence } \frac{15x}{49} = \frac{15}{7},$$
$$\text{and } x = 7,$$
$$\text{whence } y = 8.$$

25. From the first equation,

$$\frac{9}{8} \cdot \sqrt[3]{x+y} \cdot \left(\frac{1}{x} + \frac{1}{y}\right) = \frac{8}{7},$$

$$\text{or } \frac{9}{8} \cdot \frac{(x+y)^{\frac{4}{3}}}{xy} = \frac{8}{7}.$$

From the second, $\frac{7}{4} \cdot \frac{(x-y)^{\frac{4}{3}}}{xy} = \frac{1}{9}$.

Hence $\left(\frac{x+y}{x-y}\right)^{\frac{4}{3}} = \frac{8}{7} \times \frac{8}{9} \times 9 \times \frac{7}{4} = 16 = 2^4$,

$$\text{and } x + y = 8 \cdot (x - y),$$
$$\text{whence } 7x = 9y.$$

$$\text{But, } \frac{(x-y)^{\frac{4}{3}}}{xy} = \frac{4}{7 \times 9},$$

$$\therefore\ \frac{\left(\frac{2x}{9}\right)^{\frac{4}{3}}}{\frac{7x^2}{9}} = \frac{4}{7 \times 9},$$

$$\text{and } \left(\frac{2x}{9}\right)^{\frac{4}{3}} = \frac{4x^2}{81} = \left(\frac{2x}{9}\right)^2,$$

$$\therefore\ \left(\frac{2x}{9}\right)^{\frac{2}{3}} = 1,$$

$$\text{and } 2x = 9;$$
$$\therefore\ x = \frac{9}{2},$$
$$\text{and } y = \frac{7}{2}.$$

26. Squaring the second equation,

$$x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{1}{5}} + y^{\frac{2}{5}} = 36,$$
$$\text{but from the first, } 2x^{\frac{4}{3}} \qquad + 2y^{\frac{2}{5}} = 40,$$
$$\therefore \text{ by subtraction, } x^{\frac{4}{3}} - 2x^{\frac{2}{3}}y^{\frac{1}{5}} + y^{\frac{2}{5}} = 4,$$
$$\text{and extracting the square root, } x^{\frac{2}{3}} - y^{\frac{1}{5}} = \pm 2;$$
$$\text{but } x^{\frac{2}{3}} + y^{\frac{1}{5}} = 6,$$
$$\therefore \text{ by addition, } 2x^{\frac{2}{3}} = 8 \text{ or } 4,$$
$$\text{and } x^{\frac{2}{3}} = 4 \text{ or } 2,$$
$$\therefore\ x = \pm 8 \text{ or } \pm 2\sqrt{2}.$$
$$\text{by subtraction, } 2y^{\frac{1}{5}} = 4 \text{ or } 8,$$
$$y^{\frac{1}{5}} = 2 \text{ or } 4.$$
$$\therefore\ y = 32 \text{ or } 1024.$$

27. By transposition,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 1296,$$
$$\therefore \text{ extracting the root, } x + y = \pm 6;$$
$$\text{but } x - y = 4,$$
$$\therefore \text{ by addition, } 2x = 10 \text{ or } -2,$$
$$\text{and } x = 5 \text{ or } -1.$$
$$\text{by subtraction, } 2y = 2 \text{ or } -10,$$
$$\therefore\ y = 1 \text{ or } -5.$$

28. $a^{4b} . (x^{\frac{1}{2}} - 1)^2 + x - 2x^{\frac{1}{2}} + 1 = 2 . (x + 1),$

$$\therefore\ a^{4b} . (x^{\frac{1}{2}} - 1)^2 = x + 2x^{\frac{1}{2}} + 1,$$
$$\text{and } a^{2b} . (x^{\frac{1}{2}} - 1) = \pm (x^{\frac{1}{2}} + 1);$$
$$\text{by transposition, } (a^{2b} \mp 1) . x^{\frac{1}{2}} = a^{2b} \pm 1,$$

$$\text{and } x^{\frac{1}{2}} = \frac{a^{2b} \pm 1}{a^{2b} \mp 1},$$

$$\therefore\ x = \left(\frac{a^{2b} \pm 1}{a^{2b} \mp 1}\right)^2.$$

29. Multiplying the numerator and denominator by $\sqrt{a} - \sqrt{a - x}$,

$$\frac{(\sqrt{a} - \sqrt{a - x})^2}{x} = a,$$

$$\text{whence } \sqrt{a} - \sqrt{a - x} = \pm\ \sqrt{a\,x},$$

$$\text{by transposition, } \sqrt{a} \mp \sqrt{a\,x} = \sqrt{a - x},$$

$$\text{squaring both sides, } a \mp 2a\sqrt{x} + a\,x = a - x,$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ (a + 1)\,.\,x = \pm\, 2a\sqrt{x},$$

$$\text{and } (a + 1)\,.\,\sqrt{x} = \pm\, 2a,$$

$$\text{or } \sqrt{x} = \pm\frac{2a}{a + 1},$$

$$\therefore\ x = \frac{4a^2}{(a + 1)^2}.$$

30. Multiplying the numerator and denominator by $\sqrt{x} + \sqrt{x - y}$,

$$\frac{(\sqrt{x} + \sqrt{x - y})^2}{y} = 4,$$

$$\text{whence } \sqrt{x} + \sqrt{x - y} = \pm\, 2\sqrt{y},$$

$$\text{by transposition, } \sqrt{x - y} = \pm\, 2\sqrt{y} - \sqrt{x},$$

$$\text{squaring both sides, } x - y = 4y \mp 4\sqrt{x\,y} + x,$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 5y = \pm\, 4\sqrt{x\,y}.$$

But from the second equation, $y = 4\sqrt{x}$;

$$\therefore\ 5 = \pm\sqrt{y},$$

$$\text{and } y = 25.$$

$$\therefore\ 4\sqrt{x} = 25,$$

$$\text{and } x = \frac{625}{16}.$$

31. From the first equation, $x - y = \frac{xy}{4}$,

and from the second, $(x - y) \cdot xy = 16$,

$$\therefore\ \frac{x^2y^2}{4} = 16,$$

$$\text{and } \frac{xy}{2} = 4;$$

whence $x - y = 2$;

$$\therefore\ x^2 - 2xy + y^2 = 4,$$

$$\text{but } \quad 4xy \quad = 32,$$

$\therefore$ by addition, $x^2 + 2xy + y^2 = 36$,

and extracting the square root, $x + y = \pm 6$;

but $x - y = 2$,

$\therefore$ by addition, $2x = 8 \text{ or } -4$,

and $x = 4 \text{ or } -2$.

by subtraction, $2y = 4 \text{ or } -8$,

and $y = 2 \text{ or } -4$.

32. Multiplying the numerator and denominator by $\sqrt{4x + 1} + \sqrt{4x}$,

$$(\sqrt{4x + 1} + \sqrt{4x})^2 = 9,$$

$\therefore$ extracting the square root, $\sqrt{4x + 1} + \sqrt{4x} = \pm 3$,

transposing and squaring, $4x + 1 = 4x \mp 6\sqrt{4x} + 9$,

$\therefore$ (17. Cor. 3.) $6\sqrt{4x} = 8$,

and $3\sqrt{4x} = 4$,

$$\therefore\ 3\sqrt{x} = \sqrt{4},$$

$$\text{and } x = \frac{4}{9}.$$

33. Multiplying the numerator and denominator by $a + x + \sqrt{2ax + x^2}$,

$$\frac{(a + x + \sqrt{2ax + x^2})^2}{a^2} = b,$$

whence $a + x + \sqrt{2ax + x^2} = \pm a\sqrt{b}$,

transposing and squaring,

$$2ax + x^2 = a^2b \mp 2a\sqrt{b} \,.\, (a + x) + (a + x)^2,$$

$$\therefore \text{ (17. Cor. 3.) } \pm 2a\sqrt{b} \,.\, (a + x) = a^2 \,.\, (b + 1),$$

$$\text{and } a + x = \frac{a \,.\, (b + 1)}{\pm 2\sqrt{b}},$$

$$\therefore\ x = a \,.\left(\frac{b + 1}{\pm 2\sqrt{b}} - 1\right) = \pm a \,.\, \frac{(\sqrt{b} \mp 1)^2}{2\sqrt{b}},$$

34. From the second equation,

$$x^2 + y^2 : 2xy :: 17 : 15,$$

$$\therefore\ x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 32 : 2 :: 16 : 1,$$

$$\text{and } x + y : x - y :: 4 : 1,$$

$$\therefore\ x : y :: 5 : 3.$$

$$\text{But from the first, } \sqrt{\frac{2}{3}y} + \frac{1}{2}\sqrt{\frac{8}{3}y} = \frac{\frac{5}{3}y - 1}{\sqrt{\frac{2}{3}y}},$$

$$\therefore\ \sqrt{\frac{2}{3}y} \,.\, 2\sqrt{\frac{2}{3}y} = \frac{5}{3}y - 1,$$

$$\text{or } \frac{4}{3}y = \frac{5}{3}y - 1,$$

$$\text{and } \frac{1}{3}y = 1,$$

$$\text{or } y = 3;$$

$$\therefore\ x = 5.$$

35. Since $x^4y^3 - x^3y^4 = x^2y^2 \,.\, (x^2y - xy^2)$,

$$\therefore\ 6x^2y^2 = 216,$$

$$\text{and } x^2y^2 = 36,$$

$$\therefore\ xy = 6.$$

But from the second equation, $xy \,.\, (x - y) = 6$,

$$\therefore\ x - y = \pm 1.$$

$$\text{whence } x^2 - 2xy + y^2 = 1,$$

$$\text{and } \quad 4xy = 24,$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 25,$$

$$\text{extracting the square root, } x + y = \pm 5;$$

$$\text{but } x - y = \pm 1,$$

$$\therefore \text{ by addition, } 2x = \pm 6 \text{ or } \mp 4,$$

$$\therefore\ x = \pm 3 \text{ or } \mp 2.$$

$$\text{by subtraction, } 2y = \pm 4 \text{ or } \mp 6,$$

$$\text{and } y = \pm 2 \text{ or } \mp 3.$$

36. From the first equation, $x^{\frac{4}{3}} \,.\, (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 208$,

and from the second, $y^{\frac{4}{3}} \,.\, (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 1053$,

$$\therefore\ (14)\ \frac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}} = \frac{208}{1053} = \frac{16}{81},$$

$$\text{and } \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} = \frac{2}{3},$$

$$\text{or } \frac{x}{y} = \frac{8}{27}.$$

$$\text{Hence } x^{\frac{4}{3}} \,.\, (x^{\frac{2}{3}} + \frac{9}{4} x^{\frac{2}{3}}) = \frac{13x^2}{4} = 208,$$

$$\therefore\ \frac{x^2}{4} = 16,$$

$$\text{and } x = \pm 8.$$

$$\text{whence } y = \pm 27.$$

37. Dividing the second equation by the first,

$$x^{\frac{3}{2}} - x^{\frac{3}{4}} y^{\frac{3}{4}} + y^{\frac{3}{2}} = 577,$$

$$\text{but } x^{\frac{3}{2}} + x^{\frac{3}{4}} y^{\frac{3}{4}} + y^{\frac{3}{2}} = 1009,$$

$$\therefore \text{ by subtraction, } 2x^{\frac{3}{4}} y^{\frac{3}{4}} = 432,$$

$$\text{and } x^{\frac{3}{4}} y^{\frac{3}{4}} = 216;$$

whence $x^{\frac{3}{2}} + 2x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = 1225$,
and $x^{\frac{3}{2}} - 2x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = 361$,
$\therefore$ extracting the square root, $x^{\frac{3}{4}} + y^{\frac{3}{4}} = \pm 35$,
and $x^{\frac{3}{4}} - y^{\frac{3}{4}} = \pm 19$,

$\therefore$ by addition, $2x^{\frac{3}{4}} = \pm 54$ or ± 16,
and $x^{\frac{3}{4}} = \pm 27$ or ± 8,
$\therefore$ $x = 81$ or 16.
by subtraction, $2y^{\frac{3}{4}} = \pm 16$ or ± 54,
$\therefore$ $y^{\frac{3}{4}} = \pm 8$ or ± 27,
and $y = 16$ or 81.

38. From the second equation,

$x^3 + y^3 - 3x^2 - 3y^2 = 12$,
and from the first, $3xy \cdot (x + y) + 3x^2 + 3y^2 = 204$,

$\therefore$ by addition, $(x + y)^3 = 216$,
and $x + y = 6$.
Hence $x^2 + 2xy + y^2 = 36$,
but from the first, $x^2 + 6xy + y^2 = 68$,

$\therefore$ by subtraction, $4xy = 32$,
whence $x^2 - 2xy + y^2 = 4$,
and extracting the square root, $x - y = \pm 2$;
but $x + y = 6$,

$\therefore$ by addition, $2x = 8$ or 4,
and $x = 4$ or 2.
by subtraction, $2y = 4$ or 8,
and $y = 2$ or 4.

39. Squaring the first equation,

$x^2 y^2 \cdot (x^2 + 2xy + y^2) = 7056$,
but $x^2 y^2 \cdot (x^2 + y^2) = 3600$,

$\therefore$ by subtraction, $x^2 y^2 \cdot 2xy = 3456$,

$$\text{and } x^3 y^3 = 1728,$$
$$\therefore\ xy = 12;$$
$$\text{and consequently, } x + y = 7;$$
$$\text{and } x^2 + y^2 = 25;$$
$$\text{but } 2xy = 24,$$
$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1,$$
$$\text{and extracting the square root, } x - y = \pm 1;$$
$$\text{but } x + y = 7,$$
$$\therefore \text{ by addition, } 2x = 8 \text{ or } 6,$$
$$\text{and } x = 4 \text{ or } 3.$$
$$\text{by subtraction, } 2y = 6 \text{ or } 8,$$
$$\text{and } y = 3 \text{ or } 4.$$

40. Dividing the first equation by the second,

$$\frac{x^3 - y^3}{x^3 + y^3} = \frac{7}{9},$$
$$\therefore\ 9x^3 - 9y^3 = 7x^3 + 7y^3,$$
$$\text{and by transposition, } 2x^3 = 16y^3,$$
$$\therefore\ x = 2y.$$
$$\text{Whence } \frac{4y^2 + 2y^2 + y^2}{3y} = 7,$$
$$\text{and } \therefore\ y = 3;$$
$$\text{and } x = 2y = 6.$$

41. Multiplying the numerator and denominator by x^{4mn},

$$\frac{x^{(m-n)^2+4mn} + 1}{x^{(m-n)^2+4mn} - 1} = a^{\frac{r}{s}},$$
$$\text{or } \frac{x^{(m+n)^2} + 1}{x^{(m+n)^2} - 1} = a^{\frac{r}{s}},$$
$$\text{whence } (a^{\frac{r}{s}} - 1) \cdot x^{(m+n)^2} = a^{\frac{r}{s}} + 1,$$

$$\text{and } x^{(m+n)^2} = \frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1},$$

$$\therefore\ x = \left(\frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1}\right)^{\frac{1}{(m+n)^2}}.$$

SECTION IV.

Adfected Quadratics involving only one unknown Quantity.

1. Completing the square, $x^2 + 4x + 4 = 144$,
and extracting the root, $x + 2 = \pm 12$,
whence $x = 10$ or -14.

2. By transposition, $x^2 - 6x = 72$,
completing the square, $x^2 - 6x + 9 = 81$,
extracting the root, $x - 3 = \pm 9$,
and $x = 12$ or -6.

3. Adding 8 to each side of the equation, in order to complete the square, $x^2 - 10x + 25 = 9$,
extracting the root, $x - 5 = \pm 3$,
whence $x = 8$ or 2.

4. By transposition, $x^2 - x = 210$,
completing the square, $x^2 - x + \frac{1}{4} = 210 + \frac{1}{4} = \frac{841}{4}$,
extracting the root, $x - \frac{1}{2} = \pm \frac{29}{2}$,
$\therefore\ x = 15$ or -14.

5. By transposition, $3x^2 - 9x = 84$,

$$\therefore\ x^2 - 3x = 28,$$

completing the square, $x^2 - 3x + \frac{9}{4} = 28 + \frac{9}{4} = \frac{121}{4}$,

extracting the root, $x - \frac{3}{2} = \pm \frac{11}{2}$,

whence $x = 7$ or -4.

6. By transposition, $7x^2 - 21x = 280$,

$$\therefore\ x^2 - 3x = 40,$$

completing the square, $x^2 - 3x + \frac{9}{4} = \frac{169}{4}$,

extracting the root, $x - \frac{3}{2} = \pm \frac{13}{2}$,

and $\therefore\ x = 8$ or -5.

7. By transposition, $\frac{x^2}{3} + \frac{4x}{5} = 34\frac{1}{5}$,

multiplying by 3, $x^2 + \frac{12x}{5} = 102\frac{3}{5} = \frac{513}{5}$,

completing the square,

$$x^2 + \frac{12x}{5} + \frac{36}{25} = \frac{513}{5} + \frac{36}{25} = \frac{2601}{25},$$

extracting the root, $x + \frac{6}{5} = \pm \frac{51}{5}$,

$$\therefore\ x = 9 \text{ or } -\frac{57}{5}.$$

8. By transposition, $\frac{2x^2}{3} - \frac{x}{2} = 4\frac{1}{2}$,

$$\therefore\ x^2 - \frac{3x}{4} = \frac{27}{4},$$

completing the square, $x^2 - \frac{3x}{4} + \frac{9}{64} = \frac{27}{4} + \frac{9}{64} = \frac{441}{64}$,

$$\text{extracting the root, } x - \frac{3}{8} = \pm \frac{21}{8},$$

$$\therefore\ x = 3 \text{ or } -\frac{9}{4}.$$

9. (18. Cor. 1.) $x^2 + 4x + 7x - 8 = 13x$,

$$\therefore \text{ by transposition, } x^2 - 2x = 8,$$

$$\text{completing the square, } x^2 - 2x + 1 = 9,$$

$$\text{extracting the root, } x - 1 = \pm 3,$$

$$\therefore\ x = 4 \text{ or } -2.$$

10. (18. Cor. 1.) $4x^2 - 36 + x = 46x$,

$$\text{by transposition, } 4x^2 - 45x = 36,$$

completing the square,

$$4x^2 - 45x + \left(\frac{45}{4}\right)^2 = 36 + \left(\frac{45}{4}\right)^2 = \frac{2601}{16},$$

$$\text{extracting the root, } 2x - \frac{45}{4} = \pm \frac{51}{4},$$

$$\therefore\ 2x = 24 \text{ or } -\frac{3}{2},$$

$$\text{and } x = 12 \text{ or } -\frac{3}{4}.$$

11. (18. Cor. 1.) $32x - 5x + x^2 = 18 + 6x + 6x^2$,

$$\therefore \text{ by transposition, } 5x^2 - 21x = -18,$$

completing the square,

$$x^2 - \frac{21}{5}x + \left(\frac{21}{10}\right)^2 = \frac{441}{100} - \frac{18}{5} = \frac{81}{100},$$

$$\text{extracting the root, } x - \frac{21}{10} = \pm \frac{9}{10},$$

$$\therefore\ x = 3 \text{ or } \frac{6}{5}.$$

12. Multiplying by $2x - 5$,

$$x^2 + \frac{x-15}{2} + 16 - 2x = \frac{52x}{5} - 26,$$

$$\text{by transposition, } x^2 - \frac{119}{10} \cdot x = -\frac{69}{2},$$

completing the square,

$$x^2 - \frac{119}{10} \cdot x + \overline{\frac{119}{20}}\Big|^2 = \frac{14161}{400} - \frac{69}{2} = \frac{361}{400},$$

$$\text{extracting the root, } x - \frac{119}{20} = \pm \frac{19}{20},$$

$$\therefore \; x = 5 \text{ or } \frac{69}{10}.$$

13. Multiplying by $x - 7$,

$$14x - 98 + x^2 - 7x - x - 7 = \frac{4x^2 - 19x - 63}{3},$$

$$\text{or } x^2 + 6x - 105 = \frac{4x^2 - 19x - 63}{3},$$

$$\therefore \; 3x^2 + 18x - 315 = 4x^2 - 19x - 63,$$

$$\text{and (17. Cor. 3.) } x^2 - 37x = -252,$$

$$\text{completing the square, } x^2 - 37x + \overline{\frac{37}{2}}\Big|^2 = \frac{1369}{4} - 252 = \frac{361}{4},$$

$$\text{extracting the root, } x - \frac{37}{2} = \pm \frac{19}{2},$$

$$\therefore \; x = 28 \text{ or } 9.$$

14. Multiplying by 9, $3x + 12 - \dfrac{63 - 9x}{x - 3} = 4x + 7 - 9$,

$$\text{(17. Cor. 3.) } 14 - \frac{63 - 9x}{x - 3} = x,$$

$$\therefore \; 14x - 42 - 63 + 9x = x^2 - 3x,$$

$$\text{by transposition, } x^2 - 26x = -105,$$

completing the square, $x^2 - 26x + 169 = 169 - 105 = 64$,

$$\therefore\ x - 13 = \pm 8,$$

and $x = 21$ or 5.

15. Multiplying by 28,

$$105 - 7x - \frac{336 - 84x}{4x - 5} = 196x - 92x - 240,$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 345 - \frac{336 - 84x}{4x - 5} = 111x,$$

$$\text{or } 115 - \frac{112 - 28x}{4x - 5} = 37x,$$

$$\therefore\ 460x - 575 - 112 + 28x = 148x^2 - 185x,$$

by transposition, $148x^2 - 673x = -687$,

$$\therefore\ x^2 - \frac{673}{148}x = -\frac{687}{148},$$

completing the square,

$$x^2 - \frac{673}{148}x + \left(\frac{673}{296}\right)^2 = \left(\frac{673}{296}\right)^2 - \frac{687}{148} = \frac{46225}{(296)^2},$$

extracting the root, $x - \frac{673}{296} = \pm \frac{215}{296}$,

$$\therefore\ x = 3 \text{ or } \frac{229}{148}.$$

16. (18. Cor. 1.) $x^2 + 11x + 9 + 4x = 7x^2$,

$\therefore$ by transposition, $6x^2 - 15x = 9$,

$$\text{and } x^2 - \frac{5x}{2} = \frac{3}{2},$$

completing the square, $x^2 - \frac{5x}{2} + \frac{25}{16} = \frac{25}{16} + \frac{3}{2} = \frac{49}{16}$,

extracting the root, $x - \frac{5}{4} = \pm \frac{7}{4}$,

$$\therefore\ x = 2 \text{ or } -\frac{1}{2}.$$

17. Multiplying by 18,

$$4x + 18 + \frac{72x - 54}{4x + 3} = 54 + 3x - 16,$$

$$\text{(17. Cor. 3.)}\ x + \frac{72x - 54}{4x + 3} = 20,$$

$$\therefore\ 4x^2 + 3x + 72x - 54 = 80x + 60,$$

$$\text{or } 4x^2 - 5x = 114,$$

$$\text{completing the square, } 4x^2 - 5x + \overline{\frac{5}{4}}\Big|^2 = 114 + \frac{25}{16} = \frac{1849}{16},$$

$$\text{extracting the root, } 2x - \frac{5}{4} = \pm\frac{43}{4},$$

$$\therefore\ 2x = 12 \text{ or } -\frac{19}{2},$$

$$\text{and } x = 6 \text{ or } -\frac{19}{4}.$$

18. (18. Cor. 1.) $3x^2 - 5x = 7x + 420$,

$$\text{by transposition, } 3x^2 - 12x = 420,$$

$$\text{or } x^2 - 4x = 140,$$

$$\text{completing the square, } x^2 - 4x + 4 = 144,$$

$$\text{extracting the root, } x - 2 = \pm 12,$$

$$\therefore\ x = 14 \text{ or } -10.$$

12. Multiplying by $2x$,

$$6x - 14 + \frac{8x^2 - 20x}{x + 5} = 7x,$$

$$\therefore\ \text{(17. Cor. 3.)}\ \frac{8x^2 - 20x}{x + 5} = x + 14,$$

$$\text{whence } 8x^2 - 20x = x^2 + 19x + 70,$$

$$\text{(17. Cor. 3.)}\ 7x^2 - 39x = 70,$$

$$\therefore\ x^2 - \frac{39}{7}x = 10,$$

completing the square,

$$x^2 - \frac{39}{7} \cdot x + \left.\frac{39}{14}\right|^2 = \frac{1521}{196} + 10 = \frac{3481}{196},$$

$$\text{extracting the root, } x - \frac{39}{14} = \pm \frac{59}{14},$$

$$\text{and } \therefore x = 7 \text{ or } -\frac{10}{7}.$$

20. Multiplying by $2x$,

$$\frac{2x^2 + 4x}{x - 1} - 4 + x = \frac{14x}{3},$$

$$\text{by transposition, } \frac{2x^2 + 4x}{x - 1} = \frac{11x}{3} + 4.$$

$$\text{and } 2x^2 + 4x = \left(\frac{11x}{3} + 4\right) \cdot (x - 1),$$

$$= \frac{11x^2}{3} + \frac{x}{3} - 4,$$

$$\text{by transposition, } \frac{5x^2}{3} - \frac{11x}{3} = 4,$$

$$\text{and } x^2 - \frac{11}{5}x = \frac{12}{5},$$

completing the square,

$$x^2 - \frac{11}{5}x + \left.\frac{11}{10}\right|^2 = \frac{121}{100} + \frac{12}{5} = \frac{361}{100},$$

$$\text{extracting the root, } x - \frac{11}{10} = \pm \frac{19}{10},$$

$$\therefore x = 3 \text{ or } -\frac{4}{5}$$

21. Dividing the equation by 2,

$$\frac{4x}{x + 2} - 3 = \frac{10}{3x},$$

$$\therefore \text{(18. Cor. 1.) } 12x^2 - 9x^2 - 18x = 10x + 20,$$

$$(17.\ \text{Cor. } 3.)\ 3x^2 - 28x = 20,$$

$$\text{and } x^2 - \frac{28}{3}x = \frac{20}{3},$$

completing the square,

$$x^2 - \frac{28}{3}x + \frac{196}{9} = \frac{196}{9} + \frac{20}{3} = \frac{256}{9},$$

$$\text{extracting the root, } x - \frac{14}{3} = \pm\frac{16}{3},$$

$$\therefore\ x = 10, \text{ or } -\frac{2}{3}.$$

22. $(18.\ \text{Cor. } 1.)\ 40x + 27x - 135 = 13x^2 - 65x,$

$$\therefore \text{ by transposition, } 13x^2 - 132x = -135,$$

$$\text{and } x^2 - \frac{132}{13}x = -\frac{135}{13},$$

completing the square,

$$x^2 - \frac{132}{13}x + \left(\frac{66}{13}\right)^2 = \frac{4356}{169} - \frac{135}{13} = \frac{2601}{169},$$

$$\text{extracting the root, } x - \frac{66}{13} = \pm\frac{51}{13},$$

$$\therefore\ x = 9 \text{ or } \frac{15}{13}.$$

23. Multiplying by 45,

$$25x - 60 + \frac{135x - 1080}{4x - 12} = 405 - 21x + 102,$$

$$\text{by transposition, } 46x + \frac{135x - 1080}{4x - 12} = 567,$$

$$\therefore\ 184x^2 - 552x + 135x - 1080 = 2268x - 6804,$$

$$(17.\ \text{Cor. } 3.)\ 184x^2 - 2685x = -5724,$$

$$\text{and } x^2 - \frac{2685}{184} \cdot x = -\frac{5724}{184},$$

completing the square,

$$x^2 - \frac{2685}{184} \cdot x + \overline{\frac{2685}{368}}\Big|^2 = \frac{7209225}{\overline{368}|^2} - \frac{5724}{184} = \frac{2996361}{\overline{368}|^2},$$

$$\text{extracting the root, } x - \frac{2685}{368} = \pm \frac{1731}{368},$$

$$\therefore\ x = 12 \text{ or } \frac{477}{184}.$$

24. Clearing the equation of fractions,

$$6x^2 - 18x + 6x^2 - 39x + 60 = 25 \cdot (x^2 - 7x + 12),$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 13x^2 - 118x = -240,$$

$$\text{and } x^2 - \frac{118}{13}x = -\frac{240}{13},$$

completing the square,

$$x^2 - \frac{118}{13} \cdot x + \overline{\frac{59}{13}}\Big|^2 = \frac{3481}{169} - \frac{240}{13} = \frac{361}{169},$$

$$\text{extracting the root, } x - \frac{59}{13} = \pm \frac{19}{13},$$

$$\therefore\ x = 6 \text{ or } \frac{40}{13}.$$

25. Multiplying by 2 . (10 − x),

$$4x + 6 = \frac{40x - 4x^2}{25 - 3x} - 130 + 13x,$$

$$\text{by transposition, } 136 - 9x = \frac{40x - 4x^2}{25 - 3x},$$

$$\therefore\ 3400 - 633x + 27x^2 = 40x - 4x^2,$$

$$\text{by transposition, } 31x^2 - 673x = -3400,$$

$$\text{and } x^2 - \frac{673}{31} \cdot x = -\frac{3400}{31},$$

completing the square,

$$x^2 - \frac{673}{31} \cdot x + \overline{\frac{673}{62}}\Big|^2 = \frac{452929}{\overline{62}|^2} - \frac{3400}{31} = \frac{31329}{\overline{62}|^2},$$

$$\text{extracting the root, } x - \frac{673}{62} = \pm \frac{177}{62},$$

$$\therefore\ x = 13\tfrac{22}{31}, \text{ or } 8.$$

26. Multiplying by $18x$,

$$52x - 65 - \frac{39x^2 - 91x}{3x + 7} = 9x + 23,$$

$$\text{by transposition, } 43x - 88 = \frac{39x^2 - 91x}{3x + 7},$$

$$\therefore\ 129x^2 + 37x - 616 = 39x^2 - 91x,$$

$$\text{(17. Cor. 3.) } 90x^2 + 128x = 616,$$

$$\therefore\ x^2 + \frac{64}{45}x = \frac{308}{45},$$

completing the square,

$$x^2 + \frac{64}{45} \cdot x + \left(\frac{32}{45}\right)^2 = \frac{1024}{(45)^2} + \frac{308}{45} = \frac{14884}{(45)^2},$$

$$\text{extracting the root, } x + \frac{32}{45} = \pm \frac{122}{45},$$

$$\therefore\ x = 2 \text{ or } -\frac{154}{45}.$$

27. By transposition, $2x - 9 - \dfrac{8x^2 + 16}{4x + 7} = -\dfrac{12x - 11}{2x - 3}$,

multiplying by $4x + 7$,

$$8x^2 - 22x - 63 - 8x^2 - 16 = -\frac{48x^2 + 40x - 77}{2x - 3},$$

$$\text{or } 22x + 79 = \frac{48x^2 + 40x - 77}{2x - 3},$$

$$\therefore\ 44x^2 + 92x - 237 = 48x^2 + 40x - 77,$$

$$\text{(17. Cor. 3.) } 4x^2 - 52x = -160,$$

$$\text{or } x^2 - 13x = -40,$$

completing the square,

$$x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 40 = \frac{9}{4},$$

$$\text{extracting the root, } x - \frac{13}{2} = \pm\frac{3}{2},$$

$$\therefore\ x = 8 \text{ or } 5.$$

28. Multiplying by 2,

$$\frac{2x+5}{x+9} = x - \frac{x^2+20}{x+8},$$

$\therefore\ 2x^2 + 21x + 40 = x^3 + 17x^2 + 72x - x^3 - 9x^2 - 20x - 180,$

$$\text{and (17. Cor. 3.) } 6x^2 + 31x = 220,$$

$$\text{and } x^2 + \frac{31}{6}x = \frac{220}{6},$$

completing the square,

$$x^2 + \frac{31}{6}x + \frac{961}{144} = \frac{961}{144} + \frac{220}{6} = \frac{6241}{144},$$

$$\text{extracting the root, } x + \frac{31}{12} = \pm\frac{79}{12},$$

$$\therefore\ x = 4 \text{ or } -\frac{55}{6}.$$

29. Multiplying by $(3x+4)\,.\,(x+6)$,

$$3x^2 + 16x + 16 + \frac{5\,.\,(x+6)\,.\,(3x+4)}{2x+4} = 3x^2 + 25x + 42,$$

$$\text{(17. Cor. 3.) } \frac{15x^2 + 110x + 120}{2x+4} = 9x + 26,$$

$$\therefore\ 15x^2 + 110x + 120 = 18x^2 + 88x + 104,$$

$$\therefore\ \text{(17. Cor. 3.) } 3x^2 - 22x = 16,$$

$$\text{and } x^2 - \frac{22}{3}x = \frac{16}{3},$$

completing the square,

$$x^2 - \frac{22}{3} \cdot x + \frac{121}{9} = \frac{121}{9} + \frac{16}{3} = \frac{169}{9},$$

$$\text{extracting the root, } x - \frac{11}{3} = \pm \frac{13}{3},$$

$$\therefore \; x = 8 \text{ or } -\frac{2}{3}.$$

30. Multiplying by $5x \,.\, (5x + 18)$,

$$\frac{20x \,.\, (5x + 18)}{2x + 3} + 15x^2 + 30x = 15x^2 + 79x + 90,$$

$$(17.\ \text{Cor. } 3.)\ \frac{100x^2 + 360x}{2x + 3} = 49x + 90,$$

$$\therefore \; 100x^2 + 360x = 98x^2 + 327x + 270,$$

$$\text{and } (17.\ \text{Cor. } 3.)\ 2x^2 + 33x = 270,$$

$$\therefore \; x^2 + \frac{33}{2} \cdot x = 135,$$

completing the square,

$$x^2 + \frac{33}{2} \cdot x + \frac{1089}{16} = \frac{1089}{16} + 135 = \frac{3249}{16},$$

$$\text{extracting the root, } x + \frac{33}{4} = \pm \frac{57}{4},$$

$$\text{and } \therefore \; x = 6 \text{ or } -\frac{45}{2}.$$

31. Multiplying by $(2 + 4x) \,.\, (2x + 12)$,

$$\frac{8 \,.\, (2 + 4x) \,.\, (2x + 12)}{9 + 5x} + 16x^2 + 62x - 204 = 16x^2 + 20x + 6,$$

$$\therefore \; (17.\ \text{Cor. } 3.)\ \frac{8 \,.\, (8x^2 + 52x + 24)}{9 + 5x} = 210 - 42x,$$

$$\text{whence } 32x^2 + 208x + 96 = 945 + 336x - 105x^2,$$

$$\text{by transposition, } 137x^2 - 128x = 849,$$

$$\text{and } x^2 - \frac{128}{137}x = \frac{849}{137},$$

completing the square,

$$x^2 - \frac{128}{137}x + \left(\frac{64}{137}\right)^2 = \frac{4096}{\overline{137}|^2} + \frac{849}{137} = \frac{120409}{\overline{137}|^2},$$

$$\text{extracting the root, } x - \frac{64}{137} = \pm\frac{347}{137},$$

$$\therefore\ x = 3 \text{ or } -\frac{283}{137}.$$

32. Dividing every term of the equation by 4,

$$\frac{3}{5-x} + \frac{2}{4-x} = \frac{8}{x+2},$$

$\therefore\ 3\,.\,(4-x)\,.\,(x+2) + 2\,.\,(5-x)\,.\,(x+2) = 8\,.\,(5-x)$ $.\,(4-x)$,

$$\text{or } (22-5x)\,.\,(x+2) = 8\,.\,(20-9x+x^2),$$

$$\text{or } 44 + 12x - 5x^2 = 160 - 72x + 8x^2,$$

$$\text{by transposition, } 13x^2 - 84x = -116,$$

completing the square,

$$x^2 - \frac{84}{13}x + \left(\frac{42}{13}\right)^2 = \frac{1764}{169} - \frac{116}{13} = \frac{256}{169},$$

$$\text{extracting the root, } x - \frac{42}{13} = \pm\frac{16}{13},$$

$$\text{whence } x = \frac{58}{13} \text{ or } 2.$$

33. Multiplying by $2x - 2$,

$$\frac{4x^2 - 6x + 2}{3 - x} = 8 - x^2 + x^2 - x = 8 - x,$$

$$\therefore\ (18.\ \text{Cor. 1.})\ 4x^2 - 6x + 2 = 24 - 11x + x^2,$$

$$\text{by transposition, } 3x^2 + 5x = 22,$$

$$\text{completing the square, } x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} + \frac{22}{3} = \frac{289}{36},$$

$$\text{extracting the root, } x + \frac{5}{6} = \pm \frac{17}{6},$$

$$\therefore\ x = 2 \text{ or } -\frac{11}{3}.$$

34. Multiplying every term by x,

$$\frac{3}{6 - x} + \frac{6}{x + 2} = \frac{11}{5},$$

$$\therefore\ 15x + 30 + 180 - 30x = 132 + 44x - 11x^2,$$

$$\text{by transposition, } 11x^2 - 59x = -78,$$

completing the square,

$$x^2 - \frac{59}{11}x + \overline{\frac{59}{22}}\Big|^2 = \frac{3481}{\overline{22}|^2} - \frac{78}{11} = \frac{49}{\overline{22}|^2},$$

$$\text{extracting the root, } x - \frac{59}{22} = \pm \frac{7}{22},$$

$$\therefore\ x = 3 \text{ or } \frac{26}{11}.$$

35. Dividing every term of the equation by x,

$$\frac{4x + 7}{19} + \frac{5 - x}{3 + x} = \frac{4x}{9},$$

$$\therefore\ 36x + 63 + \frac{855 - 171x}{3 + x} = 76x,$$

$$(17.\ \text{Cor. } 3.)\ 63 + \frac{855 - 171x}{3 + x} = 40x,$$

$$\text{and } 189 + 63x + 855 - 171x = 120x + 40x^2,$$

$$\text{by transposition, } 40x^2 + 228x = 1044,$$

$$\text{whence } x^2 + \frac{57}{10}x = \frac{1044}{40},$$

completing the square,

$$x^2 + \frac{57}{10}x + \overline{\frac{57}{20}}\Big|^2 = \frac{3249}{400} + \frac{1044}{40} = \frac{13689}{400},$$

$$\text{extracting the root, } x + \frac{57}{20} = \pm \frac{117}{20},$$

$$\therefore\ x = 3 \text{ or } -\frac{87}{10}.$$

36. (18. Cor. 1.) $x^4 + 2x^3 + 8 = x^4 + 2x^3 + 3x^2 + 2x - 48,$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 3x^2 + 2x = 56,$$

$$\text{and } x^2 + \frac{2}{3}x = \frac{56}{3},$$

$$\text{completing the square, } x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{56}{3} + \frac{1}{9} = \frac{169}{9},$$

$$\text{extracting the root, } x + \frac{1}{3} = \pm \frac{13}{3},$$

$$\therefore\ x = 4 \text{ or } -\frac{14}{3}.$$

37. Multiplying the equation by $x \,.\, (x + 12)$,

$$x^2 + 24x + 144 + x^2 = \frac{78x^2 + 936x}{15},$$

$$\text{whence } 30x^2 + 360x + 2160 = 78x^2 + 936x,$$

$$\text{and (17. Cor. 3.) } 48x^2 + 576x = 2160,$$

$$\text{or } x^2 + 12x = 45,$$

$$\text{completing the square, } x^2 + 12x + 36 = 81,$$

$$\text{extracting the root, } x + 6 = \pm 9,$$

$$\therefore\ x = 3 \text{ or } -15.$$

38. Squaring both sides of the equation,

$$28x^2 + 39x + 5 = 900,$$

$$\therefore\ x^2 + \frac{39}{28} \,.\, x = \frac{895}{28},$$

completing the square,

$$x^2 + \frac{39}{28}x + \left(\frac{39}{56}\right)^2 = \frac{895}{28} + \frac{1521}{\overline{56}|^2} = \frac{101761}{\overline{56}|^2},$$

$$\text{extracting the root, } x + \frac{39}{56} = \pm \frac{319}{56},$$

$$\therefore\ x = 5 \text{ or } -\frac{179}{28}.$$

39. Clearing the equation of fractions,

$$81 - x = 3x - \frac{19}{5}\sqrt{x},$$

$$\text{by transposition, } 4x - \frac{19}{5}\sqrt{x} = 81,$$

completing the square,

$$4x - \frac{19}{5}\sqrt{x} + \overline{\frac{19}{20}}\Big|^2 = 81 + \frac{361}{400} = \frac{32761}{400},$$

$$\text{extracting the root, } 2\sqrt{x} - \frac{19}{20} = \pm \frac{181}{20},$$

$$\therefore\ 2\sqrt{x} = 10 \text{ or } -\frac{81}{10},$$

$$\text{and } \sqrt{x} = 5 \text{ or } -\frac{81}{20},$$

$$\therefore\ x = 25 \text{ or } \frac{6561}{400}.$$

40. Multiplying the numerator and denominator of the first fraction by $x - \sqrt{x}$,

$$\frac{x^2 - x}{(x - \sqrt{x})^2} = \frac{x^2 - x}{4},$$

$$\therefore\ (x - \sqrt{x})^2 = 4,$$

$$\text{and } x - \sqrt{x} = \pm 2;$$

$$\text{completing the square, } x - \sqrt{x} + \frac{1}{4} = \frac{1}{4} \pm 2 = \frac{9}{4} \text{ or } -\frac{7}{4},$$

$$\text{extracting the root, } \sqrt{x} - \frac{1}{2} = \pm \frac{3}{2} \text{ or } \pm \frac{\sqrt{-7}}{2},$$

$$\therefore\ \sqrt{x} = 2 \text{ or } -1;\ \text{or } \frac{1 \pm \sqrt{-7}}{2},$$

$$\text{whence } x = 4 \text{ or } 1;\ \text{or } \frac{-3 \pm \sqrt{-7}}{2},$$

41. Clearing the equation of fractions,

$$11x - 11\sqrt{x+1} = 5x + 5\sqrt{x+1},$$

$$\text{by transposition, } 6x = 16\sqrt{x+1},$$

$$\text{whence } 9x^2 = 64x + 64,$$

$$\therefore\ 9x^2 - 64x = 64,$$

completing the square,

$$9x^2 - 64x + \left(\frac{32}{3}\right)^2 = \frac{1024}{9} + 64 = \frac{1600}{9},$$

$$\text{extracting the root, } 3x - \frac{32}{3} = \pm\frac{40}{3},$$

$$\therefore\ 3x = 24 \text{ or } -\frac{8}{3},$$

$$\text{and } x = 8 \text{ or } -\frac{8}{9}.$$

42. Clearing the equation of fractions,

$$15x^{\frac{3}{2}} - 5\sqrt{x} + 2 + 10\sqrt{x} = 3x + 15x^{\frac{3}{2}},$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 3x - 5\sqrt{x} = 2,$$

$$\text{and } x - \frac{5}{3}\sqrt{x} = \frac{2}{3},$$

$$\text{completing the square, } x - \frac{5}{3}\sqrt{x} + \frac{25}{36} = \frac{2}{3} + \frac{25}{36} = \frac{49}{36},$$

$$\therefore\ \sqrt{x} = 2 \text{ or } -\frac{1}{3}.$$

$$\text{and } x = 4 \text{ or } \frac{1}{9}.$$

43. Clearing the equation of fractions, and transposing,

$$x^3 - 3x^{\frac{3}{2}} = 40,$$

completing the square, $x^3 - 3x^{\frac{3}{2}} + \frac{9}{4} = 40 + \frac{9}{4} = \frac{169}{4}$,

extracting the root, $x^{\frac{3}{2}} - \frac{3}{2} = \pm \frac{13}{2}$,

$$\therefore\ x^{\frac{3}{2}} = 8 \text{ or } -5,$$

$$\text{and } x = 4 \text{ or } \overline{-5}\big|^{\frac{2}{3}}.$$

44. Completing the square,

$$x^{\frac{4}{3}} + 7x^{\frac{2}{3}} + \frac{49}{4} = 44 + \frac{49}{4} = \frac{225}{4},$$

extracting the root, $x^{\frac{2}{3}} + \frac{7}{2} = \pm \frac{15}{2}$,

$$\therefore\ x^{\frac{2}{3}} = 4 \text{ or } -11,$$

$$\text{and } x = \pm 8, \text{ or } \pm\ \overline{-11}\big|^{\frac{3}{2}}.$$

45. Completing the square,

$$4x^{\frac{1}{3}} + x^{\frac{1}{6}} + \frac{1}{16} = 39 + \frac{1}{16} = \frac{625}{16},$$

extracting the root, $2x^{\frac{1}{6}} + \frac{1}{4} = \pm \frac{25}{4}$,

$$\therefore\ 2x^{\frac{1}{6}} = 6 \text{ or } -\frac{13}{2},$$

$$\text{and } x^{\frac{1}{6}} = 3 \text{ or } -\frac{13}{4},$$

$$\therefore\ x = 729 \text{ or } \left(-\frac{13}{4}\right)^6.$$

46. Dividing by 3,

$$x^6 + 14x^3 = 1107,$$

completing the square,

$$x^6 + 14x^3 + 49 = 1107 + 49 = 1156,$$

$$\text{extracting the root, } x^3 + 7 = \pm 34,$$
$$\therefore\ x^3 = 27 \text{ or } -41,$$
$$\text{and } x = 3 \text{ or } -\sqrt[3]{41}.$$

47. Multiplying by x^3, and transposing,
$$2x^3 - 17x^{\frac{3}{2}} = -8,$$
$$\text{or } x^3 - \frac{17}{2}.x^{\frac{3}{2}} = -4,$$
completing the square,
$$x^3 - \frac{17}{2}.x^{\frac{3}{2}} + \frac{289}{16} = \frac{289}{16} - 4 = \frac{225}{16},$$
$$\text{extracting the root, } x^{\frac{3}{2}} - \frac{17}{4} = \pm\frac{15}{4},$$
$$\therefore\ x^{\frac{3}{2}} = 8 \text{ or } \frac{1}{2}.$$
$$\text{and } x = 4 \text{ or } \sqrt[3]{\frac{1}{4}}.$$

48. Multiplying by $x^{\frac{3}{2}}$,
$$x^3 + 41 = 97 + x^{\frac{3}{2}},$$
$$\text{by transposition, } x^3 - x^{\frac{3}{2}} = 56,$$
$$\text{completing the square, } x^3 - x^{\frac{3}{2}} + \frac{1}{4} = 56 + \frac{1}{4} = \frac{225}{4},$$
$$\therefore\ x^{\frac{3}{2}} - \frac{1}{2} = \pm\frac{15}{2},$$
$$\text{and } x^{\frac{3}{2}} = 8 \text{ or } -7,$$
$$\therefore\ x = 4 \text{ or } \overline{-7}|^{\frac{2}{3}}.$$

49. Multiplying by x,
$$x^{\frac{2}{3}} + x^{\frac{4}{3}} = 3 - x^{\frac{4}{3}},$$
$$\text{by transposition, } 2x^{\frac{4}{3}} + x^{\frac{2}{3}} = 3,$$
$$\text{and } x^{\frac{4}{3}} + \frac{1}{2}x^{\frac{2}{3}} = \frac{3}{2},$$

$$\text{completing the square, } x^{\frac{2}{3}} + \frac{1}{2}x^{\frac{1}{3}} + \frac{1}{16} = \frac{3}{2} + \frac{1}{16} = \frac{25}{16},$$

$$\text{extracting the root, } x^{\frac{1}{3}} - \frac{1}{4} = \pm\frac{5}{4},$$

$$\therefore\ x^{\frac{1}{3}} = 1 \text{ or } -\frac{3}{2},$$

$$\text{and } x = 1 \text{ or } -\frac{27}{8}.$$

50. Dividing the equation by 3, and reducing,

$$x^{\frac{4n}{3}} - \frac{4}{3}x^{\frac{2n}{3}} = \frac{4}{3},$$

$$\text{completing the square, } x^{\frac{4n}{3}} - \frac{4}{3}x^{\frac{2n}{3}} - \frac{4}{9} = \frac{4}{3} + \frac{4}{9} = \frac{16}{9},$$

$$\text{extracting the root, } x^{\frac{2n}{3}} - \frac{2}{3} = \pm\frac{4}{3},$$

$$\therefore\ x^{\frac{2n}{3}} = 2 \text{ or } \frac{2}{3},$$

$$\text{and } x = 8^{\frac{1}{2n}} \text{ or } -\left(\frac{8}{27}\right)^{\frac{1}{2n}}.$$

51. (18. Cor. 1.) $6x - 2x^2 - 9\sqrt{x} + 3x^{\frac{3}{2}} = \frac{5}{3}x + \frac{10}{3} + 3x^{\frac{3}{2}} - 2x^2 + 6\sqrt{x} - 4x$,

$$\text{by transposition, } 10x - 15\sqrt{x} = \frac{5x}{3} + \frac{10}{3},$$

$$\text{dividing by 5, } 2x - 3\sqrt{x} = \frac{x}{3} + \frac{2}{3},$$

$$\therefore\ x - \frac{9}{5}\sqrt{x} = \frac{2}{5},$$

completing the square,

$$x - \frac{9}{5}\sqrt{x} + \frac{81}{100} = \frac{2}{5} + \frac{81}{100} = \frac{121}{100},$$

$$\text{extracting the root, } \sqrt{x} - \frac{9}{10} = \pm\frac{11}{10},$$

$$\therefore\ \sqrt{x} = 2 \text{ or } -\frac{1}{5},$$

$$\text{and } x = 4 \text{ or } \frac{1}{25}.$$

52. Dividing by x^3,

$$2\,.\left(1 + \frac{a^3}{x^3}\right)^{\frac{1}{2}} = 2 + \frac{4a}{x} + \frac{a^2}{x^2} - \frac{a^3}{x^3},$$

$\therefore$ adding 1, and completing the square,

$$\left(1 + \frac{a^3}{x^3}\right) + 2\,.\left(1 + \frac{a^3}{x^3}\right)^{\frac{1}{2}} + 1 = 4 + \frac{4a}{x} + \frac{a^2}{x^2},$$

$$\text{extracting the root, } \left(1 + \frac{a^3}{x^3}\right)^{\frac{1}{2}} + 1 = 2 + \frac{a}{x},$$

$$\text{and } \left(1 + \frac{a^3}{x^3}\right)^{\frac{1}{2}} = 1 + \frac{a}{x},$$

$$\therefore\ \text{squaring, } 1 + \frac{a^3}{x^3} = 1 + \frac{2a}{x} + \frac{a^2}{x^2},$$

$$\text{or } \frac{a^3}{x^3} - \frac{a^2}{x^2} = \frac{2a}{x},$$

$$\text{and } \frac{a^2}{x^2} - \frac{a}{x} = 2,$$

$$\text{completing the square, } \frac{a^2}{x^2} - \frac{a}{x} + \frac{1}{4} = \frac{9}{4},$$

$$\text{extracting the root, } \frac{a}{x} - \frac{1}{2} = \pm\frac{3}{2},$$

$$\therefore\ \frac{a}{x} = 2, \text{ or } -1,$$

$$\therefore\ x = \frac{a}{2}, \text{ or } -a.$$

53. By transposition, $a c x^2 - (a d - b c) \,.\, x = b d$,

$$\therefore\ x^2 - \left(\frac{d}{c} - \frac{b}{a}\right) . \, x = \frac{bd}{ac},$$

completing the square,

$$x^2 - \left(\frac{d}{c} - \frac{b}{a}\right) . \, x + \frac{1}{4} . \left(\frac{d}{c} - \frac{b}{a}\right)^2 = \frac{b}{a} . \frac{d}{c} + \frac{1}{4} . \left(\frac{d}{c} - \frac{b}{a}\right)^2 = \frac{1}{4} . \left(\frac{d}{c} + \frac{b}{a}\right)^2,$$

extracting the root, $x - \frac{1}{2} . \left(\frac{d}{c} - \frac{b}{a}\right) = \pm \frac{1}{2} . \left(\frac{d}{c} + \frac{b}{a}\right)$,

$$\therefore\ x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

54. By transposition, $\frac{a^2 x^2}{b^2} - 2 . \frac{b}{c} . \frac{a x}{b} = - \frac{d^2}{c^2}$,

completing the square, $\frac{a^2 x^2}{b^2} - 2 . \frac{b}{c} . \frac{a x}{b} + \frac{b^2}{c^2} = \frac{b^2 - d^2}{c^2}$,

extracting the root, $\frac{a x}{b} - \frac{b}{c} = \pm \frac{\sqrt{b^2 - d^2}}{c}$,

$$\therefore\ x = \frac{b}{a} . \frac{b \pm \sqrt{b^2 - d^2}}{c}.$$

55. Completing the square,

$$9 a^4 b^4 x^2 - 2x \,.\, 3 a^2 b^2 x + a^2 = a^2 + b^2,$$

extracting the root, $3 a^2 b^2 x - a = \pm \sqrt{a^2 + b^2}$,

$$\therefore\ x = \frac{a \pm \sqrt{a^2 + b^2}}{3 a^2 b^2}.$$

56. Multiplying every term by $(a + b)$,

$$(a + b)^2 . \, x^2 - c \,.\, (a + b) \,.\, x = a c,$$

completing the square,

$$(a+b)^2 \cdot x^2 - c \cdot (a+b) \cdot x + \frac{c^2}{4} = \frac{c^2}{4} + ac,$$

extracting the root, $(a+b) \cdot x - \frac{c}{2} = \pm \frac{\sqrt{c^2+4ac}}{2}$,

$$\therefore\ x = \frac{c \pm \sqrt{c^2+4ac}}{2 \cdot (a+b)}.$$

57. Squaring both sides of the equation,

$$9 \cdot (112 - 8x) = 361 + 38\sqrt{3x+7} + 3x + 7,$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 640 - 75x = 38\sqrt{3x+7},$$

and squaring both sides,

$$409600 - 96000x + 5625x^2 = 4332x + 10108,$$

by transposition, $5625x^2 - 100332x = -399492$,

or $625x^2 - 11148x = -44388$,

completing the square,

$$625x^2 - \frac{11148}{25} \cdot 25x + \left.\frac{5574}{25}\right|^2 = \frac{31069476}{625} - 44388 = \frac{3326976}{625},$$

$\therefore$ extracting the root, $25x - \frac{5574}{25} = \pm \frac{1824}{25}$,

and $25x = \frac{7398}{25}$ or 150,

$$\therefore\ x = \frac{7398}{625} \text{ or } 6.$$

58. Squaring both sides of the equation,

$$2x + 7 + 3x - 18 + 2\sqrt{2x+7} \cdot \sqrt{3x-18} = 7x + 1,$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 2\sqrt{2x+7} \cdot \sqrt{3x-18} = 2x + 12,$$

and $(2x+7) \cdot (3x-18) = (x+6)^2$,

or $6x^2 - 15x - 126 = x^2 + 12x + 36$,

whence $5x^2 - 27x = 162$,

and $x^2 - \frac{27}{5}x = \frac{162}{5}$,

completing the square,

$$x^2 - \frac{27}{5}x + \overline{\frac{27}{10}}\Big|^2 = \frac{162}{5} + \frac{729}{100} = \frac{3969}{100},$$

$$\text{extracting the root, } x - \frac{27}{10} = \pm\frac{63}{10},$$

$$\therefore\ x = 9 \text{ or } -\frac{18}{5}.$$

59. By transposition,

$$7\sqrt{\frac{3x}{2} - 5} - \frac{7}{4}\sqrt{10x + 56} = \sqrt{\frac{x}{5} + 45},$$

squaring both sides,

$$49 . \left(\frac{3x}{2} - 5 + \frac{10x + 56}{16} - \frac{1}{2}\sqrt{\frac{3x}{2} - 5} . \sqrt{10 + 56}\right) = \frac{x}{5} + 45,$$

$$\text{or } 49 . \left(\frac{17x - 12}{8} - \frac{1}{2}\sqrt{\frac{3x}{2} - 5} . \sqrt{10x + 56}\right) = \frac{x}{5} + 45,$$

$$\therefore\ \frac{833x}{8} - \frac{x}{5} - \frac{237}{2} = \frac{49}{2}\sqrt{\frac{3x}{2} - 5} . \sqrt{10x + 56},$$

$$\text{or } 4157x - 4740 = 980\sqrt{\frac{3x}{2} - 5} . \sqrt{10x + 56},$$

$\therefore$ squaring both sides, $17280649x^2 + 39408360 + 22467600$ $= 960400 . (15x^2 + 34x - 280)$,

and (17. Cor. 3.)

$$2874649x^2 - 72061960x = -291379600,$$

$$\text{or } x^2 - \frac{72061960}{2874649}x = -\frac{291379600}{2874649},$$

$\therefore$ completing the square,

$$x^2 - \frac{72061960}{2874649}x + \overline{\frac{36030980}{2874649}}\Big|^2 = \frac{1298231519760400}{\overline{2874649}|^2} - \frac{291379600}{2874649}$$

$$= \frac{460617444000000}{\overline{2874649}|^2},$$

$$\text{extracting the root, } x - \frac{36030980}{2874649} = \pm \frac{21462000}{2874649},$$

$$\therefore\ x = 20 \text{ or } \frac{14568980}{2874649}.$$

60. Clearing the equation of fractions,

$$4 \,.\, (16 - x) = 11 \,.\, (64 - 9x) + x^2 - 5x + 11,$$

$$\therefore \text{ by transposition, } x^2 - 100x = -651,$$

completing the square,

$$x^2 - 100x + 2500 = 2500 - 651 = 1849,$$

$$\text{extracting the root, } x - 50 = \pm 43,$$

$$\therefore\ x = 93 \text{ or } 7.$$

61. Clearing the equation of fractions,

$$9 \,.\, (36 - x) = 23 \,.\, (x^2 - 4x) + 7x^2 - 3x + 4,$$

$$\text{by transposition, } 30x^2 - 86x = 320,$$

$$\text{or } x^2 - \frac{86}{30}x = \frac{320}{30},$$

completing the square,

$$x^2 - \frac{86}{30}x + \left|\frac{43}{30}\right|^2 = \frac{1849}{900} + \frac{320}{30} = \frac{11449}{900},$$

$$\text{extracting the root, } x - \frac{43}{30} = \pm \frac{107}{30},$$

$$\therefore\ x = 5 \text{ or } -\frac{32}{15}.$$

62. (Alg. 182.) $2x : 2\sqrt{x} :: 5\sqrt{x} + 6 : \sqrt{x} + 6,$

$$\text{(Alg. 184.) } \sqrt{x} : 1 :: 5\sqrt{x} + 6 : \sqrt{x} + 6,$$

$$\therefore \text{ (21) } x + 6\sqrt{x} = 5\sqrt{x} + 6,$$

$$\text{and (17. Cor. 3.) } x + \sqrt{x} = 6,$$

$$\text{completing the square, } x + \sqrt{x} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4},$$

$$\text{extracting the root, } \sqrt{x} + \frac{1}{2} = \pm \frac{5}{2},$$

$$\therefore \sqrt{x} = 2 \text{ or } -3,$$

$$\text{and } x = 4 \text{ or } 9.$$

63. Completing the square,

$$x^2 + 11 + \sqrt{x^2 + 11} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4},$$

$$\text{extracting the root, } \sqrt{x^2 + 11} + \frac{1}{2} = \pm \frac{13}{2},$$

$$\therefore \sqrt{x^2 + 11} = 6 \text{ or } -7,$$

$$\text{and } x^2 + 11 = 36 \text{ or } 49,$$

$$\therefore x^2 = 25 \text{ or } 38,$$

$$\text{and } x = \pm 5, \text{ or } \pm \sqrt{38}.$$

64. Completing the square,

$$(x-5)^3 - 3 \cdot (x-5)^{\frac{3}{2}} + \frac{9}{4} = 40 + \frac{9}{4} = \frac{169}{4},$$

$$\text{extracting the root, } (x-5)^{\frac{3}{2}} - \frac{3}{2} = \pm \frac{13}{2},$$

$$\text{and } \therefore (x-5)^{\frac{3}{2}} = 8 \text{ or } -5,$$

$$\therefore x - 5 = 4 \text{ or } (-5)^{\frac{2}{3}},$$

$$\text{and } x = 9 \text{ or } \overline{-5}|^{\frac{2}{3}} + 5.$$

65. Adding 6 to each side and transposing,

$$(x+6) - 2\sqrt{x+6} = 8,$$

$$\text{completing the square, } (x+6) - 2\sqrt{x+6} + 1 = 9,$$

$$\text{extracting the root, } \sqrt{x+6} - 1 = \pm 3,$$

$$\therefore \sqrt{x+6} = 4 \text{ or } -2,$$

$$\text{and } x + 6 = 16 \text{ or } 4,$$

$$\therefore x = 10 \text{ or } -2.$$

66. Subtracting 20 from each side,

$$(x^2 + 5)^2 - 4 \,.\, (x^2 + 5) = 140,$$

completing the square, $(x^2 + 5)^2 - 4 \,.\, (x^2 + 5) + 4 = 144,$

extracting the root, $x^2 + 5 - 2 = \pm\ 12,$

$$\therefore\ x^2 = 9 \text{ or } -15,$$

$$\text{and } x = \pm\ 3 \text{ or } \pm\ \sqrt{-15}.$$

67. Completing the square,

$$(x^2 - 7x + 18) + \sqrt{x^2 - 7x + 18} + \frac{1}{4} = 24 + 18 + \frac{1}{4} = \frac{169}{4},$$

extracting the root, $\sqrt{x^2 - 7x + 18} + \frac{1}{2} = \pm\ \frac{13}{2},$

$$\text{and } \sqrt{x^2 - 7x + 18} = 6 \text{ or } -7;$$

$$\text{whence } x^2 - 7x + 18 = 36 \text{ or } 49,$$

$$\text{and } x^2 - 7x = 18 \text{ or } 31,$$

completing the square,

$$x^2 - 7x + \frac{49}{4} = 18 + \frac{49}{4} \text{ or } 31 + \frac{49}{4} = \frac{121}{4} \text{ or } \frac{173}{4},$$

extracting the root, $x - \frac{7}{2} = \pm\ \frac{11}{2} \text{ or } \pm\ \frac{\sqrt{173}}{2},$

$$\therefore\ x = 9 \text{ or } -2, \text{ or } \frac{7 \pm \sqrt{173}}{2}.$$

68. (17. Cor. 1.) $4x^2 - 9x - \sqrt{4x^2 - 9x + 11} = -5,$

completing the square,

$$(4x^2 - 9x + 11) - \sqrt{4x^2 - 9x + 11} + \frac{1}{4} = 11 - 5 + \frac{1}{4} = \frac{25}{4},$$

extracting the root, $\sqrt{4x^2 - 9x + 11} - \frac{1}{2} = \pm\ \frac{5}{2},$

$$\therefore\ \sqrt{4x^2 - 9x + 11} = 3 \text{ or } -2,$$

$$\text{and } 4x^2 - 9x + 11 = 9 \text{ or } 4,$$

$$\therefore\ 4x^2 - 9x = -2 \text{ or } -7,$$

completing the square,

$$4x^2 - 9x + \overline{\frac{9}{4}}\Big|^2 = \frac{81}{16} - 2, \text{ or } \frac{81}{16} - 7 = \frac{49}{16} \text{ or } -\frac{31}{16},$$

$$\text{extracting the root, } 2x - \frac{9}{4} = \pm\frac{7}{4} \text{ or } \pm\frac{\sqrt{-31}}{4},$$

$$\therefore\ 2x = 4 \text{ or } \frac{1}{2}, \text{ or } \frac{9 \pm \sqrt{-31}}{4},$$

$$\text{and } x = 2 \text{ or } \frac{1}{4}, \text{ or } \frac{9 \pm \sqrt{-31}}{8}.$$

69. By transposition, $5x + x^2 + \sqrt{5x + x^2} = 42$,
completing the square,

$$(5x + x^2) + \sqrt{5x + x^2} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4},$$

$$\text{extracting the root, } \sqrt{5x + x^2} + \frac{1}{2} = \pm\frac{13}{2},$$

$$\therefore\ \sqrt{5x + x^2} = 6 \text{ or } -7,$$

$$\text{and } x^2 + 5x = 36 \text{ or } 49,$$

$$\text{completing the square, } x^2 + 5x + \frac{25}{4} = \frac{169}{4} \text{ or } \frac{221}{4},$$

$$\text{extracting the root, } x + \frac{5}{2} = \pm\frac{13}{2} \text{ or } \pm\frac{\sqrt{221}}{2},$$

$$\therefore\ x = 4 \text{ or } -9, \text{ or } \frac{-5 \pm \sqrt{221}}{2}.$$

70. Multiplying by $2 \cdot (x + 2)^{\frac{1}{2}}$,

$$4 + (x + 2)^2 = \frac{17}{2} \cdot (x + 2),$$

$$\text{by transposition, } (x + 2)^2 - \frac{17}{2} \cdot (x + 2) = -4,$$

completing the square,

$$(x+2)^2 - \frac{17}{2} \cdot (x+2) + \frac{289}{16} = \frac{289}{16} - 4 = \frac{225}{16},$$

$$\text{extracting the root, } x + 2 - \frac{17}{4} = \pm \frac{15}{4},$$

$$\therefore\ x = 6 \text{ or } -\frac{3}{2}.$$

71. Completing the square,

$$\frac{x}{x+4} + \frac{4}{\sqrt{x+4}} + \frac{4}{x} = \frac{25}{x},$$

$$\text{extracting the root, } \sqrt{\frac{x}{x+4}} + \frac{2}{\sqrt{x}} = \pm \frac{5}{\sqrt{x}},$$

$$\therefore\ \sqrt{\frac{x}{x+4}} = \frac{3}{\sqrt{x}} \text{ or } -\frac{7}{\sqrt{x}},$$

$$\text{and } \frac{x}{x+4} = \frac{9}{x} \text{ or } \frac{49}{x},$$

$$\therefore\ x^2 - 9x = 36, \text{ or } x^2 - 49x = 196.$$

$$\text{In the first case, } x^2 - 9x + \frac{81}{4} = 36 + \frac{81}{4} = \frac{225}{4},$$

$$\text{extracting the root, } x - \frac{9}{2} = \pm \frac{15}{2},$$

$$\therefore\ x = 12 \text{ or } -3.$$

In the second case,

$$x^2 - 49x + \left|\frac{49}{2}\right|^2 = \frac{2401}{4} + 196 = \frac{3185}{4},$$

$$\text{extracting the root, } x - \frac{49}{2} = \pm \frac{\sqrt{3185}}{4},$$

$$\text{and } x = \frac{49 \pm \sqrt{3185}}{2}.$$

72. Multiplying the equation by $\frac{7+x}{7-x}$,

$$\left(\frac{7+x}{7-x}\right)^2 - \frac{29}{10}\cdot\frac{7+x}{7-x} = -1,$$

completing the square,

$$\left(\frac{7+x}{7-x}\right)^2 - \frac{29}{10}\cdot\frac{7+x}{7-x} + \overline{\frac{29}{20}}\Big|^2 = \frac{841}{400} - 1 = \frac{441}{400},$$

$$\text{extracting the root, } \frac{7+x}{7-x} - \frac{29}{20} = \pm\frac{21}{20},$$

$$\therefore \frac{7+x}{7-x} = \frac{5}{2} \text{ or } \frac{2}{5},$$

whence $14 + 2x = 35 - 5x$,

and $49 = 7x$,

$\therefore x = 7$,

or $25 + 5x = 14 - 2x$,

$\therefore 7x = -21$,

and $\therefore x = -3$.

73. Multiplying the equation by $\frac{3x+5}{3x-5}$,

$$\left(\frac{3x+5}{3x-5}\right)^2 - 1 = \frac{135}{176}\cdot\frac{3x+5}{3x-5},$$

transposing and completing the square,

$$\left(\frac{3x+5}{3x-5}\right)^2 - \frac{135}{176}\cdot\frac{3x+5}{3x-5} + \overline{\frac{135}{352}}\Big|^2 = \frac{18225}{123904} + 1 = \frac{142129}{123904},$$

$$\text{extracting the root, } \frac{3x+5}{3x-5} - \frac{135}{352} = \pm\frac{377}{352},$$

$$\therefore \frac{3x+5}{3x-5} = \frac{16}{11} \text{ or } -\frac{11}{16};$$

whence $33x + 55 = 48x - 80$,

and $15x = 135$,

$\therefore x = 9$;

or $48x + 80 = -33x + 55$,

$$\text{and } 81x = -25,$$
$$\therefore\ x = -\frac{25}{81}.$$

74. Multiplying by $\sqrt{x}$,

$$x\sqrt{x} + x + 2\sqrt{x} = x^2 + x - 4,$$
$$\therefore\ (17.\ \text{Cor. } 3.)\ (x+2).\sqrt{x} = x^2 - 4,$$
$$\therefore\ (18.\ \text{Cor. } 3.)\ \sqrt{x} = x - 2,$$
$$\text{by transposition, } x - \sqrt{x} = 2,$$
$$\text{completing the square, } x - \sqrt{x} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4},$$
$$\text{extracting the root, } \sqrt{x} - \frac{1}{2} = \pm\frac{3}{2},$$
$$\therefore\ \sqrt{x} = 2 \text{ or } -1,$$
$$\text{and } x = 4 \text{ or } 1.$$

75. Completing the square,

$$\frac{x^2}{(x^2-4)^2} + \frac{6}{x^2-4} + \frac{9}{x^2} = \frac{351 + 9.25}{25x^2} = \frac{576}{25x^2},$$
$$\text{extracting the root, } \frac{x}{x^2-4} + \frac{3}{x} = \pm\frac{24}{5x},$$
$$\therefore\ \frac{x}{x^2-4} = \frac{9}{5x} \text{ or } -\frac{39}{5x},$$
$$\text{whence } 5x^2 = 9x^2 - 36, \text{ or } = -39x^2 + 156.$$
$$\text{In the former case, } 4x^2 = 36,$$
$$\text{and } 2x = \pm 6,$$
$$\therefore\ x = \pm 3.$$
$$\text{In the latter, } 44x^2 = 156,$$
$$\text{and } x = \pm\sqrt{\frac{39}{11}}.$$

76. Transposing and completing the square,

$$\left(x+\frac{8}{x}\right)^2+\left(x+\frac{8}{x}\right)+\frac{1}{4}=\frac{169}{4},$$

$$\text{extracting the root, } x+\frac{8}{x}+\frac{1}{2}=\pm\frac{13}{2},$$

$$\therefore\ x+\frac{8}{x}=6 \text{ or } -7.$$

$$\text{In the former case, } x^2-6x=-8,$$

$$\text{completing the square, } x^2-6x+9=1,$$

$$\text{extracting the root, } x-3=\pm 1,$$

$$\text{and } x=4 \text{ or } 2.$$

$$\text{In the latter, } x^2+7x=-8,$$

$$\text{completing the square, } x^2+7x+\frac{49}{4}=\frac{49}{4}-8=\frac{17}{4},$$

$$\text{extracting the root, } x+\frac{7}{2}=\pm\frac{\sqrt{17}}{2},$$

$$\therefore\ x=\frac{-7\pm\sqrt{17}}{2}.$$

77. Completing the square,

$$(x+4)-2\sqrt{\frac{x+4}{x-4}}+\frac{1}{x-4}=\frac{4}{x-4},$$

$$\text{extracting the root, } \sqrt{x+4}-\frac{1}{\sqrt{x-4}}=\pm\frac{2}{\sqrt{x-4}},$$

$$\therefore\ \sqrt{x+4}=\frac{3}{\sqrt{x-4}} \text{ or } -\frac{1}{\sqrt{x-4}},$$

$$\text{and } x^2-16=9 \text{ or } 1,$$

$$\therefore\ x^2=25 \text{ or } 17,$$

$$\text{and } x=\pm 5 \text{ or } \pm\sqrt{17}.$$

78. By transposition, $\sqrt{12-\frac{12}{x^2}}=x^2-\sqrt{x^2-\frac{12}{x^2}},$

$$\text{by squaring, } 12 - \frac{12}{x^2} = x^4 - 2x^2\sqrt{x^2 - \frac{12}{x^2}} + x^2 - \frac{12}{x^2},$$

$$\therefore\ 12 = x^4 - 2x\sqrt{x^4 - 12} + x^2,$$

$$\text{and } (x^4 - 12) - 2x\sqrt{x^4 - 12} + x^2 = 0,$$

$$\text{extracting the root, } \sqrt{x^4 - 12} - x = 0,$$

$$\therefore\ x^4 - 12 = x^2,$$

$$\text{and } x^4 - x^2 + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4},$$

$$\therefore\ x^2 - \frac{1}{2} = \pm\frac{7}{2},$$

$$\text{and } x^2 = 4, \text{ or } -3,$$

$$\text{whence } x = \pm 2, \text{ or } \pm\sqrt{-3}.$$

79. Completing the square,

$$x^4 \cdot \left(1 + \frac{1}{3x}\right)^2 - 3x^2 \cdot \left(1 + \frac{1}{3x}\right) + \frac{9}{4} = 70 + \frac{9}{4} = \frac{289}{4},$$

$$\text{extracting the root, } x^2 \cdot \left(1 + \frac{1}{3x}\right) - \frac{3}{2} = \pm\frac{17}{2},$$

$$\therefore\ x^2 + \frac{x}{3} = 10 \text{ or } -7,$$

$$\text{completing the square, } x^2 + \frac{x}{3} + \frac{1}{36} = 10 + \frac{1}{36} = \frac{361}{36},$$

$$\text{extracting the root, } x + \frac{1}{6} = \pm\frac{19}{6},$$

$$\text{and } x = 3 \text{ or } -\frac{10}{3},$$

And in the second case,

$$x^2 + \frac{x}{3} + \frac{1}{36} = \frac{1}{36} - 7 = -\frac{251}{36},$$

$$\therefore\ x + \frac{1}{6} = \frac{\pm\sqrt{-251}}{6},$$

$$\therefore\ x = \frac{-1 \pm \sqrt{-251}}{6}.$$

80. By transposition, $x^2 + 15 + \frac{64}{x^2} = \frac{25x^2}{16} + \frac{5x}{2}$,

completing the square, $x^2 + 16 + \frac{64}{x^2} = \frac{26x^2}{16} + \frac{5x}{2} + 1$,

extracting the root, $x + \frac{8}{x} = \pm\left(\frac{5x}{4} + 1\right)$.

In the former case, $x^2 + 8 = \frac{5x^2}{4} + x$,

$\therefore \frac{x^2}{4} + x = 8$,

completing the square, $\frac{x^2}{4} + x + 1 = 9$,

extracting the root, $\frac{x}{2} + 1 = \pm 3$,

and $\frac{x}{2} = 2$ or -4,

$\therefore x = 4$ or -8.

In the latter case, $x^2 + 8 = -\frac{5x^2}{4} - x$,

$\therefore \frac{9x^2}{4} + x = -8$,

completing the square, $\frac{9x^2}{4} + x + \frac{1}{9} = \frac{1}{9} - 8 = -\frac{71}{9}$,

extracting the root, $\frac{3x}{2} + \frac{1}{3} = \pm\frac{\sqrt{-71}}{3}$,

$\therefore x = \frac{-2 \pm 2\sqrt{-71}}{9}$.

81. Multiplying by $\sqrt{x^4 - 9x^2}$,

$$34\tfrac{5}{7} + \frac{x^2 - 9}{7} = \frac{19\sqrt{x^2 - 9}}{2},$$

by transposition, $(x^2 - 9) - \frac{133}{2}\sqrt{x^2 - 9} = -250$,

completing the square,

$$(x^2 - 9) - \frac{133}{2}\sqrt{x^2 - 9} + \left|\frac{133}{4}\right|^2 = \frac{17689}{16} - 250 = \frac{13689}{16},$$

extracting the root, $\sqrt{x^2 - 9} - \frac{133}{4} = \pm \frac{117}{4}$,

$$\therefore \sqrt{x^2 - 9} = \frac{125}{2} \text{ or } 4,$$

$$\text{and } x^2 - 9 = \frac{15625}{4} \text{ or } 16,$$

$$\therefore x^2 = \frac{15661}{4} \text{ or } 25,$$

$$\text{and } x = \pm \frac{\sqrt{15661}}{2} \text{ or } \pm 5.$$

82. By transposition,

$$(\overline{x - 1}|^2 - x)^2 - \frac{2}{3} \cdot (\overline{x - 1}|^2 - x) = \frac{341}{3},$$

completing the square,

$$(\overline{x - 1}|^2 - x)^2 - \frac{2}{3} \cdot (\overline{x - 1}|^2 - x) + \frac{1}{9} = \frac{341}{3} + \frac{1}{9} = \frac{1024}{9},$$

extracting the root, $(\overline{x - 1}|^2 - x) - \frac{1}{3} = \pm \frac{32}{3}$,

$$\therefore (x - 1)^2 - (x - 1) = 12 \text{ or } -\frac{28}{3},$$

completing the square,

$$(x - 1)^2 - (x - 1) + \frac{1}{4} = \frac{49}{4} \text{ or } -\frac{109}{12},$$

extracting the root, $(x - 1) - \frac{1}{2} = \pm \frac{7}{2} \text{ or } \pm \sqrt{\frac{-109}{12}}$,

$$\therefore x = 5 \text{ or } -2 \text{ or } \frac{3\sqrt{3} \pm \sqrt{-109}}{2\sqrt{3}}.$$

83. $x^2 - 2x^{\frac{3}{2}} + x + (x - \sqrt{x}) = 6,$

$$\text{completing the square, } (x - \sqrt{x})^2 + (x - \sqrt{x}) + \frac{1}{4} = \frac{25}{4},$$

$$\text{extracting the root, } (x - \sqrt{x}) + \frac{1}{2} = \pm \frac{5}{2},$$

$$\therefore\ x - \sqrt{x} = 2 \text{ or } -3,$$

$$\text{and completing the square, } x - \sqrt{x} + \frac{1}{4} = \frac{9}{4} \text{ or } -\frac{11}{4},$$

$$\text{extracting the root, } \sqrt{x} - \frac{1}{2} = \pm \frac{3}{2} \text{ or } \frac{\pm \sqrt{-11}}{2},$$

$$\therefore\ x = 4 \text{ or } 1;\ \text{or } \frac{-5 \pm \sqrt{-11}}{2}.$$

84. By transposition, $x^4 + \frac{13x^3}{3} = 39x + 81,$

completing the square,

$$x^4 + \frac{13x^3}{3} + \left(\frac{13x}{6}\right)^2 = \left(\frac{13x}{6}\right)^2 + 39x + 81,$$

$$\text{extracting the root, } x^2 + \frac{13x}{6} = \pm \left(\frac{13x}{6} + 9\right),$$

In the former case, $x^2 = 9$,

and $x = \pm 3$.

In the latter, $x^2 + \frac{13}{3}x = -9,$

completing the square,

$$x^2 + \frac{13}{3}x + \frac{169}{36} = \frac{169}{36} - 9 = -\frac{155}{36},$$

$$\text{extracting the root, } x + \frac{13}{6} = \pm \frac{\sqrt{-155}}{6},$$

$$\therefore\ x = \frac{-13 \pm \sqrt{-155}}{6}.$$

85. Multiplying the equation by 6,

$$x^2 + 8 - \frac{24}{x} - \frac{36}{x^2} = 16 - \frac{4}{x^2},$$

by transposition, $x^2 - 4 + \frac{4}{x^2} = 4 + \frac{24}{x} + \frac{36}{x^2}$,

extracting the root, $x - \frac{2}{x} = \pm\left(2 + \frac{6}{x}\right)$;

In the former case, $x = 2 + \frac{8}{x}$,

whence $x^2 - 2x = 8$,

completing the square, $x^2 - 2x + 1 = 9$,

and extracting the root, $x - 1 = \pm 3$,

$\therefore\ x = 4$ or -2.

In the second case, $x + 2 = -\frac{4}{x}$,

$\therefore\ x^2 + 2x + 1 = -3$,

and $x + 1 = \pm\sqrt{-3}$,

$\therefore\ x = -1 \pm \sqrt{-3}$.

86. Multiplying the equation by $\sqrt{x} - 2$,

$$x - \frac{8}{\sqrt{x}} - 2\sqrt{x} + \frac{16}{x} = 7,$$

and $\left(x + 8 + \frac{16}{x}\right) - 2\left(\sqrt{x} + \frac{4}{\sqrt{x}}\right) = 15$,

completing the square,

$$\left(\sqrt{x} + \frac{4}{\sqrt{x}}\right)^2 - 2\left(\sqrt{x} + \frac{4}{\sqrt{x}}\right) + 1 = 16,$$

extracting the root, $\sqrt{x} + \frac{4}{\sqrt{x}} - 1 = \pm 4$,

$\therefore\ \sqrt{x} + \frac{4}{\sqrt{x}} = 5$ or -3.

In the former case, $x - 5\sqrt{x} = -4$,

completing the square, $x - 5\sqrt{x} + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4}$,

extracting the root, $\sqrt{x} - \frac{5}{2} = \pm\frac{3}{2}$,

$\therefore \sqrt{x} = 4$ or 1,

and $x = 16$ or 1.

In the latter case, $x + 3\sqrt{x} = -4$,

and $x + 3\sqrt{x} + \frac{9}{4} = \frac{9}{4} - 4 = -\frac{7}{4}$,

$$\therefore \sqrt{x} + \frac{3}{2} = \pm\frac{\sqrt{-7}}{2},$$

$$\text{and } \sqrt{x} = \frac{-3 \pm \sqrt{-7}}{2},$$

$$\therefore x = \frac{1 \mp 3\sqrt{-7}}{2}.$$

87. By transposition, $4x^4 - 4x^3 + \frac{x}{2} = 33$,

or $(4x^4 - 4x^3 + x^2) - \frac{1}{2} \cdot (2x^2 - x) = 33$,

completing the square,

$$(2x^2 - x)^2 - \frac{1}{2} \cdot (2x^2 - x) + \frac{1}{16} = 33 + \frac{1}{16} = \frac{529}{16},$$

extracting the root, $(2x^2 - x) - \frac{1}{4} = \pm\frac{23}{4}$,

$$\therefore 2x^2 - x = 6 \text{ or } -\frac{11}{2}.$$

whence $x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{49}{16}$ or $-\frac{43}{16}$,

$\therefore$ extracting the root, $x - \frac{1}{4} = \pm \frac{7}{4}$ or $\pm \frac{\sqrt{-43}}{4}$,

and $x = 2$ or $-\frac{3}{2}$ or $\frac{1 \pm \sqrt{-43}}{4}$.

88. Completing the square,

$$(x-2)^2 - 6x^{\frac{1}{2}} \cdot (x-2) + 9x = 24 - 5x + 15x^{\frac{1}{2}},$$

and extracting the root,

$$x - 2 - 3x^{\frac{1}{2}} = \pm \sqrt{24 - 5x + 15x^{\frac{1}{2}}},$$

and squaring both sides,

$$(x - 3x^{\frac{1}{2}})^2 - 4 \cdot (x - 3x^{\frac{1}{2}}) + 4 = 24 - 5 \cdot (x - 3x^{\frac{1}{2}}),$$

by transposition, $(x - 3x^{\frac{1}{2}})^2 + (x - 3x^{\frac{1}{2}}) = 20$,

completing the square,

$$(x - 3x^{\frac{1}{2}})^2 + (x - 3x^{\frac{1}{2}}) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4},$$

extracting the root, $x - 3x^{\frac{1}{2}} + \frac{1}{2} = \pm \frac{9}{2}$,

$\therefore x - 3x^{\frac{1}{2}} = 4$ or -5,

completing the square, $x - 3x^{\frac{1}{2}} + \frac{9}{4} = \frac{25}{4}$ or $-\frac{11}{4}$,

extracting the root, $x^{\frac{1}{2}} - \frac{3}{2} = \pm \frac{5}{2}$ or $\pm \frac{\sqrt{-11}}{2}$,

$\therefore x^{\frac{1}{2}} = 4$ or -1 or $\frac{3 \pm \sqrt{-11}}{2}$,

$\therefore x = 16$ or 1 or $\frac{-1 \pm 3\sqrt{-11}}{2}$.

89. Completing the square,

$$(4x+1)^2 + 4x^{\frac{1}{2}} \cdot (4x+1) + 4x = 1912 - (6x + 3x^{\frac{1}{2}}),$$

extracting the root,

$$4x + 1 + 2x^{\frac{1}{2}} = \pm \sqrt{1912 - 3 \cdot (2x + x^{\frac{1}{2}})},$$

and squaring both sides,

$$4 \cdot (2x + x^{\frac{1}{2}})^2 + 4 \cdot (2x + x^{\frac{1}{2}}) + 1 = 1912 - 3 \cdot (2x + x^{\frac{1}{2}}),$$

by transposition, $4 \,.\, (2x + x^{\frac{1}{2}})^2 + 7 \,.\, (2x + x^{\frac{1}{2}}) = 1911$,
completing the square,

$$4 \,.\, (2x + x^{\frac{1}{2}})^2 + 7 \,.\, (2x + x^{\frac{1}{2}}) + \frac{49}{16} = \frac{30625}{16},$$

$$\text{extracting the root, } 2 \,.\, (2x + x^{\frac{1}{2}}) + \frac{7}{4} = \pm \frac{175}{4},$$

$$\therefore\ 4x + 2x^{\frac{1}{2}} = 42 \text{ or } . - \frac{91}{2},$$

$$\text{completing the square, } 4x + 2x^{\frac{1}{2}} + \frac{1}{4} = \frac{169}{4} \text{ or } - \frac{181}{4},$$

$$\text{extracting the root, } 2x^{\frac{1}{2}} + \frac{1}{2} = \pm \frac{13}{2} \text{ or } \pm \frac{\sqrt{-181}}{2},$$

$$\therefore\ 2x^{\frac{1}{2}} = 6 \text{ or } - \frac{7}{2} \text{ or } \frac{-1 \pm \sqrt{-181}}{2},$$

$$\text{and } x = 9 \text{ or } \frac{49}{4} \text{ or } - \frac{90 \pm \sqrt{-181}}{2}.$$

90. By transposition,

$$\left(\frac{3x}{2} + 13\right) + 2x \,.\, \sqrt{\frac{3x}{2} + 13} = 8x^2,$$

completing the square,

$$\left(\frac{3x}{2} + 13\right) + 2x\sqrt{\frac{3x}{2} + 13} + x^2 = 9x^2,$$

$$\text{extracting the root, } \sqrt{\frac{3x}{2} + 13} + x = \pm\, 3x,$$

$$\therefore\ \sqrt{\frac{3x}{2} + 13} = 2x \text{ or } - 4x,$$

$$\text{and } \frac{3x}{2} + 13 = 4x^2 \text{ or } 16x^2.$$

$$\text{In the former case, } 4x^2 - \frac{3x}{2} = 13,$$

$$\text{completing the square, } 4x^2 - \frac{3x}{2} + \frac{9}{64} = \frac{841}{64},$$

$$\text{extracting the root, } 2x - \frac{3}{8} = \pm\frac{29}{8},$$

$$\therefore\ 2x = 4 \text{ or } -\frac{13}{4},$$

$$\text{and } x = 2 \text{ or } -\frac{13}{8}.$$

$$\text{In the latter case, } 16x^2 - \frac{3x}{2} = 13,$$

$$\text{completing the square, } 16x^2 - \frac{3x}{2} + \left.\frac{3}{16}\right|^2 = \frac{3337}{256},$$

$$\text{extracting the root, } 4x - \frac{3}{16} = \pm\frac{\sqrt{3337}}{16},$$

$$\therefore\ 4x = \frac{3 \pm \sqrt{3337}}{16}.$$

$$\text{and } x = \frac{3 \pm \sqrt{3337}}{64}.$$

91. Dividing every term by 3,

$$\frac{4x^2}{3} + 7x + \frac{8x}{3}\sqrt{7x-5} = 69 - \frac{4x^2}{9},$$

$\therefore$ completing the square,

$$(7x-5) + \frac{8x}{3}\,.\,\sqrt{7x-5} + \frac{16x^2}{9} = 64,$$

$$\text{extracting the root, } \sqrt{7x-5} + \frac{4x}{3} = \pm 8,$$

$$\text{and } \sqrt{7x-5} = -\frac{4x}{3} \pm 8,$$

$$\therefore\ \text{squaring both sides, } 7x - 5 = \frac{16x^2}{9} \mp \frac{64x}{3} + 64,$$

whence $\frac{16x^2}{9} - \frac{85x}{3} = -69$,

completing the square, $\left.\frac{16x^2}{9} - \frac{85}{3}x + \frac{85}{8}\right|^2 = \frac{2809}{64}$,

extracting the root, $\frac{4x}{3} - \frac{85}{8} = \pm\frac{53}{8}$,

$\therefore \frac{4x}{3} = 4 \text{ or } \frac{69}{4}$,

and $x = 3 \text{ or } \frac{207}{16}$.

Or in the second case, $\frac{16x^2}{9} + \frac{43}{3}x = -69$,

completing the square, $\left.\frac{16x^2}{9} + \frac{43}{3}\,.\,x + \frac{43}{8}\right|^2 = -\frac{2567}{64}$,

extracting the root, $\frac{4x}{3} + \frac{43}{8} = \pm\frac{\sqrt{-2567}}{8}$,

and $\frac{4x}{3} = \frac{-43 \pm \sqrt{-2567}}{8}$,

$\therefore x = \frac{-129 \pm 3\sqrt{-2567}}{32}$.

92. Multiplying every term by $\frac{2x + \sqrt{x}}{2x - \sqrt{x}}$,

$$\left(\frac{2x + \sqrt{x}}{2x - \sqrt{x}}\right)^2 = \frac{52}{15} \cdot \frac{2x + \sqrt{x}}{2x - \sqrt{x}} - 3,$$

$\therefore$ transposing and completing the square,

$$\left(\frac{2x + \sqrt{x}}{2x - \sqrt{x}}\right)^2 - \frac{52}{15} \cdot \frac{2x + \sqrt{x}}{2x - \sqrt{x}} + \left.\frac{26}{15}\right|^2 = \frac{1}{225},$$

extracting the root, $\frac{2x + \sqrt{x}}{2x - \sqrt{x}} - \frac{26}{15} = \pm\frac{1}{15}$,

$$\therefore \frac{2x + \sqrt{x}}{2x - \sqrt{x}} = \frac{9}{5} \text{ or } \frac{5}{3}.$$

In the former case, $10x + 5\sqrt{x} = 18x - 9\sqrt{x}$,

and $14\sqrt{x} = 8x$,

$\therefore 7 = 4\sqrt{x}$,

and $x = \frac{49}{16}$.

In the latter, $6x + 3\sqrt{x} = 10x - 5\sqrt{x}$,

$\therefore 8\sqrt{x} = 4x$,

and $2 = \sqrt{x}$,

$\therefore 4 = x$.

93. Multiplying the equation by $x^{-\frac{1}{m}}$,

$$a^2b^2x^{\frac{m-n}{mn}} - 4\sqrt{ab}.ab.x^{\frac{m+n}{2mn}-\frac{1}{m}} = (a-b)^2,$$

or,

$$a^2b^2x^{\frac{m-n}{mn}} - 4\sqrt{ab}.abx^{\frac{m-n}{2mn}} + 4ab = (a-b)^2 + 4ab = (a+b)^2,$$

$\therefore$ extracting the root, $abx^{\frac{m-n}{2mn}} - 2\sqrt{ab} = \pm(a+b)$,

and $ab.x^{\frac{m-n}{2mn}} = a + 2\sqrt{ab} + b = (\sqrt{a} + \sqrt{b})^2$,

or $= -(a - 2\sqrt{ab} + b) = -(\sqrt{a} - \sqrt{b})^2$;

$$\text{whence } x = \left(\frac{(\sqrt{a}+\sqrt{b})^2}{ab}\right)^{\frac{2mn}{m-n}} \text{ or } \left(-\frac{(\sqrt{a}-\sqrt{b})^2}{ab}\right)^{\frac{2mn}{m-n}},$$

94. Multiplying the equation by $2x^{-\frac{1}{p}}$,

$$2x^{\frac{p+q}{2pq}-\frac{1}{p}} - \frac{a^2-b^2}{a^2+b^2}.\left(1+x^{\frac{1}{q}-\frac{1}{p}}\right) = 0,$$

$$\therefore\ 2\,.\,\frac{a^2+b^2}{a^2-b^2}\,.\,x^{\frac{p-q}{2pq}}-\left(1+x^{\frac{p-q}{pq}}\right)=0,$$

and

$$x^{\frac{p-q}{pq}}-2\,.\,\frac{a^2+b^2}{a^2-b^2}\,.\,x^{\frac{p-q}{2pq}}+\left(\frac{a^2+b^2}{a^2-b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2-1=\frac{4a^2b^2}{(a^2-b^2)},$$

extracting the root, $x^{\frac{p-q}{2pq}}-\frac{a^2+b^2}{a^2-b^2}=\pm\frac{2ab}{a^2-b^2}$,

whence $x^{\frac{p-q}{2pq}}=\frac{(a\pm b)^2}{a^2-b^2}=\frac{a\pm b}{a\mp b}$,

$$\therefore\ x=\left(\frac{a\pm b}{x\mp b}\right)^{\frac{2pq}{p-q}}.$$

SECTION V.

Adfected Quadratics involving two unknown Quantities.

1. From the first equation, $x = 14 - 4y$;
substituting this value in the second,

$$y^2 + 56 - 16y = 2y + 11,$$

by transposition, $y^2 - 18y = -45$,

completing the square, $y^2 - 18y + 81 = 81 - 45 = 36$,

extracting the root, $y - 9 = \pm 6$,

$$\therefore\ y = 15 \text{ or } 3.$$

$$\therefore\ x = 14 - 4y = -46 \text{ or } 2.$$

2. From the first equation, $x = \frac{118 - 3y}{2}$,

substituting this value in the second,

$$\frac{5 \,.\, (118 - 3y)^2}{4} - 7y^2 = 4333,$$

$$\therefore\ 69620 - 3540y + 45y^2 - 28y^2 = 17332,$$

$$\text{or } 17y^2 - 3540y = -52288,$$

$$\text{and } y^2 - \frac{3540}{17}y = -\frac{52288}{17},$$

completing the square,

$$y^2 - \frac{3540}{17} \,.\, y + \overline{\frac{1770}{17}}\Big|^2 = \frac{3132900}{\overline{17}|^2} - \frac{52288}{17} = \frac{2244004}{17},$$

$$\text{extracting the root, } y - \frac{1770}{17} = \pm\frac{1498}{17},$$

$$\text{whence } y = 16 \text{ or } \frac{3268}{17},$$

$$\therefore\ 2x = 118 - 3y = 70 \text{ or } -\frac{7798}{17},$$

$$\therefore\ x = 35 \text{ or } -\frac{3899}{17}.$$

3. From the second equation, $4x + 3y = 16y - 32$,

$$\text{and } 13y = 4x + 32,$$

$$\therefore\ y = \frac{4x + 32}{13},$$

From the first, $2x + 7y = 2y \times 4x - (51 + 2x) \,.\, \frac{2x}{5}$,

in which let the value of y be substituted;

$$\therefore\ 2x + \frac{28x + 224}{13} = 8x \,.\, \frac{4x + 32}{13} - (51 + 2x) \,.\, \frac{2x}{5},$$

$$\text{or, } 130x + 140x + 1120 = 160x^2 + 1280x - 52x^2 - 1326x,$$

$$\therefore\ 108x^2 - 316x = 1120,$$

and completing the square,

$$x^2 - \frac{79}{27}x + \overline{\frac{79}{54}}\Big|^2 = \frac{6241}{\overline{54}|^2} + \frac{280}{27} = \frac{36481}{\overline{54}|^2},$$

$$\text{extracting the root, } x - \frac{79}{54} = \pm \frac{191}{54},$$

$$\therefore\ x = 5 \text{ or } -\frac{56}{27}.$$

$$\text{And } 13y = 4x + 32 = 52 \text{ or } \frac{640}{27},$$

$$\therefore\ y = 4 \text{ or } \frac{640}{351}.$$

4. From the second equation,

$$9x + 3y = 21x - 35y + 42,$$

$$\therefore\ 38y = 12x + 42,$$

$$\text{and } y = \frac{6x + 21}{19}.$$

From the first,

$$12xy + 9y - 9 - 15x = 12xy + 9x^2 - 6x - 90x + 5x^2,$$

$$\text{or } 14x^2 - 81x = 9y - 9 = \frac{54x + 189}{19} - 9,$$

$$\therefore\ 14x^2 - \frac{1593}{19}x = \frac{18}{19},$$

$$\text{and } x^2 - \frac{1593}{266}x = \frac{18}{266},$$

completing the square,

$$x^2 - \frac{1593}{266}\cdot x + \left(\frac{1593}{532}\right)^2 = \frac{2537649}{(532)^2} + \frac{18}{266} = \frac{2556801}{(532)^2},$$

$$\text{extracting the root, } x - \frac{1593}{532} = \pm \frac{1599}{532},$$

$$\therefore\ x = 6 \text{ or } -\frac{3}{266};$$

$$\text{and } y = \frac{6x + 21}{19} = 3 \text{ or } \frac{2784}{2527}.$$

5. From the second equation,

$$(x+b)^2+2y\,.\,(x+b)+(x+b)^2-2y\,.\,(x+b)+2y^2=2c^2,$$

$$\text{and } (x+b)^2+y^2=c^2,$$

$$\text{but from the first, } \quad x^2-y^2=a^2,$$

$$\therefore \text{ by addition, } 2x^2+2bx+b^2=a^2+c^2,$$

$$\text{and } x^2+bx=\frac{1}{2}(a^2-b^2+c^2),$$

$$\text{completing the square, } x^2+bx+\frac{b^2}{4}=\frac{1}{4}\,.\,(2a^2-b^2+2c^2),$$

$$\text{extracting the root, } x+\frac{b}{2}=\pm\frac{1}{2}\sqrt{2a^2-b^2+2c^2},$$

$$\therefore\ x=-\frac{b}{2}\pm\frac{1}{2}\sqrt{2a^2-b^2+2c^2}.$$

And

$$y^2=x^2-a^2=\frac{1}{4}(b^2+2a^2-b^2+2c^2\mp 2b\sqrt{2a^2-b^2+2c^2})-a^2$$

$$=\frac{1}{4}(2c^2-2a^2\mp 2b\sqrt{2a^2-b^2+2c^2}),$$

$$\therefore\ y=\sqrt{\frac{1}{2}\,.\,(c^2-a^2\mp b\sqrt{2a^2-b^2+2c^2})}.$$

6. (Alg. 182 and 184.) $x^2 : y :: 8 : 1,$

$$\therefore\ x^2=8y.$$

If this value be substituted in the second equation,

$$8y+1 : y+4 :: 5y+7 : 3y,$$

$$\therefore\ 24y^2+3y=5y^2+27y+28,$$

$$\therefore\ 19y^2-24y=28,$$

$$\text{and } y^2-\frac{24}{19}\,.\,y=\frac{28}{19},$$

completing the square,

$$y^2-\frac{24}{19}y+\frac{144}{361}=\frac{144}{361}+\frac{28}{19}=\frac{676}{361},$$

$$\text{extracting the root, } y - \frac{12}{19} = \pm \frac{26}{19},$$

$$\therefore\ y = 2 \text{ or } - \frac{14}{19}.$$

$$\text{And } x^2 = 16 \text{ or } - \frac{112}{19},$$

$$\therefore\ x = \pm 4 \text{ or } \pm 4\sqrt{\frac{-7}{19}}.$$

7. From the first, $x^2 + 2x^3y + x^4y^2 = 441$,

$$\text{extracting the root, } x + x^2y = \pm 21;$$

$$\therefore \text{ from the second, } x + 3x + x^2 = \pm 21,$$

$$\text{and } x^2 + 4x + 4 = 25 \text{ or } -17,$$

$$\text{extracting the root, } x + 2 = \pm 5 \text{ or } \pm \sqrt{-17},$$

$$\therefore\ x = 3 \text{ or } -7, \text{ or } -2 \pm \sqrt{-17}.$$

$$\text{And } y = \frac{3 + x}{x} = 2 \text{ or } \frac{4}{7} \text{ or } \frac{1 \pm \sqrt{-17}}{-2 \pm \sqrt{-17}};$$

which last, by multiplying numerator and denominator by $-2 \mp \sqrt{-17}$, becomes $= \dfrac{15 \mp 3\sqrt{-17}}{21} = \dfrac{5 \mp \sqrt{-17}}{7}$.

8. From the first, $x^2 + 4xy + 4y^2 = 256$,

$$\text{extracting the root, } x + 2y = \pm 16,$$

$$\text{and } x = \pm 16 - 2y.$$

If this be substituted in the second equation,

$$3y^2 - 256 \pm 64y - 4y^2 = 39,$$

$$\text{and } y^2 \mp 64y = -295,$$

completing the square,

$$y^2 \mp 64y + \overline{32}|^2 = 1024 - 295 = 729,$$

$$\text{extracting the root, } y \mp 32 = \pm 27,$$

$$\therefore\ y = \pm 5 \text{ or } \pm 59.$$

$$\text{And } x = \pm 16 - 2y = \pm 6 \text{ or } \pm 102.$$

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9. Completing the square in the first equation,

$$(x+y)^2 - 3\,.\,(x+y) + \frac{9}{4} = 28 + \frac{9}{4} = \frac{121}{4},$$

$$\text{extracting the root, } x + y - \frac{3}{2} = \pm\frac{11}{2},$$

$$\therefore\ x + y = 7 \text{ or } -4,$$

$$\text{and } y = 7 - x, \text{ or } -(4 + x).$$

Let this value of y be substituted in the second equation, and

$$\text{In the former case, } 14x - 2x^2 + 3x = 35,$$

$$\text{or } x^2 - \frac{17x}{2} = -\frac{35}{2},$$

completing the square,

$$x^2 - \frac{17x}{2} + \left|\frac{17}{4}\right|^2 = \frac{289}{16} - \frac{35}{2} = \frac{9}{16},$$

$$\text{extracting the root, } x - \frac{17}{4} = \pm\frac{3}{4},$$

$$\text{and } x = 5 \text{ or } \frac{7}{2};$$

$$\therefore\ y = 7 - x = 2 \text{ or } \frac{7}{2}.$$

$$\text{In the latter case, } -8x - 2x^2 + 3x = 35,$$

$$\text{and } x^2 + \frac{5}{2}x = -\frac{35}{2},$$

completing the square,

$$x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{25}{16} - \frac{35}{2} = -\frac{255}{16},$$

$$\text{extracting the root, } x + \frac{5}{4} = \pm\frac{\sqrt{-255}}{4},$$

$$\text{and } x = \frac{-5 \pm \sqrt{-255}}{4};$$

$$\therefore\ y = -4 - x = \frac{-11 \mp \sqrt{-255}}{4}.$$

10. From the first equation, completing the square,

$$4 \cdot (x-2y)^2 + (x-2y) + \frac{1}{16} = 5 + \frac{1}{16} = \frac{81}{16},$$

$$\text{extracting the root, } 2 \cdot (x-2y) + \frac{1}{4} = \pm \frac{9}{4},$$

$$\therefore\ 2 \cdot (x-2y) = 2 \text{ or } -\frac{5}{2},$$

$$\text{and } x - 2y = 1 \text{ or } -\frac{5}{4};$$

$$\therefore\ x^2 - 4xy + 4y^2 = 1,$$

$$\text{but } x^2 - y^2 = 8,$$

$$\therefore\ 4xy - 5y^2 = 7,$$

$$\text{and } x = \frac{5y^2+7}{4y};$$

$$\text{whence } \frac{5y^2+7}{4y} - 2y = 1,$$

$$\text{and } 5y^2 + 7 - 8y^2 = 4y,$$

$$\text{by transposition, } 3y^2 + 4y = 7,$$

$$\text{and } y^2 + \frac{4}{3}y + \frac{4}{9} = \frac{7}{3} + \frac{4}{9} = \frac{25}{9},$$

$$\therefore\ y + \frac{2}{3} = \pm \frac{5}{3},$$

$$\text{and } y = 1 \text{ or } -\frac{7}{3};$$

$$\therefore\ x = 2y + 1 = 3 \text{ or } -\frac{11}{3}.$$

11. From the first equation, $(x-y) - \frac{4}{3}\sqrt{x-y} = \frac{4}{3}$,

$$\text{completing the square, } (x-y) - \frac{4}{3}\sqrt{x-y} + \frac{4}{9} = \frac{16}{9},$$

$$\therefore \sqrt{x-y} - \frac{2}{3} = \pm\frac{4}{3},$$

$$\sqrt{x-y} = 2 \text{ or } -\frac{2}{3},$$

$$\text{and } x - y = 4 \text{ or } \frac{4}{9}.$$

$\therefore$ from the second equation, $\sqrt{x+y} = 3$,

$$\text{and } x + y = 9$$
$$\text{but } x - y = 4$$

$$\therefore 2x = 13 \qquad \text{and } x = \frac{13}{2},$$

$$2y = 5 \qquad \text{and } y = \frac{5}{2}.$$

12. Completing the square in the first equation,

$$x^2 + 2x\sqrt{y} + y + 10 \,.\, (x + \sqrt{y}) + 25 = 119 + 25 = 144,$$

$$\therefore x + \sqrt{y} = 7 \text{ or } -17,$$
$$\text{but } x + 2y = 13$$

$\therefore$ by subtraction, $2y - \sqrt{y} = 6$, or 30,

$$\therefore y - \frac{1}{2}\sqrt{y} + \frac{1}{16} = \frac{1}{16} + 3, \text{ or } = \frac{1}{16} + 15, = \frac{49}{16}, \text{ or } = \frac{241}{16},$$

$\therefore$ extracting the root, $\sqrt{y} - \frac{1}{4} = \pm\frac{7}{4}$ or $\pm\frac{\sqrt{241}}{4}$,

$$\text{and } \sqrt{y} = 2 \text{ or } -\frac{3}{2}, \text{ or } \frac{1 \pm \sqrt{241}}{4},$$

$$\therefore y = 4 \text{ or } \frac{9}{4} \text{ or } \frac{121 \pm \sqrt{241}}{8};$$

$$\therefore x = 7 - \sqrt{y} = 5 \text{ or } \frac{17}{2},$$

$$\text{or } = -17 - \sqrt{y} = \frac{-69 \mp \sqrt{241}}{4}.$$

13. Completing the square in the first equation,

$$\frac{x^4}{y^2} + \frac{2x^2}{y} + 1 = 10\frac{39}{49} = \frac{529}{49},$$

extracting the root, $\frac{x^2}{y} + 1 = \pm \frac{23}{7}$,

$$\therefore \frac{x^2}{y} = \frac{16}{7} \text{ or } -\frac{30}{7}.$$

Whence, from the second equation, $y^2 + \frac{16}{7} . y = 65$,

completing the square,

$$y^2 + \frac{16}{7} . y + \frac{64}{49} = 65 + \frac{64}{49} = \frac{3249}{49},$$

extracting the root, $y + \frac{8}{7} = \pm \frac{57}{7}$,

and $y = 7$ or $-\frac{65}{7}$;

$$\therefore x^2 = 16 \text{ or } -\frac{16 \times 65}{49},$$

and $x = \pm 4$ or $\pm \frac{4}{7}\sqrt{-65}$.

But if $\frac{x^2}{y} = -\frac{30}{7}$,

$$y^2 - \frac{30}{7} . y = 65,$$

completing the square,

$$y^2 - \frac{30}{7} . y + \frac{225}{49} = 65 + \frac{225}{49} = \frac{3410}{49},$$

extracting the root, $y - \frac{15}{7} = \pm \frac{\sqrt{3410}}{7}$,

and $y = \frac{15 \pm \sqrt{3410}}{7}$;

$$\therefore x^2 = -30 \cdot \frac{15 \pm \sqrt{3410}}{49},$$

$$\text{and } x = \pm \sqrt{\frac{-450 \mp 30\sqrt{3410}}{7}}.$$

14. Completing the square in the first equation,

$$x + y + \sqrt{x+y} + \frac{1}{4} = \frac{25}{4},$$

$$\text{extracting the root, } \sqrt{x+y} + \frac{1}{2} = \pm \frac{5}{2},$$

$$\therefore \sqrt{x+y} = 2 \text{ or } -3,$$

$$\text{and } x + y = 4 \text{ or } 9.$$

$$\text{Hence } x^2 + 2xy + y^2 = 16 \text{ or } 81,$$

$$\text{but } 2x^2 + 2y^2 = 20,$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 4 \text{ or } -61,$$

$$\text{and } x - y = \pm 2 \text{ or } \pm \sqrt{-61},$$

$$\text{but } x + y = 4 \text{ or } 9,$$

$$\therefore \text{ by addition, } 2x = 6 \text{ or } 2, \text{ or } 9 \pm \sqrt{-61},$$

$$\text{and } x = 3 \text{ or } 1, \text{ or } \frac{1}{2} \cdot (9 \pm \sqrt{-61});$$

$$\text{by subtraction, } 2y = 2 \text{ or } 6 \text{ or } 9 \mp \sqrt{-61},$$

$$\text{and } y = 1 \text{ or } 3, \text{ or } \frac{1}{2} \cdot (9 \mp \sqrt{-61}).$$

15. Transposing and completing the square in the first equation,

$$x^2 + 3y + 5 + 4\sqrt{x^2 + 3y + 5} + 4 = 64,$$

$$\text{extracting the root, } \sqrt{x^2 + 3y + 5} + 2 = \pm 8,$$

$$\text{and } \sqrt{x^2 + 3y + 5} = 6 \text{ or } -10,$$

$$\therefore x^2 + 3y + 5 = 36 \text{ or } 100,$$

$$\text{or } x^2 + 3 \,.\, \frac{6x - 16}{7} = 31 \text{ or } 95,$$

$$\text{in the former case, } x^2 + \frac{18x}{7} + \frac{81}{49} = \frac{1936}{49},$$

$$\text{and } x + \frac{9}{7} = \pm\frac{44}{7},$$

$$\therefore\ x = 5 \text{ or } -\frac{53}{7};$$

$$\text{whence } y = \frac{6x - 16}{7} = 2 \text{ or } -\frac{430}{49}.$$

$$\text{In the latter case, } x = \frac{-9 \pm \sqrt{3895}}{7},$$

$$\text{and } y = \frac{-166 \pm 6\sqrt{3895}}{49}.$$

16. Adding the two equations together,

$$x^2 + 2xy + y^2 + 4 \,.\, (x + y) = 117,$$

$$\text{completing the square, } (x + y)^2 + 4 \,.\, (x + y) + 4 = 121,$$

$$\text{extracting the root, } x + y + 2 = \pm 11,$$

$$\therefore\ x + y = 9 \text{ or } -13,$$

$$\text{and } x = 9 - y, \text{ or } -13 - y.$$

Let this value be substituted in the second equation,

$$\text{and } y^2 + 3y + 9 - y = 44,$$

$$\text{or } y^2 + 3y - 13 - y = 44;$$

$$\text{in the former case, } y^2 + 2y + 1 = 36,$$

$$\text{whence } y + 1 = \pm 6,$$

$$\text{and } y = 5 \text{ or } -7;$$

$$\text{and } \therefore\ x = 4 \text{ or } 16.$$

$$\text{In the latter, } y^2 + 2y + 1 = 58,$$

$$\text{and } y + 1 = \pm\sqrt{58},$$

$$\therefore\ y = -1 \pm \sqrt{58};$$

$$\text{whence } x = -12 \mp \sqrt{58}.$$

17. Multiplying the first equation by $(x + y)^{\frac{3}{2}}$,

$$y + \frac{(x + y)^2}{y} = \frac{17 \cdot (x + y)}{4},$$

$$\therefore \frac{(x + y)^2}{y} - \frac{17 \cdot (x + y)}{4} = -y,$$

completing the square,

$$\frac{(x + y)^2}{y} - \frac{17}{4} \cdot (x + y) + \overline{\frac{17}{8}}\Big|^2 \cdot y = \frac{225}{64} \cdot y,$$

$$\therefore \frac{x + y}{\sqrt{y}} - \frac{17}{8}\sqrt{y} = \pm \frac{15}{8} \cdot \sqrt{y},$$

$$\text{and } x + y = 4y \text{ or } \frac{1}{4}y,$$

$$\therefore x = 3y \text{ or } -\frac{3}{4}y.$$

In the former case, from the second equation,

$$y^2 - 3y = -2,$$

$$\text{and } y^2 - 3y + \frac{9}{4} = \frac{1}{4},$$

$$\therefore y - \frac{3}{2} = \pm \frac{1}{2},$$

$$\text{and } y = 2 \text{ or } 1,$$

$$\therefore x = 6 \text{ or } 3.$$

In the latter case, $y^2 + \frac{3}{4}y = -2$,

$$\text{and } \therefore y = \frac{-3 \pm \sqrt{-119}}{8},$$

$$\therefore x = -\frac{3}{4}y = \frac{9 \mp 3\sqrt{-119}}{32}.$$

18. Subtracting the equations,

$$28 - y^{\frac{1}{2}} = 16 + 4x^{\frac{1}{2}},$$

$$\therefore y^{\frac{1}{2}} = 12 - 4x^{\frac{1}{2}},$$

$$\text{and } y = 144 - 96x^{\frac{1}{2}} + 16x;$$

which being substituted in the second equation,

$$28 - 16x + 96x^{\frac{1}{2}} - 144 = x + 4x^{\frac{1}{2}},$$

$$\therefore\ 17x - 92x^{\frac{1}{2}} = -116,$$

$$\text{and } x - \frac{92}{17} \cdot x^{\frac{1}{2}} + \overline{\frac{46}{17}}\Big|^2 = \frac{2116}{289} - \frac{116}{17} = \frac{144}{289},$$

$$\therefore\ x^{\frac{1}{2}} - \frac{46}{17} = \pm\frac{12}{17},$$

$$\text{and } x^{\frac{1}{2}} = \frac{58}{17} \text{ or } 2,$$

$$\therefore\ x = \overline{\frac{58}{17}}\Big|^2 \text{ or } 4;$$

$$\text{and } y^{\frac{1}{2}} = 12 - 4x^{\frac{1}{2}} = -\frac{28}{17} \text{ or } 4;$$

$$\therefore\ y = \frac{784}{289} \text{ or } 16.$$

19. From the second equation,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625,$$

$$\text{but } x^4 + y^4 = 97,$$

$$\therefore \text{ by subtraction, } 2xy\,.\,(2x^2 + 3xy + 2y^2) = 528,$$

$$\text{but } 2xy\,.\,(2x^2 + 4xy + 2y^2) = 100xy,$$

$$\therefore \text{ by subtraction, } 2x^2y^2 = 100xy - 528,$$

$$\text{whence } x^2y^2 - 50xy = -264,$$

$$\text{and } x^2y^2 - 50xy + 625 = 625 - 264 = 361,$$

$$\therefore\ xy - 25 = \pm 19,$$

$$\text{and } xy = 44 \text{ or } 6.$$

$$\text{Now } x^2 + 2xy + y^2 = 25,$$

$$\text{and } 4xy = 24 \text{ or } 176,$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1 \text{ or } -151,$$

$$\text{and } \therefore\ x - y = \pm 1 \text{ or } \pm\sqrt{-151},$$
$$\text{but } x + y = 5$$

$$\therefore \text{ by addition, } \quad 2x = 6 \text{ or } 4, \text{ or } 5 \pm \sqrt{-151},$$
$$\text{and} \quad x = 3 \text{ or } 2, \text{ or } \frac{1}{2}\,.\,(5 \pm \sqrt{-151});$$
$$\text{by subtraction, } \quad 2y = 4 \text{ or } 6, \text{ or } 5 \mp \sqrt{-151},$$
$$\text{and} \quad y = 2 \text{ or } 3, \text{ or } \frac{1}{2}\,.\,(5 \mp \sqrt{-151}).$$

20. Multiplying the first equation by $\sqrt{\frac{3x-2y}{2x}}$,

$$\frac{3x-2y}{2x} - 2\sqrt{\frac{3x-2y}{2x}} + 1 = 0,$$
$$\text{and } \therefore\ \sqrt{\frac{3x-2y}{2x}} - 1 = 0,$$
$$\text{whence } 3x - 2y = 2x,$$
$$\text{and } x = 2y.$$
$$\therefore \text{ from the second equation, } x^2 - 18 = 2x^2 - 9x,$$
$$\text{and } x^2 - 9x + \frac{81}{4} = \frac{81}{4} - 18 = \frac{9}{4},$$
$$\therefore\ x - \frac{9}{2} = \pm\frac{3}{2},$$
$$\text{and } x = 6 \text{ or } 3;$$
$$\therefore\ y = 3 \text{ or } \frac{3}{2}.$$

21. By transposition,

$$x - 4\sqrt{xy} + 4y + 4(\sqrt{x} - 2\sqrt{y}) = 21,$$
$$\therefore\ (\sqrt{x} - 2\sqrt{y})^2 + 4(\sqrt{x} - 2\sqrt{y}) + 4 = 25,$$
$$\text{and } \sqrt{x} - 2\sqrt{y} + 2 = \pm 5,$$

$$\therefore\ \sqrt{x} - 2\sqrt{y} = 3 \text{ or } -7,$$

$$\text{but } \sqrt{x} + \sqrt{y} = 6$$

$$\therefore \text{ by subtraction, } 3\sqrt{y} = 3 \text{ or } 13,$$

$$\text{and } \sqrt{y} = 1 \text{ or } \frac{13}{3},$$

$$\therefore\ y = 1 \text{ or } \frac{169}{9},$$

$$\text{and } \sqrt{x} = 6 - \sqrt{y} = 5 \text{ or } \frac{5}{3},$$

$$\therefore\ x = 25 \text{ or } \frac{25}{9}.$$

22. From the second equation, $6x : y :: x + 2 : 3$,

$$\therefore\ 18x = xy + 2y,$$

$$\text{and } xy = 18x - 2y.$$

From the first, $3x + \frac{1}{3}y + \frac{2}{3}\sqrt{[xy\,.\,(9x + y)]} = xy$,

$$\text{or } (9x + y) + 2\sqrt{xy}\,.\,\sqrt{9x + y} + xy = 4xy,$$

$$\text{whence } \sqrt{9x + y} + \sqrt{xy} = \pm\, 2\sqrt{xy},$$

$$\text{and } \sqrt{9x + y} = \sqrt{xy} \text{ or } -3\sqrt{xy},$$

$$\therefore\ 9x + y = xy = 18x - 2y,$$

$$\text{and } 9x = 3y,$$

$$\text{or } 3x = y;$$

$$\text{whence } 3x^2 = 18x - 6x = 12x,$$

$$\therefore\ x = 4,$$

$$\text{and } y = 12.$$

23. From the first equation, $x^2 + y^2 = 25 - 2xy$,

$$\text{and } x^3 + y^3 = 125 - 3xy\,.\,(x + y) = 125 - 15xy,$$

$$\therefore\ (25 - 2xy)\,.\,(125 - 15xy) = 455,$$

$$\text{or } (25 - 2xy)\,.\,(25 - 3xy) = 91,$$

$$\text{whence } 625 - 125\,xy + 6x^2y^2 = 91,$$

$$\text{and } x^2y^2 - \frac{125}{6}\cdot xy = -\frac{534}{6},$$

completing the square,

$$x^2y^2 - \frac{125}{6}xy + \left(\frac{125}{12}\right)^2 = \frac{15625}{144} - \frac{534}{6} = \frac{2809}{144},$$

$$\therefore\ xy - \frac{125}{12} = \pm\frac{53}{12},$$

$$\text{and } xy = 6 \text{ or } \frac{89}{6};$$

$$\text{Now } x^2 + 2xy + y^2 = 25,$$

$$\text{and } \quad 4xy \quad = 24 \text{ or } \frac{178}{3},$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1 \text{ or } -\frac{103}{3},$$

$$\therefore\ x - y = \pm 1 \text{ or } \pm\sqrt{\frac{-103}{3}}$$

$$\text{and } x + y = 5$$

$$\therefore \text{ by addition, } 2x = 6 \text{ or } 4, \text{ or } 5 \pm \sqrt{\frac{-103}{3}},$$

$$\text{and } \quad x = 3 \text{ or } 2, \text{ or } \frac{1}{2}\left(5 \pm \sqrt{\frac{-103}{3}}\right)$$

$$\text{by subtraction, } 2y = 4 \text{ or } 6, \text{ or } 5 \mp \sqrt{\frac{-103}{3}},$$

$$\text{and } \quad y = 2 \text{ or } 3, \text{ or } \frac{1}{2}\left(5 \mp \sqrt{\frac{-103}{3}}\right)$$

24. Completing the square in the first equation,

$$(x^2 - y^2) - \sqrt{x^2 - y^2} + \frac{1}{4} = \frac{25}{4},$$

$$\therefore \sqrt{x^2 - y^2} - \frac{1}{2} = \pm \frac{5}{2},$$

$$\text{and } \sqrt{x^2 - y^2} = 3 \text{ or } -2,$$

$$\therefore \quad x^2 - y^2 = 9 \text{ or } 4,$$

$$\text{but } \quad x^2 + y^2 = 41,$$

$$\therefore \text{ by addition, } 2x^2 = 50 \text{ or } 45,$$

$$\text{and } x^2 = 25 \text{ or } \frac{45}{2}.$$

$$\therefore \quad x = \pm 5 \text{ or } \pm 3\sqrt{\frac{5}{2}};$$

$$\text{by subtraction, } 2y^2 = 32 \text{ or } 37,$$

$$\therefore \ y^2 = 16 \text{ or } \frac{37}{2},$$

$$\text{and } y = \pm 4 \text{ or } \pm\sqrt{\frac{37}{2}}.$$

25. By transposition, $\dfrac{x^4}{y^2} + 2xy + \dfrac{y^4}{x^2} = \dfrac{1225}{9}$,

$$\therefore \text{ extracting the root, } \frac{x^2}{y} + \frac{y^2}{x} = \pm \frac{35}{3},$$

$$\text{and } x^3 + y^3 = \pm \frac{35}{3} \cdot xy.$$

$$\text{But } x^3 + y^3 + 3xy \cdot (x + y) = 1000,$$

$$\therefore \ \pm \frac{35}{3} xy + 30xy = 1000,$$

$$\text{and } \frac{125xy}{3} = 1000 \text{ in one case,}$$

$$\text{and } \frac{55}{3} \cdot xy = 1000 \text{ in the other.}$$

In the former, $xy = 24$,

$$\text{But } x^2 + 2xy + y^2 = 100,$$
$$\text{and } \quad 4xy \quad = \quad 96,$$
$$\therefore \; x^2 - 2xy + y^2 = \quad 4,$$
$$\text{and } x - y = \pm 2,$$
$$\text{but } x + y = 10,$$
$$\therefore \text{ by addition, } \; 2x = 12 \text{ or } 8,$$
$$\text{and } x = 6 \text{ or } 4;$$
$$\text{by subtraction, } 2y = 8 \text{ or } 12,$$
$$\text{and } y = 4 \text{ or } 6.$$

$$\text{In the latter case, } xy = \frac{600}{11},$$

and ∴ by proceeding in a similar manner,

$$x = 5 \pm 5\sqrt{\frac{-13}{11}},$$

$$\text{and } y = 5 \mp 5\sqrt{\frac{-13}{11}}.$$

26. Multiplying the first equation by $\dfrac{x+y}{x-y}$,

$$\left(\frac{x+y}{x-y}\right)^2 - \frac{24}{5}\cdot\frac{x+y}{x-y} = 1,$$

completing the square,

$$\left(\frac{x+y}{x-y}\right)^2 - \frac{24}{5}\cdot\frac{x+y}{x-y} + \frac{144}{25} = \frac{169}{25},$$

$$\therefore \; \frac{x+y}{x-y} - \frac{12}{5} = \pm\frac{13}{5},$$

$$\text{whence } \frac{x+y}{x-y} = 5 \text{ or } -\frac{1}{5},$$

$$\left.\begin{aligned} \text{and } x + y &= 5x - 5y, \\ \therefore\; 4x &= 6y, \\ \text{and } 2x &= 3y, \end{aligned}\right\} \text{ or } \left\{\begin{aligned} 5x + 5y &= -x + y, \\ \therefore\; 6x &= -4y, \\ \text{and } 3x &= -2y, \end{aligned}\right.$$

From the second equation, $\frac{x-y}{x^2} + \frac{\sqrt{x-y}}{x} = \frac{4}{9}$,

completing the square,

$$\frac{x-y}{x^2} + \frac{\sqrt{x-y}}{x} + \frac{1}{4} = \frac{4}{9} + \frac{1}{4} = \frac{25}{36},$$

$$\therefore \frac{\sqrt{x-y}}{x} + \frac{1}{2} = \pm\frac{5}{6},$$

$$\text{and } \frac{\sqrt{x-y}}{x} = \frac{1}{3} \text{ or } -\frac{4}{3},$$

$$\therefore 9 \,.\, (x-y) = x^2 \text{ or } 16x^2,$$

$$\text{and } (9x - 6x =)\ 3x = x^2 \text{ or } 16x^2,$$

$$\therefore x = 3 \text{ or } \frac{3}{16}.$$

$$\text{whence } y = 2 \text{ or } \frac{1}{8}.$$

$$\text{Also } 9 \,.\, \left(x + \frac{3x}{2}\right) = \frac{45x}{2} = x^2 \text{ or } 16x^2,$$

$$\therefore x = \frac{45}{2} \text{ or } \frac{45}{32},$$

$$\text{and } y = -\frac{135}{4} \text{ or } -\frac{135}{64}.$$

27. Completing the square in the first equation,

$$\sqrt{x} + \sqrt{y} + 2\sqrt{6 \,.\, (\sqrt{x} + \sqrt{y})} + 6 = 24,$$

$$\therefore \sqrt{\sqrt{x} + \sqrt{y}} + \sqrt{6} = \pm 2\sqrt{6};$$

$$\text{and } \sqrt{\sqrt{x} + \sqrt{y}} = \sqrt{6} \text{ or } -3\sqrt{6},$$

$$\therefore \sqrt{x} + \sqrt{y} = 6 \text{ or } 54,$$

$$\text{and } \left(\frac{x-y}{\sqrt{x}+\sqrt{y}} = \right) \sqrt{x} - \sqrt{y} = 2 \text{ or } \frac{2}{9},$$

$$\text{whence by addition, } 2\sqrt{x} = 8 \text{ or } \frac{488}{9},$$

$$\text{and } \sqrt{x} = 4 \text{ or } \frac{244}{9},$$

$$\therefore\ x = 16 \text{ or } \frac{59536}{81}.$$

$$\text{by subtraction, } 2\sqrt{y} = 4 \text{ or } \frac{484}{9},$$

$$\therefore\ \sqrt{y} = 2 \text{ or } \frac{242}{9},$$

$$\text{and } y = 4 \text{ or } \frac{58564}{81}.$$

28. By transposition, $y^4 - 12xy^2 = 432$,
completing the square,

$$y^4 - 12xy^2 + 36x^2 = 36(x^2 + 12),$$

$$\therefore\ y^2 - 6x = \pm 6\sqrt{x^2 + 12}.$$

Now from the second, $y^2 - 2xy = 12$,

$$\text{and } y^2 - 2xy + x^2 = x^2 + 12,$$

$$\therefore\ y - x = \pm\sqrt{x^2 + 12};$$

$$y = x \pm \sqrt{x^2 + 12},$$

$$\text{whence } y^2 = 6y,$$

$$\text{and } y = 6;$$

$$\therefore\ 12x = y^2 - 12 = 36 - 12 = 24,$$

$$\text{and } x = 2.$$

29. From the first equation, by transposition,

$$\frac{4}{y^2} - \frac{8}{x} + \frac{4y^2}{x^2} + 2\cdot\left(\frac{2}{y} - \frac{2y}{x}\right) + 1 = \frac{16y^2}{x^2},$$

$$\text{extracting the root, } \frac{2}{y} - \frac{2y}{x} + 1 = \pm\frac{4y}{x},$$

$$\therefore\ \frac{2}{y} + 1 = \frac{6y}{x} \text{ or } -\frac{2y}{x},$$

$$\text{whence } 2x + xy = 6y^2, \text{ or } -2y^2,$$
$$\text{but } \underline{x + xy = 4y^2,}$$
$$\therefore \text{ by subtraction, } x = 2y^2 \text{ or } -6y^2,$$
$$\text{in the former case, } 2y^2 + 2y^3 = 4y^2,$$
$$\text{and } y = 1,$$
$$\therefore x = 2;$$
$$\text{in the latter, } -6y^2 - 6y^3 = 4y^2,$$
$$\therefore y = -\frac{5}{3},$$
$$\text{and } x = -6y^2 = -\frac{50}{3}.$$

30. From the first equation, by transposition,

$$\sqrt{(1+x)^2 + y^2} = 4 - \sqrt{(1-x)^2 + y^2},$$
$$\therefore (1+x)^2 + y^2 = 16 - 8\sqrt{(1-x)^2 + y^2} + (1-x)^2 + y^2,$$
$$\text{and } 4x = 16 - 8\sqrt{(1-x)^2 + y^2},$$
$$\therefore 2\sqrt{(1-x)^2 + y^2} = 4 - x,$$
$$\text{and } 4 \, . \, (1-x)^2 + 4y^2 = 16 - 8x + x^2,$$
$$\therefore 4y^2 = 12 - 3x^2;$$

hence from the second equation,

$$18 - (4 - x^2)^2 = 12 - 3x^2,$$
$$\text{and } x^4 - 11x^2 = -10,$$
$$\therefore x^4 - 11x^2 + \frac{121}{4} = \frac{81}{4},$$
$$\text{and } x^2 - \frac{11}{2} = \pm\frac{9}{2},$$
$$\text{whence } x^2 = 10 \text{ or } 1,$$
$$\text{and } x = \pm\sqrt{10} \text{ or } \pm 1;$$
$$\therefore y^2 = 3 - \frac{3}{4}x^2 = -\frac{9}{2} \text{ or } \frac{9}{4},$$
$$\text{and } y = \pm 3\sqrt{-\frac{1}{2}} \text{ or } \pm\frac{3}{2}.$$

31. From the second equation,

$$y^2 - \sqrt{x} \cdot y + \frac{x}{4} = \frac{25x}{36},$$

$$\therefore\ y - \frac{\sqrt{x}}{2} = \pm \frac{5\sqrt{x}}{6},$$

$$\text{and } y = \frac{4\sqrt{x}}{3} \text{ or } -\frac{\sqrt{x}}{3}.$$

Multiplying the first equation by $\dfrac{x + \sqrt{x} + y}{x - \sqrt{x} + y}$,

$$\left(\frac{x + \sqrt{x} + y}{x - \sqrt{x} + y}\right)^2 - \frac{89}{40} \cdot \frac{x + \sqrt{x} + y}{x - \sqrt{x} + y} = -1,$$

$$\therefore\ \left(\frac{x + \sqrt{x} + y}{x - \sqrt{x} + y}\right)^2 - \frac{89}{40} \cdot \frac{x + \sqrt{x} + y}{x - \sqrt{x} + y} + \left|\frac{89}{80}\right|^2 = \frac{7921}{\overline{80}|^2} - 1 = \frac{1521}{6400},$$

$$\text{and } \frac{x + \sqrt{x} + y}{x - \sqrt{x} + y} - \frac{89}{80} = \pm \frac{39}{80},$$

$$\therefore\ \frac{x + \sqrt{x} + y}{x - \sqrt{x} + y} = \frac{8}{5} \text{ or } \frac{5}{8},$$

$\therefore$ in the former case,

$$5x + 5\sqrt{x} + 5y = 8x - 8\sqrt{x} + 8y,$$

$$\text{and } 13\sqrt{x} = 3x + 3y,$$

$$\therefore\ 13\sqrt{x} = 3x + 4\sqrt{x},$$

$$\text{and } 9\sqrt{x} = 3x,$$

$$\therefore\ \sqrt{x} = 3,$$

$$\text{and } x = 9,$$

$$\text{whence } y = 4:$$

but if the second value of y be taken,

$$13\sqrt{x} = 3x - \sqrt{x},$$

$$\therefore\ x = \frac{196}{9},$$

$$\text{and } y = -\frac{14}{9}.$$

In the other case, $8x + 8\sqrt{x} + 8y = 5x - 5\sqrt{x} + 5y$,

$$\therefore\ 3x + 3y = -13\sqrt{x};$$

whence, if the first value of y be taken,

$$3x + 4\sqrt{x} = -13\sqrt{x},$$

$$\text{and } x = \frac{289}{9}, \text{ and } y = -\frac{68}{8};$$

but if the second value be taken,

$$3x - \sqrt{x} = -13\sqrt{x},$$

$$\therefore\ x = 16, \text{ and } y = \frac{4}{3}.$$

32. Multiplying the first equation by $\dfrac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}$,

and completing the square,

$$\left(\frac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}\right)^2 - \frac{17}{4} \cdot \frac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}} + \frac{289}{64} = \frac{289}{64} - 1 = \frac{225}{64},$$

$$\therefore\ \frac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}} - \frac{17}{8} = \pm\frac{15}{8},$$

$$\text{and } \frac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}} = 4 \text{ or } \frac{1}{4},$$

in the former case, $x + \sqrt{x^2 - y^2} = 4x - 4\sqrt{x^2 - y^2}$,

$$\text{or } 5\sqrt{x^2 - y^2} = 3x,$$

$$\therefore\ 25x^2 - 25y^2 = 9x^2,$$

$$\text{and } 16x^2 = 25y^2,$$

$$\therefore\ 4x = 5y.$$

In the latter case, $4x + 4\sqrt{x^2 - y^2} = x - \sqrt{x^2 - y^2}$,

whence $4x = 5y$.

Completing the square in the second equation,

$$(x^2 + xy + 4) + \sqrt{x^2 + xy + 4} + \frac{1}{4} = \frac{225}{4},$$

$$\therefore\ \sqrt{x^2 + xy + 4} + \frac{1}{2} = \pm\frac{15}{2},$$

$$\text{and } \sqrt{x^2 + xy + 4} = 7 \text{ or } -8,$$

$$\therefore\ x^2 + xy = 49 \text{ or } 64,$$

$$\text{and } x^2 + xy = 45 \text{ or } 60,$$

$$\therefore\ x^2 + \frac{4}{5}x^2 = 45 \text{ or } 60,$$

$$\text{and } x^2 = 25 \text{ or } \frac{100}{3},$$

$$\therefore\ x = \pm 5 \text{ or } \pm\frac{10}{\sqrt{3}};$$

$$\text{and } y = \pm 4 \text{ or } \pm\frac{8}{\sqrt{3}}.$$

33. Multiplying the first equation by 3, and transposing,

$$x^2 - 15y - \frac{3}{5}\sqrt{x^2 - 15y - 14} = 108,$$

completing the square,

$$x^2 - 15y - 14 - \frac{3}{5}\sqrt{x^2 - 15y - 14} + \frac{9}{100} = 94 + \frac{9}{100} = \frac{9409}{100},$$

extracting the root, $\sqrt{x^2 - 15y - 14} - \frac{3}{10} = \pm\frac{97}{10},$

$$\therefore\ \sqrt{x^2 - 15y - 14} = 10 \text{ or } -\frac{47}{5},$$

$$\text{and } x^2 - 15y = 114 \text{ or } \frac{2559}{25}.$$

From the first equation,

$$\frac{x^2}{8y} + \frac{2x}{3} = \frac{x}{\sqrt{y}} \cdot \sqrt{\frac{x}{3} \pm \frac{y}{4}} - \frac{y}{2},$$

$$\therefore\ \left(\frac{x}{3} + \frac{y}{4}\right) - \frac{x}{2\sqrt{y}} \cdot \sqrt{\frac{x}{3} + \frac{y}{4}} + \frac{x^2}{16y} = 0,$$

extracting the root, $\sqrt{\frac{x}{3}+\frac{y}{4}}-\frac{x}{4\sqrt{y}}=0,$

and $\frac{x}{3}+\frac{y}{4}=\frac{x^2}{16y},$

completing the square, $\frac{x^2}{16y}-\frac{x}{3}+\frac{4y}{9}=\frac{y}{4}+\frac{4y}{9}=\frac{25y}{36},$

extracting the root, $\frac{x}{4\sqrt{y}}-\frac{2\sqrt{y}}{3}=\pm\frac{5\sqrt{y}}{6},$

and $\frac{x}{4\sqrt{y}}=\frac{3\sqrt{y}}{2}$, or $\frac{\sqrt{y}}{6},$

$\therefore x=6y$ or $\frac{2}{3}y.$

Whence $36y^2-15y=114$ or $\frac{2559}{25},$

$\therefore 36y^2-15y+\frac{25}{16}=114+\frac{25}{16}$ or $\frac{2559}{25}+\frac{25}{16},$

$=\frac{1849}{16}$ or $\frac{41569}{25\times 16},$

$\therefore 6y-\frac{5}{4}=\pm\frac{43}{4}$ or $\pm\frac{\sqrt{41569}}{20},$

and $6y=12$ or $-\frac{19}{2}$ or $\frac{25\pm\sqrt{41569}}{20},$

$\therefore y=2$ or $-\frac{19}{12}$ or $\frac{25\pm\sqrt{41569}}{120},$

and $x=12$ or $-\frac{19}{2}$ or $\frac{25\pm\sqrt{41569}}{20}.$

Or, $\frac{4}{9}y^2-15y+\overline{\frac{45}{4}}\Big|^2=114+\overline{\frac{45}{4}}\Big|^2$ or $\frac{2559}{25}+\overline{\frac{45}{4}}\Big|^2$

$=\frac{3849}{16}$ or $\frac{91569}{25\times 16},$

$$\therefore \frac{2}{3}y - \frac{45}{4} = \pm \frac{\sqrt{3849}}{4} \text{ or } \pm \frac{\sqrt{91569}}{20},$$

$$\text{and } \frac{2}{3}y = \frac{45 \pm \sqrt{3849}}{4} \text{ or } \frac{225 \pm \sqrt{91569}}{20},$$

$$\therefore y = \frac{135 \pm 3\sqrt{3849}}{8} \text{ or } \frac{675 \pm 3\sqrt{91569}}{40};$$

$$\text{and } x = \frac{45 \pm 3\sqrt{3849}}{4} \text{ or } \frac{225 \pm \sqrt{91569}}{20}.$$

34. Multiplying the first equation by $\sqrt{y^2 + x}$,

$$\frac{y^2 + x}{4x} + \frac{y}{\sqrt{4x}} = \frac{y^2}{4},$$

$$\text{completing the square, } \frac{y^2}{x} + \frac{2y}{\sqrt{x}} + 1 = y^2,$$

$$\therefore \frac{y}{\sqrt{x}} + 1 = \pm y,$$

$$\text{whence } 1 = \pm y - \frac{y}{\sqrt{x}} = y \cdot \frac{\pm\sqrt{x} - 1}{\sqrt{x}},$$

$$\therefore y = \frac{\sqrt{x}}{\pm\sqrt{x} - 1}.$$

Multiplying the numerator and denominator of the fraction in the second equation by $\sqrt{x} + \sqrt{x - y - 1}$,

$$\frac{(\sqrt{x} + \sqrt{x - y - 1})^2}{y + 1} = y + 1,$$

$$\text{whence } \sqrt{x} + \sqrt{x - y - 1} = \pm (y + 1),$$

$$\therefore \sqrt{x - y - 1} = \pm (y + 1) - \sqrt{x},$$

$$\text{and } x - y - 1 = (y + 1)^2 \mp 2\sqrt{x} \cdot (y + 1) + x,$$

$$\therefore \pm 2\sqrt{x} = (y + 1) + 1 = y + 2,$$

$$\text{and } 2(\pm\sqrt{x} - 1) = y = \frac{\sqrt{x}}{\pm\sqrt{x} - 1},$$

$$\therefore\ x \mp 2\sqrt{x} + 1 = \frac{\sqrt{x}}{2},$$

$$\text{and } x \mp 2\sqrt{x} - \frac{\sqrt{x}}{2} = -1;$$

$$\text{In the former case, } x - \frac{5}{2}\sqrt{x} + \frac{25}{16} = \frac{9}{16},$$

$$\text{and } \sqrt{x} - \frac{5}{4} = \pm\frac{3}{4},$$

$$\therefore\ \sqrt{x} = 2 \text{ or } \frac{1}{2};$$

$$\therefore\ x = 4 \text{ or } \frac{1}{4};$$

$$\text{and } y = 2 \text{ or } -6;\ \text{or } -1 \text{ or } -3.$$

$$\text{In the latter case, } x + \frac{3}{2}\sqrt{x} = -1,$$

$$\text{and } x + \frac{3}{2}\sqrt{x} + \frac{9}{16} = -\frac{7}{16},$$

$$\therefore\ \sqrt{x} + \frac{3}{4} = \pm\frac{\sqrt{-7}}{4},$$

$$\text{and } \sqrt{x} = \frac{-3 \pm \sqrt{-7}}{4},$$

$$\therefore\ x = \frac{1 \mp 3\sqrt{-7}}{8};$$

$$\text{and } y = \frac{-7 \pm \sqrt{-7}}{2} \text{ or } \frac{-1 \mp \sqrt{-7}}{2}.$$

35. From the first equation,

$$\frac{(x + y + \sqrt{x^2 - y^2})^2}{2xy + 2y^2} = \frac{9}{8y}\,.\,(x + y),$$

$$\therefore\ x + y + \sqrt{x^2 - y^2} = \pm\frac{3}{2}\,.\,(x + y),$$

$$\text{and } \sqrt{x^2 - y^2} = \frac{1}{2} \cdot (x + y) \text{ or } -\frac{5}{2} \cdot (x + y),$$

$$\therefore \sqrt{x - y} = \frac{1}{2}\sqrt{x + y} \text{ or } -\frac{5}{2}\sqrt{x + y},$$

$$\text{and } x - y = \frac{1}{4} \cdot (x + y) \text{ or } \frac{25}{4} \cdot (x + y),$$

$$\text{whence } 4x - 4y = x + y \text{ or } = 25x + 25y,$$

$$\text{and } 3x = 5y \text{ or } 21x = -29y,$$

$$\therefore y = \frac{3x}{5} \text{ or } -\frac{21x}{29}.$$

From the second equation,

$$(x^2 + y)^2 - 2x \cdot (x^2 + y) + x^2 = 506 + x^2 - x + y,$$

$$\therefore x^2 + y - x = \pm\sqrt{506 + x^2 + y - x},$$

$$\text{whence } (x^2 + y - x)^2 - (x^2 + y - x) + \frac{1}{4} = 506 + \frac{1}{4} = \frac{2025}{4},$$

$$\therefore x^2 + y - x - \frac{1}{2} = \pm\frac{45}{2},$$

$$\text{and } x^2 + y - x = 23 \text{ or } -22;$$

$$\text{in the former case, } x^2 + \frac{3}{5}x - x = 23,$$

$$\text{or } x^2 - \frac{2}{5}x + \frac{1}{25} = 23 + \frac{1}{25} = \frac{576}{25},$$

$$\text{and } x - \frac{1}{5} = \pm\frac{24}{5},$$

$$\therefore x = 5 \text{ or } -\frac{23}{5}.$$

$$\text{and } y = 3 \text{ or } -\frac{69}{25};$$

$$\text{in the latter case, } x^2 - \frac{2x}{5} + \frac{1}{25} = -22 + \frac{1}{25} = -\frac{1209}{25},$$

$$\therefore x = \frac{1 \pm \sqrt{-1209}}{5},$$

$$\text{and } y = \frac{3 \pm 3\sqrt{-1209}}{25}.$$

The cases in which $y = -\frac{21x}{29}$ are solved in a similar manner.

36. From the first equation, by completing the square,

$$\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{1}{2} \cdot \frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}} + \frac{1}{16} = 5 + \frac{1}{16} = \frac{81}{16},$$

$$\therefore \frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}} + \frac{1}{4} = \pm \frac{9}{4},$$

$$\text{and } \frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}} = 2 \text{ or } -\frac{5}{2},$$

$$\therefore \frac{y}{x} = 16 \text{ or } \frac{625}{16}.$$

From the second equation,

$$\frac{x^2}{y} - \frac{x}{6\sqrt{y}} + \frac{1}{144} = \frac{1}{6} + \frac{1}{144} = \frac{25}{144},$$

$$\therefore \frac{x}{\sqrt{y}} = \frac{1}{12} = \pm \frac{5}{12},$$

$$\text{and } \frac{x}{\sqrt{y}} - \frac{1}{2} \text{ or } -\frac{1}{3},$$

$$\therefore \sqrt{y} = 2x \text{ or } -3x.$$

$$\text{Hence } \sqrt{y} = \frac{y}{8},$$

$$\text{and } y = 64, \ \therefore \ x = 4.$$

$$\text{Also } \sqrt{y} = -\frac{3y}{16},$$

$$\therefore y = \frac{256}{9}, \text{ and } x = \frac{16}{9}.$$

$$\text{Also } \sqrt{y} = \frac{32y}{625},$$

$$\therefore y = \left(\frac{625}{32}\right)^2, \text{ and } x = \frac{625}{64}.$$

Also $\sqrt{y} = -\frac{48y}{625}$,

$$\therefore\ y = \overline{\frac{625}{48}}\Big|^2, \text{ and } x = \frac{625}{144}.$$

37. From the first equation, $x + y = 61 + \sqrt{xy}$;

and from the second, $x^{\frac{1}{4}}y^{\frac{1}{4}} \cdot (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 78$,

$$\therefore\ \sqrt{xy} \cdot (x + y + 2\sqrt{xy}) = 6804,$$

in which let the value of $x + y$ be substituted, from the first equation,

$$\therefore\ \sqrt{xy} \cdot (61 + 3\sqrt{xy}) = 6804;$$

$$\text{hence } xy + \frac{61}{3}\sqrt{xy} = \frac{6804}{3},$$

$$\text{and } xy + \frac{61}{3}\sqrt{xy} + \overline{\frac{61}{6}}\Big|^2 = \frac{3721}{36} + \frac{6804}{3} = \frac{76729}{36},$$

$$\therefore\ \sqrt{xy} + \frac{61}{6} = \pm\frac{277}{6},$$

$$\text{and } \sqrt{xy} = 36 \text{ or } -\frac{169}{3};$$

$$\text{Hence } x + y = 97,$$

$$\text{and } \quad 2\sqrt{xy} = 72 \text{ or } -\frac{338}{3},$$

$$\text{by subtraction, } x - 2\sqrt{xy} + y = 25 \text{ or } -\frac{47}{3},$$

$$\text{and } \sqrt{x} - \sqrt{y} = \pm 5 \text{ or } \pm\sqrt{\frac{-47}{3}};$$

$$\text{also } \sqrt{x} + \sqrt{y} = 13 \text{ or } 6\sqrt{-3},$$

$$\therefore\ \text{by addition, } 2\sqrt{x} = 18 \text{ or } 8, \text{ or } 6\sqrt{-3} \pm \sqrt{\frac{-47}{3}},$$

$$\therefore \sqrt{x} = 9 \text{ or } 4, \text{ or } 3\sqrt{-3} \pm \frac{1}{2}\sqrt{\frac{-47}{3}},$$

$$\text{and } x = 81 \text{ or } 16, \text{ or } -\frac{371}{12} \pm 3\sqrt{47};$$

$$\text{and also } y = 16 \text{ or } 81, \text{ or } -\frac{371}{12} \mp 3\sqrt{47}.$$

38. Multiplying the first equation by 2, and transposing,
$2y^2 + 2x - 11 + 2\sqrt{(3y^2 + 2x - 11)} = 14 - 11 + 4y = 4y + 3$,
completing the square,

$$(3y^2 + 2x - 11) + 2\sqrt{(3y^2 + 2x - 11)} + 1 = y^2 + 4y + 4,$$

extracting the root, $\sqrt{(3y^2 + 2x - 11)} + 1 = y + 2$,

whence $3y^2 + 2x - 11 = (y + 1)^2 = y^2 + 2y + 1$,

and $2y^2 - 2y = 12 - 2x$,

or $y^2 - y = 6 - x$,

$\therefore\ y^2 - y + 1 = 7 - x$,

which being substituted in the second equation,

$$\sqrt{(y^2 + 2y + 1)} = \frac{x + y}{x - y},$$

$$\text{or } y + 1 = \frac{x + y}{x - y},$$

whence $xy + x - y^2 - y = x + y$,

$\therefore\ xy = y^2 + 2y$,

or $x = y + 2$,

whence $y^2 - y = 6 - x = 6 - y - 2 = 4 - y$,

$\therefore\ y^2 = 4$,

and $y = \pm 2$;

$\therefore\ x = y + 2 = 4$.

39. From the first equation,

$$x^4 - 2x^2y^2 + y^4 = 1 + 2xy + x^2y^2,$$

$\therefore$ extracting the root, $x^2 - y^2 = 1 + xy$,

and $x^2 - xy + y^2 = 2y^2 + 1$.

R 2

Substituting this in the second equation,

$$x^3 + y^3 = (x^2 - x\,y + y^2) \cdot (x + 1),$$

$$\therefore \text{ dividing by } (x^2 - x\,y + y^2),$$

$$x + y = x + 1,$$

$$\text{and } \therefore \; y = 1;$$

$$\text{whence } x^2 - 1 = 1 + x,$$

$$\text{and } x^2 - x = 2,$$

$$\therefore \; x^2 - x + \frac{1}{4} = \frac{9}{4},$$

$$\text{and } x - \frac{1}{2} = \pm \frac{3}{2},$$

$$\therefore \; x = 2 \text{ or } - 1.$$

40. From the first equation,

$$x^2 y + x\,y^2 + \frac{y^3}{4} = x\,y^2 + 4x^{\frac{1}{2}} y + 4,$$

$$\therefore \text{ extracting the root, } y^{\frac{1}{2}} \cdot \left(x + \frac{y}{2}\right) = x^{\frac{1}{2}} y + 2,$$

$$\text{and } x\,y^{\frac{1}{2}} - x^{\frac{1}{2}} y + \frac{1}{2} y^{\frac{3}{2}} = 2,$$

$$\text{or } \quad 2x\,y^{\frac{1}{2}} - 2x^{\frac{1}{2}} y + y^{\frac{3}{2}} = 4,$$

$$\text{but from the second, } \quad x^{\frac{3}{2}} - x\,y^{\frac{1}{2}} + x^{\frac{1}{2}} y = 3,$$

$$\therefore \text{ by subtraction, } \quad y^{\frac{3}{2}} - 3x^{\frac{1}{2}} y + 3x\,y^{\frac{1}{2}} - x^{\frac{3}{2}} = 1,$$

$$\text{and extracting the cube root, } y \quad - x^{\frac{1}{2}} = 1,$$

$$\text{or } x^{\frac{1}{2}} = y^{\frac{1}{2}} - 1.$$

Let this value be substituted in the second equation,

$$\text{and } y^{\frac{3}{2}} - 2y + 2y^{\frac{1}{2}} - 1 = 3,$$

$$\text{or } y^{\frac{3}{2}} + 2y^{\frac{1}{2}} = 2y + 4,$$

$$\therefore \; y^{\frac{1}{2}} = 2,$$

$$\text{and } y = 4.$$

$$\text{whence } x^{\frac{1}{2}} = y^{\frac{1}{2}} - 1 = 1,$$

$$\text{and } \therefore \; x = 1.$$

41. From the second equation, $7 - 10\sqrt{xy} = xy - 16y$,

$$\therefore\ xy - 10\sqrt{xy} + 25 = 16y + 32 = 16 \cdot (y + 2),$$

extracting the root, $\sqrt{xy} + 5 = 4\sqrt{y + 2}$.

Substituting this value of $\sqrt{xy}$ in the first equation,

$$5 - 2\sqrt{y + 2} = \frac{9x^2}{64} - x - 9y + 6\sqrt{xy} = \frac{9x^2}{64} - x - 9y + 24\sqrt{y + 2} - 30,$$

$$\therefore\ 9y + 35 - 26\sqrt{y + 2} = \frac{9x^2}{64} - x,$$

$$\text{or } 9 \cdot (y + 2) - 26\sqrt{y + 2} + \frac{169}{9} = \frac{9x^2}{64} - x + \frac{16}{9},$$

$$\text{extracting the root, } 3\sqrt{y + 2} - \frac{13}{3} = \frac{3x}{8} - \frac{4}{3},$$

$$\text{and } 3\sqrt{y + 2} = \frac{3x}{8} + 3,$$

$$\therefore\ \sqrt{y + 2} = \frac{x}{8} + 1,$$

$$\text{and } y + 2 = \frac{x^2}{64} + \frac{x}{4} + 1,$$

$$\therefore\ y = \frac{x^2}{64} + \frac{x}{4} - 1,$$

$$\text{and } xy = \frac{x^3}{64} + \frac{x^2}{4} - x.$$

$$\text{But } \sqrt{xy} = 4\sqrt{y + 2} - 5 = \frac{x}{2} - 1,$$

$$\therefore\ xy = \frac{x^2}{4} - x + 1;$$

$$\text{whence } \frac{x^3}{64} + \frac{x^2}{4} - x = \frac{x^2}{4} - x + 1,$$

$$\therefore \frac{x^3}{64} = 1,$$

$$\text{and } \frac{x}{4} = 1, \text{ or } x = 4.$$

$$\therefore \sqrt{y+2} = \frac{1}{2} + 1 = \frac{3}{2},$$

$$\text{and } y + 2 = \frac{9}{4},$$

$$\therefore y = \frac{1}{4}.$$

42. From the second equation,

$$y^4 - 4y^2x + 4x^2 = 4x^2 - 4,$$

$$\therefore y^2 - 2x = 2\sqrt{x^2 - 1},$$

$$\text{and } y^2 - 2\sqrt{x^2-1} = 2x;$$

$$\therefore \text{ from the first, } \frac{y}{2x} + \frac{2}{3} \cdot \frac{y - \sqrt{x-1}}{2x} = \frac{\sqrt{x+1}}{x},$$

$$\text{and } \frac{1}{2}y + \frac{1}{3}y - \frac{1}{3}\sqrt{x-1} = \sqrt{x+1},$$

$$\therefore \frac{5}{6}y = \sqrt{x+1} + \frac{1}{3}\sqrt{x-1},$$

$$\text{and } \frac{25}{36} \cdot y^2 = x + 1 + \frac{1}{9} \cdot (x-1) + \frac{2}{3}\sqrt{x^2-1} = \frac{10x+8}{9} + \frac{2}{3}\sqrt{x^2-1},$$

$$\text{or } \frac{25}{12} \cdot y^2 = \frac{10x+8}{3} + 2\sqrt{x^2-1};$$

$$\text{whence } \frac{25}{6} \cdot (x + \sqrt{x^2-1}) = \frac{10x+8}{3} + 2\sqrt{x^2-1},$$

$$\text{and } 25x + 25\sqrt{x^2-1} = 20x + 16 + 12\sqrt{x^2-1},$$

$$\therefore 13\sqrt{x^2-1} = 16 - 5x,$$

$$\text{and } 169x^2 - 169 = 256 - 160x + 25x^2,$$

$$\therefore\ 144x^2 + 160x + \overline{\frac{20}{3}}\Big|^2 = 425 + \frac{400}{9} = \frac{4225}{9},$$

$$\therefore\ 12x + \frac{20}{3} = \pm\frac{65}{3},$$

$$\text{and } x = \frac{5}{4} \text{ or } -\frac{85}{36},$$

$$\text{whence } y = \pm 2 \text{ or } \pm\sqrt{-\frac{85}{18} + \frac{\sqrt{5929}}{648}}.$$

43. From the second equation,

$$x^2y^2 - 18xy = 4\sqrt{xy} - 48,$$

completing the square,

$$x^2y^2 - 14xy + 49 = 4xy + 4\sqrt{xy} + 1,$$

$$\text{extracting the root, } xy - 7 = 2\sqrt{xy} + 1,$$

$$\therefore\ xy - 2\sqrt{xy} + 1 = 9,$$

$$\text{and } \sqrt{xy} - 1 = \pm 3,$$

$$\therefore\ \sqrt{xy} = 4, \text{ or } -2.$$

From the first equation,

$$x.(y+1) - 2y.(y+1) = 4.(y^2-1),$$

$$\therefore\ x - 2y = 4y - 4,$$

$$\text{and } x = 6y - 4,$$

$$\therefore\ xy = 6y^2 - 4y,$$

$$\text{whence } 6y^2 - 4y = 16,$$

$$\text{and } y^2 - \frac{2}{3}y + \frac{1}{9} = \frac{8}{3} + \frac{1}{9} = \frac{25}{9},$$

$$\therefore\ y - \frac{1}{3} = \pm\frac{5}{3},$$

$$\text{and } y = 2 \text{ or } -\frac{4}{3},$$

$$\therefore\ x = 8 \text{ or } -12.$$

44. By adding 2 to each side of the first equation,

$$3x + 2 - x\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} = 4 - y,$$

multiplying by 2, adding $\frac{5x^2}{4}$ to each side, and transposing,

$$\left(\frac{5x^2}{4} - 2y + 8\right) + 2x\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} = \frac{5x^2}{4} + 6x + 4,$$

completing the square,

$$\left(\frac{5x^2}{4} - 2y + 8\right) + 2x\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} + x^2 = \frac{9x^2}{4} + 6x + 4,$$

extracting the root, $\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} + x = \frac{3x}{2} + 2,$

$$\therefore \frac{5x^2}{4} - 2y + 8 = \left(\frac{x}{2} + 2\right)^2 = \frac{x^2}{4} + 2x + 4,$$

whence $x^2 - 2y + 4 = 2x,$

and $x^2 - 2x + 4 = 2y.$

From the second,

$$\frac{x + y}{2x} - \frac{3}{4} \cdot x\sqrt{x + y} = 2x - 3 - \frac{3y}{2x}\sqrt{x + y},$$

$$\text{or } (x + y) - \left(\frac{3x^2}{2} - 3y\right)\sqrt{x + y} = (2x - 3) \cdot 2x,$$

$$\text{and } (x + y) - \frac{3}{2} \cdot (x^2 - 2y)\sqrt{x + y} = 4x^2 - 6x.$$

$\therefore$ from the former,

$$(x + y) - 3 \cdot (x - 2) \cdot \sqrt{x + y} = 4x^2 - 6x,$$

$$\text{and } (x + y) - (3x - 6)\sqrt{x + y} + \left(\frac{3x - 6}{2}\right)^2 =$$

$$\frac{9x^2 - 36x + 36}{4} + 4x^2 - 6x = \frac{25x^2 - 60x + 36}{4},$$

$$\therefore \sqrt{x + y} - \frac{3x - 6}{2} = \frac{5x - 6}{2},$$

$$\text{and } \sqrt{x+y} = 4x - 6,$$
$$\text{whence } 2x + 2y = 2 \,.\, (4x-6)^2 = 32x^2 - 96x + 72,$$
$$\text{or } x^2 - 4 = 32x^2 - 96x + 72,$$
$$\text{and } 31x^2 - 96x = -68,$$

completing the square,

$$x^2 - \frac{96}{31}x + \overline{\frac{48}{31}}\Big|^2 = \frac{2304}{\overline{31}|^2} - \frac{68}{31} = \frac{196}{\overline{31}|^2},$$
$$\therefore\ x - \frac{48}{31} = \pm\frac{14}{31},$$
$$\text{and } x = 2, \text{ or } \frac{34}{31},$$
$$\text{whence } y = 2.$$

45. From the first equation, $x^4 + y^4 = 9 + 2x^2y^2$,

$$\therefore \text{ from the second, } (9 + 2x^2y^2)^2 + 9x^2y^2 + 3 = 328,$$
$$\text{or } 81 + 36x^2y^2 + 4x^4y^4 + 9x^2y^2 + 3 = 328,$$
$$\therefore\ 4x^4y^4 + 45x^2y^2 = 244,$$
$$\text{and } 4x^4y^4 + 45x^2y^2 + \overline{\frac{45}{4}}\Big|^2 = \frac{5929}{16},$$
$$\text{whence } 2x^2y^2 + \frac{45}{4} = \pm\frac{77}{4},$$
$$\text{and } x^2y^2 = 4 \text{ or } -\frac{61}{4}.$$

$$\text{And since } x^4 - 2x^2y^2 + y^4 = 9,$$
$$\text{and } \quad 4x^2y^2 = 16 \text{ or } -61,$$
$$\therefore \text{ by addition, } x^4 + 2x^2y^2 + y^4 = 25 \text{ or } -52,$$
$$\text{and } \quad x^2 + y^2 = \pm 5 \text{ or } \pm 2\sqrt{-13},$$
$$\text{but } \quad x^2 - y^2 = 3,$$
$$\therefore \quad 2x^2 = 8 \text{ or } -2, \text{ or } 3 \pm 2\sqrt{-13},$$
$$\therefore\ x = \pm 2 \text{ or } \pm\sqrt{-1}, \text{ or } \pm\sqrt{\frac{1}{2}\,.\,(3 \pm 2\sqrt{-13})};$$
$$\text{and } y = \pm 1 \text{ or } \pm 2\sqrt{-1}, \text{ or } \pm\sqrt{\frac{1}{2}\,.\,(-3 \pm 2\sqrt{-13})}.$$

46. From the first equation,

$$x^2y^2 - 8 - 80y^2 = 280 - 2y\sqrt{(x^2y^2 - 272)},$$

by transposition, and completing the square,

$$(x^2y^2 - 272) + 2y\sqrt{(x^2y^2 - 272)} + y^2 = 81y^2,$$

extracting the root, $\sqrt{(x^2y^2 - 272)} + y = \pm 9y$,

$$\therefore \sqrt{(x^2y^2 - 272)} = 8y \text{ or } - 10y.$$

From the second equation, $x^2y^2 - 6 - 30xy = 30 + 5xy$,

$$\therefore x^2y^2 - 35xy = 36,$$

completing the square,

$$x^2y^2 - 35xy + \overline{\frac{35}{2}}\Big|^2 = \frac{1225}{4} + 36 = \frac{1369}{4},$$

extracting the root, $xy - \frac{35}{2} = \pm \frac{37}{2}$,

$$\therefore xy = 36 \text{ or } - 1;$$

whence $\sqrt{1024} = 8y$ or $- 10y$,

and $y = 4$ or $- \frac{16}{5}$;

$$\therefore x = 9 \text{ or } - \frac{45}{4}.$$

47. From the first equation,

$$(y^2 - 4\sqrt{x}) + \sqrt{x} \,.\, \sqrt{y^2 - 4\sqrt{x}} + \frac{x}{4} = x,$$

$$\therefore \sqrt{y^2 - 4\sqrt{x}} + \frac{\sqrt{x}}{2} = \pm \sqrt{x},$$

$$\therefore \sqrt{y^2 - 4\sqrt{x}} = \frac{\sqrt{x}}{2} \text{ or } - \frac{3\sqrt{x}}{2},$$

and $y^2 - 4\sqrt{x} = \frac{x}{4}$ or $\frac{9x}{4}$.

From the second,

$$8 \,.\, (y - \sqrt{x}) - 8\sqrt{8 \,.\, (y - \sqrt{x}) - 4} = - 8,$$

and $[8 \,.\, (y - \sqrt{x}) - 4] - 8\sqrt{8 \,.\, (y - \sqrt{x}) - 4} + 16 = 4$,

$$\therefore \sqrt{8 \, . \, (y - \sqrt{x}) - 4} - 4 = \pm 2,$$

$$\text{and } 8 \, (y - \sqrt{x}) - 4 = 36 \text{ or } 4,$$

$$\therefore y - \sqrt{x} = 5 \text{ or } 1,$$

$$\text{and } y = \sqrt{x} + 5, \text{ or } \sqrt{x} + 1.$$

Hence, 1st, since $y^2 = 4\sqrt{x} + \frac{x}{4}$, and $y = \sqrt{x} + 1$,

$$\therefore x + 2\sqrt{x} + 1 = 4\sqrt{x} + \frac{x}{4},$$

$$\text{and } \frac{3x}{4} - 2\sqrt{x} = -1,$$

$$\therefore x - \frac{8}{3}\sqrt{x} + \frac{16}{9} = \frac{16}{9} - \frac{4}{3} = \frac{4}{9},$$

$$\text{and } \sqrt{x} - \frac{4}{3} = \pm \frac{2}{3},$$

$$\therefore \sqrt{x} = 2 \text{ or } \frac{2}{3},$$

$$\text{and } x = 4 \text{ or } \frac{4}{9},$$

$$\therefore y = \sqrt{x} + 1 = 3 \text{ or } = \frac{5}{3}.$$

2d, since $y^2 = 4\sqrt{x} + \frac{x}{4}$, and $y = \sqrt{x} + 5$,

$$x + 10\sqrt{x} + 25 = 4\sqrt{x} + \frac{x}{4},$$

$$\therefore x + 8\sqrt{x} = -\frac{100}{3},$$

$$\text{and } x + 8\sqrt{x} + 16 = -\frac{52}{3},$$

$$\therefore \sqrt{x} = -4 \pm 2\sqrt{\frac{-13}{3}},$$

$$\text{and } x = -\frac{4}{3} \mp 16\sqrt{\frac{-13}{3}};$$

$$\therefore\ y = 1 \pm 2\sqrt{\frac{-13}{3}}.$$

3d, since $y^2 = 4\sqrt{x} + \frac{9x}{4}$, and $y = \sqrt{x} + 1$,

$$\therefore\ x + 2\sqrt{x} + 1 = 4\sqrt{x} + \frac{9x}{4},$$

$$\text{and } \frac{5x}{4} + 2\sqrt{x} = 1,$$

$$\therefore\ x + \frac{8\sqrt{x}}{5} + \frac{16}{25} = \frac{4}{5} + \frac{16}{25} = \frac{36}{25},$$

$$\text{and } \sqrt{x} + \frac{4}{5} = \pm\frac{6}{5},$$

$$\therefore\ \sqrt{x} = \frac{2}{5} \text{ or } -2,$$

$$\text{and } x = \frac{4}{25} \text{ or } 4;$$

$$\therefore\ y = \frac{7}{5} \text{ or } -1.$$

4th, since $y^2 = 4\sqrt{x} + \frac{9x}{4}$, and $y = \sqrt{x} + 5$,

$$\therefore\ x + 10\sqrt{x} + 25 = 4\sqrt{x} + \frac{9x}{4},$$

$$\text{whence } x - \frac{24}{5}\sqrt{x} + \frac{144}{25} = \frac{644}{25},$$

$$\text{and } \sqrt{x} = \frac{12 \pm \sqrt{644}}{5},$$

$$\therefore\ x = \frac{788 \pm 24\sqrt{644}}{25};$$

$$\text{and } y = \frac{37 \pm \sqrt{644}}{5}.$$

48. From the first equation,

$$\frac{x}{y^{\frac{1}{3}}} - \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} - \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \cdot (x^{\frac{1}{3}} + y^{\frac{1}{3}}) + \frac{3y}{x^{\frac{1}{3}}} + 2,$$

$$\therefore\ x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{2}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + 4y^{\frac{2}{3}} \cdot (x^{\frac{1}{3}} + y^{\frac{1}{3}}) + 3y^{\frac{4}{3}},$$

$$\text{and } x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}} = (x^{\frac{1}{3}} + y^{\frac{1}{3}})^2 + 4y^{\frac{2}{3}} \cdot (x^{\frac{1}{3}} + y^{\frac{1}{3}}) + 4y^{\frac{4}{3}},$$

$$\therefore \text{ extracting the root, } x^{\frac{2}{3}} + y^{\frac{2}{3}} = x^{\frac{1}{3}} + y^{\frac{1}{3}} + 2y^{\frac{2}{3}},$$

$$\text{and } x^{\frac{2}{3}} - y^{\frac{2}{3}} = x^{\frac{1}{3}} + y^{\frac{1}{3}},$$

$$\text{whence } x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1.$$

From the second equation,

$$\frac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}} - \frac{2x^{\frac{2}{3}}}{y} + \frac{1}{y^{\frac{2}{3}}} - \frac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}} + \frac{2}{x^{\frac{1}{3}}} + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{169}{36} \cdot \frac{1}{y^{\frac{2}{3}}},$$

$$\therefore \text{ extracting the root, } \frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} - \frac{1}{y^{\frac{1}{3}}} - \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \pm \frac{13}{6} \cdot \frac{1}{y^{\frac{1}{3}}},$$

$$\text{whence } \frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} - \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{19}{6y^{\frac{1}{3}}}, \text{ or } \frac{-7}{6y^{\frac{1}{3}}},$$

$$\text{or } x - y = \frac{19}{6}x^{\frac{1}{3}}y^{\frac{1}{3}}, \text{ or } -\frac{7}{6}x^{\frac{1}{3}}y^{\frac{1}{3}}.$$

$$\text{But } x - y - 3x^{\frac{1}{3}}y^{\frac{1}{3}} \cdot (x^{\frac{1}{3}} - y^{\frac{1}{3}}) = 1,$$

$$\therefore\ \frac{19}{6}x^{\frac{1}{3}}y^{\frac{1}{3}} - 3x^{\frac{1}{3}}y^{\frac{1}{3}} = 1,$$

$$\text{or } x^{\frac{1}{3}}y^{\frac{1}{3}} = 6,$$

$$\text{but } x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1,$$

$$\therefore\ x^{\frac{1}{3}} = 3, \text{ and } x = 27;$$

$$\text{and } y^{\frac{1}{3}} = 2,\ \therefore\ y = 8.$$

SECTION VI.

Problems producing Simple Equations involving only one unknown Quantity.

1. Let $x =$ the number,

$$\therefore\ 3x - 18 = 6,$$

$$\text{by transposition, } 3x = 24,$$

$$\text{and } \quad x = 8.$$

2. Let $x =$ the number,

$$\therefore\ 2x - \frac{4}{5} \cdot \frac{x}{2} = 40,$$

$$\text{and } 10x - 2x = 200,$$

$$\therefore\ x = 25.$$

3. Let $x =$ the number of days,

$\therefore\ 9x =$ the number of yards fenced by the one,

and $6x =$ the number fenced by the other,

$$\therefore\ (9x + 6x =)\ 15x = 450,$$

$$\text{and } x = 30.$$

4. Let $x =$ the number of yards in the first,

$\therefore\ 2x =$ the number in the second,

$3x =$ . . the third,

$4x =$ . . the fourth,

$\therefore\ 10x = 50,$

and $\quad x = 5.$

And the lengths of the pieces were 5, 10, 15, 20 yards, respectively.

5. Let $x =$ the price of a bushel,
$\therefore\ 13x =$ the sum received for the first,
and $17x =$ the sum received for the second,

$$\therefore\ (17x - 13x =)\ 4x = 36,$$
$$\text{and } x = 9.$$

6. Let $x =$ the number of gallons the fourth held,
$\therefore\ 3x =$ . . . . the third .
$6x =$ . . . the second .
$12x =$ . . . . the first .

$22x = 198,$
and $x = 9.$

$\therefore$ the number of gallons held by them were 108, 54, 27, 9, respectively.

7. Let $x =$ the number of ounces the first weighed,
$\therefore\ x + 12 =$ . . . the second .
$x + 21 =$ . . . . the third .

$\therefore\ 3x + 33 = 48,$
and $x + 11 = 16,$
$\therefore\ x = 5.$

And the weights were 5, 17, 26 ounces, respectively.

8. Let $x =$ the number of gallons of brandy,
$\therefore\ 20 + x =$ . . . . of water,
$37 + x =$ . . . . of wine,

$57 + 3x = 96,$
and $19 + x = 32,$
$\therefore\ x = 13.$

And the number of gallons of brandy, water, and wine, were 13, 33, 50, respectively.

9. Let x = the price of the first,

$\therefore x + 12 =$ . . second,

$x + 18 =$ . . . . third,

$x + 20 =$ . . fourth,

$\therefore 4x + 50 = 230,$

and $4x = 180,$

$\therefore x = 45.$

And the prices were 45, 57, 63, and 65.

10. Let x = the sum earned by the youngest,

$\therefore x + 5 =$ . . . . . fifth,

$x + 9 =$ . . . . . fourth,

$x + 15 =$ . . . . . third,

$x + 23 =$ . . . . . second,

$x + 30 =$ . . . . . eldest,

$\therefore 6x + 82 = 130,$

and $6x = 48,$

$\therefore x = 8.$

And the sums were 38, 31, 23, 17, 13, 8, respectively.

11. Let x = the number of miles he travelled the 1st day,

$\therefore x - 5 =$ 2d

$x - 9 =$ 3d

$x - 14 =$ 4th

$\therefore 4x - 28 = 240,$

and $x - 7 = 60,$

$\therefore x = 67.$

And the numbers were 67, 62, 58, 53, respectively.

12. Let x = the distance from A to B,

$\therefore x + 10 =$ . . . B to C,

$x - 15 =$ . . . C to D,

$x + 17 =$ . . . D to E.

$\therefore 4x + 12 = 80,$

and $x + 3 = 20$,
$\therefore x = 17$.

And the distances were 17, 27, 2, and 34, respectively.

13. Let x = the number of shillings given to one,
$\therefore x + 4 =$ the number given to the other,
and $2x + 5 = 27$,
$\therefore 2x = 22$,
and $x = 11$.
$\therefore$ he gave 11 and 16.

14. Let $2x =$ the number,
$\therefore 6x - 40 = 51 - x$,
and $7x = 91$,
$\therefore x = 13$,
and the number is 26.

15. Let $x =$ the sum,
$\therefore x + 15 = 3 \cdot (x - 9)$,
and $2x = 42$,
$\therefore x = 21$.

16. Let $x =$ the less gain,
$\therefore x + 54 =$ the greater,
and $2x + 54 = 3x - 49$,
$\therefore x = 103$,
and the gains were £103 and £157.

17. Let $x =$ the length of the base,
$\therefore x - 11$, and $x - 16$, are the lengths of the sides,
and $(x + x - 11 + x - 16 =)\ 3x - 27 = 75$,
$\therefore x - 9 = 25$,
and $x = 34$;
and the sides were 23 and 18 feet.

18. Let $x =$ the number,

$$\therefore\ 8x = \text{the sum paid,}$$
$$\text{and } 8x = 7\,.\,(x + 4),$$
$$\therefore\ x = 28.$$

19. Let $x =$ one part,

$$\therefore\ 46 - x = \text{the other,}$$
$$\text{and } \frac{x}{3} + \frac{46 - x}{7} = 10,$$
$$\text{whence } 7x + 138 - 3x = 210,$$
$$\text{or } 4x = 72,$$
$$\text{and } x = 18;$$
$$\therefore\ \text{the parts are 28 and 18.}$$

20. Let $x =$ the less payment (in shillings),

$\therefore\ x + 19 =$ the greater;

$$\text{and } 3x + 5 = 2x + 38,$$
$$\text{whence } x = 33,$$
$$\text{and the sums are 33 and 52 shillings.}$$

21. Let $x =$ the number of gallons of brandy,

$\therefore\ x + 9 =$ . . . . of rum,

and $19x$, and $15\,.\,(x + 9)$, are their prices respectively,

$$\therefore\ 19x = 15x + 135 + 1,$$
$$\text{or } 4x = 136,$$
$$\therefore\ x = 34,$$

and there were 34 gallons of brandy, and 43 of rum.

22. Let $x =$ the sum spent by B,

$\therefore\ x + 40 =$ . . by A;

and $400 - x =$ sum saved by B,

and $360 - x =$ . . by A.

$$\text{Whence } 4\,.\,(760 - 2x) = 400,$$
$$\text{and } 760 - 2x = 100,$$

$$\therefore\ 2x = 660,$$
$$\text{and } x = 330.$$
$$\therefore\ \text{A spent £370, and B £330.}$$

23. Let $x =$ the loss sustained by one,
$\therefore\ x + 6 =$ . . . by the other;
$$\text{and } 2x + 6 = 3x - 5,$$
$$\therefore\ x = 11,$$
and the losses were 11 and 17.

24. Let $x =$ the number of acres,
$$\therefore\ 5 \,.\, (x - 6) = \text{the sum received} = 50s.$$
$$\text{and } \therefore\ x - 6 = 10,$$
$$\text{and } x = 16.$$

25. Let $x =$ the number sunk,
$\therefore\ x + 7 =$ . . taken,
$x - 2 =$ . . burnt,
$$\therefore\ x + x + 7 + x - 2 + 15 = 8x,$$
$$\text{and } 5x = 20,$$
$$\therefore\ x = 4;$$
and the fleet consisted of 32.

26. Let $x =$ the number of acres of arable,
$\therefore\ x - 5 =$ . . . . the rest;
$$\text{whence } 8x + 5 \,.\, (x - 5) = 703,$$
$$\text{or } 13x = 728,$$
$$\therefore\ x = 56, \text{ arable},$$
and the rest was 51.

27. Let $x =$ the number through the third,
$\therefore\ x + 10$, and $x - 5$, are the numbers through the first and second.
$$\text{Whence } 20 \,.\, (3x + 5) = 820,$$
$$\text{or } 3x + 5 = 41,$$

$\therefore\ 3x = 36,$

and $x = 12,$

and the numbers are 22, 7, and 12.

28. Let $x =$ the number of cavalry,

$\therefore\ 3x =$ . . artillery,

and $9x =$ . . infantry,

$\therefore\ 13x = 2600,$

and $x = 200,$

$\therefore$ there were 200 cavalry, 600 artillery, and 1800 infantry.

29. Let $9x =$ the sum which B had at first,

$\therefore\ 4x =$ . . A . .

and $4x + 10 = 9x - 10,$

or $5x = 20,$

$\therefore\ x = 4,$

and A had £16, B £36.

30. Let $x =$ the number at livery,

$\therefore\ 4x =$ . . at grass,

$\therefore\ 4x + 15 = 7x,$

and $3x = 15,$

$x = 5,$

and the whole number is 35.

31. Since they travel at rates which are in the proportion of 3 to 7, let $3x =$ the number of miles one goes,

$\therefore\ 7x =$ the number the other goes;

and $(3x + 7x =)\ 10x = 150,$

$\therefore\ x = 15,$

and one goes 45, the other 105 miles.

32. Let $5x =$ the number A had,

$\therefore\ 11x =$ the number B had,

and $16x =$ the number C had.

$$\text{Whence } 32x = 864,$$
$$\text{and } x = 27.$$
$$\therefore \text{ A had 135, B 297, and C 432.}$$

33. Let $4x =$ the number of women,

$\therefore 7x =$ . . of children,

also, $24x =$ the number of shillings the women received,

and $14x =$ . . . . . children .

$$\therefore 38x = 114,$$
$$\text{and } x = 3,$$
$$\therefore \text{ there were 12 women, and 21 children.}$$

34. Let $x =$ B's stock,

$\therefore 3x =$ A's,

$$\text{Hence } 3x + 50 : x + 50 :: 7 : 3,$$
$$\text{and (Alg. 180), } 2x : x + 50 :: 4 : 3,$$
$$\text{(Alg. 185), } x : x + 50 :: 2 : 3,$$
$$\text{(Alg. 180), } x : 50 :: 2 : 1,$$
$$\therefore x = 100,$$
and A's stock was £300, B's £100.

35. Let $3x =$ one number,

$\therefore 4x =$ the other,

$$\text{whence } 3x + 6 : 4x + 5 :: \frac{2}{5} : \frac{1}{2} :: 4 : 5,$$
$$\text{and } 15x + 30 = 16x + 20,$$
$$\therefore x = 10,$$
and the numbers are 30 and 40.

36. Let $9x =$ the number of loads it contained,

$4x$ being the quantity sold, and $5x$ the quantity remaining,

$$\therefore 5x - 15 : 4x :: 1 : 2,$$
$$\text{(Alg. 185.) } 5x - 15 : 2x :: 1 . 1,$$
$$\text{and } 5x - 15 = 2x,$$

$$\therefore\ 3x = 15,$$
$$\text{and } x = 5,$$
the number in the stack $\therefore$ was 45.

37. Let $3x$, $5x$, and $7x$, be the lengths,
and since the whole quantity is diminished in the proportion of $20 : 17$, the whole quantity is to the part cut off $:: 20 : 3$,

$$\text{or } 15x : 18 :: 20 : 3,$$
$$\therefore \text{(Alg. 185)},\quad 3x : 18 :: 4 : 3,$$
$$\text{and}\quad x : 6 :: 4 : 3,$$
$$\therefore\ x = 8;$$

and the lengths are 24, 40, and 56, yards.

38. Let $4x$, $5x$, $6x$, $7x$, be the number of days employed,

$$\therefore\ 3 . (4x + 5x) + 36 = 3 . (6x + 7x),$$
$$\text{or } 9x + 12 = 13x,$$
$$\therefore\ 4x = 12,$$
$$\text{and } x = 3.$$

and the sums received were 36, 45, 54, 63, shillings.

39. Let $6x$ and $7x$ be the number of gallons drawn off,

$$\therefore\ 7x - 16 = 3x,$$
$$4x = 16,$$
$$\text{and } x = 4,$$

$\therefore$ the quantities drawn off were 24 and 28 gallons.

40. Let $6x$ and $11x =$ the longer sides,

$$\therefore\ 4x = \text{the shorter side of the less};$$
$$\text{and } 2 . (6x + 4x) = 11x + 135,$$
$$\therefore\ 9x = 135,$$
$$\text{and } x = 15,$$

$\therefore$ 90 and 60 were the sides of the less; and the longer side of the greater was 165.

41. Let $x =$ sum taken from B,
$\therefore 2x + 5 =$ sum taken from A;
$\therefore 75 - 2x =$ sum left with A,
and $80 - x =$ sum left with B.

$$\therefore 2 . (75 - 2x + 13) = 80 - x,$$
$$\text{or } 176 - 4x = 80 - x,$$
$$\therefore 3x = 96,$$
$$\text{and } x = 32,$$

$\therefore$ the sums taken were £69 and £32.

42. Let x and $50 - x$ be the numbers,

$$\therefore 9x + 15 . (50 - x) = 12 \times 40,$$
$$\text{or } 3x + 250 - 5x = 160,$$
$$\therefore 2x = 90,$$
$$\text{and } x = 45,$$

$\therefore$ 45 received 9*d.* each, and 5 received 15*d.* each.

43. Let $x =$ the sum,

$$\therefore \frac{x}{20} = \text{the interest for one year,}$$
$$\text{and } \frac{x}{20} \times \frac{13}{2} + 185 = \frac{x}{25} \times 12\tfrac{3}{4} = \frac{x}{25} \times \frac{51}{4},$$
$$\text{and } 130x + 74000 = 204x,$$
$$\therefore 74x = 74000,$$
$$\text{and } x = 1000.$$

44. Let $3x$ and $5x$ be the numbers raised by A and B.

$$\therefore 8 : 7 :: 5x : \text{the contingent of C, } = \frac{35x}{8},$$
$$\text{whence } 3x + 5x + \frac{35x}{8} = 594,$$
$$\text{and } (64x + 35x =)\ 99x = 8 \times 594,$$
$$\therefore x = 48,$$

and the numbers are 144, 240, 210.

45. Let $4x =$ A's principal, and $\therefore x =$ his gain,
$4x - 50 =$ B's principal.

$$\therefore 6 \times 4x : 9 . (4x - 50) :: x : \text{B's gain} = \frac{3 . (2x - 25)}{4},$$

$$\text{and } 5 : 4 :: 5x : 4x - 50 + \frac{3 . (2x - 25)}{4} - \frac{25}{4},$$

$$\therefore 4x = 4x - 50 + \frac{3 . (2x - 25)}{4} - \frac{25}{4},$$

$$\text{and } 6x = 300,$$

$$\therefore x = 50,$$

and their principals were £200 and £150.

46. Let $2x$ and $3x =$ the sums A and B received,

$$\therefore 4 : 5 :: 3x : \text{sum C received} = \frac{15x}{4},$$

$$\text{and } 6 : 7 :: \frac{15x}{4} : \text{sum D received, which} \therefore \text{is} = \frac{35x}{8},$$

$$\therefore 2x + 3x + \frac{15x}{4} + \frac{35x}{8} = 21000,$$

$$\text{or } (16x + 24x + 30x + 35x =) \; 105x = 8 \times 21000,$$

$$\therefore x = 1600,$$

and the sums were £3200, £4800, £6000, £7000.

47. Let $4x =$ the number of bushels of the first,
$\therefore 3x =$ the number of the second;

$$\text{and } 14 \times 4x + 9 \times 3x = 10 \times 7x + 156,$$

$$\text{or } 83x = 70x + 156,$$

$$\therefore 13x = 156,$$

$$\text{and } x = 12,$$

$\therefore$ the numbers bought were 48 and 36 bushels.

48. Let $4x =$ the part paid by A,
$\therefore x =$ . . B,
and $5x =$ the reckoning,

$$\therefore\ x + 3 = 4x - 3,$$
$$3x = 6,$$
$$\text{and } x = 2,$$

$\therefore$ the reckoning was 10*s*. A paid 8*s*. and B 2*s*.

49. Let $12x =$ the sum divided,

$\therefore\ 6x - 3000 =$ the sum A receives,

$4x - 1000 =$ . . B . .

$3x + \ \ 800 =$ . . C . .

$$\therefore\ 13x - 3200 = 12x,$$
$$\text{and } x = 3200,$$

$\therefore$ the whole = £38400. A receives £16200, B £11800, C 10400.

50. Let $x =$ the number,

$\therefore\ \dfrac{30}{x} =$ the price of one;

and $\dfrac{15}{2x} =$ the price of one of the first.

$$\therefore\ 8 \cdot \frac{30}{x} + 4 = 40 \cdot \frac{15}{2x},$$
$$\text{or } \frac{60}{x} + 1 = \frac{75}{x},$$
$$\therefore\ 1 = \frac{15}{x}, \text{ and } x = 15,$$

and the prices were £2, and 10*s*.

51. Let $4x =$ the number won by A,

$\therefore\ 3x - 1 =$ the number won by B.

$$\text{and } 4 \times 4x = 5 \cdot (3x - 1) + 10,$$
$$\text{or } 16x = 15x + 5,$$
$$\therefore\ x = 5,$$

and A won 20, and B 14.

52. Let $x =$ the number,

$$\therefore\ 6 \times 7 \times 10x = 8 \times 7 \times 12 \times (x - 1200),$$
$$\text{or } 5x = 8x - 9600,$$
$$\therefore\ 3x = 9600,$$
$$\text{and } x = 3200.$$

53. Let $x =$ the number of ounces of copper,
$\therefore\ 505 - x =$ the number of ounces of tin.

$$\text{and } \frac{4x}{21} + \frac{4\,.\,(505 - x)}{17} = 100,$$
$$\text{or } 17x + 21\,.\,(505 - x) = 17 \times 21 \times 25,$$
$$\therefore\ 4x = 1680,$$
$$\text{and } x = 420,$$

$\therefore$ there were 420 of copper, and 85 of tin.

54. Let $x =$ the number of hours the first travels before they meet,
$\therefore\ x - 8 =$ the number the second travels,

$$\text{and } 4x + 5x - 40 = 131,$$
$$\therefore\ 9x = 171,$$
$$\text{and } x = 19.$$

$\therefore$ they meet 76 miles from A, and 55 from B.

55. Let $2x =$ the sum,
$\therefore\ x - 48 =$ what he lent,

and $\frac{4}{5}\,.\,(x - 48) =$ what remained after spending.

$$\therefore\ \frac{4}{5}\,.\,(x - 48) = \frac{x}{5},$$
$$\text{and } 4x - 192 = x,$$
$$\therefore\ 3x = 192,$$
$$\text{and } x = 64,$$

and the sum was £128.

56. Let $x =$ the number of men in front,
$\therefore\ 4x =$ the number of spectators,
and $40x =$ the army.

$$\text{Hence } 45 \,.\, (x - 100) = 44x,$$
$$\text{and } x = 4500,$$
$$\therefore \text{ the army consisted of } 18000.$$

57. Let $x =$ the quantity raised by A,
$\therefore\ 235 - x =$ the quantity raised by B;

$$\text{and } 86x = 86 \,.\, (235 - x) + 4214,$$
$$\text{or } x = 235 - x + 49,$$
$$\therefore\ 2x = 284,$$
$$\text{and } x = 142, \text{ the quantity raised by A},$$
$$\text{and } 93 = \text{the quantity raised by B}.$$

58. Let $x =$ the length of the shorter,
$\therefore\ 4x - 12 =$ the length of the longer.

$$\text{and } 5 \,.\, (x - 5) + 6 \,.\, (4x - 35) = 142,$$
$$\therefore\ (5x + 24x =)\ 29x = 377,$$
$$\text{and } x = 13,$$
$$\text{the lengths } \therefore \text{ were 40 and 13 yards.}$$

59. Let $x =$ the number of years he was old in 1799,
$\therefore\ (2x + 7) \,.\, 4 =$ the sum he received in groats.

$$\text{and } 4 \,.\, (2x + 7) = 476 \text{ groats},$$
$$\therefore\ 2x + 7 = 119,$$
$$2x = 112,$$
$$\therefore\ x = 56.$$

And $1799 - 56 = 1743$, the year in which A was born,
and $56 + 7 = 63$, the number of years completed.

60. Let $6x =$ the number of hours the second travels,
$\therefore\ 6 \,.\, (x + 2) =$ . . . . first . .

and $3 : 26 :: 6x$: the distance the second travels after passing A,
which $\therefore = 52x$,

In the same manner, the distance which the first travels after passing A, is $= 39 \,.\, (x + 2)$,

$$\therefore\ 52x = 39x + 78,$$
$$\text{and } 13x = 78,$$
$$\therefore\ x = 6,$$

and he must $\therefore$ travel 36 hours, and 312 miles.

61. Let $x =$ the sum daily demanded at first,
$\therefore\ 3x =$ the daily demand afterwards;
also $3x =$ sum paid by A in the three first days,
and $6x =$ that paid in the two following days;
whence $9x =$ the whole sum paid by A.

And if they had joined capitals, $2x$ would have been the sum paid by B in four days from A's stock.

$$\text{whence } 2x = 4000,$$
$$\text{and } x = 2000.$$

62. Let $x =$ the breadth,

$$\therefore\ x + 13 = \text{the length of A's shot},$$
$$\text{and } x + 22\tfrac{2}{7} = \text{the length of B's shot};$$
$$\text{and } 8 \,.\, (x + 13) + 7 \,.\, (x + 22\tfrac{2}{7}) = 1760,$$
$$\text{or } 15x = 1760 - 260 = 1500,$$
$$\therefore\ x = 100.$$

63. Let $x =$ A's subscription,
$\therefore\ 4x + 10 =$ B's,
and $2x + 35 =$ C's,

$$2x = \text{A's gain},$$
$$\text{and } 5x : 2x :: (4x + 10 : 148 ::)\ 2x + 5 : 74,$$
$$\therefore\ 4x + 10 = 370,$$

$$\text{and } 4x = 360,$$
$$\therefore\ x = 90.$$

Also $5 : 2 :: 180 + 35 : \text{C's gain}, = \frac{2}{5} \cdot 215 = 86.$

$\therefore$ the sums subscribed were 450, 370, 215.
and the whole gain $= 180 + 148 + 86 = 414$.

64. Let $6x =$ the whole value of the produce,
$\therefore\ x =$ the tenant's share of the profit,
and $3x =$ the whole profit,
$2x =$ the landlord's share,
After falling, $2x =$ the expense of cultivation,
and $\frac{18x}{5} =$ the whole value of the produce,
$\therefore\ \frac{8x}{5} =$ the whole profit,
and $\frac{2}{3} \cdot \frac{8x}{5} =$ the landlord's share.

$$\therefore\ \frac{16x}{15} = 400,$$
$$\text{and } x = 375,$$

whence the original value of the produce is £2250.

65. Let $3x =$ B's share,

$$\therefore\ 6 + x : 30 + x :: 30 + x : 78 + x,$$
$$\text{and } 6 + x : 24 :: 30 + x : 48,$$
$$\therefore\ 6 + x : 30 + x :: 1 : 2,$$
$$\text{and } 6 + x : 24 :: 1 : 1,$$
$$\therefore\ 6 + x = 24,$$
$$\text{and } x = 18.$$

$\therefore$ 24, 48, 96, are the sums required.

66. Let $3x =$ the number of which the regiment consisted,
$\therefore\ x =$ the number in the detachment,

$$\text{whence } x - 50 + \frac{2x}{3} = \frac{3x}{2},$$

$$\text{and } 6x - 300 + 4x = 9x,$$

$$\therefore\ x = 300,$$

and the number required is 900.

67. Let $4x =$ the number of hours,

$\therefore\ 2 : 3 :: 4x :$ the number of miles the first travelled $= 6x$,

and $5x =$ the number the second travelled,

$$\therefore\ 11x = 154,$$

$$\text{and } x = 14,$$

$\therefore$ the number of hours $= 56$.

68. Let $x =$ the number of hours,

$\therefore\ 5 : 7 :: x :$ the number of miles A travels $= \frac{7x}{5}$,

In the same way, $\frac{5 \cdot (x-8)}{3} =$ the number B travels,

$$\text{and } \frac{5 \cdot (x-8)}{3} = \frac{7x}{5},$$

$$\text{and } 25x - 25 \times 8 = 21x,$$

$$\therefore\ 4x = 25 \times 8,$$

$$\text{and } x = 50;$$

and the number of miles $= 70$.

69. Let $6x =$ his annual expenditure,

$\therefore\ 6x + 30 =$ value of his produce in the first year,

$x =$ the amount of assessed taxes,

and $5x =$ the remaining expenses.

He incurred a debt of £20, and the rent in the second year was £40, and taxes $\frac{x}{2}$; and the value of the produce $\frac{4}{3} \cdot (6x + 30) = 8x + 40$.

$$\text{whence } 40 + 20 + 5 + \frac{x}{2} + 5x = 8x + 40,$$

$$\text{or } \frac{5x}{2} = 25,$$

$$\text{and } x = 10.$$

∴ the expenditure was £60 the first year, and £55 the second; the value of the produce was £90 the first year, and £120 the second.

70. Let $x =$ the sum,

$$\therefore \left(\frac{8x}{100} = \right)\frac{2x}{25} = \text{the interest for one year,}$$

$$\text{and } \frac{24x}{25} = \text{the interest for 12 years,}$$

$$\text{Hence } \frac{49x}{25} = \text{the sum put out the second time,}$$

$$\text{and the annual interest} = \frac{2}{25} \cdot \frac{49x}{25},$$

$$\therefore 2 \cdot \frac{49x}{\overline{25}|^2} = \frac{2x}{25} + \frac{192}{5},$$

$$\text{and } \frac{49x}{5 \times 25} = \frac{x}{5} + 96,$$

$$\therefore \frac{24x}{5 \times 25} = 96,$$

$$\text{and } x = 500,$$

and £980 the second time.

71. Let $5x =$ the rate required,

6 : 4 :: 18 : the distance he can row with the tide, per hour, = 12,

$$\therefore 12 - 5x = \text{the distance without the tide,}$$

and 9 : 4 :: 18 : the distance he can row up the stream, per hour, against the tide, = 8,

$\therefore\ 8 + 3x =$ the distance per hour without the tide,

$$\text{and } 8 + 3x = 12 - 5x,$$
$$\therefore\ 8x = 4,$$
$$\text{and } x = \tfrac{1}{2},$$

its rate $\therefore$ is $2\frac{1}{2}$ miles per hour.

72. Let $3x =$ the weight of flour.
$\therefore\ 2x - 5 =$ the weight of rice,
and $x - 1 =$ the weight of water,

$$\therefore\ 6x - 6 = 15,$$
$$\text{and } 6x = 21,$$
$$\therefore\ x = 3\tfrac{1}{2},$$

$\therefore$ there were 2lbs. of rice, $10\frac{1}{2}$ of flour, and $2\frac{1}{2}$ of water.

73. Let $7x =$ the number of times the second was fired,
$\therefore\ 8x + 36 =$ . . . . first . .
and since the quantity of powder consumed is the same,

$$8x + 36 : 7x :: 4 : 3,$$
$$\text{or } 2x + 9 : 7x :: 1 : 3,$$
$$\therefore\ 7x = 6x + 27,$$
$$\text{and } x = 27,$$

$\therefore$ the number is 189.

74. Let $x =$ the number of hours,
$\therefore$ the first makes x revolutions,
and $\frac{60x}{61} =$ the number of revolutions the second makes,

$$\therefore\ x = \frac{60x}{61} + 1,$$
$$\text{and } \therefore\ x = 61.$$

75. Let $3x =$ the number of yards in the first,

$$\therefore\ \frac{23}{x} = \text{the price of a yard;}$$

$$\text{and } 2x - \frac{25}{3} = \text{number in the second,}$$

$$\text{and } \frac{63}{6x - 25} = \text{the price of a yard;}$$

$$\text{whence } \frac{23}{x} = \frac{63}{6x - 25},$$

$$\text{and } 23 \,.\, (6x - 25) = 63x,$$

$$\therefore\ 75x = 23 \times 25,$$

$$\text{and } x = \frac{23}{3},$$

∴ there were 23 yards in the first, and 7 in the second, and the price was £3.

76. Let $5x =$ the number of apples,

∴ $5x - 180 =$ the number of oranges,

and $5 : 3 :: 5x :$ the price of the apples, $= 3x$,

In the same way, the price of 35 apples $= 21d.$;

$$\text{and } \frac{45}{2} \div 15 = \frac{3}{2} = \text{price of an orange;}$$

$$\therefore\ 3x + \frac{3}{2} \,.\, (5x - 180) = 234,$$

$$\text{or } x + \frac{5x}{2} - 90 = 78,$$

$$\therefore\ \frac{7x}{2} = 168,$$

$$\text{and } x = 48,$$

∴ the number was 240; and the oranges were 60, at $1\frac{1}{2}d.$ each.

77. Let $x =$ the fourth,

∴ $20x =$ the fifth,

$4x + 3 =$ the third,

$4x - 2 =$ the second,

$4x - 1 =$ the first,

$$\text{hence } 33x = 198,$$
$$\text{and } x = 6,$$
$$\therefore \text{ the parts are } 23,\ 22,\ 27,\ 6, \text{ and } 120.$$

78. Let $7x =$ the first,

$\therefore\ 3x =$ the second,

$\frac{9x}{4} =$ the third,

and $4x =$ the fourth,

$$\text{hence } 7x = \frac{9x}{4} + 4x + 15,$$
$$\text{and } \frac{3x}{4} = 15,$$
$$\therefore\ x = 20,$$

and the casks hold 140, 60, 45, and 80, gallons.

79. Let $x =$ the rate of sailing in going,
and $x - 6 =$ the rate in returning,
$2x =$ the distance,

$\therefore\ \frac{x}{x-6}$ and $\frac{x}{x-4} =$ the times of going halfway before and after the change of the wind,

$$\therefore \frac{1}{x-6} + \frac{1}{x-4} : \frac{1}{x-6} :: 12 : 7,$$
$$\text{and } \frac{1}{x-4} : \frac{1}{x-6} :: 5 : 7,$$
$$\therefore\ x - 6 : x - 4 :: 5 : 7,$$
$$\text{and } 2 : x - 4 :: 2 : 7,$$
$$\therefore\ x - 4 = 7,$$
$$\text{and } x = 11,$$

$\therefore$ the distance is 22 miles, and the rates in returning are 5 and 7 miles per hour.

80. Let $300 + x =$ the winning party at first,

$\therefore\ 300 - x =$ the losing party,

and $2x =$ the number by which it was lost.

Hence $4x =$ the number by which it was afterwards carried,

and $\left.\begin{matrix}300 + 2x \\ 300 - 3x\end{matrix}\right\}$ are the numbers voting on the $\left\{\begin{matrix}\text{winning} \\ \text{losing}\end{matrix}\right\}$ side.

$$\text{Hence } 300 + 2x : 300 + x :: 8 : 7,$$

$$\therefore\ x : 300 + x :: 1 : 7,$$

$$\text{and } x : 300 :: 1 : 6,$$

$$\therefore\ 6x = 300,$$

$$\text{and } x = 50.$$

$\therefore\ 3x = 150$ the number that changed their mind.

81. Let $5x =$ the cost of a large burner per night,

$4x =$ that of a small one.

He paid $\therefore$ £1 11*s.* for 1 large and 8 small for 52 nights, and 2 small for 52 weeks;

$$\therefore\ 52 \,.\, (5x + 32x) + 52 \,.\, 56x = 31 \times 12,$$

$$\text{or } 52 \times 93x = 12 \times 31,$$

$$\text{and } 13 \times 3x = 3,$$

$$\therefore\ x = \frac{1}{13}.$$

$$\therefore\ \frac{5}{13}d = \text{cost of a large burner per night},$$

$$\text{and } \frac{35}{13} = \text{cost per week},$$

$$\therefore\ \frac{280}{13} = \text{cost of 10 small ones per week},$$

and whole cost per annum $= \dfrac{315}{13} \times 52d. = 1260d. = 5$ guineas.

82. Let $x = CD$,

$$\therefore\ \frac{x}{7} = \text{the time of travelling } CD,$$

$$\text{and } \frac{x}{7} + \frac{3}{20}DE = \frac{DE}{5},$$

$$\text{or DE} = \frac{20x}{7}.$$

$$\text{Now } \frac{\text{CE} - 5}{7} = \frac{\text{DE} - 5}{5},$$

$$\text{or } \left(x + \frac{20x}{7} - 5\right) . 5 = 7 . \left(\frac{20x}{7} - 5\right),$$

$$\therefore x + \frac{20x}{7} - 5 = 4x - 7,$$

$$\text{and } \frac{x}{7} = 2,$$

$x = 14$, the distance from C to D,
and the distance from D to E = 40.

83. Let x = the sum bought into the 4 per cents,

$$\therefore \frac{x}{22} = \text{the interest for one year,}$$

$$\text{and } \frac{2x}{11} = \text{the eldest's fortune;}$$

also $\dfrac{x - 3500}{21}$ = interest for one year in the three per cents,

$$\therefore \frac{x - 3500}{3} = \text{the youngest's fortune,}$$

$$\text{and } \frac{x - 3500}{3} = \frac{2x}{11},$$

$$\therefore 11x - 11 \times 3500 = 6x,$$

$$\text{and } 5x = 11 \times 3500,$$

$\therefore x = 7700$ = sum bought into the 4 per cents,
and 4200 = sum bought into the 3 per cents,
and 1400 = their fortune.

84. Let $6x$ = the length of the course,

then $\dfrac{6x}{440}$ = the distance of the horses at the end of 4 minutes, and $\frac{6x}{1760}$ = the distance gained by A each minute;

$$\therefore\ 20 - 6 \cdot \frac{6x}{1760} = 2,$$

$$\text{and } 18 = 6 \cdot \frac{6x}{1760},$$

$$\therefore\ 2x = 1760,$$

$$x = 880,$$

$\therefore$ the course $= 3 \times 1760$ or 3 miles.

85. Let x = the number dealt to each,

$$\text{then } 3x : x - 1 :: 10 : 3,$$

$$\text{or } 9x = 10x - 10,$$

$$\therefore\ x = 10.$$

86. Let x = the number,

$$\therefore\ x - 50 + \frac{5}{7} \cdot (x - 20) - 30 = \frac{1}{2} \cdot \left\{ x - 20 + 46 + \frac{3}{5} \cdot (x - 50) - 20 \right\},$$

$$\text{or } \frac{1}{7} \cdot (24x - 1320) = \frac{1}{5} \cdot (8x - 120),$$

$$\therefore\ 64x = 5760,$$

$$\text{and } x = 90.$$

87. Let $2x$ = the number killed the first year,

$\therefore\ 2 : 3 :: 2x :$ the number killed in the second, $+ 50$,

$\therefore\ 3x - 50$ = the number killed in the second,

and $\frac{3}{2} \cdot (3x - 50) - 50 = \frac{9x - 250}{2}$ = number killed in the third,

also $\frac{3}{2} \cdot \left(\frac{9x - 250}{2}\right) - 50$ = the number killed in the fourth,

$$\therefore\ \frac{27x - 950}{4} = 6x - 170,$$

$$\text{and } (27x - 24x =)\, 3x = 270,$$
$$\therefore\ x = 90,$$
$$\text{and the number} = 180.$$

88. Let $x =$ the number of balls,

$\therefore\ 72 + \frac{x - 72}{9} =$ the number taken by the first detachment,

$\therefore\ x - 72 - \frac{x - 72}{9} = \frac{8}{9} . (x - 72) =$ the number remaining,

and $144 + \frac{1}{9} . \left\{\frac{8}{9} . (x - 72) - 144\right\} =$ the number taken by the second detachment,

$$\therefore\ 72 + \frac{x - 72}{9} = 144 + \frac{8}{81} . (x - 72) - 16 = 128 + \frac{8}{81} . (x - 72),$$

$$\text{and } \frac{x - 72}{81} = 56,$$

$$\therefore\ x = 72 + 56 \times 81 = 4608,$$

$$\text{and the number of detachments} = \frac{4608}{72 + 504} = \frac{4608}{576} = 8.$$

89. Let $x =$ the price of an original share,

$\therefore\ 5x + 595 =$ the profits of the first speculation,

and $\frac{5x + 595}{15} = \frac{x + 119}{3} =$ each man's share of the gains;

$\therefore\ \frac{x + 119}{3} - 173 = \frac{x - 400}{3} =$ what each man ventured upon the second supposition,

and $\frac{8 . (x - 400)}{3} =$ the whole venture on the steam-boats;

$$\therefore\ \frac{8 . (x - 400)}{3} + \frac{8 . (x + 119)}{3} + 368 = \text{whole loss} = 8 \times 419,$$

$$\text{or } \frac{x - 400 + x + 119}{3} + 46 = 419,$$

$$2x - 281 = 1119,$$
$$\text{and } 2x = 1400,$$
$$\therefore\ x = 700.$$

The price of a share $\therefore$ in the first speculation was £700, in the second £100.

90. Let $5x =$ the number of bags,

$\therefore\ 8x =$ their price, $=$ the sum sent,

and $5x - 18 =$ the number actually bought,

$$\therefore\ \frac{8x}{5x - 18} = \text{the price of a bag,}$$

$$\text{and } \left(\frac{5x}{3} + 5\tfrac{1}{4}\right) \cdot \left(\frac{8x}{5x - 18} - \frac{8}{5}\right) = 10\tfrac{7}{20},$$

$$\text{or } \frac{20x + 63}{12} \times \frac{144}{5 \cdot (5x - 18)} = \frac{207}{20},$$

$$\therefore\ \frac{(20x + 63) \cdot 12}{5x - 18} = \frac{207}{4},$$

$$\text{and } (20x + 63) \cdot 16 = 69 \cdot (5x - 18),$$
$$\text{or } 320x + 1008 = 345x - 1242,$$
$$\therefore\ 25x = 2250,$$
$$\text{and } x = 90,$$

$\therefore$ the quantity purchased $= 5x - 18 = 432$ bags.

91. Let $x =$ what the first had,

$\therefore\ x + 50 =$ . . . second,

$\frac{x + 170}{3} =$ . . . third,

$\frac{x + 170}{6} + 60 =$ . . . fourth,

$$\therefore\ \frac{x + 170}{6} + 60 + 50 = 3 \cdot (x - 50) + 5,$$

$$\text{or } x + 170 + 660 = 18x - 900 + 30,$$

$$\text{and } 17x = 1700,$$
$$\therefore\ x = 100,$$

and they had 100, 150, 90, and 105, respectively.

92. Let $x =$ the number,

$$\text{then } 15 : 4 :: x : \text{the proper weight} = \frac{4x}{15};$$

$$\text{in the same way, } \frac{36}{15} = \text{the proper weight of 9 guineas,}$$

$$\text{and } \frac{4x - 36}{15} = \text{the apparent weight;}$$

also, $x : \dfrac{4x - 36}{15} :: \dfrac{x + 21}{2}$: the apparent weight of $\dfrac{x + 21}{2}$ guineas $= \dfrac{(4x - 36) \cdot (x + 21)}{30x}$,

$$\text{and } 15 : 4 :: \frac{x + 21}{2} : \text{their real weight} = \frac{2 \cdot (x + 21)}{15},$$

$$\therefore \frac{(2x - 18) \cdot (x + 21)}{15x} + \frac{4}{3} = \frac{2 \cdot (x + 21)}{15},$$

$$\therefore \frac{4}{3} = \frac{18}{15x} \cdot (x + 21) = \frac{6}{5x} \cdot (x + 21),$$

$$\text{and } 10x = 9x + 189,$$
$$\therefore\ x = 189.$$

93. Let $8x =$ the number of bushels he bought,

$$\therefore\ 4x = \text{the quantity reserved,}$$
$$\text{and } 3x + 5 = \text{the number first sold,}$$
$$\text{and } x - 5 = \text{the number second sold;}$$

$$\text{also } \frac{200}{8x} = \frac{25}{x} = \text{the buying price of a bushel,}$$

and $(100 : 140 ::)\ 5 : 7 :: \dfrac{25}{x}$: the first selling price of a bushel $= \dfrac{35}{x}$, and (100 : 260 ::) 5 : 13 :: $\frac{25}{x}$: the second selling price $= \frac{65}{x}$;

$$\therefore (3x+5) \cdot \frac{35}{x} + (x-5) \cdot \frac{65}{x} - 100 = 70 - \frac{150}{x} = \text{the gain};$$

$$\text{and } 100 : 67 :: 100 : 70 - \frac{150}{x},$$

$$\therefore 3x = 150,$$

$$\text{and } x = 50,$$

and the number of bushels = 400,

also the first selling price $= \frac{35}{50} = 14s.$

and the second $= \frac{65}{50} = 26s.$

94. Let $x =$ the number of gallons of strong beer,

and as £12½ = the duty on the ale, and £50 what he sells the ale for, ∴ £17½ = the gain,

$\frac{x}{40} =$ the duty on the strong,

$\frac{500-x}{160} =$ the duty paid on the small,

and $20 + \frac{500+3x}{160} =$ sum expended in the second case,

and $50 - 20 - \frac{500+3x}{160} =$ second gain;

$$\text{also } 17\tfrac{1}{2} : 30 - \frac{500+3x}{160} :: 7 : 10,$$

$$\therefore \frac{5}{2} : 30 - \frac{500+3x}{160} :: 1 : 10,$$

$$\text{and } 25 = 30 - \frac{500+3x}{160},$$

$$\therefore 500 + 3x = 800,$$

$$\text{and } 3x = 300,$$

$$\therefore x = 100.$$

SECTION VII.

Problems producing Simple Equations involving two unknown Quantities.

1. Let x and y be the lengths,

$$\therefore\ 8x \text{ and } 9y = \text{what they cost,}$$
$$\text{or } 8x + 9y = 253,$$
$$\text{but } 2x + 2y = 60,\ \therefore\ 8x + 8y = 240,$$
$$\text{and } \therefore \text{ by subtraction, } y = 13,$$
$$\text{whence } x = 30 - y = 17.$$

2. Let $2x$ = the number of apples,
and $3y$ = the number of pears,

$$\therefore\ \frac{1}{2}x = \text{the price of apples,}$$
$$\text{and } \frac{3}{5}y = \text{the price of pears,}$$
$$\text{and } \frac{x}{4} + \frac{y}{5} = 13,$$
$$\text{and } \therefore\ \frac{x}{2} + \frac{2y}{5} = 26,$$
$$\text{but } \frac{x}{2} + \frac{3y}{5} = 30,$$
$$\therefore \text{ by subtraction, } \frac{y}{5} = 4,$$
$$\text{and } y = 20.$$
$$\therefore\ \frac{x}{2} = 18, \text{ and } x = 36.$$

3. Let $2x =$ the number of half guineas,
and $y =$ the number of crowns,

$$\therefore\ 21x + 5y = 525,$$
$$\text{and}\ \ 4x - 3y = 17,$$

multiplying the former by 3, and the latter by 5,

$$63x + 15y = 1575,$$
$$20x - 15y = 85,$$
$$\therefore \text{ by addition, } 83x = 1660,$$
$$\text{and } x = 20,$$
$$\therefore\ 3y = 63, \text{ and } y = 21.$$

4. Let $x =$ A's, $y =$ B's } daily wages,

$$\therefore\ 15x + 14y = 117,$$
$$\text{and } 4x - 3y = 11,$$

multiplying the former by 3, and the latter by 14,

$$45x + 42y = 351,$$
$$56x - 42y = 154,$$
$$\therefore \text{ by addition, } 101x = 505,$$
$$\text{and } x = 5;$$
$$\therefore\ 3y = 9, \text{ and } y = 3.$$

5. Let $x =$ number of gallons held by the larger,
and $y =$ number held by the less,

$$\therefore\ 5x + 22y = 332,$$
$$\text{and } 22x + 5y = 451,$$
$$\therefore \text{ by addition, } 27x + 27y = 783,$$
$$\text{and } x + y = 29,$$
$$\therefore\ 5x + 5y = 145,$$
$$\text{but } 5x + 22y = 332,$$
$$\therefore \text{ by subtraction, } 17y = 187,$$
$$\text{and } y = 11;$$
$$\therefore\ x = 18.$$

6. Let $2x =$ what A had,
and $y =$ what B had,

$$\therefore\ 3x + 15 = 3 . (y - x),$$
$$\text{whence } 2x + 5 = y;$$
$$\text{also } 2 . (3x - 10) = y - x + 10,$$
$$\text{and } \therefore\ 6x - 20 = 2x + 5 - x + 10,$$
$$\text{or } 5x = 35,$$
$$\therefore\ x = 7;$$
$$\text{and } y = 19;$$

$\therefore$ A had 14, and B 19 shillings.

7. Let $x =$ the sum,
and $y =$ the rate,

$$\therefore\ \frac{xy}{100} = \text{the interest for one year,}$$
$$\text{and } \frac{xy}{150} = \text{the interest for 8 months,}$$
$$\text{and } \frac{xy}{80} = \text{the interest for 15 months,}$$
$$\therefore\ \frac{xy}{150} + x = 297\tfrac{3}{5},$$
$$\text{and } \frac{xy}{80} + x = 306,$$
$$\text{whence } (297\tfrac{3}{5} - x) . 150 = (306 - x) . 80,$$
$$\text{or } 4464 - 15x = 2448 - 8x,$$
$$\therefore\ 7x = 2016,$$
$$\text{and } x = 288,$$
$$\therefore\ \frac{288y}{80} = 18, \text{ or } 16y = 80, \text{ and } \therefore\ y = 5.$$

8. Let $x =$ the number of quarters,
and $y =$ the price of one,

$$\therefore\ (x + 8) . (y + 7) = xy + 235,$$
$$\text{and } (x + 7) . (y + 8) = xy + 237,$$

$$\text{or } 7x + 8y = 179,$$
$$\text{and } 8x + 7y = 181,$$
$$\text{hence } 56x + 64y = 1432,$$
$$\text{and } 56x + 49y = 1267,$$
$$\therefore \text{ by subtraction, } \quad 15y = 165,$$
$$\text{and } y = 11;$$
$$\therefore 7x = 179 - 88 = 91,$$
$$\text{and } x = 13.$$

9. Let $10x + y$ be the number,

$$\therefore \frac{10x + y}{x + y} = 4,$$
$$\text{or } 10x + y = 4x + 4y,$$
$$\therefore 2x = y;$$
$$\text{also, } \frac{10y + x}{y - x + 2} = 14,$$
$$\therefore 10y + x = 14y - 14x + 28,$$
$$\text{or } 15x - 4y = 28,$$
$$\therefore (15x - 8x =)\ 7x = 28,$$
$$\text{and } x = 4,$$
$$\therefore y = 8,$$

and the number is 48.

10. Let $\frac{x}{y}$ be the fraction,

$$\therefore \frac{2x}{y + 7} = \frac{2}{3},$$
$$\text{and } \therefore 3x = y + 7;$$
$$\text{also, } \frac{x + 2}{2y} = \frac{3}{5},$$
$$\therefore 5x + 10 = 6y = 18x - 42,$$
$$\text{whence } 13x = 52,$$

$$\text{and } x = 4;$$
$$\therefore\ y = 5,$$
$$\text{and the fraction is } \frac{4}{5}.$$

11. Let $x =$ the price of a horse,
and $y =$ that of a cow;

$$\therefore\ 9x + 7y = 300 = 6x + 13y,$$
$$\text{or } 3x = 6y,$$
$$\text{and } x = 2y;$$
$$\text{whence } (12y + 13y =)\ 25y = 300,$$
$$\text{and } y = 12,$$
$$\therefore\ x = 24.$$

12. Let $x =$ the number of acres of arable,
and $y =$ the number of pasture,

$$\therefore\ x : \frac{1}{2}.(x - y) :: 28 : 9,$$
$$\text{or } x : x - y :: 14 : 9,$$
$$\therefore\ x : y :: 14 : 5,$$
$$\text{and } 5x = 14y;$$
$$\text{but } 2x + \frac{7}{5}y = 245,$$
$$\text{or } 2x + \frac{x}{2} = 245;$$
$$\therefore\ x = 98,$$
$$\text{and } y = 35.$$

13. Let $11x$ and $6y =$ the debts,

$$\therefore\ 4x + y + 3 = 53,$$
$$\text{and } 3x + \frac{5y}{3} - 1 = 42,$$

$$\begin{aligned} \text{or } 20x + 5y &= 250, \\ \text{and } 9x + 5y &= 129, \\ \therefore \text{ by subtraction, } 11x &= 121, \\ \text{whence } y = 50 - 4x &= 6, \\ \text{and } 6y &= 36. \end{aligned}$$

14. Let $x =$ the number A won,
and $y =$ the number B won;

$$\begin{aligned} &\therefore\ 3y - 2x = 17, \\ &\text{and } x + 3 : y - 3 :: 5 : 4, \\ &\therefore\ 4x + 12 = 5y - 15, \\ &\text{and } 5y - 4x = 27, \\ &\text{but } 6y - 4x = 34, \\ &\therefore \text{ by subtraction, } y = 7; \\ &\text{whence } x + 3 = 5, \text{ and } \therefore\ x = 2. \end{aligned}$$

15. Let x and $y =$ the number of gallons each holds,

$$\begin{aligned} &\therefore\ x - 15 : y - 11 :: 8 : 3, \\ &\text{and } 3x - 45 = 8y - 88, \\ &\text{or } 3x = 8y - 43; \\ &\text{also, } \frac{x}{2} + 10 : \frac{y}{2} + 10 :: 9 : 5, \\ &\therefore\ 5x + 100 = 9y + 180, \\ &\text{and } 5x = 9y + 80; \\ &\text{whence } 40y - 215 = 27y + 240, \\ &\text{or } 13y = 455, \\ &\therefore\ y = 35; \\ &\text{and } x = 79. \end{aligned}$$

16. Let x and $y =$ the sides,

$$\begin{aligned} &\therefore\ x + 4 : y + 4 :: 5 : 4, \\ &\text{and } x - 4 : y - 4 :: 4 : 3, \\ &\text{hence } x + 4 : x - y :: 5 : 1, \\ &\text{and } x - y : x - 4 :: 1 : 4, \\ &\therefore \text{ ex æquali, } x + 4 : x - 4 :: 5 : 4, \end{aligned}$$

$$\text{and } x : 4 :: 9 : 1,$$
$$\therefore\ x = 36;$$
$$\text{and } 36 - y : 32 :: 1 : 4,$$
$$\therefore\ 36 - y = 8,$$
$$\text{and } y = 28.$$

17. Let $9x$, $8x$, and y, be the numbers,

$$\therefore\ 8x - y : x + 7 :: 7 : 12,$$
$$\text{and } 96x - 12y = 7x + 49,$$
$$\text{or } 89x - 12y = 49;$$
$$\text{also, } \frac{9x + y + 1}{2} = 8x + 7,$$
$$\text{or } 9x + y + 1 = 16x + 14,$$
$$\therefore\ 7x - y = -13,$$
$$\text{and } 84x - 12y = -156,$$
$$\text{but } 89x - 12y = 49,$$
$$\therefore \text{ by subtraction, } 5x = 205,$$
$$\text{and } x = 41,$$
$$\therefore\ y = 7x + 13 = 300;$$

and the numbers are 369, 328, 300.

18 Let $2x - 6$, $3x - 6$, and y, be the numbers,

$$\therefore\ 2x - 1 : y + 5 :: 7 : 11,$$
$$\text{and } 22x - 11 = 7y + 35,$$
$$\text{or } 22x - 7y = 46;$$
$$\text{also, } 3x - 42 : y - 36 :: 6 : 7,$$
$$\text{and } 7x - 98 = 2y - 72,$$
$$\text{or } 7x - 2y = 26;$$
$$\text{whence } 49x - 14y = 182,$$
$$\text{but } 44x - 14y = 92,$$
$$\therefore \text{ by subtraction, } 5x = 90,$$
$$\text{and } x = 18;$$
$$\therefore\ 2y = 7x - 26 = 100,$$
$$\text{and } y = 50;$$

∴ the numbers are 30, 48, 50.

19. Let x and y be the numbers,

$$\therefore\ x + 80 : y - 20 :: 8 : 3,$$

$$\text{and } 3x + 240 = 8y - 160,$$

$$\text{or } 3x - 8y = -400;$$

$$\text{also } x - 20 : y + 90 :: 7 : 10,$$

$$\therefore\ 10x - 200 = 7y + 630,$$

$$\text{or } 10x - 7y = 830;$$

$$\therefore\ 30x - 21y = 2490,$$

$$\text{but } 30x - 80y = -4000,$$

$$\therefore \text{ by subtraction, } 59y = 6490,$$

$$\text{and } y = 110;$$

$$\therefore\ 3x = 480,$$

$$\text{and } x = 160.$$

20. Let x and y be the number of days,

$$\therefore\ \frac{16}{x} + \frac{16}{y} = 1;$$

$$\text{and } \frac{4}{x} + \frac{4}{y} + \frac{36}{y} = 1,$$

$$\text{or } \frac{4}{x} + \frac{40}{y} = 1,$$

$$\text{but } \frac{4}{x} + \frac{4}{y} = \frac{1}{4},$$

$$\therefore \text{ by subtraction, } \frac{36}{y} = \frac{3}{4},$$

$$\text{and } y = 48;$$

$$\therefore\ \frac{16}{x} = \frac{2}{3}; \text{ and } x = 24.$$

21. Let x, x, and y, be the numbers,

$$\therefore\ \frac{8}{x} + \frac{4}{y} = \frac{5}{12};$$

$$\text{and } \frac{32}{3x} + \frac{32}{3y} = \frac{7}{9},$$

$$\therefore \frac{8}{x} + \frac{8}{y} = \frac{7}{12},$$

$$\text{but } \frac{8}{x} + \frac{4}{y} = \frac{5}{12},$$

$$\therefore \text{ by subtraction, } \quad \frac{4}{y} = \frac{1}{6},$$

$$\text{and } y = 24,$$

$$\therefore \frac{8}{x} = \frac{1}{4}, \text{ and } x = 32.$$

22. Let $x =$ the number the first goes,
and $y =$ the number the second goes,

$$\therefore \frac{17}{x} + \frac{56}{y} = \frac{41}{3};$$

$$\text{and } \frac{147}{x} - \frac{147}{y} = 28,$$

$$\text{or } \frac{21}{x} - \frac{21}{y} = 4;$$

$$\text{whence } \frac{168}{x} - \frac{168}{y} = 32,$$

$$\text{but } \frac{51}{x} + \frac{168}{y} = 41,$$

$$\therefore \text{ by addition, } \quad \frac{219}{x} = 73,$$

$$\text{and } \frac{3}{x} = 1,$$

$$\therefore x = 3;$$

$$\text{and } 3 = \frac{21}{y},$$

$$\therefore y = 7.$$

23. Let $4x$ and $5x$ = their weights,

and $6y$ and $7y$ = the parts taken out;

$$\therefore\ 4x - 6y : 5x - 7y :: 2 : 3,$$

$$\text{and } 2x - 3y : x - y :: 1 : 1,$$

$$\therefore\ x = 2y;$$

$$\text{also } 9x - 13y = 10,$$

$$\therefore\ (18y - 13y =)\ 5y = 10,$$

$$\text{and } y = 2,$$

$$\therefore\ x = 4.$$

And their weights were 16 and 20 tons.

24. Let x = the number of shillings one man received,

and y = the number one woman received,

$$\therefore\ 14x + 15y = \text{the whole sum},$$

$$\text{and } \frac{7x + 15y - 12}{15} = x + 2,$$

$$\therefore\ 15y - 8x = 42;$$

$$\text{also, } \frac{14x + 7y}{14} = 2y,$$

$$\text{or } 2x + y = 4y,$$

$$\therefore\ 2x = 3y;$$

$$\text{whence } (15y - 12y =)\ 3y = 42,$$

$$\text{and } y = 14,$$

$$\therefore\ x = 21.$$

25. Let x = the number of bushels of wheat he must buy,

and y = the number of bushels of rye,

$\therefore\ 5x + 3y$ = his money;

$$\text{and } \frac{5x + 3y - 21}{5} + 7 = x + y - 2,$$

$$\text{or } 5x + 3y - 21 + 35 = 5x + 5y - 10,$$

$$\text{or } 24 = 2y,$$

$$\therefore\ y = 12;$$

$$\text{also } 30 + 3 \,.\, (x + y - 6) = 5x + 3y - 6,$$
$$\text{or } 2x = 18,$$
$$\therefore\ x = 9.$$

26. Let $x =$ the number of outside places,
$y =$ the fare inside;
then $6y$ and $13x =$ the whole fares, inside and outside,

$$\text{and } \frac{13x}{3} - \frac{6y}{5} = 21\tfrac{11}{15},$$
$$\therefore\ 65x - 18y = 326;$$
$$\text{also } 4 \,.\, (y - 5) + \frac{13}{2} \,.\, (x - 3) = 6y + 13x - 140\tfrac{1}{2},$$
$$\text{whence } 4y + 13x = 202,$$
$$\text{and } \therefore\ 117x + 36y = 1818,$$
$$\text{but } \quad 130x - 36y = 652,$$
$$\therefore \text{ by addition,} \quad 247x = 2470,$$
$$\text{and } x = 10;$$
$$\text{hence } 650 - 18y = 326,$$
$$\text{and } 18y = 324,$$
$$\therefore\ y = 18.$$

27. Let $x =$ the number of yards of the finer,
and $y =$ the number of the coarser;
$\therefore\ y =$ the price of a yard of the coarser,
and $\frac{6 . (y+2)}{5} =$ price of a yard of the finer;

$$\therefore\ \frac{6x \,.\, (y + 2)}{5} + y \,.\, (x + 6) = 744;$$
$$\text{also } \frac{24 \,.\, (y + 2)}{5} + 12y : 744 :: 20 : 31,$$
$$\text{or } \frac{24 \,.\, (y + 2)}{5} + 12y : 24 :: 20 : 1,$$

$$\therefore \frac{2y+4}{5} + y = 40,$$

$$\text{or } 7y = 196,$$

$$\therefore y = 28;$$

$$\text{whence } 36x + 28 . (x+6) = 744,$$

$$\text{and } 64x = 576,$$

$$\therefore x = 9.$$

28. Let $x =$ the number contained in the shorter,
and $y =$ the number contained in the larger;

$$\therefore \frac{25}{x} \text{ and } \frac{25}{y} = \text{the price of a yard of each};$$

$$\text{and } \frac{50}{x} - \frac{75}{y} = \frac{1}{3};$$

$$\text{also } \frac{30}{x} . (x-2) + \frac{30}{y} . (y-2) = 53\tfrac{3}{5},$$

$$\therefore 60 - \frac{60}{x} - \frac{60}{y} = 53\tfrac{3}{5},$$

$$\text{whence } \frac{10}{x} + \frac{10}{y} = \frac{16}{15},$$

$$\text{but } \frac{10}{x} - \frac{15}{y} = \frac{1}{15},$$

$$\therefore \text{by subtraction,} \quad \frac{25}{y} = 1,$$

$$\text{and } y = 25;$$

$$\therefore x = 15.$$

29. Let $x =$ the number of miles A went per hour,
$\therefore x + 2 =$ the number B went;
let $y =$ the number of hours B travelled,
$\therefore y + 3 =$ the number A travelled;

$$\text{And } x . (y+3) : y . (x+2) :: 13 : 15,$$

$$\therefore\ 15xy + 45x = 13xy + 26y,$$
$$\text{or } 2xy = 26y - 45x;$$
$$\text{again } x\,.\,(y-2) : y\,.\,(x+4) :: 2 : 5,$$
$$\therefore\ 5xy - 10x = 2xy + 8y,$$
$$\text{or } 3xy = 10x + 8y;$$
$$\text{whence } 20x + 16y = 78y - 135x,$$
$$\text{or } 155x = 62y,$$
$$\text{and } 5x = 2y,$$
$$\therefore\ 3xy = 4y + 8y = 12y,$$
$$\text{and } x = 4,$$
$$\therefore\ y = 10.$$

30. Let $4x =$ the first revenue,

$\therefore\ 9x =$ the increased revenue;

hence the expenses of collecting being in the ratio of $\sqrt{4x} : \sqrt{9x}$, or as $2 : 3$, $2y$ and $3y$ may $=$ those expenses.

Let $z =$ the interest of the national debt,

$\therefore$ the available incomes are $4x - 2y - z$, and $9x - 3y - z$;

$$\text{hence } 4x - 2y - z : 9x - 3y - z :: 23 : 81,$$
$$\text{and } 4x - 2y - z : 5x - y :: 23 : 58.$$

Again, $\frac{9x}{4} =$ the reduced income,

and $\frac{3y}{2} =$ the expense of collecting,

$$\text{and } \frac{9x}{4} - \frac{3y}{2} - z = 4;$$
$$\text{also, } \frac{9x}{4} - \frac{3y}{2} - z : 4x - 2y - z :: 3 : 23,$$
$$\text{and } \frac{7x}{4} - \frac{y}{2} : 4x - 2y - z :: 20 : 23,$$
$$\text{but } 4x - 2y - z : 5x - y :: 23 : 58,$$

$$\therefore \frac{7x}{4} - \frac{y}{2} : 5x - y :: 20 : 58 :: 10 : 29,$$

$$\text{hence } 29 . \left(\frac{7x}{4} - \frac{y}{2}\right) = 10 . (5x - y),$$

$$\text{or } 203x - 58y = 200x - 40y,$$

$$\text{and } 3x = 18y,$$

$$\text{or } x = 6y;$$

$$\text{also } \left(\frac{9x}{4} - \frac{3y}{2} - z =\right) 4 : \frac{7x}{4} - \frac{y}{2} :: 3 : 20,$$

$$\therefore \frac{7x}{4} - \frac{y}{2} = \frac{80}{3},$$

$$\text{or } \frac{21y}{2} - \frac{y}{2} = \frac{80}{3},$$

$$\therefore y = \frac{8}{3},$$

and $x = 16$ millions.

The first revenue $\therefore$ was 64, and the increased 144 millions;

also $36 - 4 - z = 4$,

$\therefore z = 28$ millions, the interest of the national debt.

31. Let $3x =$ A's daily work,

$\therefore 2x =$ B's,

let $z =$ C's,

and $y =$ the number of days C worked,

$\therefore (3x + 2x) . 12 + yz =$ whole quantity of work done;

and $(3x + 2x + z) . 9 =$ the same;

$$\therefore (3x + 2x) . 12 + yz = (3x + 2x + z) . 9,$$

$$\text{and } 15x + yz = 9z;$$

$$\text{now } 3x + z :: 2x + z : 8 : 7,$$

$$\text{or } x : 2x + z :: 1 : 7,$$

$$\therefore 7x = 2x + z,$$

$$\text{and } 5x = z,$$

$$\therefore 3z + yz = 9z,$$

$$\text{or } y + 3 = 9,$$
$$\therefore\ y = 6;$$
and C was called in after six days.

32. Let $4x$ and $3x$ be the quantities of brandy,

$\therefore$ ($12x : 6x$ ::) $2 : 1$, the ratio of the quantities of sherry;

let $\therefore$ $2y$ and y be the quantities of sherry;

$$\therefore\ y = x + 25;$$
$$\text{also } 4x + y : 3x + 2y :: 5 : 6,$$
$$\text{and } 24x + 6y = 15x + 10y,$$
$$\therefore\ 9x = 4y = 4x + 100,$$
$$\therefore\ 5x = 100,$$
$$\text{and } x = 20;$$

$\therefore$ the quantities of brandy are 80 and 60, and of sherry 90 and 45.

33. Let x = value of a bushel of coals in pence, and y = value of a basket of turf.

Now in the first case, A consumes $\frac{2}{3}$ of coals,

$\therefore\ \frac{2}{3} \cdot 5x$ = value of coals consumed by him,

and $\frac{7y}{2}$ = value of turf consumed by him;

$$\therefore\ \frac{10x}{3} + \frac{7y}{2} = 3x + 34.$$

In the second case, B consumes $\frac{3}{4}$ of coals,

$\therefore\ \frac{3}{4} \cdot 6x$ = value of coals consumed by him,

$$\text{and } \frac{3}{4} \cdot 6x + \frac{6y}{2} = x + 222,$$

$$\therefore\ 7x + 6y = 444,$$

$$\text{but } 7x + \frac{147}{2} \cdot y = 714,$$

$$\therefore\ \frac{135}{2} \cdot y = 270,$$

$$\text{and } y = 4;$$

$$\therefore\ x = 60.$$

34. Let $x =$ the price of a loaf in pence,
and $y =$ the price of a bottle of wine.
$\therefore\ 3x =$ the price of A's loaves,
and $2x =$ the price of B's,

$$\text{and } 12x + 4 = 6x + 4y,$$

$$\therefore\ y = \frac{3x + 2}{2}.$$

They all ate equal portions; $\therefore$ each ate $\frac{1}{3}$ of 5 loaves. A $\therefore$ ate $\frac{5}{3}$ and gave $\frac{4}{3}$ to the stranger; and B having 2 loaves, gave $\frac{1}{3}$ to the stranger. But B had a bottle of wine, $\frac{1}{3}$ of which he gave to the stranger.

Hence $\frac{4x - y}{3}$ is the price of the provisions A furnished to the stranger:

$$\text{and } \frac{x + 2y}{3} = \text{the price of what B furnished.}$$

$$\text{Now A receives } \frac{55}{2} \text{ pence, and B 50 pence,}$$

$$\therefore\ \frac{4x - y}{3} : \frac{x + 2y}{3} :: \frac{55}{2} : 50,$$

$$\text{or } 4x - y : x + 2y :: 11 : 20,$$

$$\therefore\ 3y - 3x :: x + 2y :: 9 : 20,$$

$$\text{or } y - x : x + 2y :: 3 : 20,$$

$$\therefore\ 20y - 20x = 3x + 6y,$$

$$\text{and } 14y = 23x,$$

$$\therefore\ y = \frac{23x}{14}.$$

$$\text{Hence } \frac{23x}{14} = \frac{3x+2}{2},$$

$$\text{and } 46x = 42x + 28,$$

$$\therefore\ 4x = 28,$$

$$\text{and } x = 7;$$

$$\therefore\ y = 11\tfrac{1}{2}.$$

SECTION VIII.

Problems producing Pure Equations.

1. Let $5x$ and $8x$ be the numbers,

$$\therefore\ 40x^2 = 360,$$

$$x^2 = 9,$$

$$\text{and } x = \pm 3;$$

$\therefore$ the numbers are ± 15 and ± 24.

2. Let $8x$ = their sum,

$\therefore\ x$ = their difference,

whence $\frac{9x}{2}$ = the greater, and $\frac{7x}{2}$ = the less.

$$\therefore\ \frac{1}{4}.(81x^2 - 49x^2) = \frac{1}{4}.32x^2 = 8x^2 = 128,$$

$$\text{and } x^2 = 16,$$

$$\therefore\ x = \pm 4;$$

and the numbers are ± 18 and ± 14.

3. Let x = a side of the one,

$\therefore\ x + 10$ = a side of the other,

$$\text{and } (x + 10)^2 : x^2 :: 25 : 9,$$
$$\therefore x + 10 : x :: 5 : 3,$$
$$\text{and } 10 : x :: 2 : 3,$$
$$\therefore x = 15,$$
and the sides are 15 and 25.

4. Let x and $36 - x$ be the lengths,

$$\therefore x^2 : (36 - x)^2 :: 4 : 1,$$
$$\text{and } x : 36 - x :: 2 : 1,$$
$$\text{and } x : 36 :: 2 : 3,$$
$$\therefore x = 24,$$
and the lengths are 24 and 12.

5. Let $2x =$ the less,

$\therefore 3x =$ the greater,

$$\text{and } x \,.\, 5x^2 = 135,$$
$$\therefore x^3 = 27,$$
$$\text{and } x = 3;$$
$\therefore$ the numbers are 9 and 6.

6. Let $3x$ and $2x$ be the numbers,

$$\therefore 81x^4 - 16x^4 : 27x^3 + 8x^3 :: 26 : 7,$$
$$\text{or } 65x : 35 :: 26 : 7,$$
$$\therefore 5x : 5 :: 2 : 1,$$
$$\text{and } x = 2;$$
$\therefore$ the numbers are 6 and 4.

7. Let $6x$ and $5x =$ the sides,

$$\therefore \text{the area} = 30x^2,$$
$$\text{and } 25x^2 = 625,$$
$$\therefore 5x = 25,$$
$$\text{and } x = 5,$$
$\therefore$ the sides are 30 and 25.

8. Let $x =$ the number,

$x^2 =$ the number of servants,

and $2x^2 =$ the number of pounds each took,

$$\therefore 2x^3 = 3456,$$

$$\text{and } x^3 = 1728,$$

$$\therefore x = 12.$$

9. Let x and $49 - x$ be the numbers,

$$\therefore \frac{x}{49 - x} : \frac{49 - x}{x} :: \frac{4}{3} : \frac{3}{4},$$

$$\text{and } x^2 : (49 - x)^2 :: 16 : 9,$$

$$\therefore x : 49 - x :: 4 : 3,$$

$$\text{and } x : 49 :: 4 : 7,$$

$$\therefore x = 28,$$

and the parts are 28 and 21.

10. Let $x =$ the number,

$\therefore 4x^2 =$ the number first furnished,

$$\text{and } 4x^2 : 3x :: 16 : 1,$$

$$\therefore x : 3 :: 4 : 1,$$

$$\text{and } x = 12.$$

11. Let $4x =$ the number of men,

$\therefore 5x =$ the number of women,

also $3x =$ the sum each man received,

and $2x =$ the sum each woman received;

$$\therefore 12x^2 = 18 + 10x^2,$$

$$\text{and } 2x^2 = 18,$$

$$\therefore x^2 = 9,$$

$$\text{and } x = 3;$$

the numbers $\therefore$ were 12 and 15.

12. Let $3x =$ the number at the stables,

$\therefore 7x =$ the number at home;

and $6x =$ the number of shillings one at home cost,

$\therefore \frac{15x}{2} =$ the price of one at the stables ;

$$\text{and } 3x \times \frac{15x}{2} = 90,$$

$$\therefore 45x^2 = 180,$$

$$\text{and } x^2 = 4,$$

$$\therefore x = 2,$$

$\therefore$ there were 6 in the stables and 14 at home.

13. Let $x =$ the number of bargemen,

$\therefore x^2 =$ the number of gentlemen,

$(x + 1)^2 =$ the number of ladies,

$x + 1 =$ the number of turtles,

$\therefore (x + 1)^3 - 361 =$ the number of bottles of wine,

$$\text{and } \frac{(x + 1)^3 - 361}{2 \cdot (x + 1)^2} = \frac{x}{2},$$

$$\text{or, } x^3 + 3x^2 + 3x + 1 = x^3 + 2x^2 + x + 361,$$

$$\text{and } x^2 + 2x + 1 = 361,$$

$\therefore x + 1 = 19$, the number required.

14. Let $5x =$ the number of miles A travels,

$\therefore 3x + 35 =$ the number B travels;

and $3x + 35 : 5x :: 20\frac{5}{6} :$ the number of hours A has travelled $= \frac{625x}{6 \cdot (3x + 35)}$;

In the same way, the number B has travelled $= \frac{6 \cdot (3x + 35)}{x}$,

$$\therefore \frac{625x}{6 \cdot (3x + 35)} = \frac{6 \cdot (3x + 35)}{x},$$

$$\text{and } 625x^2 = 36 \cdot (3x + 35)^2,$$

$$\therefore 25x = 6 \cdot (3x + 35),$$

$$\text{and } 7x = 6 \times 35,$$
$$\therefore\ x = 30;$$
and the distance is 235 miles.

15. Let x and $x + 18$ be the numbers,

then $x : x + 18 :: 6 :$ price of 6 of the first flock $= \frac{6 . (x + 18)}{x}$,

$$\text{and the price of 7 of the second} = \frac{7x}{x + 18},$$
$$\therefore\ \frac{6 . (x + 18)}{x} : \frac{7x}{x + 18} :: 7 : 6,$$
$$\text{and } 36 . (x + 18)^2 = 49x^2,$$
$$\therefore\ 6 . (x + 18) = 7x,$$
$$\text{and } x = 108;$$
$\therefore$ the numbers are 108 and 126.

16. Let $x =$ the number of turkies,

$\therefore\ x + 8 =$ the number of ducks,

and $\frac{1}{2}x . (x + 8) =$ the prices of each set:

$$\therefore\ x^2 + 8x + 16 = 4 . (x - 4)^2,$$
$$\text{and } x + 4 = 2 . (x - 4),$$
$$\therefore\ x = 12;$$
and the numbers were 12 and 20.

17. Let $9x =$ A's stock,

and $8x =$ B's stock,

$\therefore\ 3x =$ A's gain,

and $6x =$ the number of years;

also $9x : 3x :: 8x :$ B's gain was $= \frac{8x}{3}$,

$$\therefore\ 6x . \left(3x + \frac{8x}{3}\right) = 1666,$$

$$\text{or } 34x^2 = 1666,$$
$$\therefore\ x^2 = 49,$$
$$\text{and } x = 7,$$

$\therefore$ A contributed £63, B £56; and the number of years is 42.

18. Let $x =$ the longer side of the parallelogram,

$$\therefore\ \sqrt{x^2 + 3600} = \text{the diagonal},$$
$$\text{and } 60x = \text{the area of the parallelogram},$$
$$\text{and } 30\sqrt{x^2 + 3600} = \text{the area of the triangle};$$
$$\therefore\ 60x : 30\sqrt{x^2 + 3600} :: 8 : 5,$$
$$\text{or } x : \sqrt{x^2 + 3600} :: 4 : 5,$$
$$\therefore\ x^2 : x^2 + 3600 :: 16 : 25,$$
$$\text{and } x^2 :: 3600 :: 16 : 9,$$
$$\therefore\ x : 60 :: 4 : 3,$$
$$\text{and } x : 20 :: 4 : 1,$$
$$\therefore\ x = 80,$$
$$\text{and the area} = 60 \times 80 + 30\sqrt{6400 + 3600} = 7800,$$

19. Let $x =$ the sum,

$\therefore\ x + 69 =$ the stock at the beginning of the second year,

$$\text{and } x : 69 :: x + 69 : \text{the gain the second year} = \frac{69 . (x + 69)}{x},$$

$$\therefore \text{ the stock the third year} = (x + 69) + \frac{69}{x} . (x + 69) = \frac{(x+69)^2}{x},$$

$$\text{and the gain the third year} = \frac{69}{x} . \frac{(x + 69)^2}{x};$$

$$\text{hence the stock at the beginning of the 5th year} = \frac{(x + 69)^4}{x^3},$$

$$\therefore\ \frac{(x + 69)^4}{x^3} : x :: 81 : 16,$$
$$\text{and } x + 69 : x :: 3 : 2,$$
$$\therefore\ 69 : x :: 1 : 2,$$
$$\text{and } x = 138.$$

20. Let $10x + y$ be the number,

$$\therefore\ 10x^2 + xy = 46,$$
$$\text{and}\quad x^2 + xy = 10,$$
$$\therefore \text{ by subtraction, } 9x^2 = 36,$$
$$\text{and } 3x = 6,$$
$$\therefore\ x = 2;$$
$$\text{whence } 2y = 6, \text{ and } y = 3;$$
$$\therefore \text{ the number is } 23.$$

21. Let x = the number A went,
and y = the number B went;

$$\therefore\ x^2 - xy = 216.$$
$$\text{and } xy - y^2 = 180,$$
$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 36,$$
$$\text{and } x - y = 6,$$
$$\therefore\ x = 36, \text{ and } y = 30.$$

22. Let x and y be the numbers,

$$\therefore\ x + y : 40 :: x : y,$$
$$\text{and } y : x + y :: x : 90,$$
$$\therefore \text{ ex æquali, } y : 40 :: x^2 : 90y,$$
$$\text{and } 4x^2 = 9y^2,$$
$$\therefore\ 2x = 3y;$$
$$\text{and } x + \frac{2}{3}x = \frac{40x}{\frac{2}{3}x},$$
$$\text{or } \frac{5x}{3} = 60,$$
$$\therefore\ x = 36;$$
$$\text{and } y = 24.$$

23. Let x and y = the numbers,

$$\therefore\ xy^3 : yx^3 :: 4 : 9,$$

$$\text{or } y^2 : x^2 :: 4 : 9,$$
$$\therefore y : x :: 2 : 3,$$
$$\text{or } y = \frac{2}{3}x;$$
$$\text{also } x^3 + y^3 = 35,$$
$$\text{or } x^3 + \frac{8}{27} . x^3 = 35,$$
$$\therefore 35x^3 = 27 \times 35,$$
$$\text{and } x^3 = 27,$$
$$\therefore x = 3,$$
$$\text{and } y = 2.$$

24. Let x and y = the lengths,
$$\therefore x^2y + xy^2 = 205 \times 20 \times 4,$$
$$\text{and } x + y = 41;$$
$$\therefore 41\,xy = 16400,$$
$$\text{and } xy = 400;$$
$$\text{hence } x - y = 9,$$
$$\text{but } x + y = 41,$$
$$\therefore 2x = 50, \text{ and } x = 25;$$
$$2y = 32, \text{ and } y = 16.$$

25. Let x = the number of apples,
and y = the number of pears,
$$\text{then } \frac{10x}{y} = 2 . \frac{45y}{x},$$
$$\text{and } x^2 = 9y^2,$$
$$\therefore x = 3y;$$
$$\text{whence } (3y + y =)\ 4y = 80,$$
$$\text{and } y = 20;$$
$$\therefore x = 60.$$

26. Let x = the number of gallons of brandy,
and y = the number of rum;

$$\therefore\ x^2 = \text{the price of the brandy,}$$
$$\text{and } y^2 = \text{the price of the rum;}$$
$$\therefore\ x^2 - y^2 = 225,$$
$$\text{and } x^2 + xy = 1125;$$
$$\text{whence } 2x^2 + 2xy = 2250,$$
$$\text{but } x^2 - y^2 = 225,$$
$$\therefore \text{ by subtraction, } x^2 + 2xy + y^2 = 2025,$$
$$\text{and } x + y = 45;$$
$$\text{hence } 45x = 1125,$$
$$\text{and } x = 25;$$
$$\therefore\ y = 20.$$

27. Let x and y = the depths,

$$\therefore\ x^2y - xy^2 = 20,$$
$$\text{and } x^2y : xy^2 :: 5 : 4,$$
$$\text{or } x : y :: 5 : 4, \quad \therefore\ y = \frac{4x}{5};$$
$$\text{hence } \frac{4x^3}{5} - \frac{16x^3}{25} = 20,$$
$$\text{or } \frac{4x^3}{25} = 20, \quad \therefore\ x = 5;$$
$$\text{and } y = 4.$$

28. Let x and y = the lengths of the sides,

$$\therefore\ x^3 + y^3 = 1241,$$
$$\text{and } x^2y + xy^2 = 1224;$$

adding three times the second equation to the first,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 4913,$$
$$\therefore\ x + y = 17,$$
$$\text{and } xy = \frac{1224}{x+y} = 72,$$
$$\text{whence } x^2 - 2xy + y^2 = 1,$$

$$\text{and } x - y = \pm 1,$$
$$\text{but } x + y = 17,$$
$$\therefore\ x = 9 \text{ or } 8,$$
$$\text{and } y = 8 \text{ or } 9.$$

29. Let x^2 and y^2 = the numbers,

$$\therefore\ x + y = 84,$$
$$\text{and } \frac{x^2}{y} + \frac{y^2}{x} = 91;$$
$$\text{whence } x^3 + y^3 = 91\,xy,$$
$$\text{but } x^3 + y^3 + 3\,xy\,.\,(x + y) = \overline{84}|^3,$$
$$\therefore\ 91\,xy + 252\,xy = \overline{84}|^3,$$
$$\text{or } 343\,xy = \overline{84}|^3,$$
$$\therefore\ xy = \overline{12}|^3 = 1728,$$
$$\text{and since } x + y = 84,$$
$$\therefore\ x - y = \pm 12,$$
$$\text{and } x = 48 \text{ or } 36,$$
$$y = 36 \text{ or } 48.$$

30. Let x = the less side,
and y = the greater,
$\therefore$ $4xy$ = the number of trees;

and $40 : \frac{1}{3}\sqrt{x^2+y^2} :: 4xy :$ the price of planting $= \frac{xy\sqrt{x^2+y^2}}{30}$,

$$\therefore\ \frac{xy\sqrt{x^2+y^2}}{30} - \frac{x^2 y}{25} = 4480,$$
$$\text{and } 5\,xy\sqrt{x^2+y^2} - 6x^2y = 150 \times 4480;$$
$$\text{but } x^2 + y^2 = \frac{8}{3}x^2 + (y - x)^2 = x^2 + y^2 + \frac{8}{3}x^2 - 2\,xy,$$
$$\therefore\ 2\,xy = \frac{8}{3}x^2,$$

$$\text{and } y = \frac{4x}{3};$$

$$\text{hence } \frac{20x^2}{3} \times \frac{5x}{3} - 6x^2 \cdot \frac{4x}{3} = 150 \times 4480,$$

$$\therefore\ 28x^3 = 9 \times 150 \times 4480,$$

$$\text{and } x^3 = 9 \times 150 \times 160 = 216000,$$

$$\therefore\ x = 60,$$

$$\text{and } y = 80.$$

31. Let $x =$ the distance from A to B,
and $y =$ the rate per hour;

$\therefore\ \dfrac{x}{y+4}$ and $\dfrac{x}{y-4} =$ the times of going down the stream, and returning,

and if there had been no stream, $\dfrac{2x}{y}$ would have been the whole time;

$$\therefore\ \frac{x}{y+4} + \frac{x}{y-4} - \frac{2x}{y} = \frac{39}{60} = \frac{13}{20},$$

$$\text{and } \frac{1}{y+4} + \frac{1}{y-4} - \frac{2}{y} = \frac{13}{20x},$$

$$\text{or } \frac{2y}{y^2-16} - \frac{2}{y} = \frac{13}{20x},$$

$$\therefore\ \frac{20}{13} \cdot \frac{2 \cdot 16}{y \cdot (y^2-16)} = \frac{1}{x}.$$

Now, when he has another waterman, he rows $y + \frac{1}{2}y = \frac{3}{2}y$ per hour;

$$\text{and in this case, } \frac{x}{\frac{3}{2}y+4} + \frac{x}{\frac{3}{2}y-4} - \frac{4x}{3y} = \frac{8}{60} = \frac{2}{15},$$

$$\therefore\ \frac{1}{3y+8} + \frac{1}{3y-8} - \frac{2}{3y} = \frac{1}{15x},$$

$$\text{and } \frac{6y}{9y^2 - 64} - \frac{2}{3y} = \frac{1}{15x},$$

$$\therefore \frac{2 \times 64 \times 15}{3y \cdot (9y^2 - 64)} = \frac{1}{x};$$

$$\text{whence } \frac{20}{13} \cdot \frac{2 \times 16}{y \cdot (y^2 - 16)} = \frac{2 \times 64 \times 15}{3y \cdot (9y^2 - 64)},$$

$$\text{whence } 13y^2 - 13 \times 16 = 9y^2 - 4 \times 16,$$

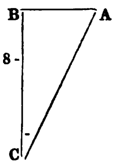
$$\text{and } 4y^2 = 9 \times 16,$$

$$\therefore 2y = 3 \times 4,$$

$$\text{and } y = 6;$$

$$\text{hence } \left(\frac{20}{13} \cdot \frac{2 \times 16}{6 \times 20} = \right) \frac{16}{39} = \frac{1}{x},$$

$$\text{and } \therefore x = 2\tfrac{7}{16}.$$



32. Let $AB = x$,

$$BC = y,$$

$$\therefore AC = \sqrt{x^2 + y^2}.$$

Let $z =$ the rate of the pedestrian,

$\therefore 3z =$ the rate of the coach ;

$$\text{and } \frac{x + 8}{z} = \frac{x + 8}{3z} + 4,$$

$$\therefore 2 \cdot \frac{x + 8}{3z} = 4,$$

$$\text{and } x + 8 = 6z,$$

$$\therefore x = 6z - 8;$$

$$\text{also, } \frac{x + y}{z} = \frac{\sqrt{x^2 + y^2}}{z} + \frac{8}{3},$$

$$\text{and } 4 + 2 \cdot \frac{x + y + \sqrt{x^2 + y^2}}{3z} + 6\tfrac{2}{3} = \frac{x + y + \sqrt{x^2 + y^2}}{z} + 4,$$

$$\therefore \frac{x + y + \sqrt{x^2 + y^2}}{3z} = 6\tfrac{2}{3},$$

$$\text{or } \frac{x+y+\sqrt{x^2+y^2}}{z} = 20;$$

$$\text{but } \frac{x+y-\sqrt{x^2+y^2}}{z} = \frac{8}{3},$$

$$\therefore\ 2 \cdot \frac{x+y}{z} = \frac{68}{3},$$

$$\text{and } x+y = \frac{34}{3}z;$$

$$\text{also, } \frac{2\sqrt{x^2+y^2}}{z} = \frac{52}{3},$$

$$\text{and } \sqrt{x^2+y^2} = \frac{26z}{3};$$

$$\text{hence } x^2+2xy+y^2 = \frac{1156}{9}z^2,$$

$$\text{and } 2x^2 \qquad + 2y^2 = \frac{1352}{9}z^2,$$

$$\therefore\ x^2-2xy+y^2 = \frac{196}{9}z^2,$$

$$\text{and } y-x = \frac{14}{3}z,$$

$$\text{but } y+x = \frac{34}{3}z,$$

$$\therefore\ 2y = 16z, \quad \text{and } y = 8z;$$

$$\text{and } 2x = \frac{20z}{3}, \quad \therefore\ x = \frac{10z}{3};$$

$$\text{whence } \frac{10z}{3} = 6z - 8,$$

$$\text{and } \frac{8z}{3} = 8,$$

$$\therefore\ z = 3;$$

whence $x = 10$,
and $y = 24$.

$\therefore$ the sides of the triangle are 10, 24, and 26; and the rates of travelling of the pedestrian and the coach are 3 and 9 miles per hour.

SECTION IX.

Problems producing Adfected Quadratics.

1. Let x and $19 - x$ be the numbers,

$$\therefore 2x^2 - 19x = 60,$$

$$\therefore x^2 - \frac{19}{2} \cdot x + \left|\frac{19}{4}\right|^2 = 30 + \frac{361}{16} = \frac{841}{16},$$

$$\text{and } x - \frac{19}{4} = \pm \frac{29}{4},$$

$$\therefore x = 12 \text{ or } -\frac{5}{2}.$$

2. Let $x =$ the number,

$$\therefore \frac{(\sqrt{40 - x^2} + 10) \cdot 2}{x} = 4,$$

$$\text{and } \sqrt{40 - x^2} = 2x - 10,$$

$$\therefore 40 - x^2 = 4x^2 - 40x + 100,$$

$$\text{and } x^2 - 8x = -12,$$

$$\therefore x^2 - 8x + 16 = 4,$$

$$\text{and } x - 4 = \pm 2,$$

$$\therefore x = 6 \text{ or } 2.$$

3. Let x and $x - 16$ be the length and breadth,

$$\therefore x^2 - 16x = 960,$$

$$\text{and } x^2 - 16x + 64 = 1024,$$
$$\therefore\ x - 8 = 32,$$
$$x = 40,$$
$\therefore$ the length is 40, and the breadth 24 yards.

4. Let $x =$ his age,

$$\therefore\ \frac{x}{2} + \sqrt{x} - 12 = 0,$$
$$\text{and } x + 2\sqrt{x} + 1 = 25,$$
$$\therefore\ \sqrt{x} + 1 = 5,$$
$$\sqrt{x} = 4,$$
$$\text{and } x = 16.$$

5. Let $3x =$ the number of gallons in the less,
$\therefore\ 3x + 5 =$ the number in the greater,
and $x - 2 =$ the price of a gallon;

$$\therefore\ (6x + 5) \,.\, (x - 2) = 58,$$
$$\text{and } 6x^2 - 7x = 68,$$
$$\therefore\ x^2 - \frac{7}{6}x + \frac{49}{144} = \frac{68}{6} + \frac{49}{144} = \frac{1681}{144},$$
$$\text{and } x - \frac{7}{12} = \frac{41}{12},$$
$$\text{and } x = 4,$$
$\therefore$ the numbers are 12 and 17, and the price 2*s*.

6. Let $2x =$ the number B went per day,
$\therefore\ 2x + 8 =$ the number A went,
and $x =$ the number of days;

$$\therefore\ 4x^2 + 8x = 320,$$
$$\text{and } x^2 + 2x + 1 = 81,$$
$$\therefore\ x + 1 = 9,$$
$$\text{and } x = 8,$$
$\therefore$ A went 24, and B 16 miles per day, and the distances travelled by them were 128 and 192 miles.

7. Let $x =$ the hypothenuse,

$\therefore x - 6 =$ the base,

and $x - 3 =$ the perpendicular,

$$\text{and } x^2 = (x-6)^2 + (x-3)^2 = 2x^2 - 18x + 45,$$
$$\therefore x^2 - 18x + 45 = 0,$$
$$\text{and } x^2 - 18x + 81 = 36,$$
$$\text{whence } x - 9 = \pm 6,$$
$$\text{and } x = 15,$$

and the sides are 15, 12, and 9.

8. Let x and $24 - x$ be the numbers,

$$\therefore 2x \,.\, (24 - x) = 18 \times 12,$$
$$\text{or } x^2 - 24x = -108,$$
$$\therefore x^2 - 24x + 144 = 36,$$
$$\text{and } x - 12 = \pm 6,$$
$$\therefore x = 18 \text{ or } 6,$$

and there were 18 of one, and 6 of the other.

9. Let $x =$ the number of bushels of wheat,

$\therefore x + 16 =$ the number of barley,

also $\dfrac{288}{x} =$ the price of a bushel of wheat,

and $\dfrac{288}{x+16} =$ the price of a bushel of barley,

$$\therefore \frac{288}{x} = \frac{288}{x+16} + 3,$$
$$\text{or } \frac{96}{x} = \frac{96}{x+16} + 1,$$
$$\therefore 96 \,.\, (x + 16) = 96x + x^2 + 16x,$$
$$\text{and } x^2 + 16x = 96 \times 16,$$
$$\therefore x^2 + 16x + 64 = 1600,$$
$$x + 8 = 40,$$
$$\text{and } x = 32;$$

$\therefore$ there were 32 bushels of wheat, and 48 of barley.

10. Let x = the number of miles A went per hour,

$\therefore x - 1$ = the number B went,

$$\text{and } \frac{90}{x} = \frac{90}{x-1} - 1,$$

$$\text{or } 90 . (x - 1) = 90x - x^2 + x,$$

$$\therefore x^2 - x + \frac{1}{4} = 90 + \frac{1}{4} = \frac{361}{4},$$

$$\text{and } x - \frac{1}{2} = \pm \frac{19}{2},$$

$$\therefore x = 10,$$

and A went 10, and B 9 miles per hour.

11. Let x = the number of quartos,

$\therefore x^2$ = their value,

$3x^2$ = the value of the folios,

and $8x$ = the value of octavos,

$$\therefore 4x^2 + 8x = 1932,$$

$$\text{and } x^2 + 2x + 1 = 484,$$

$$\therefore x + 1 = 22,$$

$$\text{and } x = 21,$$

and each folio cost $4\frac{1}{2}$ guineas, each quarto 1; and each octavo *5s. 3d.*

12. Let x = the number,

$$\therefore \frac{105}{x} = \frac{105}{x-2} - 6,$$

$$\text{or } 35 . (x - 2) = 35x - 2 . (x^2 - 2x),$$

$$\therefore x^2 - 2x = 35,$$

$$x^2 - 2x + 1 = 36,$$

$$x - 1 = 6,$$

$$\text{and } x = 7.$$

13. Let x and $x + 5$ be the numbers,

$$\therefore x^2 + (x + 5)^2 = 1313,$$

$$\text{or } 2x^2 + 10x + 25 = 1313,$$

$$\therefore\ x^2 + 5x + \frac{25}{4} = \frac{2601}{4},$$

$$\text{and } x + \frac{5}{2} = \frac{51}{2},$$

$$\therefore\ x = 23,$$

and the numbers were 23 and 28.

14. Let x and $x + 4$ be the numbers,

$$\text{then } x^2 + (x + 4)^2 = 1066,$$

$$\text{and } x^2 + 4x + 4 = 529,$$

$$\therefore\ x + 2 = 23,$$

$$\text{and } x = 21,$$

and the numbers are 21 and 25.

15. Let $x =$ the number,

$$\therefore\ \sqrt{x + 24} = x - 18,$$

$$\text{and } x + 24 = x^2 - 36x + 324,$$

$$\therefore\ x^2 - 37x = -300,$$

$$x^2 - 37x + \left(\frac{37}{2}\right)^2 = \frac{1369}{4} - 300 = \frac{169}{4},$$

$$\therefore\ x - \frac{37}{2} = \pm\frac{13}{2},$$

$$\text{and } 6 = 25 \text{ or } 12.$$

16. Let $x =$ the number in the first,

$\therefore\ 4x^2 =$ the number in the second,

and $\dfrac{4x^2 + x}{2} - 5 =$ the number in the third,

$$\text{and } \frac{3}{2}\,.\,(4x^2 + x) - 5 = 220,$$

$$\therefore\ 4x^2 + x = 150,$$

$$\text{and } 4x^2 + x + \frac{1}{16} = \frac{2401}{16},$$

$$\therefore\ 2x + \frac{1}{4} = \frac{49}{4},$$

$$\text{and } 2x = 12, \quad \therefore\ x = 6;$$

and the numbers were 6, 144, and 70.

17. Let x = the number of yards,

$\therefore\ \frac{147}{x}$ = the buying price per yard,

and $\frac{481}{4.(x-12)}$ = the selling price;

$$\therefore\ \frac{147}{x} = \frac{481}{4.(x-12)} - \frac{1}{4},$$

$$\text{and } 588x - 7056 = 481x - x^2 + 12x,$$

$$\therefore\ x^2 + 95x + \left(\frac{95}{2}\right)^2 = 7056 + \frac{9025}{4} = \frac{37249}{4},$$

$$\text{and } x + \frac{95}{2} = \frac{193}{2},$$

$$\therefore\ x = 49.$$

18. Let x = the number,

$\therefore\ \frac{216}{x}$ = the number each was to send,

$$\text{and } (x-3).\left(\frac{216}{x} + 12\right) = 216,$$

$$\therefore\ 216 - \frac{648}{x} + 12x - 36 = 216,$$

$$\text{and } 12x^2 - 36x = 648,$$

$$\therefore\ x^2 - 3x + \frac{9}{4} = 54 + \frac{9}{4} = \frac{225}{4},$$

$$\text{and } x - \frac{3}{2} = \frac{15}{2},$$

$$\therefore\ x = 9;$$

and each was to send 24 men.

19. Let $x =$ the price of a duck,

$$\left(\frac{105 - 15x}{12} =\right) \frac{35 - 5x}{4} = \text{the price of a turkey,}$$

$$\text{and } x : 1 :: 18 : \text{the number of ducks for } 18s. = \frac{18}{x},$$

$$\text{and } \frac{4 \times 4}{7 - x} = \text{the number of turkeys for } 20s.$$

$$\therefore \frac{18}{x} = \frac{16}{7 - x} + 2,$$

$$\text{or } \frac{9}{x} = \frac{8}{7 - x} + 1,$$

$$\therefore 63 - 9x = 8x + 7x - x^2,$$

$$\text{or } x^2 - 24x + 144 = 144 - 63 = 81,$$

$$\text{and } x - 12 = \pm 9,$$

$$\therefore x = 3 \text{ or } 21;$$

and the prices were 3*s.* and 5*s.*

20. Let $x =$ the sum,

$\therefore x + 15 =$ B's subscription,

$$\text{and } 16 . (2x + 15) + 12 \times 50 : 16x :: 159 : 88 - x,$$

$$\text{or } 2 . (2x + 15) + 75 : 2x :: 159 : 88 - x,$$

$$\therefore (4x + 105) . (88 - x) = 318x,$$

$$\text{and } 4x^2 + 71x + \left|\frac{71}{4}\right|^2 = 9240 + \frac{3041}{16} = \frac{152881}{16},$$

$$\text{whence } 2x + \frac{71}{4} = \frac{391}{4},$$

$$\text{and } 2x = 80, \quad \therefore x = 40.$$

21. Let $x =$ the height,

$\therefore$ the sides are $8x - 2$ and $6x - 5$,

$$\therefore (8x - 2) . (6x - 5) = 2x . (14x - 7) + 178,$$

$$\text{or } (4x - 1) . (6x - 5) = 14x^2 - 7x + 89,$$

$$\text{hence } 10x^2 - 19x = 84,$$

$$\text{and } x^2 - \frac{19}{10}x + \overline{\frac{19}{20}}\Big|^2 = \frac{84}{10} + \frac{361}{400} = \frac{3721}{400},$$

$$\therefore\ x - \frac{19}{20} = \frac{61}{20},$$

$$\text{and } x = 4;$$

or the sides were 30 and 19, and the height 4 yards.

22. Let $3x =$ the number of soldiers,

$\therefore\ x =$ what each sailor received,

and $74x + 3x^2 - 9x + 768 =$ the value of the prize;

$$\therefore\ \frac{3x^2 + 65x + 768}{3x + 74} = \frac{3x}{2},$$

$$\therefore\ 6x^2 + 130x + 1536 = 9x^2 + 222x,$$

$$\text{or } 3x^2 + 92x = 1536,$$

$$\therefore\ x^2 + \frac{92}{3} \cdot x + \overline{\frac{46}{3}}\Big|^2 = \frac{1536}{3} + \frac{2116}{9} = \frac{6724}{9},$$

$$\therefore\ x + \frac{46}{3} = \frac{82}{3}, \text{ and } x = 12;$$

there were 36 soldiers; each soldier received £9, and each sailor £12.

23. Let $x =$ the number in the first,

$\therefore\ 3\sqrt{2x} + 6 =$ the number in the second,

and $3 \cdot (x + 3\sqrt{2x} + 6) =$ the number in the third,

and $(x + 3\sqrt{2x} + 6)^2 + 6 =$ the number in the fourth,

$$\therefore\ (x + 3\sqrt{2x} + 6)^2 + 4 \cdot (x + 3\sqrt{2x} + 6) + 4 = 1936,$$

$$\text{and } x + 3\sqrt{2x} + 6 + 2 = 44,$$

$$\therefore\ x + 3\sqrt{2x} = 36,$$

$$\text{and } 2x + 6\sqrt{2x} + 9 = 72 + 9 = 81,$$

$$\therefore\ \sqrt{2x} + 3 = 9,$$

$$\sqrt{2x} = 6, \text{ and } x = 18;$$

$\therefore$ the numbers were 18, 24, 126, and 1770.

24. Let x = number in a side of the first triangle,

$$\therefore 3 \,.\, (x - 1) = \text{number in the first,}$$

$$\text{and } 3 \,.\, (x - 3) - 3 = \text{number in the second,}$$

$$3 \,.\, (x - 6) - 3 = \text{number in the third,}$$

$$\therefore 9x - 36 = \text{number of men.}$$

Now $x^{\frac{1}{2}} + 1$ = number of men in a side of the hollow square,

$$\therefore (x^{\frac{1}{2}} + 1)^2 - (x^{\frac{1}{2}} - 7)^2 = \text{number of men in the hollow square,}$$

$$\text{and } 9x - 36 = 16x^{\frac{1}{2}} - 48 + 597,$$

$$\therefore 9x - 16x^{\frac{1}{2}} = 585,$$

$$\text{and } 9x - 16x^{\frac{1}{2}} + \frac{64}{9} = 585 + \frac{64}{9} = \frac{5329}{9},$$

$$\therefore 3x^{\frac{1}{2}} - \frac{8}{3} = \frac{73}{3},$$

$$\text{and } 3x^{\frac{1}{2}} = 27,$$

$$\therefore x^{\frac{1}{2}} = 9, \text{ and } x = 81,$$

$$\therefore \text{the number is } 9 \times 77 = 693.$$

25. Let x^2 = the sum to be divided,

$$\therefore x = \text{what C received,}$$

$$\text{and } \frac{x^2 - x}{2} = \text{what A received;}$$

$$4 : 1 :: \frac{1}{2} \,.\, (x^2 - x) : \text{their daily pay} = \frac{1}{8} \,.\, (x^2 - x),$$

$$\text{and } \frac{1}{8} \,.\, (x^2 - x) : x :: 1 : \text{the time C worked} = \frac{8}{x - 1};$$

$$\text{also } \frac{7x}{5} = \text{C's pay on the second supposition,}$$

$$\therefore \frac{5x^2 - 7x}{10} = \text{A's receipt,}$$

$$\text{and } 4 : 1 :: \frac{5x^2 - 7x}{10} : \text{their daily pay} = \frac{5x^2 - 7x}{40},$$

and $\frac{5x^2 - 7x}{40} : \frac{7x}{5} :: 1 :$ the time C worked in the second case, $= \frac{56}{5x - 7}$,

$$\therefore \frac{56}{5x - 7} = \frac{8}{x - 1} + \frac{10}{9},$$

$$\text{or } \frac{28}{5x - 7} = \frac{4}{x - 1} + \frac{5}{9},$$

$$\therefore 9 \times 28 \times (x - 1) = 4 \times 9 . (5x - 7) + 5 . (5x - 7) . (x - 1),$$

$$\text{or } 25x^2 - 132x = -35,$$

$$\therefore 25x^2 - 132x + \left(\frac{66}{5}\right)^2 = \frac{4356}{25} - 35 = \frac{3481}{25},$$

$$\text{and } 5x - \frac{66}{5} = \frac{59}{5},$$

$$\therefore 5x = 25, \text{ and } x = 5,$$

$\therefore$ he worked 2 days, and received 5*s*.

26. Let $x =$ the quantity,

$\therefore 20 - x =$ the quantity remaining, or the quantity of water in the second,

and $20 : x :: x :$ quantity of spirit returned to the first, $= \frac{x^2}{20}$,

$$\text{and } x - \frac{x^2}{20} = \text{quantity in the second,}$$

and $20 : 20 - x + \frac{x^2}{20} :: \frac{20}{3} :$ quantity of spirit in $6\frac{2}{3}$ gallons $= \frac{x^2 - 20x + 400}{60}$,

$$\therefore \frac{20x - x^2}{20} + \frac{x^2 - 20x + 400}{60} = \frac{2}{3} . \frac{x^2 - 20x + 400}{20},$$

$$\text{or } 60x - 3x^2 = x^2 - 20x + 400,$$

$$\text{and } 4x^2 - 80x + 400 = 0,$$

$$\text{or } x^2 - 20x + 100 = 0,$$
$$\therefore\ x - 10 = 0,$$
$$\text{or } x = 10.$$

27. Let $\left.\begin{matrix} x - y \\ x \\ x + y + 5 \end{matrix}\right\}$ be the numbers,

$$\therefore\ 3x + 5 = 20,$$
$$\text{and } x = 5;$$
$$\text{hence } (5 - y) \,.\, 5 \,.\, (10 + y) = 130,$$
$$\text{or } 50 - 5y - y^2 = 26,$$
$$\therefore\ y^2 + 5y + \frac{25}{4} = 24 + \frac{25}{4} = \frac{121}{4},$$
$$\text{and } y + \frac{5}{2} = \pm\frac{11}{2},$$
$$\therefore\ y = 3 \text{ or } -8,$$
and the numbers are 2, 5, 13.

28. Let $x - y$, x, and $x + y + 3$, be the numbers,

$$\therefore\ 3x + 3 = 21,$$
$$\text{and } x = 6;$$
$$\text{hence } (6 - y)^2 + (9 + y)^2 = 137,$$
$$\therefore\ 2y^2 + 6y = 20,$$
$$\text{and } y^2 + 3y + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4},$$
$$\therefore\ y + \frac{3}{2} = \pm\frac{7}{2},$$
$$\text{and } y = 2 \text{ or } -5,$$
and the numbers are 4, 6, 11.

29. Let $10x + y$ be the number,

$$\text{then } \frac{10x + y}{x + y} = x + 2,$$
$$\text{and } 10x + y = x^2 + 2x + xy + 2y,$$

$$\therefore\ 8x - y = x^2 + xy;$$

$$\text{also } \frac{10y + x}{x + y + 1} = x + 4,$$

$$\text{and } 10y + x = x^2 + xy + 5x + 4y + 4,$$

$$\therefore\ 6y - 4x - 4 = x^2 + xy = 8x - y,$$

$$\text{and } 7y = 12x + 4,$$

$$\therefore\ y = \frac{12x + 4}{7};$$

$$\text{and } 8x - \frac{12x + 4}{7} = x^2 + \frac{12x^2 + 4x}{7},$$

$$\text{or } 56x - 12x - 4 = 7x^2 + 12x^2 + 4x,$$

$$\text{and } 19x^2 - 40x = -4,$$

$$\therefore\ x^2 - \frac{40}{19}x + \overline{\frac{20}{19}}\Big|^2 = \frac{400}{361} - \frac{4}{19} = \frac{324}{361},$$

$$\text{and } x - \frac{20}{19} = \pm\frac{18}{19},$$

$$\therefore\ x = 2,$$

$$\text{and } y = 4.$$

30. Let $2x$ = A's stock,

$\therefore\ x + 100$ = B's stock;

$$\text{and A's gain} = \frac{3}{20}\,.\,(x + 100) = \frac{3x}{20} + 15,$$

$$\therefore\ \text{B's gain} = 100 - \left(\frac{3x}{20} + 15\right) = 85 - \frac{3x}{20},$$

$$\text{and } 3x + 100 : 100 :: 2x : \frac{3x}{20} + 15,$$

$$\therefore\ 200x = \frac{9x^2}{20} + 60x + 1500,$$

$$\text{and } \frac{9x^2}{20} - 140x = -1500,$$

$$\therefore\ 9x^2 - 2800x + \frac{\overline{1400}\big|^2}{9} = \frac{1960000}{9} - 30000 = \frac{1690000}{9},$$

$$\text{whence } 3x - \frac{1400}{3} = \pm \frac{1300}{3},$$

$$\text{and } 3x = \frac{2700}{3} = 900,$$

$$\therefore\ x = 300,$$

and A's stock was 600, and his gain was 60,
B's stock was 400, and his gain 40.

31. Let $x =$ height or length of A's large storehouse,
and $\therefore =$ width of B's large one;
$y =$ length and height of B's large one,
and $\therefore =$ length, width, and height, of A's small one;
also $x =$ length and height of B's small one,
and $y =$ its width;
$\therefore$ x^3 and y^3 are the solid contents of those built by A,
and x^2y and xy^2 of those built by B,

$$\text{and } x^3 + y^3 - x^2y - xy^2 = 73728;$$

also $x^2 - y^2 =$ ground plot of C's warehouse,

$$\therefore\ x^2 - y^2 + 8\sqrt{x^2 - y^2} = 2688;$$

$$\text{and } x^2 - y^2 + 8\sqrt{x^2 - y^2} + 16 = 2704,$$

$$\therefore\ \sqrt{x^2 - y^2} + 4 = \pm 52,$$

$$\text{and } \sqrt{x^2 - y^2} = 48,$$

$$\therefore\ x^2 - y^2 = 2304;$$

dividing the first equation by this, $x - y = 32$,

$$\text{but } \left(\frac{x^2 - y^2}{x - y} =\right) x + y = \frac{2304}{32} = 72,$$

$$\text{and } x - y = 32,$$

$$\therefore\ 2x = 104, \text{ and } x = 52,$$

$$\text{also } 2y = 40, \text{ and } \therefore\ y = 20;$$

hence the width of A's and B's large warehouse was 52 feet,
of A's and B's small one, 20,
and of C's 48.

32. Let $3x$ = the number of acres A had,

$\therefore 7x$ = the number B had,

y = the number C had,

$$\therefore 3x + y + 36 = 3y + 2x,$$

$$\text{and } x = 2y - 36;$$

$$\text{also } \frac{10x + y}{xy} = \frac{3}{4},$$

$$\text{or } \frac{21y - 360}{y \cdot (2y - 36)} = \frac{3}{4},$$

$$\therefore \frac{7y - 120}{y^2 - 18y} = \frac{1}{2},$$

$$\text{or } y^2 - 18y = 14y - 240,$$

$$y^2 - 32y + \overline{16}|^2 = 256 - 240 = 16,$$

$$\therefore y - 16 = \pm 4,$$

$$\text{and } y = 20 \text{ or } 12;$$

$$\text{and } x = 4,$$

$\therefore$ the numbers are 12, 28, and 20; and the sum was £3.

33. Let x = the number of calves,

and y = the number of sheep,

$$\therefore xy + \frac{1}{4}y^2 + 140 = x \cdot (y + 4) + \left(\frac{1}{4}y + 2\right) \cdot y,$$

$$= xy + 4x + \frac{1}{4}y^2 + 2y,$$

$$\therefore 70 = 2x + y,$$

$$\text{and } x = \frac{70 - y}{2};$$

$$\text{also } (x + y) \cdot y = 1128,$$

$$\text{or } y^2 + \frac{70y - y^2}{2} = 1128,$$

$$\therefore y^2 + 70 + \overline{35}|^2 = 2256 + 1225 = 3481,$$

$$\therefore y + 35 = 59,$$

$$\text{and } y = 24;$$

$$\therefore x = 23.$$

34. Let $3x$ = the price of an ox,

$\therefore\ x$ = the price of a sheep;

let y = the number of oxen,

$\therefore\ 2y$ = the number of sheep;

$$\text{and } (3xy + 2xy =)\ 5xy = 100,$$

$$\therefore\ xy = 20;$$

now, $3x - 1$ = the price of an ox in the evening,

and $x - \frac{1}{3}$ = the price of a sheep;

also, $3y + 10$ = the whole stock;

$\therefore\ \frac{1}{4}\,.\,(3y + 10)$ = the number of oxen,

$\frac{3}{4}\,.\,(3y + 10)$ = the number of sheep;

$$\text{and } \frac{1}{4}\,.\,(3y + 10)\,.\,(3x - 1) + \frac{3}{4}\,.\,(3y + 10)\,.\,\frac{3x - 1}{3} = 100,$$

$$\text{or } (3y + 10)\,.\,(3x - 1) = 200,$$

$$9xy + 30x - 3y - 10 = 200,$$

$$\text{and } 30x - 3y = 30,$$

$$10x^2 - xy = 10x,$$

$$\text{whence } 10x^2 - 20 = 10x,$$

$$\text{and } x^2 - x = 2,$$

$$\therefore\ x^2 - x + \frac{1}{4} = \frac{9}{4},$$

$$\text{and } x - \frac{1}{2} = \pm\frac{3}{2},$$

$$\therefore\ x = 2 \text{ or } -1,$$

$$\text{and } y = 10;$$

$\therefore$ there were 10 oxen and 30 sheep bought,
and the prices were £5, and £1 13*s*. 4*d*.

35. Let x = A's daily wages,

and y = B's;

$$\therefore \left(x + \frac{7y}{4}\right) . y = 48,$$

$$\text{and } (x + 2) . \frac{7y}{2} = 98,$$

$$\therefore 7xy + \frac{49}{4} . y^2 = 336,$$

$$\text{and } 7xy + \quad 14y = 196,$$

$$\therefore \text{by subtraction, } \frac{49}{4} . y^2 - 14y = 140,$$

$$\text{and } \frac{49}{4}y^2 - 14y + 4 = 144,$$

$$\therefore \frac{7y}{2} - 2 = 12,$$

$$\text{and } \frac{7y}{2} = 14, \quad \therefore y = 4,$$

$$\text{and } x = 5.$$

36. Let $x = AB$,
$y = BC$,

$$\therefore x - y = CD + y + 4,$$

$$\text{or } CD = x - 2y - 4;$$

$$x - y - 4 = \frac{2}{3} . (x + y),$$

$$\text{and } x = 5y + 12;$$

$$\text{also } x : x - 2y - 4 :: 7y : 26,$$

$$\text{or } 5y + 12 : 3y + 8 :: 7y : 26,$$

$$\therefore 130y + 312 = 21y^2 + 56y,$$

$$\text{and } 21y^2 - 74y = 312,$$

$$\text{or } y^2 - \frac{74}{21}y + \overline{\frac{37}{21}}\Big|^2 = \frac{1369}{\overline{21}|^2} + \frac{312}{21} = \frac{7291}{\overline{21}|^2},$$

$$\therefore y - \frac{37}{21} = \frac{89}{21},$$

$$\text{and } y = 6,$$
$$\therefore\ x = 42,$$
$$\text{and AB} = 42,\ \text{BC} = 6,\ \text{CD} = 26 \text{ miles.}$$

37. Let $x =$ the number of better,

$y =$ the number of worse,

$$\text{then } x^2 + xy : xy - y^2 :: 72 : 7,$$
$$\therefore\ x^2 + xy : (x^2 + 2xy - y^2 =)\ 158 :: 72 : 79,$$
$$\text{or } x^2 + xy = 144,$$
$$\therefore\ xy - y^2 = 14,$$
$$\text{hence } x + y = \frac{144}{x},$$
$$x - y = \frac{14}{y},$$
$$\therefore \text{ by subtraction, } 2y = \frac{144}{x} - \frac{14}{y},$$
$$\text{and } y = \frac{72}{x} - \frac{7}{y} = \frac{72y}{y^2 + 14} - \frac{7}{y},$$
$$\therefore\ y^2 = \frac{72y^2}{y^2 + 14} - 7,$$
$$\text{or } y^4 + 14y^2 = 72y^2 - 7y^2 - 98,$$
$$\text{or } y^4 - 51y^2 + \overline{\frac{51}{2}}\Big|^2 = \frac{2601}{4} - 98 = \frac{2209}{4},$$
$$\therefore\ y^2 - \frac{51}{2} = \frac{47}{2},$$
$$\therefore\ y^2 = 49,$$
$$\text{and } y = 7,\quad \therefore\ x = 9.$$

38. Let $x =$ the number in a handful,

$\therefore\ x^2 =$ the number remaining in the greater,

and $x^{\frac{2}{3}} =$ the number remaining in the less.

The contents of the greater and less will be $x^2 + x$, and $x + x^{\frac{2}{3}}$;

and $x^{\frac{4}{3}}$ = the number remaining in the greater after the second drawing;

$$\text{whence } x^{\frac{4}{3}} + x^{\frac{2}{3}} = \frac{5}{3} \cdot (x + x^{\frac{2}{3}}),$$

$$\text{and } x^{\frac{2}{3}} + 1 = \frac{5}{3} \cdot (x^{\frac{1}{3}} + 1),$$

$$\text{whence } x^{\frac{2}{3}} - \frac{5}{3} \cdot x^{\frac{1}{3}} + \frac{25}{36} = \frac{2}{3} + \frac{25}{36} = \frac{49}{36},$$

$$\therefore\ x^{\frac{1}{3}} - \frac{5}{6} = \pm \frac{7}{6},$$

$$\text{and } x^{\frac{1}{3}} = 2,$$

$$\therefore\ x = 8;$$

and the numbers are 72 and 12.

39. Let x = the number of bushels of barley,
and y = the price of a bushel,
$\therefore\ 2y - 3$ = the price of a bushel of wheat,

$$\text{and } xy + 20y - 30 = 159,$$

$$\text{or } xy + 20y = 189;$$

$$\text{now } (x + 4) \cdot (y + 2) + 15 \cdot (2y - 1) + 24 = 318,$$

$$\therefore\ xy + 2x + 34y = 301,$$

$$\text{but } xy \qquad + 20y = 189,$$

$$\therefore \text{ by subtraction, } \quad 2x + 14y = 112,$$

$$\text{and } x = 56 - 7y;$$

$$\text{whence } 56y - 7y^2 + 20y = 189,$$

$$\therefore\ y^2 - \frac{76}{7}y + \left(\frac{38}{7}\right)^2 = \frac{1444}{49} - \frac{189}{7} = \frac{121}{49},$$

$$\therefore\ y - \frac{38}{7} = \frac{11}{7},$$

$$\text{and } y = 7, \quad \therefore\ x = 7.$$

40. Let x = the number of bushels,
and y = the price of a bushel;

$$\therefore \frac{xy}{40} = \text{interest for 6 months};$$

$$\text{and } x.(y+3) - \frac{xy}{40} - xy = \text{expected gain} = 3x - \frac{xy}{40};$$

also, $xy + \dfrac{xy}{40} - x.(y-1) = \dfrac{xy}{40} + x =$ loss on first supposition $= 5y$, by prob.

also, $xy + \dfrac{xy}{20} - x.(y-2) = \dfrac{xy}{20} + 2x =$ loss on second supposition $= 3x - \dfrac{xy}{40} - 10$,

$$\text{whence } \frac{3xy}{40} = x - 10;$$

$$\text{but from the first equation, } \frac{3xy}{40} = 15y - 3x,$$

$$\therefore 15y - 3x = x - 10,$$

$$\text{and } x = \frac{15y+10}{4};$$

$$\text{whence } 10y + 15y^2 = (800y - 400 - 600y =)\ 200y - 400,$$

$$\text{or } 3y^2 - 38y = -80,$$

$$\therefore y^2 - \frac{38}{3}y + \left|\frac{19}{3}\right|^2 = \frac{361}{9} - \frac{80}{3} = \frac{121}{9},$$

$$\therefore y - \frac{19}{3} = \pm\frac{11}{3},$$

$$\text{and } y = 10, \text{ or } \frac{8}{3};$$

$$\therefore x = \frac{10+150}{4} = 40;$$

$\therefore$ the quantity of corn laid up was 40 bushels, and the price per bushel was 10*s*.

41. Let $x =$ the number of coins secreted,

$\therefore 60 - x =$ the number remaining;

$y =$ the number of urns secreted,

$\therefore 9 - y =$ the number remaining;

$$\text{and } x : 60 - x :: y : 9 - y,$$

$$\therefore x : 60 :: y : 9,$$

$$\text{and } 3x = 20y;$$

$$\text{also } x - 4 : 64 - x :: y^2 : 20 . (9 - y) - y^2,$$

$$\therefore x - 4 : 60 :: y^2 : 20 . (9 - y),$$

$$\text{and } (x - 4) . (9 - y) = 3y^2;$$

$$\text{or } (20y - 12) . (9 - y) = 9y^2;$$

$$\therefore 29y^2 - 192y = -108,$$

$$\text{and } y^2 - \frac{192}{29}y + \overline{\frac{96}{29}}\Big|^2 = \frac{9216}{\overline{29}|^2} - \frac{108}{29} = \frac{6084}{\overline{29}|^2},$$

$$\therefore y - \frac{96}{29} = \frac{78}{29},$$

$$\therefore y = 6, \text{ and } x = 40.$$

42. Let $x =$ the number B went per day,

$\therefore \frac{5x}{3} =$ the number A went;

$y =$ the number of days B travelled,

$\therefore y + 5 =$ the number A travelled;

$$\text{and } \frac{5x}{3} . (y + 5) - xy = \frac{2xy}{3} + \frac{25x}{3} = 259,$$

$$\text{and } \frac{5x}{3} . (y - 1) - (x + 2) . y = \frac{2xy}{3} - \frac{5x}{3} - 2y = 37,$$

$$\therefore \text{by subtraction, } 10x + 2y = 222,$$

$$\text{and } y = 111 - 5x,$$

$$\therefore \frac{2x}{3} . (111 - 5x) + \frac{25x}{3} = 259,$$

$$\text{or } 222x - 10x^2 + 25x = 777,$$

$$\text{and } x^2 - \frac{247}{10}x + \overline{\frac{247}{20}}\Big|^2 = \frac{61009}{400} + \frac{777}{10} = \frac{29929}{400},$$

$$\therefore\ x - \frac{247}{20} = \frac{173}{20},$$

and $x = 21$, $\therefore\ y = 6$.

43. Let $y =$ the time in which Bacchus would empty the cask,
and $3x =$ the time in which Silenus would empty it,
$\therefore\ 2x =$ the time Bacchus drank,

and $y : 1 :: 2x :$ the quantity Bacchus drank $= \dfrac{2x}{y}$,

$\therefore\ 1 - \dfrac{2x}{y} =$ the quantity Silenus drank;

and $1 : 3x :: 1 - \dfrac{2x}{y}$: the time Silenus was drinking $= 3x - \dfrac{6x^2}{y}$,

$\therefore\ 5x - \dfrac{6x^2}{y} =$ the time of emptying.

Also $\dfrac{1}{2} - \dfrac{x}{y} =$ the quantity Bacchus would have had,

and $1 : y :: \dfrac{1}{2} - \dfrac{x}{y}$: the time of Bacchus drinking that quantity $= \dfrac{y}{2} - x$,

$$\therefore\ 5x - \frac{6x^2}{y} = \frac{y}{2} - x + 2;$$

also $1 : 3x :: \dfrac{1}{2} + \dfrac{x}{y}$: time of Silenus drinking $= \dfrac{3x}{2} + \dfrac{3x^2}{y}$,

$$\therefore\ \frac{y}{2} - x = \frac{3x}{2} + \frac{3x^2}{y},$$

and $6x^2 = y^2 - 5xy$,

$$\therefore\ y^2 - 5xy + \frac{25x^2}{4} = \frac{49x^2}{4},$$

and $y - \dfrac{5x}{2} = \dfrac{7x}{2}$,

or $y = 6x$;

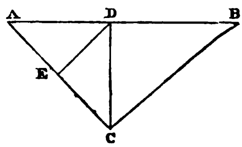
$$\text{hence } 5x - x = 3x - x + 2,$$
$$\text{or } 2x = 2, \text{ and } x = 1.$$

And Silenus would empty the cask in 3 hours, and Bacchus in 6.

44. Let x and y be the numbers,

$$\therefore x^2 - y^2 = 5,$$
$$\text{and } (x^4 + y^4)^2 + x^2y^2 . (x^2 - y^2)^2 + x^2y^2 = 10345,$$
$$\text{or } 625 + 100x^2y^2 + 4x^4y^4 + 26x^2y^2 = 10345,$$
$$\therefore 4x^4y^4 + 126x^2y^2 + \overline{\frac{63}{2}}\Big|^2 = 9720 + \frac{3969}{4} = \frac{42849}{4},$$
$$\therefore 2x^2y^2 + \frac{63}{2} = \frac{207}{2},$$
$$\text{and } 2x^2y^2 = 72,$$
$$\therefore x^2y^2 = 36;$$
$$\text{now } x^4 - 2x^2y^2 + y^4 = 25,$$
$$\text{and } 4x^2y^2 = 144,$$
$$\therefore (x^2 + y^2)^2 = 169,$$
$$\text{and } x^2 + y^2 = 13,$$
$$\text{but } x^2 - y^2 = 5,$$
$$\therefore x^2 = 9, \text{ and } x = 3,$$
$$y^2 = 4, \text{ and } y = 2.$$

45. Let $CA = y$,



$x =$ the length of the sewer,

$DB = x - 11$,

and $xy =$ the expense;

also $AC : CB : 6 : x - 11$,

$$\therefore CB = \frac{y}{6} . (x - 11),$$
$$\text{and } AD : DC :: AE : EC,$$
$$\text{or } 6 : x :: 4 : y - 4,$$
$$\therefore 2x = 3y - 12;$$
$$\text{also } 9y + \frac{3}{2} . (x - 11) . y = xy + 54,$$

$$\text{and } xy - 15y = 108;$$
$$\text{whence } \frac{3y^2}{2} - 21y = 108,$$
$$\text{and } y^2 - 14y = 72,$$
$$\therefore\ y^2 - 14y + 49 = 121,$$
$$\text{and } y - 7 = 11,$$
$$\therefore\ y = 18,$$
$$\text{and } x = 21;$$
$$\therefore\ \text{DB} = 10,$$
$$\text{and CB} = 30.$$

46. Let x^2 = the number of poles in the square,
$\therefore\ x^4$ = the number of trees;
and $\frac{3x^2}{2} - 6$ = the area of the oblong,

$$\text{hence } \left(\frac{3x^2}{2} - 6\right) . 4x + 144 = x^4,$$
$$\text{or } 6x^3 - 24x + 144 = x^4,$$
$$\therefore\ 6x^3 + 144 = x^4 + 24x,$$
$$\text{or } 6 . (x^3 + 24) = x . (x^3 + 24),$$
$$\therefore\ x = 6,$$
and the number = 1296.

47. Let x = the length of the building,
and $2y$ = the base of either triangle = the height of the walls,
$\therefore\ \sqrt{x^2 - y^2}$ = the perpendicular altitude of the triangle,
and $4y^2$ = the area of the wall,
$\therefore\ 4y^2x$ = the content of the body,
and $4y^2x + xy\sqrt{x^2 - y^2}$ = content of the whole barn;

$$\therefore\ 4y^2x + xy\sqrt{x^2 - y^2} + 6x^3 : 4y^2x :: 11 : 2,$$
$$\text{and } xy\sqrt{x^2 - y^2} + 6x^3 : 4y^2x :: 9 : 2,$$
$$\therefore\ y\sqrt{x^2 - y^2} + 6x^2 = 18y^2,$$
$$\text{and } x^2 + \frac{y}{6}\sqrt{x^2 - y^2} = 3y^2,$$

$$\therefore\ (x^2 - y^2) + \frac{y}{6}\sqrt{x^2 - y^2} + \frac{y^2}{144} = 2y^2 + \frac{y^2}{144} = \frac{289}{144}y^2,$$

$$\text{and } \sqrt{x^2 - y^2} + \frac{y}{12} = \frac{17}{12} \cdot y,$$

$$\therefore\ \sqrt{x^2 - y^2} = \frac{4y}{3},$$

$$\text{whence } x^2 - y^2 = \frac{16y^2}{9},$$

$$\text{and } x^2 = \frac{25}{9}y^2,$$

$$\therefore\ x = \frac{5}{3}y, \text{ or } y = \frac{3}{5}x.$$

Now, one square foot in the roof cost x pence, and there are $2x^2$ feet; $\therefore$ the whole expense is $2x^3$. And the area of the floor $= 2xy$, $\therefore$ its expense $= 2x^2y$,

$$\text{and } (2x^3 + 2x^2y =)\ 2x^3 + \frac{6x^3}{5} = 50000,$$

$$\therefore\ 8x^3 = 125000,$$

$$\text{and } 2x = 50,$$

$$\therefore\ x = 25, \text{ and } y = 15.$$

48. Let $1 =$ the weight of gold in each mixture,

$x =$ the weight of silver in the first,

and $y =$ the weight in the second;

$$\therefore\ \frac{x + 13}{x + 1} = \text{value of weight 1 of the first,}$$

$$\text{and } \frac{y + 13}{y + 1} = \text{value of weight 1 of the second;}$$

$$\therefore\ \frac{x + 13}{x + 1} : \frac{y + 13}{y + 1} :: 11 : 17,$$

$$\text{or } (x + 13) \cdot (y + 1) : (y + 13) \cdot (x + 1) :: 11 : 17,$$

$$\text{whence } y = \frac{21x - 13}{x + 35};$$

$$\text{also } \frac{x+26}{x+2} : \frac{y+26}{y+2} :: 7 : 11,$$

$$\text{or } (x+26) \,.\, (y+2) : (y+26) \,.\, (x+2) :: 7 : 11,$$

$$\text{whence } y = \frac{40x-52}{x+68};$$

$$\text{hence } \frac{21x-13}{x+35} = \frac{40x-52}{x+68},$$

$$\text{and } 21x^2 + 1415x - 884 = 40x^2 + 1348x - 1820,$$

$$\text{or } 19x^2 - 67x = 936,$$

$$\therefore\ x^2 - \frac{67}{19} \,.\, x + \overline{\frac{67}{38}}\Big|^2 = \frac{936}{19} + \frac{4489}{\overline{38}|^2} = \frac{75625}{\overline{38}|^2},$$

$$\text{and } x - \frac{67}{38} = \frac{275}{38},$$

$$\therefore\ x = 9,$$

$$\text{and } y = 4,$$

and the proportion of gold to silver is 1 : 9 in the first mixture, and 1 : 4 in the second.

49. Let x = a side of the white one,

y = a side of the black one,

$$\text{then } (x+2) \,.\, 4y = 4y^2 - 3x^2 + 3,$$

$$\text{and } 3x^2 + 4xy = 4y^2 - 8y + 3,$$

$$\therefore\ x^2 + \frac{4}{3}xy + \frac{4}{9}y^2 = \frac{16}{9}y^2 - \frac{8}{3}y + 1,$$

$$\text{and } x + \frac{2}{3}y = \frac{4}{3}y - 1,$$

$$\therefore\ x = \frac{2}{3}y - 1;$$

$$\text{also } 2x^3 + 2y^3 = \frac{22}{3}y - 11 + 9 + 12y = \frac{58}{3}y - 2,$$

$$\text{or } \frac{16y^3}{27} - \frac{8y^2}{3} + 4y - 2 + 2y^3 = \frac{58}{3} \,.\, y - 2,$$

$$\text{or } 70y^2 - 72y = 414,$$

$$\therefore\ y^2 - \frac{36}{35}\cdot y + \overline{\frac{18}{35}}\Big|^2 = \frac{207}{35} + \frac{324}{\overline{35}\,|^2} = \frac{7569}{\overline{35}\,|^2},$$

$$\text{and } y - \frac{18}{35} = \frac{87}{35},$$

$$\therefore\ y = 3, \text{ and } x = 1.$$

50. Let x = the rate at which A or B travels,

the geese travel at the rate $\frac{3}{2}$ per hour, and the waggon at the rate $\frac{9}{4}$,

B approaches the waggon at the rate $x + \frac{9}{4}$, and he overtakes the geese $\frac{10}{3}$ hours after A.

$$\frac{10x}{3} - 5 = \text{B's distance from A.}$$

A meets the waggon $50 - 2x$ miles from London,

and B meets $31 + \frac{2x}{3}$ miles from London,

$\therefore$ it had travelled in the interim $\frac{8x}{3} - 19$ miles,

$$\text{and } \therefore \text{ the interval} = \left(\frac{8x}{3} - 19\right)\cdot\frac{4}{9}.$$

$$\text{Also A's distance from B} = \left(\frac{8x}{3} - 19\right)\cdot\frac{4}{9}\cdot\left(x + \frac{9}{4}\right),$$

$$\therefore\ \frac{4}{9}\cdot\left(\frac{8x}{3} - 19\right)\cdot\left(x + \frac{9}{4}\right) = \frac{10x}{3} - 5,$$

$$\text{and } \frac{16x^2}{9} - \frac{41x}{3} = 21,$$

$$\therefore\ \left(\frac{4x}{3}\right)^2 - \frac{41}{4}\cdot\left(\frac{4x}{3}\right) + \overline{\frac{41}{8}}\Big|^2 = \frac{3025}{64},$$

$$\text{and } \frac{4x}{3} - \frac{41}{8} = \frac{55}{8},$$

$$\therefore \frac{4x}{3} = 12, \text{ and } x = 9;$$

$$\therefore \text{the distance required} = 25.$$

51. Since A takes 6 strokes while B takes 4, but 4 of B's throw out as much as 5 of A's, the water thrown out in a given time by A is to that thrown out by B :: 6 : 5.

Let z = the number of gallons in the hold at first,
y = the influx of gallons at the leak per hour,
x = the time B worked;
$\therefore x$ = the time in which A would clear the hold.

In x hours the influx at the leak would be xy gallons,
$\therefore 6 : 5 :: z + xy$: quantity thrown out by B on the first supposition, $= \frac{5}{6} \cdot (z + xy)$,

and $z + \frac{40}{3} \cdot y$ = the whole quantity pumped out in $13^h\ 20'$,

$\therefore z + \frac{40}{3}y - \frac{5}{6} \cdot (z + xy)$ = the quantity pumped out by A on the first supposition;

Now, $\frac{40}{3} - x$ = the time A worked,

$\therefore x : \frac{40}{3} - x :: z + xy$: the quantity pumped out by A on the first supposition, $= \frac{(40 - 3x) \cdot (z + xy)}{3x}$,

$$\therefore \frac{(40 - 3x) \cdot (z + xy)}{3x} = \frac{z + 80y - 5xy}{6},$$

$$\text{or } 80z + 80xy - 6xz - 6x^2y = xz + 80xy - 5x^2y,$$

$$\text{and } 80z - 7xz = x^2y,$$

$$\therefore\ z = \frac{x^2 y}{80 - 7x}.$$

But had they worked together, the hold would have been cleared in $\frac{15}{4}$ hours, $\therefore\ z + \frac{15y}{4}$ would equal the whole quantity pumped out on this supposition;

and $x : \frac{15}{4} :: \frac{5}{6} . (z + xy)$: quantity pumped out by B in this case, $= \frac{5 \times 5 . (z + xy)}{4 \times 2 . x}$;

and $x : \frac{15}{4} :: z + xy$: quantity pumped out by A in this case, $= \frac{15}{4x} . (z + xy)$.

$$\therefore\ \frac{25 . (z + xy)}{8x} + \frac{15}{4x} . (z + xy) = z + \frac{15y}{4},$$

$$\text{or } 55 . (z + xy) = 8xz + 30xy,$$

$$\text{whence } 8xz - 55z = 25xy,$$

$$\text{and } z = \frac{15xy}{8x - 55}.$$

$$\text{Hence } \frac{25xy}{8x - 55} = \frac{x^2 y}{80 - 7x},$$

$$\text{and } 25 . (80 - 7x) = 8x^2 - 55x,$$

$$\text{and } 8x^2 + 5 \times 24x = 2000,$$

$$\text{and } x^2 + 15x + \frac{15}{2}^2 = \frac{225}{4} + \frac{1000}{4} = \frac{1225}{4},$$

$$\therefore\ x + \frac{15}{2} = \frac{35}{2},$$

$$\text{and } x = 10;$$

$$\text{again, } \frac{15}{4x} . (z + xy) - 100 = \frac{z + 80y - 5xy}{6},$$

$$\therefore\ \frac{3}{8} . (z + 10y) - 100 = \frac{z + 30y}{6},$$

$$\text{whence } 9z + 90y - 2400 = 4z + 120y,$$
$$\text{or } z - 6y = 480;$$
$$\text{now } z = \frac{x^2 y}{80 - 7x} = \frac{100y}{10} = 10y,$$
$$\text{whence } (10y - 6y =)\ 4y = 480,$$
$$\text{and } y = 120;$$
$$\therefore\ z = 1200.$$

SECTION X.

Problems in Arithmetical and Geometrical Progressions.

1. Let $x - y$, x, and $x + y$, be the numbers,

$$\therefore\ 3x = 21,$$
$$\text{and } x = 7;$$
$$\text{also } 2x - y : 2x + y :: 3 : 4,$$
$$\therefore\ 2x : y :: 7 : 1,$$
$$\text{whence } y = 2,$$

and the numbers are 5, 7, 9.

2. Let $x - 2y$, $x - y$, x, $x + y$, $x + 2y$, be the distances,

$$\therefore\ 2x - 3y = 16,$$
$$\text{and } 2x + y = 24,$$
$$\therefore \text{ by subtraction, } \quad 4y = 8,$$
$$\text{and } y = 2; \quad \therefore\ x = 11;$$

and the distances are 7, 9, 11, 13, 15.

3. Let $x - y$, x, $x + y$, be the quantities,

$$\therefore\ 3x = 51,$$
$$\text{and } x = 17;$$

$$\text{also } 2x - y : 2x + y :: 8 : 9,$$
$$\text{and } 4x : 2y :: 17 : 1,$$
$$\text{or } 2 \times 17 : y :: 17 : 1,$$
$$\therefore y = 2;$$
and the quantities are 15, 17, 19, gallons.

4. Let $x + y$, x, $x - y$, be the digits,

$$\therefore \frac{100 . (x + y) + 10x + x - y}{3x} = 48,$$
$$\text{or } 111x + 99y = 3 \times 48x,$$
$$\therefore 37x + 33y = 48x,$$
$$\text{and } 33y = 11x,$$
$$\therefore x = 3y;$$

also $100 . (x + y) + 10x + x - y - 198 = 100 . (x - y) + 10x + x + y$,

$$\text{or } 198y = 198,$$
$$\text{and } y = 1,$$
$$\therefore x = 3,$$
and the number is 432.

5. Let $x - y$, x, and $x + y$, be the quantities,

$$\therefore 3x = 24,$$
$$\text{and } x = 8;$$
$$\text{also } 8 - y + \frac{2}{5} . (8 + y) : 8 + \frac{3}{5} . (8 + y) :: 5 : 7,$$
$$\text{or } 56 - 3y : 64 + 3y :: 5 : 7,$$
$$\therefore 56 - 3y : 120 :: 5 : 12,$$
$$\text{and } 56 - 3y = 50,$$
$$\therefore 3y = 6,$$
$$\text{and } y = 2,$$
$\therefore$ the quantities were 6, 8, and 10.

6. Let xy^3, xy^2, xy, and x, be the numbers,

$$\therefore (xy^3 - xy^2 =) \, xy^2 . (y - 1) = 36,$$

$$\text{and } (xy - x =)\, x\,.\,(y-1) = 4,$$
$$\text{whence } \left(\frac{xy^2}{x} =\right) y^2 = 9,$$
$$\text{and } y = 3,$$
$$\therefore\ x = 2,$$
and the numbers are 54, 18, 6, and 2.

7. Let $x - y$, x, and $x + y$, be their wages,
$$\text{whence } 3x^2 = 147,$$
$$x^2 = 49, \text{ and } x = 7;$$
$$\text{and } (x^2 + xy) - (x^2 - xy) = 2xy = 28,$$
$$\therefore\ xy = 14,$$
$$\text{or } y = 2;$$
and their wages were 5, 7, and 9, shillings.

8. Let x, xy, xy^2, be the numbers,
$$\therefore\ x + xy = 9,$$
$$\text{and } x + xy^2 = 15,$$
$$\text{whence } \frac{9}{1+y} = \frac{15}{1+y^2},$$
$$\text{and } y^2 - \frac{5}{3}y + \frac{25}{36} = \frac{2}{3} + \frac{25}{36} = \frac{49}{36},$$
$$\therefore\ y - \frac{5}{6} = \frac{7}{6},$$
$$\text{and } y = 2,$$
$$\therefore\ x = 3,$$
and the numbers are 3, 6, 12.

9. Let $\frac{x}{y}$, x, and xy, be the numbers,
$$\therefore\ \frac{x}{y} + x : x + xy :: 1 : 2,$$
$$\text{or } \frac{1}{y} + 1 : 1 + y :: 1 : 2,$$

$$\therefore \frac{1}{y} : 1 :: 1 : 2,$$

and $y = 2$,

hence $\left(\frac{x}{2} + x + 2x =\right) \frac{7x}{2} = 14$,

$$\therefore x = 4,$$

and the numbers are 2, 4, 8.

10. Let $\frac{x}{y}$, x, xy, be the numbers,

$$\therefore x^2 = 64, \text{ and } x = 4;$$

also $\frac{x^3}{y^3} + x^3 + x^3y^3 = 584$,

and $\frac{1}{y^3} + 1 + y^3 = \frac{584}{64} = \frac{73}{8}$,

$$\therefore y^6 - \frac{65}{8} \cdot y^3 + \overline{\frac{65}{16}}\Big|^2 = \frac{4225}{256} - 1 = \frac{3969}{256},$$

and $y^3 - \frac{65}{16} = \pm \frac{63}{16}$,

$$\therefore y^3 = 8 \text{ or } \frac{1}{8},$$

and $y = 2$ or $\frac{1}{2}$,

and the numbers are 2, 4, 8.

11. Let x, xy, xy^2, xy^3, be the numbers,

$$\therefore x + xy^3 : xy + xy^2 :: 7 : 3,$$

and $1 - y + y^2 : y :: 7 : 3$,

$$\therefore 1 + y^2 : y :: 10 : 3,$$

and $3y^2 + 3 = 10y$,

$$\therefore y^2 - \frac{10}{3} \cdot y + \frac{25}{9} = \frac{25}{9} - 1 = \frac{16}{9},$$

$$\therefore y - \frac{5}{3} = \pm \frac{4}{3},$$

$$\text{and } y = 3 \text{ or } \frac{1}{3};$$

$$\text{and } (xy^3 - xy =)\ 24x = 24,$$

$$\therefore\ x = 1,$$

and the numbers are 1, 3, 9, 27.

12. Let $x =$ the number of days,

$$\therefore\ [6 + (x - 1) . 2] . \frac{x}{2} = x^2 + 2x = \text{number of miles A went,}$$

$$\text{and } [8 + (x - 1) . 2] . \frac{x}{2} = x^2 + 3x = \text{number B went,}$$

$$\therefore\ 2x^2 + 5x = 168,$$

$$\text{and } x^2 + \frac{5}{2} . x + \frac{25}{16} = \frac{168}{2} + \frac{25}{16} = \frac{1369}{16},$$

$$\therefore\ x + \frac{5}{4} = \frac{37}{4},$$

$$\text{and } x = 8.$$

13. Let $x =$ the number of days the first travels,

$$\therefore\ [2 + (x - 1) . 2] . \frac{x}{2} = x^2 = \text{the number of miles he travels,}$$

and $[24 + (x - 4) . 1] . \dfrac{x - 3}{2} = \dfrac{(x + 20) . (x - 3)}{2} =$ the number the second travels;

$$\therefore\ 2x^2 = x^2 + 17x - 60,$$

$$\text{and } x^2 - 17x + \left(\frac{17}{2}\right)^2 = \frac{289}{4} - 60 = \frac{49}{4},$$

$$\therefore\ x - \frac{17}{2} = \pm\frac{7}{2},$$

$$\text{and } x = 5 \text{ or } 12,$$

$\therefore$ the number required is 2 or 9.

14. Let $2x =$ a side of the triangle,

$$\therefore\ x . \frac{x + 1}{2} = x . (x - 4) - 5,$$

$$\text{or } x^2 + x = 2x^2 - 8x - 10,$$

$$\therefore\ x^2 - 9x + \frac{81}{4} = \frac{81}{4} + 10 = \frac{121}{4},$$

$$\text{and } x - \frac{9}{2} = \pm\frac{11}{2},$$

$$\therefore\ x = 10,$$

and the sides of the triangle are 20, and of the parallelogram 20 and 12 yards.

15. Let $x - 3y$, $x - y$, $x + y$, $x + 3y$, be the numbers,

$$\therefore\ 4x = 28, \text{ and } x = 7;$$

$$\text{also } (x^2 - 9y^2) \,.\, (x^2 - y^2) = 585,$$

$$\text{and } \therefore\ 9y^4 - 490y^2 + 2401 = 585,$$

$$\text{and } 9y^4 - 490y^2 + \left.\frac{245}{3}\right|^2 = \frac{60025}{9} - 1816 = \frac{43681}{9},$$

$$\therefore\ 3y^2 - \frac{245}{3} = \pm\frac{209}{3},$$

$$\therefore\ 3y^2 = 12,$$

$$y^2 = 4, \text{ and } y = 2;$$

and the numbers are 1, 5, 9, 13.

16. Let $x - 3y$, $x - y$, $x + y$, $x + 3y$, be the numbers,

$$\therefore\ 2x^2 - 8xy + 10y^2 = 34,$$

$$2x^2 + 8xy + 10y^2 = 130,$$

$$\therefore \text{ by addition, } 4x^2 + 20y^2 = 164,$$

$$\text{and } x^2 + 5y^2 = 41,$$

$$\text{also } 16xy = 96,$$

$$\therefore\ xy = 6;$$

$$\text{hence } \frac{36}{y^2} + 5y^2 = 41,$$

$$\text{and } 5y^4 - 41y^2 = -36,$$

$$\text{and } y^4 - \frac{41}{5} \,.\, y^2 + \left.\frac{41}{10}\right|^2 = \frac{1681}{100} - \frac{36}{5} = \frac{961}{100},$$

$$\therefore\ y^2 - \frac{41}{10} = \pm\frac{31}{10},$$
$$y^2 = 1, \text{ and } y = 1,$$
$$\therefore\ x = 6,$$
and the numbers are 3, 5, 7, 9.

17. Let x, xy, xy^2, xy^3, be the shares,

$$\therefore\ xy^3 - x : xy^2 - xy :: 37 : 12,$$
$$\text{or } y^2 + y + 1 : y :: 37 : 12,$$
$$\therefore\ y^2 + 1 : y :: 25 : 12,$$
$$\text{and } y^2 - \frac{25}{12}y + \overline{\frac{25}{24}}\Big|^2 = \frac{625}{576} - 1 = \frac{49}{576},$$
$$\therefore\ y - \frac{25}{24} = \pm\frac{7}{24},$$
$$\text{and } y = \frac{4}{3} \text{ or } \frac{3}{4};$$
$$\text{hence } x + \frac{4}{3}x + \frac{16}{9}\,.\,x + \frac{64}{27}\,.\,x = 700,$$
$$\therefore\ 175x = 27 \times 700,$$
$$\text{and } x = 108;$$
$\therefore$ their shares are £108, £144, £192, £250.

18. Let $x-2y$, $x-y$, x, $x+y$, $x+2y$, = the number of days,

$$\therefore\ 5x = 20, \text{ and } x = 4;$$
$$\text{now } \frac{1}{x-2y} + \frac{1}{x-y} + \frac{1}{x+y} + \frac{1}{x+2y} = \frac{87}{60} - \frac{1}{4} = \frac{6}{5},$$
$$\text{whence } \left.\begin{array}{l} (x^2-y^2)\,.\,(x+2y) \\ +\,(x^2-4y^2)\,.\,(x+y) \\ +\,(x^2-4y^2)\,.\,(x-y) \\ +\,(x^2-y^2)\,.\,(x-2y) \end{array}\right\} = \frac{6}{5}(x^2-y^2)\,.\,(x^2-4y^2),$$
or $[2x\,.\,(x^2-y^2) + 2x\,.\,(x^2-4y^2) =]\ 2x\,.\,(2x^2-5y^2) =$
$\frac{6}{5}\,.\,(x^2-y^2)\,.\,(x^2-4y^2),$

$$\text{and } x\,(2x^2 - 5y^2) = \frac{3}{5}\,.\,(x^4 - 5x^2y^2 + 4y^4),$$

$$\text{or since } x = 4,\ 2x^2 - 5y^2 = \frac{3}{5}\,.\,(x^3 - 5\,x\,y^2 + y^4),$$

$$\text{hence } y^4 - \frac{35}{3}\,.\,y^2 + \overline{\frac{35}{6}}\Big|^2 = \frac{1225}{36} - \frac{32}{3} = \frac{841}{36},$$

$$\text{and } y^2 - \frac{35}{6} = \pm\frac{29}{6},$$

$$\therefore\ y^2 = \frac{64}{6} \text{ or } 1,$$

$$\text{and } y = 1\,;$$

and the number of days $= 3, 4, 5, 6$.

19. Since the difference of the squares of the extremes is equal to 16 times the mean, if $x =$ the mean, the extremes will be $x - 4$, and $x + 4$.

Now $x - 4$ being drawn off, there remain $28 - x$,

$\therefore\ 24 : 28 - x :: x :$ the spirit drawn off the second time $=$ $\frac{x}{24}\,.\,(28 - x)$, and there now remain $28 - x - \frac{x}{24}\,.\,(28 - x) =$ $\frac{(24 - x)\,.\,(28 - x)}{24}$; and this by the supposition, $= \frac{24}{6} = 4$,

$$\text{whence } x^2 - 52x + 672 = 96,$$

$$\text{and } x^2 - 52x + 676 = 100,$$

$$\therefore\ x - 26 = \pm\,10,$$

and $x = 16$ or 36, the latter of which cannot answer the conditions; $\therefore\ x - 4 = 12 =$ the quantity first drawn,

and $(24 - x)\,.\,\frac{x}{24} = 8 =$ the quantity drawn the second time,

whence $4 =$ the quantity now remaining,

and $24 : 4 :: (x + 4 =)\ 20 :$ spirit drawn the third time $= 3\frac{1}{3}$.

20. Let $x =$ the sum paid by the youngest,

and $2y =$ the number of persons,

$$\therefore\ [2x + (2y - 1) \,.\, 5] \,.\, y = \text{value of the field} = 345,$$

$$\text{and } [2x + (y - 1) \,.\, 5] \,.\, \frac{y}{2} = 22y,$$

$$\therefore\ 2x + (y - 1) \,.\, 5 = 44,$$

$$\text{and } 2x + 5y = 49,$$

$$\therefore\ 2xy + 5y^2 = 49y,$$

$$\text{but } 2xy + 10y^2 - 5y = 345,$$

$$\therefore \text{ by subtraction, } 5y^2 - 5y = 345 - 49y,$$

$$\text{and } y^2 + \frac{44}{5}y + \overline{\frac{22}{5}}\Big|^2 = \frac{484}{25} + \frac{345}{5} = \frac{2209}{25},$$

$$\therefore\ y + \frac{22}{5} = \frac{47}{5},$$

$$\therefore\ y = 5;$$

$$\text{and } x = 12;$$

and the number of persons was 10.

21. Let a = each man's daily provision,

x = the number of men at first,

$\therefore\ a \,.\, (2x - 42) \,.\, 4 = (8x - 168) \,.\, a$ = stock of provisions,

$a \,.\, (2x - 30) \,.\, 3 = (6x - 90) \,.\, a$ = stock exhausted at the end of the 6th day;

$$\therefore\ (2x - 78) \,.\, a = \text{remainder} = 366a,$$

$$\therefore\ x = 222;$$

and $222 - 136 = 86$ = number of men after the sally.

Let n = the number of days the provision lasted afterwards,

$$\therefore\ [172 - (n - 1) \,.\, 10] \,.\, \frac{n}{2} \,.\, a = 366a,$$

$$\text{or } 91n - 5n^2 = 366,$$

$$\therefore\ n^2 - \frac{91}{5} \,.\, n + \overline{\frac{91}{10}}\Big|^2 = \frac{8281}{100} - \frac{366}{5} = \frac{961}{100},$$

$$\text{and } n - \frac{91}{10} = \pm \frac{31}{10},$$

$$\therefore n = 6;$$

whence 86 − 60 = 26 = number of men remaining after the provisions were exhausted.

22. Let x = the number of days the voyage was expected to last,

∴ $175x$ = the quantity of water laid in, supposing each man to drink daily a pint of water.

On the 31st day, the quantity drunk was 172; on the 32d, 169; and thus the quantity of water consumed daily, after 30 days, forms a decreasing arithmetical progression, whose common difference is 3, and number of terms $x - 9$.

∴ the quantity drunk after 30 days $= [344 - (x - 10) \cdot 3] \cdot \frac{x-9}{2}$, which must $= (x - 30) \cdot 175$,

$$\therefore 374x - 3x^2 - 3366 + 27x = 350x - 10500,$$

$$\text{or } 51x - 3x^2 = -7134,$$

$$\text{and } x^2 - 17x + \overline{\frac{17}{2}}\Big|^2 = 2378 + \frac{289}{4} = \frac{9801}{4},$$

$$\therefore x - \frac{17}{2} = \frac{99}{2},$$

$$\text{and } x = 58,$$

$$\therefore 58 + 21 = 79 \text{ days the voyage lasted;}$$

and $172 - (x - 10) \cdot 3 = 28$, the number of men alive when the vessel entered the harbour.

23. Let xy^2, xy, x, = the sums they had at first,

and z = what B lost,

∴ $xy - z$ = what he had remaining;

and $xy - z : z :: xy + x : xy - x$,

∴ $xy : z :: 2xy : xy - x$,

∴ $z = \frac{1}{2} \cdot (xy - x)$ = what B lost,

and ∴ $\frac{1}{2} \cdot (xy + x)$ = what he had remaining;

also as C had x, $\therefore$ A had xy, and lost $xy^2 - xy$,

whence $xy^2 - xy + x = 64 + xy$,

and $xy^2 - 2xy + x = 64$;

also, $xy + \frac{1}{2}(xy + x) + x : xy^2 - xy + \frac{1}{2}(xy - x) :: 6 : 7$,

or $\frac{3}{2}(xy + x) : xy^2 - \frac{1}{2}(xy + x) :: 6 : 7$,

and $\frac{1}{2}(xy + x) : xy^2 - \frac{1}{2}(xy + x) :: 2 : 7$,

$\therefore \frac{1}{2}(xy + x) : xy^2 :: 2 : 9$,

and $y + 1 : y^2 :: 4 : 9$,

$\therefore y^2 - \frac{9}{4}y = \frac{9}{4}$,

and $y^2 - \frac{9}{4}y + \frac{81}{64} = \frac{9}{4} + \frac{81}{64} = \frac{225}{64}$,

whence $y - \frac{9}{8} = \pm \frac{15}{8}$,

and $y = 3$, or $-\frac{3}{4}$;

also $9x - 6x + x = 64$,

$\therefore 4x = 64$,

and $x = 16$,

$\therefore$ they had 144, 48, and 16, respectively.

24. Let $1 =$ the circumference of the fore wheel of the fly,

$x =$ the common ratio,

$\therefore 1, x, x^2$, are the proportional lengths of the three wheels. The distances described by the two coaches in the same time are as $12 : 2 = 6 : 1$. The number of revolutions of a wheel in a given time $\propto \frac{\text{distance}}{\text{circumference}}$,

$\therefore \frac{1}{x}, 1, \frac{6}{x^2}$, are the revolutions made by the three wheels in the same time; x, x^2, 6, are also revolutions made in the same time, since they are equimultiples of the former. But these revolutions increase in an arithmetical progression, whose common difference is x,

$$\therefore\ x^2 = x + x = 2x,$$

$$\text{and } x = 2,$$

$\therefore$ 1, 2, 4, are the proportional lengths of the wheels, and 2, 4, 6, the revolutions in a given time.

24. Let $x =$ the number of merchants,

and $y =$ the number of months at the end of which the captain was entitled to a £100 share;

$$\therefore\ y : 25 :: 1 : \frac{25}{y} = \text{the number of the captain's shares,}$$

$$25x = \text{the middle term, and } y = \text{the common ratio,}$$

$$\therefore\ \frac{25x}{y} + 25x + 25xy = 100x + 1375,$$

$$\text{or } \frac{x}{y} + x + xy = 4x + 55,$$

$$\therefore\ \frac{x}{y} - 3x + xy = 55;$$

also $100x + 500 =$ the sum to be divided after deducting prize-money,

$$\text{now, captain's share} = \frac{1}{5} \text{ company's share,}$$

$$\text{and } \therefore\ = \frac{1}{6} \text{ whole number of shares;}$$

$$\text{hence } \frac{100x + 500}{6} : \frac{500x + 2500}{6} :: \frac{25}{y} : x,$$

$$\therefore\ x = \frac{125}{y},$$

$$\text{and } \frac{125}{y^2} - \frac{375}{y} = -70,$$

$$\text{or } \frac{25}{y^2} - 15 \cdot \frac{5}{y} + \overline{\frac{15}{2}}\Big|^2 = \frac{225}{4} - 14 = \frac{169}{4},$$

$$\therefore \frac{5}{y} - \frac{15}{2} = \pm \frac{13}{2},$$

$$\text{and } \frac{5}{y} = 1 \text{ or } 14,$$

$$\therefore y = 5,$$

$$\text{and } x = 25.$$

25. Let $x =$ the sum saved by the eldest child, or 3d in family,
$y =$ the sum saved by the seventh, or 9th in family,

$$\therefore \frac{y}{6} = \text{the number of bushels,}$$

$$\text{and the sum saved by the 5th child} = \frac{9-7}{9-3} \cdot x + \frac{7-3}{9-3} \cdot y = \frac{x+2y}{3},$$

$$\therefore \text{ a bushel cost } \frac{1}{2} \cdot \left(\frac{4x + 2y}{3} + 120\right) = \frac{2x + y}{3} + 60,$$

and the sum monthly saved $= \frac{y}{6} \cdot \frac{2x + y + 180}{3} - 39 =$
$\frac{2xy + y^2}{18} + 10y - 39.$

$$\text{Now, after the rise, a bushel cost } \frac{2x + y}{3} + 84,$$

$$\text{and the number of bushels} = \frac{y}{6} - 2,$$

$$\therefore \text{ the sum saved} = \frac{2xy + y^2}{18} + 14y - \frac{4x + 2y}{3} - 168,$$

$$\text{which } \therefore \text{ is} = \frac{2xy + y^2}{18} + 10y - 39 - 105,$$

$$\text{whence } 4y - \frac{4x + 2y}{3} = 24,$$

$$\text{and } \therefore 2x = 5y - 36.$$