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KANT'S DOCTRINE OF THE BASIS OF MATHEMATICS[1]

THE treatment of the philosophy of mathematics by Kant, in the 'Critique' and in the 'Prolegomena,' is equally characteristic of his philosophy in general, and of the age in which he did his work. The age in question was one of a rapid development in certain relatively advanced regions of mathematical research. But it was also an age of disillusionment regarding the power of mathematical science to demonstrate metaphysical truth. It was furthermore a time when mathematical inventiveness was decidedly more noticeable than mathematical rigor; and when construction had for the moment outrun logical reflection in mathematics. On the other hand, it was a time when the philosophers had learned many lessons concerning the importance of experience for their own constructions. Consequently it was a period when mathematics and philosophy were further apart, in spirit and in interest, than they had been during a portion of the seventeenth century. The mathematicians were in a sense more disposed to novel speculation and researches. The philosophers were less confident of the success of *a priori* constructions. Since the death of Leibnitz no thorough-going effort towards a philosophy of mathematics had been made. The efforts of Leibnitz himself regarding this topic were very imperfectly known in the age when Kant wrote. In brief, it was a moment when a sharp sundering of the task of the mathematician and of the philosopher appeared especially called for. That a critical philosopher should lay stress upon the contrast was, therefore, extremely natural. Even as a rationalist Kant had to feel that reason in philosophy had other offices than it had in mathematics. And the mathematicians of the time were too little possessed of an insight into the philosophy of their own science to give him any aid in bridging the chasm that seemed to him to divide the two kinds of activity. Furthermore, the way in which his own critical thought came to him, namely, through the reflections which first culminated in the year 1769, was a way which served to emphasize the contrast be-

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tween mathematical and philosophical truth. The discovery of the essay of 1769, on 'The Reason for the Distinction between Regions in Space,' was such as very soon to lead Kant to the conclusion which characterizes the dissertation of 1770, namely, to the conclusion that space and time predetermine the form of the phenomenal world of our percepts, but do not throw any light upon the ultimate nature of things. When Kant passed from the doctrine of the dissertation to the later deduction of the categories, he indeed learned to abandon all methods of obtaining knowledge of the noumena; but the failure to win this knowledge did not bring philosophy any nearer again to the position of mathematics. The business of philosophy remains for Kant the criticism of fundamental concepts, and the determination of their range of validity. The business of mathematics he conceived as the construction of those objects whose laws are determined by the forms of our perceptual faculty. This contrast of the two is henceforth, therefore, absolute. Philosophy, as the *Methodenlehre* of the 'Critique' explains to us, possesses no axiom of a theoretical character, and can justify its concepts only by an explicit proof of their necessity as the conditions of a possible experience. Philosophy, therefore, can never construct its objects of synthetic knowledge. Such objects, constructed and presented by the mind for itself and to itself, are the topics of mathematical science alone. The certainty of mathematical science depends entirely upon the necessity which for us our forms of perception possess. Kant never falters in his assurance that these, our forms of perception, are determinate, are finished, are for us absolutely predetermined by our constitution. In the 'Prolegomena' he triumphantly shows how the possibility of an exact knowledge of mathematical truth is explicable upon his theory and not upon any other. In emphasizing the contrast between mathematical and philosophical method, he expressly does so on the ground that no form of pure thinking can ever present to the mind an object, or can ever demonstrate the properties of this object otherwise than by a mere analysis of a prevously given concept. And Kant always confidently speaks as if mere analysis must necessarily lead to comparatively barren and unprogressive scientific precedure. By pure thought I can discover that man is rational, only in case my definition of man has already included rationality amongst the marks which are to be characteristic of man. If mathematical science is able to know objects *a priori* and in ways which are both synthetic and instructive, that must be because mathematical science depends upon something quite different from pure thinking. As this something is *a priori* and necessary, it can only be, in Kant's opinion, a form of perception, and our power to construct objects in accordance with this form. So much, then, for the view of mathematics which Kant took.

The development of mathematical science since the time of Kant has followed a path which his influence has no doubt affected, but whose direction he was entirely unable to foresee. The mathematicians of Kant's time did indeed make unhesitating use of generalizations derived from the observations of objects constructed in space; and they made this use in a way which rigorous mathematicians no longer regard as justifiable. The mathematicians since the time of Kant have tended more and more to follow the very direction which he would have warned them not to follow. Namely, they have, on the whole, increasingly forsaken the method of trusting to perceptual construction as a means of mathematical demonstration. Geometry without diagrams is now the order of the day amongst the most rigorous students of the bases of geometry. Where diagrams are introduced, the reader is especially warned (as in Hilbert's recent lectures on the foundations of geometry, autographed for his hearers)—the reader is expressly warned, I say, to take as it were no logical notice of the diagrams, to regard them merely as hints, illustrations, suggestions of a relational structure whose consequences are to be developed without any use of the perceived properties of diagrams. In this sense it would seem as if the ideal of modern mathematics were the ideal of a science of pure concepts—the very ideal that Kant expressly declared to be impossible for mathematical science. Kant warned the philosophers that they must not attempt to use the methods of mathematics, just because they could not construct their concepts *a priori*. The modern mathematician is warned that he must not put his trust in the properties of visible figures, just because the ideal of his science, the ideal of the search for necessary conclusions, is an ideal which perceptual intuition rather confuses than directly furthers. As Kant interprets the business of mathematics, the mathematician has seen, and therefore believes. He believes because he has seen *a priori*. The modern logicians of mathematics would rather seem to say, Blessed is he who has not seen, in Kant's sense of the *reine Anschauung,* but has yet learned rationally to believe; for he alone has learned with true rigidity to grasp the meaning of his fundamental concepts.

As an incident of this whole development of modern mathematical logic there have appeared various doctrines concerning the bases of geometry which appear to be remote enough from those which Kant explicitly recognized. The tridimensionality of space is for Kant a result of the *a priori* form of intuition. The modern geometer would in general admit that we can indeed see no conceptual reason why space must be limited to three dimensions. But instead of saying like Kant, that the limitation of space to three dimensions is something *a priori*, necessary and certain, the modern geometer would regard this limitation as something which, from the point of view of pure mathematics, is not necessary at all. The properties of a tridimensional space can be, with sufficient definition of the other properties of space, rationally developed. But the form of a tridimensional space is, logically speaking, only one of countless possible forms, whose logically definable properties are precisely as justified a topic of pure mathematics as are the properties of the space of our ordinary geometry. If you reply that tridimensional space is alone worthy to be called space, because that is the only kind of space that we happen to have, then the modern mathematician replies that this limitation may be as important as you please for philosophy, but is an empirical limitation, which makes tridimensional geometry of great importance for applied mathematics, but which, just because of the limitation, has nothing whatever that is mathematically necessary about it. In brief, show me a form of intuition of the Kantian type—so the modern logician of mathematics might say,—and if I accepted your account of it, I should regard it merely as a character belonging to a specifically defined human experience, a character which for that very reason would have no sort of mathematical necessity about it, and, therefore, no authority which need limit in the least mathematical generalizations which may be suggested by this form, but which may vary from it in any given way.

So much for tridimensionality. But of decidedly greater importance for modern theoretical geometry than the merely formal possibility of doing away with the limitation of geometry to three dimensions is that study of geometrical implications, necessities, and possibilities which has appeared in connection with the non-Euclidean geometry, and which still continues, in the form of constantly new additions to our present list of possible geometries. The geometry which Euclid found it convenient to work out explained the relations present in a large number of observable diagrams, by means of certain simple principles. As a fact, this explanation of the observed phenomena by the assumed principle was, in Euclid's case, incomplete, since there are demonstrations in Euclid which do not follow from the axioms alone, but which depend upon the observation of special diagrams. The modern geometer regards such demonstrations as unsatisfactory, just because they make use of principles which the diagram more or less unconsciously suggests, and which the Greek geometer did not make explicit in his list of axioms and postulates. In other words, that very use of intuition which Kant regarded as geometrically ideal, the modern geometer regards as scientifically defective, because surreptitious. No mathematical exactness without explicit proof from assumed principles —such is the motto of the modern geometer. But suppose the reasoning of Euclid purified of this comparatively surreptitious appeal to intuition. Suppose that the principles of geometry are made quite explicit at the outset of the treatise, as Pieri and Hilbert or Professor Halstead or Dr. Veblen makes his principles explicit in his recent treatment of geometry. Then, indeed, geometry becomes for the modern mathematician a purely rational science, so far as any one special form of geometry is concerned. But hereupon a question of great philosophical interest becomes only all the more insistent. Any one form of geometry, such, for instance, as the Euclidean geometry, depends upon assuming the simultaneous truth of a number of distinct fundamental principles. It is possible to show, and in recent mathematical treatises it has been distinctly shown, that such principles can be so stated as to be logically quite independent of one another, so that no one of them could be deduced from the others. A system of objects which should conform to some of these principles and not to the others is therefore perfectly definable, is mathematically possible. This has been indubitably shown in case of the system of principles assumed by Euclid, and is all the more obvious when to Euclid's explicit principles are added such statements as make explicit the meaning of those principles which, guided by surreptitious appeals to intuition, he more or less unconsciously assumed. Under these circumstances, that very indifference to what we perceptually find present in this or in that diagram, that indifference, I say, which modern mathematical method encourages makes all the more inevitable the question: What necessity is there of assuming precisely that system of mutually independent first principles which Euclid found it convenient to assume, and which, with some supplements, the modern expositors of Euclidean geometry employ? Since it is demonstrable that no sort of logical inconsistency would be involved in supposing the existence of systems of objects which satisfy some of these principles and not others, whence, if from anywhere, is derived the authority of this particular system? As is well known, the modern logicians of mathematics differ a good deal in the theories of knowledge which they use in their answer to this question. But it may be said that very few students of the logic of mathematics at the present time can see any warrant in the analysis of geometrical truth for regarding just the Euclidean system of principles as possessing any discoverable necessity. The facts of the world of experience seem to be economically describable, so many say, in the terms of Euclidean geometry. But in this sense Euclidean geometry differs in no whit from the concepts of the theory of energy. Mathematical necessity belongs to the deductions from the principles, and to them only. For those who take this view a considerable range of difference of opinion still remains open, regarding the sense in which this convenience of Euclidean geometry as a means of describing the world is forced upon us by experience. Some are disposed to say, No other system of geometry seems to be probably applicable to our physical world. Some would insist that, for reasons upon which I need not here dwell, the known phenomena might be characterized in non-Euclidean terms, if only we could agree to accept certain conventions which actually run counter to our present mental habits. There are some mathematical logicians who are disposed to accept the Kantian view far enough to admit that the Euclidean space form is the expression of what we men, so long as we remain true to our present perceptual nature, must needs find the most natural way of interpreting spatial experience. But about one matter nearly all the modern students who approach the subject without a distinct preexisting Kantian bias are agreed, namely, that whatever necessity belongs to Euclidean geometry, apart from the necessity of its deductions, is in no sense mathematical necessity, any more than the present necessity that bitter tastes, if sufficiently strong, should be disagreeable is a mathematical necessity. With this view I myself agree. It determines our judgment as to the positive value of Kant's view of the basis of mathematics. For mathematics, from the modern point of view, is *concerned with necessary inferences.* If the field within which necessary inference is itself a possible matter is in any way a restricted field, that is, if there are some subjects that admit of systematic mathematical treatment, while other subjects altogether forbid such treatment, then the field within which systems of exact mathematical inference are possible is determined by the categories of thought, and not by the forms of any intuition. The ideal of mathematical science is the exact development of the consequences of all those ideal forms which it is possible to subject to exact treatment at all.

Thus, if the *ego* has a determinate relational structure, and if this relational structure admits of mathematical treatment, then, and just in so far, the science of the *ego* will become a branch of mathematics. What will make it so, if at all, will not be the necessity under which we now stand of appreciating the presence of the *ego,* but the capacity which our concepts of the *ego* may possess of development in terms of a precisely definable system of relationships. In the same way, if our spatial experience presents a character which admits, as it does, of precise relational definition and development, then we shall have, as we have, a mathematical geometry. But the mathematical necessity of this geometry will belong solely to the field where the exact development of the relational structure of the ideal entity called, for instance, Euclidean space, is possible. Mathematical necessity will in no wise be possessed by this entity itself as distinct from any other entity (say a non-Euclidean space), which can be treated with equal exactness. God may have made our space-perceiving nature on the lines of Euclid's geometry. If he did, that is a matter of experience, not of mathematical necessity.

If your boots have a relationally exact structure, there may be a mathematical science of this structure, precisely so far as the relations in question are exact. But it will be no part of mathematical science to determine whether or why you have any boots at all. If you insist that the form of your boots is determined *a priori* by the form of your feet (a proposition which may be regarded rather as advisable than as necessary), then the form of your feet will be a topic for mathematical science, precisely so far as the relations involved in this form constitute a system reducible to certain fundamental principles, and such that the characters of this system can be deduced from these principles. But mathematical science will have nothing to do with the question why you have any feet at all, or why you have not fins instead. If one conceives an absolute being possessed of a totality of perfect mathematical knowledge, so that it defined with absolute adequacy all possible relational systems, even such a being, so far as it was merely mathematical, would define only general types of ideal objects, and not individual objects such as *these* boots and feet and fins, unless, indeed, it added to its mathematical determination such will-decisions as distinguish the individual deed that we do from the possibility that we leave undone. In brief, a form of intuition, if such exists, is precisely a character belonging to the individual nature of man as a real being, and is not a mathematical necessity. It is a mathematical necessity that an ideal entity defined in general as 'a spendthrift' will become bankrupt *if* his capital is so much and *if* he regularly exceeds his income by so much a year. You can provisionally predict when such a spendthrift will become bankrupt. But there is no mathematical necessity that in the real world anybody should be a spendthrift. That result, if it happens, is due to free will, or to inherited disposition, or to training, or to the devil, or to whatever other existence you decide to take into account. As I regard this distinction between the general definition of an ideal necessity and the individual decision of a will as valid from an idealistic point of view just as much as from an empiristic point of view, as I regard the Absolute as subject to this distinction quite as much as we are, just because it is an absolute distinction, I should myself fully agree that a Kantian form of intuition, if you can prove its existence in our own nature, has absolutely no interest as the foundation of any mathematical science, except in so far as it may suggest to some mathematician the particular ideal topics upon which he finds it convenient to build up a mathematical theory.

So much for the way in which the whole modern mathematical development is distinctly opposed to the Kantian conception that something called a form of intuition, distinct from a conceptual system, is a necessary basis of mathematical investigation. But there is indeed quite another aspect of the Kantian doctrine to be considered. Kant, after defining with a natural, but to us no longer interesting, narrowness the business which he calls the mere analysis of pure concepts, decided very correctly that so barren an undertaking as declaring that a rational animal is rational could be of no service for enlarging our knowledge. He accordingly maintains, in the form of the famous distinction between synthetic and analytic judgments, that every significant science which truly enlarges our knowledge depends upon a genuinely constructive and synthetic process. He also very correctly pointed out that every productive type of reasoning depends upon its own sort of experience. Whoever reasons, unquestionably observes something. That is, whoever considers some ideal object, and yet enlarges his knowledge as he does so, gets this knowledge from actually observing what happens to his idea as he works over it. Now observing what happens to one's ideas as one works over them is indeed definable as a kind of rational perception. But the possibility of such rational perception exists quite as much when you are considering the idea of God, or Kant's favorite idea of the possibility of experience, as when you are observing facts of spatial experience, or your boots. There is indeed a great difference between observing an ideal process, and making a decision as to which one of two ideas, whose consequences you have ideally observed, you shall henceforth allow to be individuated as the deed that you choose. There is also a sharp difference between observing such an ideal process, and looking to see whether that which the natural world permits to exist does or does not accord with your ideas. In either case you are observing a process which expresses a purpose. But abstract ideal processes without final and individual decisions, are observed in a way which differs from the way in which decisive and individual facts are identified as actually or as probably real facts of existence. For the ideal processes, with whose consequences mathematical science is alone concerned, are universals in the abstract sense. What you observe as the consequences of such an abstract idea may or may not accord with what your own personal decision, or the decision of the world-will permits to exist as individual fact. Hence observing a mathematical necessity is never the same as observing an individual existence. And for the same reason observing a mathematical necessity is never the same as observing what we call a phenomenon of nature. Nevertheless, observing a mathematical necessity is indeed observing a process of ideal construction, and its results. And every such process of ideal construction unquestionably has a form. This form is, however, not what Kant meant by a form of intuition as distinct from a form of thought, for what you observe when you observe mathematical truth is a precisely and abstractly definable and general *necessity,* which is neither the perception of a fact in the natural world, nor yet a final decision of your own will, nor yet a metaphysically individual thing; but which is precisely *the general way in which this idea has to express itself.* The principle that guides one in such observations is unquestionably what Kant meant by the principle of contradiction, when he called this principle the principle of analytical judgment. So far as an idea is defined as a type of action, a plan, a way of behavior, it necessarily implies whatever is such that the contradictory of this consequence would be opposed to the idea itself. Whenever you observe such implications you observe a system of truth which comes to you as the system of the consequences of certain ideal processes. Such an observation is, however, an observation of synthesis, quite as much as it is an observation of the truth of Kantian analytical judgments. And as a fact, the lesson of Kant's whole deduction of the categories is that analysis apart from synthesis is impossible, and *vice versa.* In consequence, mathematical truth is indeed truth relating to a system of possible experience. And the mathematician observes the structure of this system empirically. Only because what he observes is an abstract process of construction, not an individual phenomenon, the truth that he discovers is of abstractly universal application to all things, whatever they may prove to be—and if such there be—that conform to his ideal constructions. Mathematical insight is, then, not without experience, and, if you please to use the term, not without intuition. But the intuition is not of perceived diagrams, nor of the special conditions of human experience, but of the relational structure of an ideal system.

Mathematical science has nothing to say, for instance, as to whether either human beings or the inhabitants of Mars are necessarily forced to count. Mathematical science defines the eternal validity of numerical truth. This truth is true for us. It is also true for the inhabitants of Mars, *if* there are any such. We experience it to be true because we try an ideal experiment, and see that what was true in this ideal case must needs be true of an infinity of other ideal cases, precisely because of the abstract nature of what we have observed. This which we have observed must be true for all beings and at all times and places, because the opposite would be contradictory. But mathematical science has nothing to say as to whether or no we, or the inhabitants of Mars, must be beings such as are able to perceive this truth. Kant, however, was quite wrong in supposing that the application of the principle of contradiction would give us an analysis only of commonplaces. He was quite right in supposing that whenever we think we engage in a constructive process and observe something. But what we observe when we think is, as the non-Euclidean geometers show, frequently so general that we can define vast numbers of objects that we never hope or even desire to perceive; precisely as a moral agent is capable of conceiving accomplished plans of action that he never hopes to carry out, and that in many cases he deliberately forbids himself to carry out. Yet every consideration of a plan of action is an ideal sort of acting, which simply does not carry itself out into the individual deed, but remains abstract and general.

It is a perfectly fair question to ask, What is the universal form of that abstract type of ideal experience upon which all reasoning processes depend? This, however, is the question, not of the Kantian forms of intuition, but of the categories. The forms of thought are unquestionably the forms of mathematical science. That is what the whole recent mathematical theory has made manifest. On the other hand, the immortal soul of the Kantian doctrine of the forms of intuition remains this, that thinking itself is a kind of experience, that true thinking is synthetic as well as analytic, is engaged in construction of a peculiar kind, and not in mere barren analyses such as the statements that all rational animals are rational. Kant was right that the novelties of mathematical science are due to the observation of the results of constructive processes. He was even right that the observation of a diagram, in so far as the diagram is simply the expression of an idea, may be an admirable guide in the thinking process. He was wrong in supposing that a special form of intuition, such as that of Euclidean space, can have any other necessity than that which every individual fact in the world possesses. Every fact, in my opinion, is what some will decides it to be. Every fact is individual. But that does not determine the range of ideal possibilities, nor the range of mathematical truth. For mathematical truth is concerned with the consequences of ideas in advance of, or apart from, the decision whether those ideas which are then taken as what I have elsewhere called internal meanings, are expressed in individual realities. Mathematical science is abstract, and can, therefore, never define the whole truth. For the whole truth of things is always individual, and is never expressible in terms merely of abstraction, nor in terms of merely logical implication. On the other hand, as soon as you consider any individual fact, as, for instance, the fact that this man has this form of intuition, you consider what, if true, is no topic of mathematical science. I conclude, then, that Kant's theory of the basis of mathematics has been in one respect wholly abandoned, and properly so, by the modern logic of mathematics. In another respect, precisely in so far as Kant declared that constructive synthesis and observation of its ideal results are both necessary for mathematics, Kant was unquestionably right. And as nobody before him had so clearly seen this fact, and as the progress of mathematical logic since his time has been so profoundly influenced by his criticisms, we owe to him an enormous advance in our reflective insight in this field.

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SOME OUTSTANDING PROBLEMS FOR PHILOSOPHY

FOR a long space of time the domains of philosophy and mathematics were regarded less as intersecting spheres of interest than as adjacent fields separated by a kind of 'dead line' that no worker in either might venture to cross without risking the loss both of his identity and of the respect of his fellows. I once heard a distinguished mathematician say that the study of mathematics acts on the metaphysical instinct like sulphur on the itch. Undoubtedly that savant had a philosophy, but, like the Irishman's snake, he was unconscious of it, and he was the less tolerant on that account. On the other hand, the mathematician has not infrequently been compelled to forgive such disrespect as that of Sir Wm. Hamilton's on ground analogous to the good old Catholic principle of invincible ignorance. Happily, the tokens more and more abound that the uncanny day of such misunderstandings is rapidly passing away. It may return again, but not for some generations. A new era has begun that shall be distinguished by intellectual sympathy and cooperation, by increasing wholesomeness of scholarship. The indicia have reference to old records. They indicate the reestablishment of broken traditions of an older time when philosopher and mathematician were often united in a single personality. Such men as C. S. Peirce and Pearson and Mach and Couturat and Poincaré and Georg Cantor, exemplify clearly enough that the larger incarna-