DIFFICULTIES

OF

ELEMENTARY GEOMETRY.

LONDON:
RICHARD CLAY, PRINTER, BREAD STREET HILL.

THE DIFFICULTIES

OF

ELEMENTARY GEOMETRY,

ESPECIALLY THOSE WHICH CONCERN

THE STRAIGHT LINE, THE PLANE, AND THE
THEORY OF PARALLELS.

BY

FRANCIS WILLIAM NEWMAN,

FORMERLY FELLOW OF BALLIOL COLLEGE, OXFORD.

LONDON:
WILLIAM BALL AND CO.
34, PATERNOSTER ROW.
1841.

The reader is requested, on reading Art. 123, to peruse Appendix II., and at Artt. 55, 150, to refer to the Note at the end of the Volume.

CONTENTS.

PART I.

ON THE STRAIGHT LINE AND PLANE.

PART II.

ON PARALLEL STRAIGHT LINES.

PART III.

ON SOLID ANGLES.

PART IV.

ON PLANE CURVES.

PART V.

ON DOUBLE CURVATURE.

PART VI.

ON CURVED SURFACES.

PART VII.

SHORTEST PATH ON A SPHERE.

Appendix I.

Appendix II.

INTRODUCTION.

THIS book consists of extracts from one which was intended to form a continuous system of Elementary Geometry; but as the author finds no reasonable ground for hoping that any one would adopt his system as a whole, he has determined on selecting those parts which are either wholly new, or wanting in the common treatises. In this form they may be read as supplementary, by a student who has gone through Euclid; yet the endeavour has been made so to arrange them that no part shall be unintelligible to a person who may have no previous acquaintance at all with Geometry.

He anticipates that objections will be made on two heads to the methods which he has employed: to the introduction of *Motion* into Geometry, and to the early use of the doctrine of *Limits*. It is said by many that Motion belongs solely to Mechanics, and not to Geometry; but this is a mere dogma, to which it is difficult to find reason for deferring. It is true that in Mechanics the doctrine of Motion is treated, but is treated on a perfectly different footing. In Geometry we pay no attention to Velocity, nor are we concerned with *Measurements* of Time, nor do we consider Motion as an effect of Force. We regard merely the successive changes of position which a body undergoes; and although we know that such changes require both Time and Force, we are not concerned to estimate that Time nor that Force; but we abstract these considerations as irrelevant to the subject. Now the points which are purposely omitted in Geometry are specially discussed by Mechanics; nay, form the sole business of that branch of Mechanics which contemplates Motion at all. Hence we are perfectly clear from the charge of intruding on the province of Mechanics.

This method has been deliberately preferred, from the conviction that no definition of a geometrical figure is so vivid to the understanding, or so satisfactory in a logical point of view, as that which states *how the figure is to be generated.* Unless this can be done, the mind is justly in suspense, whether the definition may not have laid down something self-contradictory and absurd. It may be added that the common systems, from Euclid downwards, introduce the same thing in disguise, and cannot do without it. Geometers call it *Supraposition;* and in the very first theorem of the science it is employed. Yet it might seem as if Euclid had been ashamed of it; for he does not employ it afterwards, in numerous cases where it would have made his proofs clearer and more concise. If, however, it may be used once, it may be used a thousand times; and ought to be used, whenever such advantages are to be gained.

If any one objects to the early use of the doctrine of Limits, it will not be as though it were illogical, but because it is imagined to belong rather to the higher Geometry. If this remark means merely to state the *fact,* that hitherto it has not been used in the Elements, the writer can see no reason why the beaten track should be held sacred, if a better offer itself. To him it appears that the notion of a Limit enters into the very first conceptions of Geometry, (as of a surface, a line, and a point,) and is essential to the establishment of those Laws, on which he believes the science to rest. It is equally essential to an understanding of the doctrine of Ratio and

Proportion, if incommensurable quantities are to be treated with logical accuracy. Nor does it seem to involve any difficulty comparable to those with which the Elements of Geometry abound.

The Lemma concerning Proportions has been added, as appearing to him the most convenient link between the doctrines of Proportion and Magnitudes. It consists in nothing but the first and simplest problem of the Integral Calculus in disguise.

He has to acknowledge his obligation to Mr. Perronet Thompson's "Geometry without Axioms," (4th edition,) for two very valuable hints, which have materially influenced the form in which the First Part of these discussions now appears; namely, First, to regard the doctrine of the Sphere as prior to that of the Plane; Secondly, to pay peculiar attention to such lines and surfaces as are capable of sliding along themselves. But in the actual execution of the plan there is here little or nothing in common with the method which Mr. T. has chosen.

GENERAL PLAN OF THE FOLLOWING TREATISE.

We design to discuss successively certain points which appear to be defective in the Elements of Geometry.

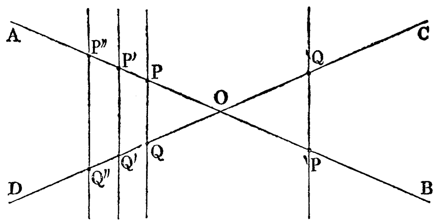
FIRST, The doctrine of the Straight Line and Plane.

In the common treatises a Straight Line is defined as "one which lies *evenly* between its extreme points;" but as the word *evenly* has not been explained, this is not more instructive than to be told that it lies *straightly* between its extreme points. A confession of the uselessness of the definition is found in the device of an axiom, that "two straight lines cannot enclose a space;" which ought to be a corollary from the definition, if the latter were adequate.

Some have defined a Straight Line to be "the shortest path between two given points;" but this is not legitimate, till it have been proved that there is always some *one* path shorter than all other paths. If any one choose to resort to a very simple experiment in proof of this, he will at once cut short all difficulties attending the doctrine both of Straight Line and of Plane. And this is, perhaps, the course which all our minds secretly follow. But it is thought right to appeal to experiment as little as possible; and perhaps the above appeal is not necessary. The method used below of explaining the term *evenly* is fundamentally the same with that which Professor Leslie suggested; but is much more developed.

The definition of the Plane found in Simson's Euclid and elsewhere, labours under the serious fault of being *redundant*. A Plane (say they) is a surface, such that, if *any two* points *whatever* be joined by a straight line, this line shall lie wholly upon the surface. But how are we to know that such a surface is possible? Let us try to *generate* such a one. Take two straight lines, (*Fig.* 1.) AOB, COD, intersecting each other in O. In OA, OD, (or else in their prolongations OB, OC,) take two points, P and Q; and *first*, let the distances OP, OQ, be in all cases equal; and through P and Q pass a straight line. Then if the distance OP increase indefinitely, and again diminish indefinitely, the straight line PQ, moving with it, traces out a surface; which surface has the property, that "if any two points in it *that lie in the same generatrix* PQ be joined by a straight line, this line lies wholly upon the surface." Now it remains to prove that the same will be true when the points joined are neither on the same generatrix, nor in the lines AB, CD. But *secondly*, if to meet this difficulty we suppose OP and OQ not to be equal, but to bear some other ratio, or to vary

Fig. 1.

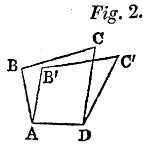


independently, it will then be no longer manifest that the locus or surface, generated by the motion of PQ, is a *single* continuous sheet, and not an infinity of different surfaces. This, which needs to be proved, is assumed, and that, covertly, in the common definition.

Some have put into its place the definition, that "a plane is a surface, which lies *evenly* between its extreme boundaries." But this is doubly objectionable, both from the vagueness of the term *evenly*, and from the want of proof that there is any one such surface; to say nothing of our inability to decide what is meant by "extreme boundaries." The outline may be called "the extreme boundary;" but what are the "boundar*ies*" in an oval curve, for example?

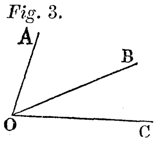
It is to meet these difficulties, about the Straight Line and Plane, that our First Part is intended. To follow the methods employed in the common treatises is impossible; for they *assume*, from the beginning, the very properties which we want to prove. It is well, however, here to remark, that the proposition at which we are secretly driving is, that which Euclid has made his 8th; namely, that when the *lengths* of the three sides of a triangle are given, the *shape* of the triangle is hereby entirely determined. The importance of this will be clearly understood when it is remarked, that the same thing is not true of a four-sided figure; for if the *lengths* AB, BC, CD, DA, (*Fig.* 2,) were *alone* given, there is nothing to hinder the figure from assuming different shapes, as the diagram shows, by a change of the size of the angles. If any one choose to resort to experiment to establish this peculiarity of the triangle, this would be a way equally effectual with that suggested above, of cutting short our First Part.

Fig. 2.



There is another defect, less fundamental, yet not unimportant, in this part of the common treatises, in their neglecting to establish any satisfactory principles to regulate the addition and subtraction of angles. It is taken for granted that if two angles, AOB, BOC, (*Fig.* 3,) be laid down, side by side, *on a plane*, the angle AOC may fitly be regarded as a *sum* of the other two. But why on a " plane " in particular? The very word *sum* implies that angles are quantities; or are resolvable into parts as small as we please, all homogeneous to each other; and conversely, that angles may be generated from the accumulation of parts indefinitely small. But nothing is laid down or proved concerning them in the Elements, as usually treated, to justify and establish this view.

Fig. 3.

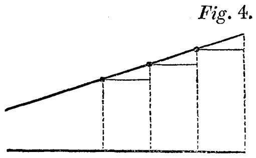


The subject is closely connected with that of *the Shortest Path that joins two points on a Sphere*, and that of the contact of two cones. The addition of angles may be founded on either of these doctrines, if it be judged convenient to abstain so long from every allusion to angles as *quantities* made up of parts.

Secondly, The endeavour has been made to remove the celebrated difficulties embarrassing the doctrine of Parallel Straight Lines. Euclid's method of disposing of it would be honest, and so far good, if the ambiguity of the Greek term *Axiom* had not led to the annexing of the 12th axiom (so called) to others perfectly unlike it in kind. It might be called ἀξίωμα, a " Postulate," or Assumption, with much propriety, and no student would demur to grant it. Yet it must tend to throw light on the philosophical basis of the science, either to demonstrate this, or to prove that no demonstration is to be looked for; neither of which seems yet to have been done, so as to satisfy geometers generally. To the writer it had always appeared that the *illustration* offered by Professor Playfair, of the equality of the three external angles of a triangle to four right angles, contained the germ of an unexceptionable *demonstration* of the same. This, accordingly, he has endeavoured here to exhibit. As the proof is concise enough, it is, if logically unimpeachable, practically deserving of acceptance.

At the same time he is so deeply convinced that every geometer secretly settles all questionings in his own mind concerning the truth of Euclid's 12th axiom, by appealing to the doctrine of proportions, as to induce him to suspect that future inquirers may succeed in obviating every objection which has been urged against Le Gendre's method. That in descending a sloping path, (*Fig.* 4,) we make equal vertical descents, by traversing equal distances along the path, is a truth of which the mind seems to possess itself *before* it attains to the belief that the slope may be carried so far as to descend to any required level; which latter is substantially Euclid's 12th axiom. And whether the former proposition can or cannot be established abstractedly, as by Le Gendre's triangles; in any case, the writer is persuaded that the latter should be proved by the former, and not, in the reverse order which Euclid follows, the former by the latter.

Fig. 4.



THIRDLY, The method of Measuring the Solid Angle is treated; not because it has any real difficulty, but because it is rather unceremoniously slurred over in the common treatises.

FOURTHLY, FIFTHLY, and SIXTHLY, Some propositions concerning Plane Curves, Double Curvature, and Curved Surfaces, have been demonstrated, which are generally assumed without proof, and are of no little importance in the higher Geometry.

SEVENTHLY, The Shortest Path on a Sphere has been treated, with a view chiefly to the question concerning the Addition of Angles.

The intelligent reader will probably remark of himself, that as the main difficulty of Parallel Straight Lines is identical with that of proving that no finite arc of a curve can have its curvature every where infinitely less than that of a circle; so the difficulty of demonstrating the evenness of the plane, is here virtually reduced to that of proving that no finite arc can have an infinity of cusps.

Throughout, it has been endeavoured to handle every topic in such a way as to prepare the mind for that large view which must be taken in the higher mathematics; for which, naturally and necessarily, the works of a Greek geometer are wholly unfit. Especially, the undue contraction of definitions has been avoided; nor has the writer felt it requisite so to press on towards the end mainly sought, as not to tarry on collateral subjects which would tend to illustrate the matter in hand, and which in themselves are worthy of being known. Least of all can he persuade himself, that the unbending formality which characterises the Greek geometers,—the affected disdain to notice difficulties, to obviate misconceptions, to offer illustrations,—in any degree conduces to soundness of demonstration. Certainly, whatever helps the student to get vivid and distinct notions, helps towards logical reasoning, which must be in the mind, not on the paper; for the dead letter does but give hints for the mind to seize; and it is a strange feature in modern mathematics, that, as if to discourage beginners, a more repulsive and unexplanatory style is adopted in Geometry than in any other branch. Yet, with all this sacrifice for the attainment of imaginary "rigour," there is no department of exact science so full of fundamental flaws. If the reason be asked, perhaps none better will be found, than that we assiduously cultivate our Mechanics and Hydrostatics, our Algebra, our Calculus, anxiously removing their defects in successive generations, by help of the fresh light constantly poured in; while in Geometry we have set up one of the ancients for our idol, and have cramped the science in its adult state by the trammels of its infancy.

DIFFICULTIES

OF

ELEMENTARY GEOMETRY.

PART I.

ON SPACE GENERALLY.

1. GEOMETRY is a particular branch of the science of Quantity, namely, that which is concerned with *Space*.

2. The difference in principle between this science and Mechanics or Hydrostatics, is perhaps not so great as is often supposed. In the two latter, the mathematician speaks of bars absolutely inflexible, threads wholly inextensible, balls perfectly elastic, fluids void of viscidity, and so on; although he does not expect actually to meet such things. But he seizes a few of the prominent and most influential properties of matter, and stipulates to drop the rest, at least for a while, and argue as if they had no existence. Thus the things of which he speaks are not such as are found in nature, but are imaginary *limits*, towards which nature only approaches more or less. Afterwards, in adapting his science to practice, he has to make allowance for the deviation, and, if possible, complete his theory by taking in the circumstances before omitted.

Just so the geometer proceeds. He finds before him bodies of different material,—stone and wood, iron and silver,—but he drops all consideration of this point. They differ in weight and in colour; but this too he neglects.

He regards solely their size and shape. Again: he sees some to be round and others square; and although on close inspection each may be found to have irregularities, being neither *quite* round nor *quite* square, he drops this circumstance also. He invents for himself shapes simpler than any found in nature, and which are mere *limits* more or less distant from the realities of the world. In consequence, his reasonings may possibly mislead him, when applied to practice, because what he actually encounters proves to be not precisely that of which he has been treating.

As no one, without the experience of sensible Forces and sensible Fluids, could form the notion of mathematical Forces and Fluids, so neither, without the use of the senses, to give us experience of actual matter, could we arrive at the mathematical conceptions which are at the basis of Geometry. It is not without touch that we gain first the idea of Extension; as indeed also of Length, Breadth, and Thickness; of Protuberance, of Flatness, of Hollowness, of Pointedness. The earliest exercise of a baby's fingers and lips is, to assist in acquiring such notions; which gives a fair apology to those who would call geometrical notions innate, as they are probably the first that enter the mind at all.

3. We find, moreover, that those objects which resist our touch do also mutually resist each other. Hence rises the apprehension that each occupies a certain *Space* of its own, into which a second body cannot intrude, without displacing the former. We are farther thus enabled to neglect all consideration of the material of which bodies consist, and even the fact of substantial existence. For solid bodies we substitute the empty space which they *might* occupy. Consequently no absurdity is involved in speaking of two "solids" as penetrating each other, when neither has corporeal substance.

4. It is likewise allowable to imagine a solid to be transferred from one position into any other that can be described. For as by experience we learn the possibility

of this in the case of numerous light and small bodies, we infer that a power may, without absurdity, be conceived, which might wield at pleasure the greatest and the heaviest. Much more, then, if we dispense with the idea of substance and weight in the body transferred, does all difficulty vanish. But, as was stated in the Introduction, it is no business of the geometer to treat on Time, (nor consequently on the *Velocity* of Motion,) any more than on Force.

5. The word *Magnitude* is employed universally to represent Geometrical Quantity, of whatever kind the quantity may be.

All Quantity, and therefore Magnitude, is generally regarded as differing from Number, in being *continuous.* It is impossible to *count,* without leaving finite gaps between the numbers, as in 1, 2, 3, 4, where we proceed by units, or as in 1, 1·01, 1·02, 1·03, 1·04, &c. where we proceed by hundredths of a unit: and this is Discontinuity. Whereas in weights, we conceive of every intermediate grade between one pound and two pounds; and in size, of every intermediate bulk between a cannon ball and the globe on which we stand. But the supposed difference is fictitious, and a needless source of perplexity. In the realities of life, Quantity as well as Number is discontinuous; while in theory, neither Quantity nor Number need be regarded as such. The mind which can suppose a quantity to increase from one value to another by finite increments *as small as it pleases,* can pass to an imaginary limit by a successive diminution of the increments, till it arrives at the idea of continuity. And in Arithmetic, we with equal ease conceive of continuous number; though we devise modes of *expressing* the intermediate values only so far as practical convenience dictates.

Thus, " Continuity of Magnitude " is a theoretic limit invented by the mind.

6. *Relative Magnitude.* No magnitudes can be regarded as absolutely great nor absolutely little. There is no object so great but we can imagine its double; and of this

latter the double again, and so on, until a magnitude is attained, such as to exceed the first in any proposed ratio. To suppose a termination of space bewilders the mind; and amazing as is the thought of space infinitely extended on all sides, yet we are incapable of conceiving a boundary beyond which space should not exist.

Again: the least molecule that we can see or imagine, has opposite sides, separated by a determinate interval; and is in conception divisible into any number of parts. Hence, also, an object is conceivable, which shall be *less* in any required ratio than a given solid.

7. Actually to exhibit such multiples or submultiples, is sometimes an important problem with the practical geometer or mechanist. To graduate the arc of a large circle is a most delicate affair, of the highest value to astronomy. The fine screw which measures minute distances, is equally essential for accurate observations. But to theoretic geometry such matters are quite irrelevant. Appeal is made to the mind alone; diagrams are meant to assist the imagination; but expertness of manipulation is wholly needless, *as far as the logical texture of the argument is concerned.* Hence we should be perfectly at liberty to say:—"*Let* the circumference of a circle be divided into 360 equal parts;" although we had not suggested by what instruments it could be done, even without gross and sensible inaccuracy. For the pure science, it suffices that no absurdity is involved in the conception.

VOLUME, AREA, LENGTH.

8. By a *Solid* is then understood, any limited portion of space. Sometimes, however, it is convenient to attribute to it material existence, and, accordingly, to name it hard and inflexible, or to attribute to it joints, breakages, and such like.

9. The exterior boundary of a solid is called a *Surface.*

The boundary of a surface is called a *Line*. The extremity of a line is called a *Point*.

10. All three terms merely express a "limit," which the mind invents. The most obvious is the Surface, because we suppose that we *see* and *touch* the surfaces of all bodies, not being aware of that which Optics and Mechanics teach, that to the exercise of each sense some *thickness* is requisite in the surface which is to be seen or felt. But since the thickness may be lessened perpetually, and in any required proportion, the mind has no difficulty in imagining it wholly to vanish.

Again: we conceive the diameter of a rope or string to be continually lessened, and the limit apprehended by the mind is a line.

Similarly, we may suppose a solid to be perpetually diminished, till it attains a size barely appreciable to our senses. Thus, if from being as large as a cocoa-nut, it shrink successively into the size of a walnut, a pin's head, a grain of sand, we hereby readily pass to the limit, and form an idea of *position independent of magnitude*, in which consists the notion of a geometrical point.

11. By *Volume*, or *Bulk*, is understood the magnitude or capacity of a given solid, in comparison with that of some other, which is assumed as a standard or unit. Thus a numerical measure of volume is attained; just as when we say that a cask holds forty-two gallons; in which case the cask or jar of one gallon is the standard, or arbitrary unit.

That any two solids admit of numerical comparison with each other is easy to perceive. For that all the parts of a solid are homogeneous to the whole, appears by considering, that if we repeatedly take away from the whole an exceedingly small magnitude, we may at last, as nearly as we please, exhaust the whole. And as any two solids, when placed side by side, may be regarded as one, it follows that they also are homogeneous to each other.

12. *Area* is the magnitude or extent of a given surface, compared with that of some other, assumed as a standard.

It is not altogether so easy to show, in this stage, that any two areas admit of numerical comparison; or, what comes to the same thing, that all the parts of a surface are homogeneous magnitudes. Yet, by regarding a surface as entirely cut up into very small portions, we presently are able to pronounce that any one portion (A), in comparison with any other portion (B), must needs be either greater, or equal, or finally less. And if this were established, it would follow that all its parts are homogeneous. But a full proof of this is neither possible nor necessary, before the curvature of surfaces has been discussed.

13. *Length* is the magnitude of one line compared with that of another. It is of immediate importance to us to show that any two lines admit of numerical comparison.

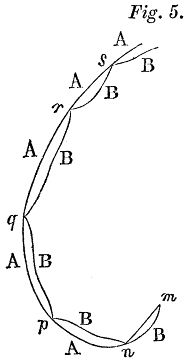
(Let it be observed, that although for the sake of illustration we may already speak of Length, Breadth, and Thickness, these are terms which cannot be at present employed with scientific propriety in opposition to each other. All three are at present merged in the one word, Length.)

Since no magnitude is affected by change of position, neither will *part of a line* be hereby affected. We may then, without altering the magnitude of a line, suppose any part to be bent aside at any point; or, what is the same, we may imagine a joint to be introduced at any point, about which each portion may freely play. Now let several joints be supposed; in short, let the number of joints at every part of the line be continually increased; and neither does this imply any change of magnitude. Thus the mind approximates to the idea of a line, which is the theoretic limit of the above; namely, one which is perfectly flexible at every point. Such a line is called a *Thread;* and preserves the same magnitude (or length) in every position.

In the place of any line under consideration, we may thus substitute a thread of equal length. Any two threads admit of direct comparison; and consequently any two lines (A, B,) are homogeneous magnitudes. A numerical measure of them is obtained, by assuming one (as A) for the unit, which determines for the length of B some other number, whole, fractional, or approximate.

14. If two lines, *A*, *B*, (*Fig.* 5,) are equal in length, and we suppose, *first*, a limited number of joints introduced in *B*; and *then* *B* to be so applied on *A* as that one extremity of both shall coincide (in *m*,) and that every joint moreover in *B* shall (as far as possible) fall on the line *A*; it is manifest that by increasing perpetually the number of joints in *B*, the line *B* (which tends more and more to become a thread,) will finally lie altogether along the line *A*. Let *m*, *n*, *p*, *q*, *r*, *s*, . . . be the successive points of coincidence of the two lines; all of these points, except the first, being joints in *B*. It follows from the above, that the two paths which unite any two consecutive joints, (as *qAr*, *qBr*, which unite *q* and *r*,) tend more and more to perfect coincidence; so that the *limit* of the ratio of the two lengths, *qAr* : *qBr*, is absolutely 1 : 1; when the number of joints perpetually increases in all parts of the line.

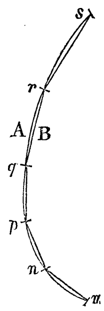
Fig. 5.



15. This conclusion, it must be observed, holds, whatever may be the nature of the line *B*. It will not be vitiated, should *B* happen to be what we shall afterwards call a Straight Line.

16. Moreover if *s* is the farthest joint from *m*, (*Fig.* 6,) the ratio of the line *mAs* : broken line *mBs*; (or again, of the whole line *A* to *mBs*,) has for its limit, 1 : 1. For *mAs* tends to *A* as its limit, and *mBs* to the whole line *B*, while *A* and *B*, by hypothesis, are equal.

Fig. 6.



17. It is convenient here to add, what is evidently comprised in the above, that if *mnpqrs* be any line soever, and *mn*, *np*, *pq*, &c. be joined also by short lines, such as we shall hereafter

call Straight, then the sum of all the straight lines joining *m* to *s* has for its limit the length of the curve line *mqs*, if the number of points *n*, *p*, *q*, &c. intermediate to *m* and *s*, be perpetually increased in every part of the line.

ON EQUALITY IN UNLIKE SHAPES.

18. This is perhaps the best place for bringing forward the various circumstances under which Equality is found among geometrical quantities.

The three notions, Equal, Greater, Less, arise simultaneously in the mind. Each implies the others, nor is it possible to say which of the three ought to be defined first, were definition possible. But no definition of any can be given.

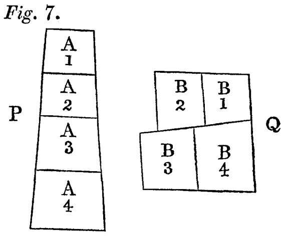
19. I. The simplest case of Equality is, when two objects have the same shape, as well as equal size; that is, when they are such that by a mere change of position one may be made precisely to occupy the place which was before held by the other.

[Of this kind is the equality of two straight lines, or of two rectilinear angles, or of two circles, or of two curves that are equally curved the one with the other.]

Magnitudes thus related are often called *Identical*, each being a perfect counterpart of the other.

20. II. When two magnitudes are separable into an equal number of parts, such that each of the one set has a fellow in the other set, to which it is identical.

Fig. 7.



Thus, let P and Q be two magnitudes (*Fig.* 7); also, let

$$P = A_1 + A_2 + A_3 + \ldots\ldots + A_n$$
$$Q = B_1 + B_2 + B_3 + \ldots\ldots + B_n$$

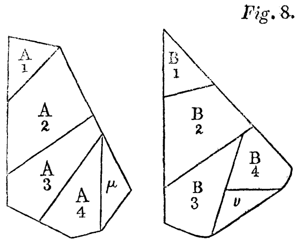
If then it be found that every A is identical with (and therefore equal to) its fellow B; we of course pronounce that $P = Q$. For "the sums of equals are equal."

[Such is the equality of two rectilinear plane areas, or of two plane-sided solids. In the Greek geometry the two cases of equality hitherto mentioned were regarded as the most rigid, and a silent effort was made to reduce all to these.]

21. III. When two magnitudes are the limits to which equal series perpetually approach.

Thus, suppose that from P and Q, (*Fig.* 8,) are taken parts which are equal or identical; as A_1 from P, B_1 from Q; where $A_1 = B_1$. Again, from what remains take the equals A_2 and B_2. From the remainders again take the equals A_3 and B_3, and so on. If, by repeating this process continually, we can reduce both remainders to be *as small as we please*, then P must needs be equal to Q.

Fig. 8.



By way of proof: Let μ and ν be the remainders after (n) subtractions; so that

$$P = (A_1 + A_2 + A_3 + \ldots + A_n) + \mu$$
$$Q = (B_1 + B_2 + B_3 + \ldots + B_n) + \nu$$

Then, from the equality of each A to its fellow B, we get,

$$P \sim Q = \mu \sim \nu$$

[or, the *difference* of P & Q is equal to the *difference* of μ & ν.]

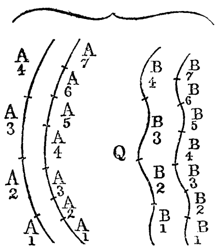
Now since μ and ν are each susceptible of being made as small as we please by increasing (n), much more is their difference. Hence P and Q have either no difference at all, or a difference which can be made as small as we please by increasing (n). But neither of them, nor therefore their difference, depends at all upon (n), which is the arbitrary number of subtractions. They have therefore no difference; or are equal.

More concisely: P is the limit of the sum of the A's, and Q is the limit of the sum of the B's. Hence $P = Q$, because " the limits of equal sums are equal."

[Such is the equality which subsists between any two plane areas, two areas on the surface of the same sphere, two solid angles, or two curve-sided solids.]

22. IV. When two magnitudes are separable into an equal, but variable, number of parts; such, that, (by increasing the number,) every ratio, of each part in the one set, to its fellow in the other set, approximates to the ratio, 1 : 1, as its limit.

Fig. 9.



Thus, as in the second case of equality (*Fig.* 9,) let

$$P = A_1 + A_2 + \ldots + A_n; \text{ and}$$
$$Q = B_1 + B_2 + \ldots + B_n$$

But instead of supposing (n) a fixed number, and the A's and B's fixed magnitudes, let (n) increase indefinitely, and every A and every B diminish indefinitely. And instead of supposing every A to be precisely equal to its fellow B, let every one of the ratios $\frac{A_1}{B_1}, \frac{A_2}{B_2}, \frac{A_3}{B_3} \ldots\ldots$ approximate towards $\frac{1}{1}$ as their limit. And we assert still that $P = Q$.

The detail of proof belongs rather to Arithmetic than Geometry. It is, however,* easy to see that the ratio

$$\frac{P}{Q} \text{ or } \frac{A_1 + A_2 + A_3 + \ldots + A_n}{B_1 + B_2 + B_3 + \ldots + B_n}$$

must always be intermediate between the least and greatest of the partial ratios, $\frac{A_1}{B_1}, \frac{A_2}{B_2}, \frac{A_3}{B_3} \ldots \frac{A_n}{B_n}$, and as none

* If ϵ is the greatest, and $\acute{\epsilon}$ the least, of the partial ratios, then $A_1, A_2, A_3 \ldots A_n$, are not greater than $\epsilon B_1, \epsilon B_2, \epsilon B_3 \ldots \epsilon B_n$; but some are less: consequently $A_1 + A_2 + \ldots + A_n$ is less than $\epsilon(B_1 + B_2 + \ldots + B_n)$; that is, $P < \epsilon Q$. Similarly, $P > \acute{\epsilon} Q$, or $\frac{P}{Q}$ is less than ϵ, but greater than $\acute{\epsilon}$.

of these have any finite difference from their limit $\frac{1}{1}$, neither can $\frac{P}{Q}$ have any. But the last ratio does not change with (n); hence it is absolutely $= \frac{1}{1}$. That is, $P = Q$.

Two such equal magnitudes may be popularly said to consist of *an equal infinitely great number of equal infinitely small parts.*

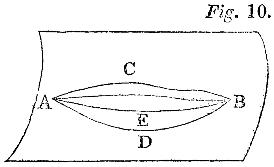
[Such are two equal lines or areas, of different curvature.]

DISTANCE.

23. By *Distance* is understood "least length," under various circumstances.

I. Between two points along a given surface. If AB are given points on a given surface, (*Fig.* 10,) and along the surface numerous paths ACB, ADB, AEB, are drawn to connect them, some of these paths may be shorter than others. Yet there must exist one or more paths, than which none shorter can be found. Just as if A and B were two towns, to be joined by a road; there must needs be *some* limit to the possible diminution of the length of the road, unless indeed the towns were somewhere in contact. The length then of the path, than which no path joining A and B can be shorter, is called the Distance between A and B along the given surface.

Fig. 10.



II. Between two points, when the path is not restricted to any particular surface. In the former case, perhaps, AEB may have been as short a path as possible; but now, (*Fig.* 11,) by drawing the path so as not to lie along the surface, it is conceivable that a yet shorter may be found. If AFB is as short as any possible, (and there

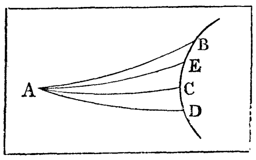
Fig. 11.

must be some least length,) then the length of AFB is, absolutely, the Distance, (or the Distance in Space,) between A and B.

Observe: We are not at present competent to assert, that there is necessarily *one*, and *only one* path, AFB, such as to be of this least length; although the mind readily persuades itself of this. But in fact, while the paths are restricted to a given surface, there may be many which have the shortest length. Thus on the globe, every meridian, joining the north and south poles, is equal to every other meridian.

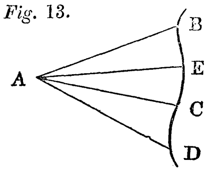
III. Between a point (A) and a line ($DCEB$) upon a given surface (*Fig.* 12). From A to the several points of the line, let there be drawn (along the surface) paths as short as possible. If, then, of all these paths none is shorter than AC, the length of AC is the Distance (along the surface) of A from the line DB.

Fig. 12.



IV. Between a point and a line, when the paths are not restricted to any surface (*Fig.* 13). The last case applies equally here, dropping the restriction of the surface; in consequence of which the Distance (or least length AC,) is probably shortened.

Fig. 13.



V. Between two lines (AAA, BBB,) along a given surface (*Fig.* 14). From every point A in the one line, let there be drawn along the surface a path as short as possible, to meet the other line. If of all these paths none be shorter than $\alpha\beta$, the length of $\alpha\beta$ is the Distance between the lines along the surface.

Fig. 14.

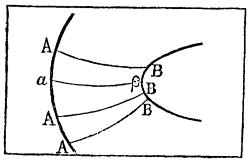
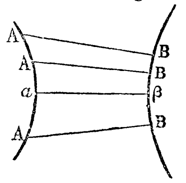


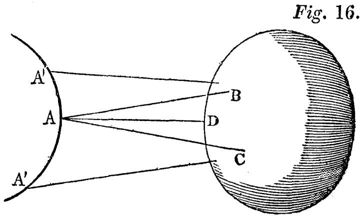
Fig. 15.



VI. Between two lines, when no surface is given (*Fig.* 15). The explanation of the last case will include this, if the surface be left out.

VII. Between a point (A) and a surface (*Fig.* 16). If points B, C, D, on the surface, are assumed at random, and they are joined to A by paths AB, AC, AD, as short as possible; then out of all such conceivable paths one (or more) is shorter than any other. If AD is as short as any of them, its length is the distance of A from the surface.

Fig. 16.



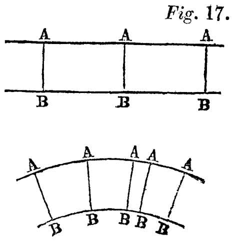
VIII. Between a line ($A'AA'$) and a surface (*Fig.* 16). If from every point of $A'AA'$ as short a path as possible is drawn to the surface, any one of these which proves to be as short as any of the rest expresses the distance between the *line* and surface.

IX. Between two surfaces. In place of the line $A'AA'$, in the last case, let a surface be introduced, and all which is there said will apply here.

It is all along supposed that the points, lines, or surfaces, between which we are estimating the distances, have no part actually in common; not so much as a single point in common. Otherwise there is no shortest path, but the distance is said to vanish, or to be *zero*.

24. *Parallelism.* By this word is understood "Equality of Distance," under several circumstances.

Fig. 17.



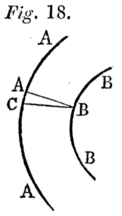
I. Parallelism of a line to line: as of AAA to BBB (*Fig.* 17). By this it is understood, that every point A in the one line lies at the same distance as every other point *A*, from the other line *BBB*.

II. Parallelism of a surface to a surface. The same definition may be given here as in the former case, supposing only that *AAA*, *BBB*, now represent surfaces instead of lines.

25. We have as yet no way of ascertaining, whether to a given line or surface a second line or surface can be conceived such as to be parallel. But we shall very soon see examples of parallel lines and surfaces, and this will at once manifest that there is no intrinsic incongruity in that for which we have been inventing a name.

Meanwhile it may be observed, that when two lines, or two surfaces, are parallel, the distance of the first from the second is the same as the distance of the second from the first. In fact, if *AB* be a path, expressing the distance of *A* from *BBB*, the same path *BA*, in an opposite direction, expresses likewise the distance of *B* from *AAA*. Else, let *BC* be shorter than *BA*, *C* being a point in *AAA*. Then *CB* would be shorter than the distance of every point in *AAA* from *BBB*, which is self-contradictory.

Fig. 18.



It follows that the Parallelism is *reciprocal;* or that if *AAA* is everywhere equidistant from *BBB*, then so is *BBB* from *AAA: provided that every point in* BBB *is the extremity of some shortest path drawn from* AAA.

III. Parallelism of a line to a surface. This means, that every point in the line lies at the same distance from the surface. But, in this case, there is no reciprocation; for not all points in the surface necessarily lie at the same distance from the line.

[We might here proceed to explain the nature of Asymptotism; when two infinite lines have evanescent distance, without actually meeting. But it is of no importance to our present objects.]

LAWS OF ROTATION.

26. Let one end of a stick be thrust into the sand, and any motion given to the opposite end. Next, suppose the point to be made sharper and sharper, and that it is allowed to move as little as possible out of its place. In this way the mind passes to the conception of a body which has *one point fixed;* and it appears that the body may nevertheless turn on every side round this point.

Secondly, let some mechanical method be used of fastening a body as nearly as can be, at two points only. Thus, it may be wedged between two walls, so as to touch each wall in but a very small part. Now in every case there is really a small *surface* in contact with the wall; and if we attempt to move the body, perhaps the friction so resists us that no motion can be produced. But if parts of the surface in contact are successively cut away, we shall at last be able to produce a sort of motion, even while the contact at the two sides continues in nearly the same spot of the wall. Such motion is called *Rotatory.* (See *Fig.* 19.)

Many ways are conceivable of producing a more and more perfect Rotation; but it suffices here to enounce as fact, that if a continual approach be made towards an accurate fixing of *two* and *only two* points in an inflexible body, we arrive at the result, that " not all motion of the system is hereby hindered."

27. It remains to state the peculiar character of the motion which it then undergoes, wherein consist the Laws of Rotation.

I. The motion of every particular point of the inflexible system is then *constrained* to one determinate path; which path, if the motion be continued in one direction, at length *rejoins itself.* The whole system has then regained its original position, and is said to have made one Revolution

II. If another revolution be given to it, the same point must needs describe the very same path as before; and this, *although the motion be reversed.*

III. If any revolving point be fixed, all motion is stopped.

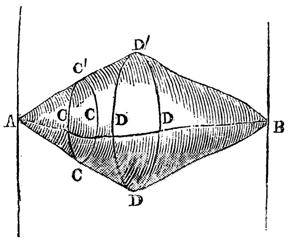
It appears to the writer, that our knowledge of these laws is as truly based upon experiment as is our knowledge that Water seeks its level. When he endeavours to assign to himself a reason, *why* the motion must needs be constrained, he finds himself making an inward appeal to the remembered sensation, that if an oblique pressure be applied, tending to cause motion along a new path, it is violently resisted, and cannot produce any effect, until some part of the system is crushed, or is lengthened, or slips. Easy as it is to satisfy oneself of this truth, by mechanical and experimental considerations, he has hitherto wholly failed in the attempt to prove it more abstractedly. Until, therefore, others shall have supplied a deductive proof, similar to the other demonstrations of Geometry, he will hold these laws to be experimental, and that Geometry stands on a like basis with Statics.

CIRCLES.

28. *Circle.* In any case of rotation, the self-rejoining path ($CCC'C$), described by any one point (C), is called a Circle.

29. *Parallel Circles.* Any second point D of the system may simultaneously describe another circle DDD'; and it is easy to see that this must be a line parallel to the former circle.

Fig. 19.



For if $D'C'$ be as short a path as possible from one circle to the other, then by supposing the system to revolve, D' and C' will describe the two circles, and the path $D'C'$ accompanying D' expresses the shortest distance of *every* point D from the circle CCC. (*See* Art. 24.)

30. *Sliding of the whole circle on its own ground.* If a circle (CCC) be regarded as a hoop of unappreciable thickness, connected with the fixed points of the system (A and B) by inflexible lines, the rotation makes the circle turn along itself; so that while every point C is moving round, the circle as a whole does not change its position.

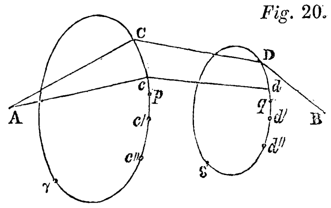
Hence it is like itself all round: in Homer's language, πανтόσε ἴση, "on all sides equal."

31. *Surface of Revolution.* If two of the circles, CCC, DDD, be joined by a line CD, that revolves with the rest of the system, this line will describe a self-rejoining surface, which may popularly be regarded as a collection of Parallel Rings, indefinitely thin, since *every point* in the line CD describes a circle.

Such a surface is called a Surface of Revolution; and if it close on all sides, so as to contain a solid, this is called a Solid of Revolution.

32. *Proportional Arcs.* It is manifest that any two points (C, D) in the system must complete the revolution, so as to regain their original position, simultaneously. For if C be fixed, as well as A and B, the whole is fixed. Hence any other point, as D, has its position determined by A, B, and C; and when C has regained its original place, D cannot be in any other place than that which it held at first.

Fig. 20.



Let γ be the point opposite to C, in the circle of C, so that the portion (or *Arc*) $C\gamma$ is equal to the opposite arc γC; for there must be some middle point of the path $C\gamma C$. Then when the system has been carried by rotation so far that C has reached γ, it may be said to have performed *half a revolution.* For it is evident that D will simultaneously describe half of its circle, and reach its opposite point δ; and the like may be said of every moving point in the system.

To prove this more distinctly, we will take a larger proposition. Let C reach c, at the same moment that D reaches d; then, *whatever ratio the arc* C c *bears to the entire length of the circumference* C γ C, *such likewise is the ratio of* D d *to* D δ D.

Suppose Cc, Dd, to be inflexible lines, attached to $ACDB$; then when the rotation brings C and D to c and d, let Cc have the position cc', and Dd the position dd'. Then Cc', Dd', are doubles of Cc, Dd; and, moreover, when C reaches c', D will reach d'. Similarly, if along the one circle we take any number of arcs equal to Cc, and along the other the same number of arcs equal to Dd, the points C and D will, during the rotation, describe the two sets of arcs simultaneously.

If now Cc were any fraction of the circumference, say $\frac{1}{12}$th, then Dd must be likewise $\frac{1}{12}$th of its circumference. For by taking Cc twelve times we complete the circle; or by passnig over twelve times Cc, the point C regains its position. Therefore by passing over twelve times Dd, the point D regains its position; which proves that twelve times Dd is equal to the circumference; or that $Dd = \frac{1}{12}$th of its circumference. And the like would apply, if for $\frac{1}{12}$th we substituted $\frac{1}{13}$th or $\frac{1}{14}$th, or any other submultiple; from which the mind instantly collects that the arcs described simultaneously by C and D are *always Proportional.*

But to elucidate this conclusion the better, it is desirable to make a short digression.

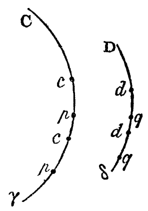
DIGRESSION CONCERNING PROPORTIONALS.

33. A geometrical quantity may be supposed to *vary* independently, under numerous circumstances; as, when we suppose an object to become larger or smaller. But often it happens that two magnitudes *vary together;* as, just now, did the two arcs Cc, Dd. For if by a motion of the system the length of Cc change, the length of Dd instantly changes likewise.

34. When two magnitudes thus vary together, the *law* connecting them may be very different under different circumstances. Sometimes, whatever increases the one diminishes the other; sometimes, on the contrary, they increase together and diminish together. Yet, even then, it may happen that their *rates* of increase are very different. While one doubles itself, the other may become five times as great; and while the former triples itself, the latter may become twenty times as great. But the simplest case of connected variables is, when they increase and diminish Proportionally; which is also of chief importance in the Elements of Geometry.

35. It is requisite for Proportionality, that the two variables *vanish together.* Thus, by taking Cc as small as we please, Dd may likewise be made as small as we please; and when Cc is actually *nothing*, c being at C, then d is at D; or Dd is *nothing*. Regarding the magnitudes as increasing instead of diminishing, we may say that Cc, Dd, *begin together from nothing;* which is obviously necessary for Proportionality.

Fig. 21.



36. Again, when Cc increases by the portion cp, (*Fig.* 21,) let Dd increase by the corresponding portion dq. Then, *in the case of proportional variables,* since Cc is to Cp as Dd is to Dq, it follows that Cc is to its increment cp as Dd to its increment dq. Suppose $C\gamma$, $D\delta$, to be any other corresponding values of the variables, which of

course are (by hypothesis) proportional to Cc, Dd; then we infer that the increments cp, dq, are proportional to the fixed magnitudes $C\gamma$, $D\delta$. Consequently, if the increment cp be a given magnitude, the magnitude of dq is instantly thereby determined, *be the magnitude of* C c *what it may*. Thus, if a new value be assumed for Cc, as in the diagram, and consequently a new value for Dd, and yet cp be assumed just as great as before, the dq likewise will be just as great as before.

37. This last property of Proportional Variables admits of being concisely expressed by saying, that they *increase uniformly*. For if any number of successive increments to the former be all equal, then the corresponding successive increments to the latter will be also equal to one another.

38. So much being premised, it will be easier to understand the following LEMMA, which is the converse of all this: viz. that "Magnitudes which begin together from nothing, and increase uniformly, are Proportional."

PROOF. Let x and y be two such variables, which have increased together from nothing. Then x has been formed by an aggregation of small increments, every one of which may be regarded as equal to the first of them, viz. $= h$; in which case, by hypothesis, y will have been produced by the aggregation of the *same* number of increments, each equal to the first of them, or $= k$. Thus, x is the same multiple of h, as y is of k; and consequently x and y are proportional to h and k; $(x : h = y : k)$. This holds, however many fresh increments are added to both; so that if x' and y' are *new* values of the variables, these also are proportional to h and k, and consequently to x and y. That is, $x : x' = y : y'$. Which was to be proved.

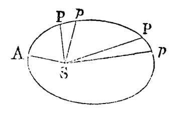
39. The only objection to this is, that if x and y increase by *finite* additions, they will not receive all conceivable values intermediate to the first and last. The reply is, that the increments h and k may be in imagination lessened as much as we please; and in this way x and y may be made

to approach ever so close to any intermediate magnitude. If farther satisfaction be desired on this head, let it be supposed that x' and y' do not exactly contain h and k; but that x' contains h a certain number of times, and μ over; then y' must contain k the same number of times, and ν over. Consequently $(x' - \mu)$ is the same multiple of h, as $(y' - \nu)$ is of k; and, reasoning as above, we find that $x : x' - \mu = y : y' - \nu$. Here μ and ν may be called the *errors* incident to the 2d and 4th terms of the proportion which we are aiming to establish. But by diminishing h and k as much as we please, we may make the errors less than any proposed value; since μ is less than h, and ν than k. And such diminution leaves the values of $x\,x'$, $y\,y'$, unchanged. Omitting, therefore, the errors as unreal, because they have no assignable fixed value, we have as before, $x : x' = y : y'$.

In fact, in all cases, it is needless and useless to refine concerning Geometrical quantities, as though, in respect of *Continuity*, they required a different treatment from Numbers. Every conclusion drawn generally, by treating them as discontinuous, may unceremoniously be received as universally proved: for *an error which has no finite value* (as μ and ν just now) *has evidently no existence at all.*

40. The Lemma may be also modified conveniently, as follows. "Let δx, δy, represent the *new* increments which x and y are just about to receive: if, then, the values of δx and δy are *determined solely by one another*, without any reference* to the values of x and y, supposing also that x

* This may become yet clearer by considering the opposite case. Let S be a point inside an oval curve; $x =$ an arc AP, $y =$ area ASP, contained between the curve, and two straight lines. Let $Pp = \delta x$; then area $PSp = \delta y$. Now a little thought shows, that if AP be made longer, (or P be taken at a more distant point in the curve,) although Pp should have the *same* length assigned to it as before, yet the area PSp would ordinarily be *different*; so that here δy would depend not only on δx, but also on x, or on the length AP. Thus here, x does *not* vary proportionally to y.



and y begin together from nothing, it follows that these last vary proportionally."

For since the values of x and y do not affect those of δx and δy, a succession of increments to x, each equal to δx, would produce a succession of increments to y, each equal to δy, and thus x and y would *increase uniformly*.

41. Returning to the case of the arcs Cc, Dd, in Art. 32, it is there plain that the two increments cp, dq, are determined solely by one another, and are nowise affected by the lengths Cc, Dd, *already attained*. Moreover Cc, Dd, begin together from nothing. Hence we infer that they throughout "vary proportionally."

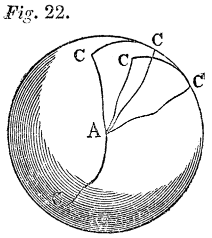
42. In like manner it appears that the *surface* passed over by the line CD, (*Fig.* 20,) which revolves with C, varies proportionally to the arc described by C. For they begin together from nothing; and so long as cp, the increment of the arc, is uniform, the corresponding increment of the surface is likewise uniform.

43. Similarly, if A be a fixed point of the system, and AC be a line revolving with C, the surface generated by AC is proportional to the arc described by C.

SPHERES.

44. If AC be a stiff line, (*Fig.* 22,) of which one end A is fixed, the other end C is free to play round A in all directions. The *locus* in which C then lies is a surface. For if CC' be any path described by C, suppose CC' to be a rigid line attached to AC and AC', and let the whole revolve about A. Then CC' sweeping round A on all sides, traces out a surface, into any point of which the point C may evidently be brought. And as CC' may play round on all

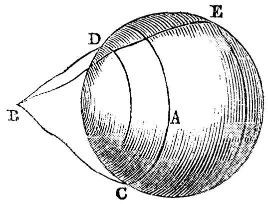
Fig. 22.



sides of A, the surface rejoins itself, and encloses a Solid. This Solid is called a *Sphere*, and A its *Centre*.

45. *Parallel Circles on the Sphere.* If B be a second fixed point, (*Fig.* 23,) and BC an inflexible line, the place of C becomes restricted to a circle. But this circle must lie upon the sphere, since AC is still the same. Change the length BC to BD and BE; D and E being still on the sphere, and BD, BE, inflexible; then D and E generate new circles on the sphere, which are parallel to the former circle described by C.

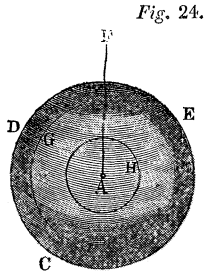
Fig. 23.



It is immaterial whether the second fixed point B be within or without the sphere; or upon its surface.

46. *A Spherical Surface is the* LOCUS *of all the points that lie at one particular distance from the centre.* For, first, that all the points *on* the surface, as C, D, E, &c. lie at one and the same distance from the centre A, is manifest; because the paths AC, AD, AE, &c. may be made wholly to coincide. Next, (*Fig.* 24,) that a point F, which is *outside* the sphere, lies at a *greater* distance from the centre than do the points on the surface, follows from this; that no path can join F and A, without piercing the surface. Lastly, if G be a point *within* the sphere, we may conceive a new sphere to be described from the same centre A, by means of the line AG; so that G may be on its surface. Therefore all points on the surface CDE are such, that the shortest path connecting them with A pierces the surface of G, and consequently G is *nearer* to A than are C, D, E, &c.

Fig. 24.



It thus appears that every point in the surface CDE, and no point not in this surface, lies at the same distance as C from the centre A; and this is what is meant by calling that surface the LOCUS of all points which lie at that distance.

COR. We moreover infer that a Spherical Surface consists of *but one sheet*. No part of the surface is overlapped, or comprised within another part.

47. *Parallel Spherical Surfaces*. The concentric surfaces described by C and G are obviously Parallel. Moreover, the surface G is entirely contained within the surface C; and every point in G is nearer to the surface C than is the centre A.

48. *Spherical Shell*. The solid intercepted between two concentric spherical surfaces is called a Shell. It is evident that, about a given sphere, a shell may be conceived to be added, such as to be less than any solid proposed. For the inner surface of the shell having a determinate magnitude, it is evident that if the thickness be perpetually diminished, the magnitude of the shell becomes evanescent.

49. *Continuous Increase of the Sphere*. Hence a Sphere in increasing from one size to another may be supposed to pass through all intermediate magnitudes. For the successive accessions of magnitude may be made as small as we please.

Moreover, since a Sphere is readily conceivable *less* than any proposed solid, and another Sphere *greater* than the same; if the former increase till it reaches the magnitude of the latter, it must pass through a state in which it was *equal* to the solid in question. Or, " There is some Sphere equal to any proposed solid."

50. *No Second Centre to a Sphere*. The Centre of a Sphere is equidistant from all points on the surface. Conversely, if a point within a sphere be thus equidistant, it will possess all the properties of a Centre; for there would be no impropriety in conceiving the sphere to have been generated from it. But, we say, " there cannot be *two* such Centres." For it was shown that the centre A is more remote from the outer surface than is any other point (G) within the sphere, (Art. 47.) But it is obvious that there

cannot be two points within the sphere, *each* more remote from the surface *than any besides itself.*

51. In fact, suppose the surface CDE to be given, we may thus approximate towards one determinate centre (*Fig.* 24). Take G within the body of the sphere, and through it pass a new spherical surface, parallel to the former; for we know that there may be one parallel. But it is evident that no second surface can pass through G, also *parallel* to CDE; for the mere principle of equidistance suffices to fix the surface of G, when the first surface CDE and the point G are given. Within the surface of G take a new point H, and it in like manner appears that through H may pass one and only one spherical surface parallel to that of G. By continuing this process, we form a series of spheres, less and less, each interior to the former, and tending continually to shrink into a single determinate point, which of course must be the centre A.

This suggests another mode of stating the result of the last Article, viz. "If a spherical surface be given, the centre is determined."

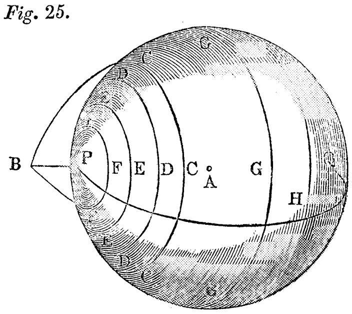
52. *Sliding of the Sphere on its own ground.* If the surface be stiff, and inflexibly connected with the centre, which we may suppose to be fixed, rotatory motions in various directions may be given to the sphere, by which its surface will move, but only along itself; so that the sphere, as a whole, will suffer no change of place.

During such sliding, every point G in the interior, (supposing the sphere to be solid,) of course moves along a concentric spherical surface, so that the centre is the *only* point not susceptible of motion, while the outer surface, as a whole, retains its position.

Thus every one of the concentric surfaces, G, H, &c. is constrained to keep its own place, and slide along its own ground, if the outermost surface is thus constrained; and conversely. Hence it is obvious that *if* by any means we can secure that *one surface*, as CDE *retains its place*, we may infer that G, and that H, and every other yet more interior surface, does the same, and consequently that *the centre* A *remains fixed,* since this series of surfaces may be conceived to approach as near to the centre as we please.

53. *Poles of a Sphere.* Let the centre A, and some other point B, be fixed, (*Fig.* 25,) B being either within or without the sphere, or on the surface, but connected with it inflexibly. Then the only motion of which the system is susceptible is, a rotation about the fixed points A and B. Of course every point C describes a circle CCC upon the surface, which slides on its own ground; and all such circles are Parallel.

Fig. 25.



Now any one circle, as CCC, divides the spherical surface into two parts. On either side of CCC take a point D on the surface, and let DDD be the circle parallel to CCC. This cuts off a less portion of the sphere's surface on that side, and leaves a larger portion on the other. Within the area cut off by DDD on the opposite side from C, take a new point E, and let EEE be its circle parallel to the former circles. Within EEE, and towards the same side again, take F, and describe the parallel FFF. The process may be repeated continually, and each portion of surface intercepted is wholly interior to the preceding. Nor is there any internal *area* which is a limit towards which these areas converge; for within any such area a new circle might be placed. Hence the series of circles tends towards a determinate point, which we may call P, intermediate to all such possible areas. P will then have the peculiarity of *turning about itself,* so as not to change its position during the rotation.

Again; on the other side of CCC we may suppose a succession of parallel circles, GGG, HHH, &c., and it may similarly be proved that they tend towards a point Q on the surface, which has similar properties to P.

The points P and Q are called *Opposite Poles.*

54. If B be *on* the surface, it must obviously coincide either with P, or else with Q. Thus one pole (say B or P) being given, the opposite pole Q is determined.

55. *Poles of a Circle on the Sphere.* Let $CC'C''$ be *any* circle upon a sphere, and suppose that we do not know from what two fixed points the circle was generated. The poles P and Q may nevertheless be determined just as before.

For we may conceive the circle as a line painted upon the solid sphere, which sphere is hung upon an immovable centre. Now by turning the sphere about, any point on its surface may be guided along any line soever drawn on its surface, but immovable in space. Wherefore any one point C may be made to describe the circle $CC'C''$, *without shifting the sphere's centre.* As in this motion C successively takes the place C', C'', &c., it is clear that C', C'', &c. *also* move round in the same path. Thus the circle slides along its own ground, and the rotation is constrained, just as when a fixed point B was given. We have therefore only to proceed as before, and determine the points P and Q, poles of the Circle.

56. It is easy to see that (*counting distances along the surface*) any two parallel circles are equidistant; as FFF from CCC. Hence that P is more distant (*along the surface*) from CCC, than is any other point F within the same area $CPCC$.

57. Since P and Q remain fixed during the revolution of the circle, they may be regarded as the fixed points from which it was generated; and they are at once poles of the Sphere, and of the Circle *upon the Sphere.*

[Here we might proceed to explain the notions of Latitude and Longitude, and others connected with them, if a complete Treatise of Geometry were aimed at.]

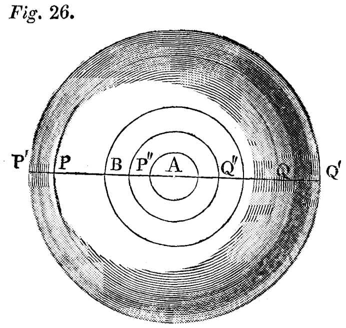
58. *Points that lie evenly.* It has appeared, that when the points A, B, are fixed, the points P and Q turn about

themselves, or are fixed likewise. It is manifest that the very same rotation of the system would be effected if B were free, but P and A were fixed; or if B and A were free, but P and Q were fixed. In fact, if all are inflexibly connected, any two being fixed, the rest are fixed likewise. Three or more points thus related are said *to lie evenly*.

AXES.

59. Suppose, as originally, that A and B are given points (*Fig.* 26). If the size of the sphere change, many new pairs of poles $P' Q'$, $P'' Q''$, &c. are attainable, which will all lie evenly with A and B. Let the sphere change its magnitude by a continuous motion; then by its perpetual *increase*, the opposite poles move away from each other, tracing out two lines, PP', QQ', stretching to an unlimited extent both ways. If the sphere perpetually *diminish*, the poles move towards each other, and tend to meet in the centre A. When the sphere is such that B is on its surface, B is itself one of the poles. Hence the whole *locus* in which the poles lie, forms a single continuous line, passing through A and B, and susceptible of indefinite prolongation each way.

Fig. 26.



All the points in this line lie evenly with A and B, so as to suffer no change of place by the rotation of the system, of which this line is called the *Axis*.

60. It is now manifest that to fix any two points in the axis is equivalent to fixing all its points. Also, to fix all its points offers no impediment to rotation. But if, besides the axis, a point likewise is fixed which is not in the axis,

then by the third Law of Rotation no farther motion is possible.

61. *Axis of a Circle, or of a Surface of Revolution.* All the above applies to any rotatory system soever, in which A and B are two fixed points. For it is evident that the very same axis is produced, whether A or B is made the centre of spheres; and that such spheres may be introduced arbitrarily into any rotatory system.

62. *No second Axis to a Circle.* If a circle be given, but the fixed points from which it was generated are not given; or, what is the same, if the axis be not given; it may be inquired whether more than one axis is conceivable, from which it might have been generated. We shall therefore show that *only one* is conceivable; so that, when the circle is given, the axis is determined.

Let CCC be a given circle, and (since it must needs have at least *one* axis,) let one axis pass through A and B (*Fig.* 25). A sphere is conceivable, large enough to contain the circle within it, and consequently too large to pass *through* the circle as through a hoop. Let such a sphere have its centre placed upon the axis, and be moved along the axis until it strikes against the circle in *one* point C, the centre of the sphere being then at A. Then, since neither the circle nor the sphere changes its place by rotation about the axis AB, it follows that *every* point C of the circle lies on the surface of the sphere.

Now the axis AB pierces the sphere's surface in two opposite points, P, Q, which are poles to the circle, upon the sphere, and each of which is equidistant from all parts of the circle, counting the distances along the surface. But if the circle could have any second axis, this must pierce the sphere in some other two points than P and Q. Suppose, for an instant, it pierces the sphere in a second point F, on the same side of CCC as P is, and we shall see the absurdity of it. For F would need to be equidistant from all the points C, C, C, of the circle, (counting distance along the surface,) a property which, it is evident by

Arts. 55, 56, no point on the surface but P can possess. There is then no second axis to the Circle.

63. *The Axis of a Circle is the* Locus *of the Points, each of which is equidistant from all points in the Circle.* For, first, that any one point in the axis is thus equidistant, is manifest from the nature of rotation. But, next, we have to prove that every point which has this property is in the axis. Let A be equidistant from all points in the circle CCC, then A may be the centre of a sphere, on the surface of which the circle CCC lies. Hence, reasoning as in Art. 55, it will appear that A remains fixed, while CCC revolves on its own ground; and consequently A is a point in the axis, which was to be proved.

64. *Axes of the Sphere.* In contrast to the circle, it is manifest that a sphere possesses an infinity of axes, all uniting in its centre. For A being the centre, B may be chosen arbitrarily, and an axis AB may be determined, piercing the sphere in opposite Poles. Thus, also, a Sphere is a Surface of Revolution.

65. That portion of a Sphere's Axis which is intercepted between opposite poles, is called a Diameter, and the half of it, between the centre and surface, a Radius.

It is manifest that all the diameters of a sphere are equal, as likewise all the radii.

66. The mind here naturally guesses that the sphere is the *only* rotatory system which has more than one axis; which is nearly the truth. The exception is, that a Plane, of which we shall soon speak, has likewise an infinity of axes, all Parallel to each other.

67. *Straight Line.* It is with reference to the rotation of bodies that the word Axis is used; but when rotation is not immediately contemplated, a line, which has all its points lying evenly, is called Straight.

68. The following properties of a straight line are manifest from the mode of generating it.

I. If A, B, are points within a finite solid, the straight line AB may be prolonged so as to pass out of the solid;

so also as to return no more into it if prolonged yet farther. This is evident from the fact, that a sphere of centre A is conceivable, large enough to envelop the whole solid, and of course having the poles P, Q, which lie evenly with A and B, exterior to the solid; which must be true equally of all larger concentric spheres.

II. In the same way it appears that if a straight line lie on the surface of a finite solid, it can be prolonged so far as no longer to lie on the surface.

III. Any two points in a straight line of unlimited length determine the whole.

IV. Hence, also, two straight lines of unlimited length being applied together, so as to have two points in common, will entirely coincide.

V. Between two given points but one straight line can be drawn.

VI. A straight line has but one prolongation each way.

VII. The parts of straight lines are straight.

VIII. A straight line may *slide* along another, or along its own direction; and if inverted it will, as a whole, still occupy the same position.

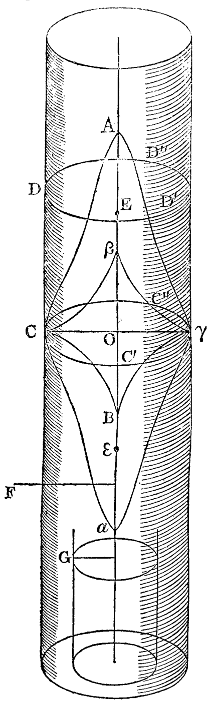
CYLINDERS.

69. Let the points A, B, be inflexibly connected, as before, with the circle or hoop $C\,C'\,C''$, and an axis pass through A, B. We have before remarked, Art. 63, that any one point in the axis is at the same distance from all points in the circle. But, on the other hand, not all points in the axis lie at the same distance from the circle; else the whole axis would lie on the surface of a sphere whose centre is any one point C in the circle—an infinite straight line on the surface of a finite solid.

Hence if A slide to an indefinite distance along the axis, carrying B with it, and the connexion of $C\,C'\,C''$ with A

and B be preserved inflexibly, the circle $C\,C'\,C$ cannot remain fixed, but must at length move also. This remark being alike true, whatever point on the axis A may have reached, it follows that by moving A as far as we please, we shall cause $C\,C'\,C''$ also to move indefinitely. By such motion $C\,C'\,C''$ traces out, or generates, a certain surface, which forms *a continuous sheet surrounding the axis,* and prolonged indefinitely in both directions. This surface is called a Cylinder, and the circle's axis, the Axis of the Cylinder.*

Fig. 27.



70. It is well to add here, that since the system $A\,C\,B\,C'$ is inflexible, and like itself in all positions, $C\,C'\,C''$ cannot remain motionless, if A has any *ever so small* a motion along the axis.

In fact, (since the circle is determined, when one point in it is given, and the axis is given,) all the points of the circle either move together or are at rest together; and not all its points lie in one straight line. Hence, if the circle be fixed, the whole system is immovable (Art. 60); wherefore no motion whatever can take place in A, unless the circle move with it.

71. *A Cylinder is the* Locus *of all the Points which lie at one particular distance from the Axis.* For, first, that all points C, D, *on* the cylinder, lie at the same distance, is manifest. Next, that a point F, which is outside, is farther from the axis than are C and D, appears by this, that every path from F to the axis must pierce the surface. Lastly,

* Here, then, is an example of a *line* (the Axis) which is Parallel to a *surface*, (that of the Cylinder.) Also there is here a reciprocation found, such as as not calculated on in Art. 25, No. III.

if G be within the cylinder, we may suppose G to generate round the axis an inner circle, and this circle to generate an inner cylinder; whence it will follow that C is more distant from the axis than is G. Thus *no* point *not* on the surface is at the same distance from the axis as are *all* the points *on* the surface; which completes the proof of our statement.

72. *The Cylinder is a Surface of Revolution.* For if $D D' D''$ be a new position of the circle, and $D C$* be a line drawn *upon* the cylinder, every point in $D C$ lies in one of the generating circles. Hence the revolution of $D C$ round the axis would generate the cylinder.

It also thus appears that the cylinder may revolve about its axis, without changing its place as a whole; this being common to all surfaces of revolution.

73. *Sliding of the Cylinder.* Regarding the surface as indefinitely extended each way in the direction of the axis; if the axis slide along itself, carrying the surface with it, the cylinder, as a whole, does not change its place, but slides along itself. This appears from the circumstance that each particular circle, such as $C C' C''$, $D D' D''$, will thus be made to slide along the surface.

74. *Inversion of the Cylinder.* Let the whole system be removed, and be then replaced with the axis inverted, but holding its former place. Then, since the axis is where it was, and the distance of the points on the surface of the cylinder from the axis is as before, the surface also will have regained its position as a whole.

75. After the inversion, let the axis slide up till the circle $C C' C''$ has regained its former place; and then let the cylinder revolve till the point C has regained its own place upon the circle. Let γ be the point in the circumference which is opposite to C, so that γ bisects the circumference, counting round from C to C again. It will

* The mind readily perceives that if $D C$ be as short as possible between the two circles, it will be a straight line. But to *prove* this, involves the whole difficulty of Parallel Straight Lines. The reader ought to be aware that we have not yet proved it possible to draw a straight line on a cylindrical surface.

follow that γ also has, after the inversion, regained its proper place; but the opposite halves of the circle have exactly changed places. Thus it appears that *every circle is susceptible of being doubled about itself*, by a half revolution about two opposite points in the circumference; so that the opposite halves coincide.

76. *Centre of the Circle.* Suppose that after the inversion, and after C and γ have regained their own places, A and B have the new places α and β on the axis. Let also E be any point on the axis between A and B; and after the inversion, let E be found at ε. Of course $AE = \alpha\varepsilon$; but ordinarily E and ε do not coincide, and AE is not equal to *half* of $A\alpha$. Let AE increase or diminish, till it $= \frac{1}{2} A\alpha$, and then it is evident that E and ε will coincide, as at O. Thus if O be connected with A, and $AO = \alpha O$, the point O, like C and γ, regains its own place after the inversion.

This amounts to saying, that if the circle perform half a revolution about the points C and γ, O turns about itself, or lies evenly between C and γ, that is, between *any two opposite points* of the circle. Thus all the straight lines which join opposite points in the circle meet the axis, and meet it in the same point.

This point (O) is called the Centre of the Circle.

77. The whole straight line $CO\gamma$ is called the Diameter of the Circle, and its half, CO, the Radius. That all the diameters of a circle are equal, and all the radii are equal, is evident.

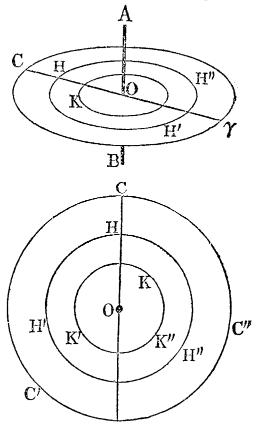
78. While A slides along the axis, and $CC'C''$ slides with it (Art. 70,) along the surface, the centre O accompanies $CC'C''$ in its motion; so that *every* point (as O or E) on the axis, is centre to *one*, and *only one*, of the generating circles.

PLANES.

79. If C revolve through the circumference, and carry the diameter $CO\gamma$ with it, the two halves, CO, $O\gamma$, trace

out or generate one and the same surface, since, after half a revolution C and γ exchange places. This surface is the LOCUS of all the diameters, and is called the *Plane of the Circle.* It is a species of Surface of Revolution.

Fig. 28.



80. *Inversion of the Plane.* After inverting the system as above, the plane, as a whole, exactly regains its own position. Thus it is like itself at both sides, and cannot be said to bend either way.

81. *Concentric Circles.* While OC, revolving, generates the plane, any point H in OC (*Fig.* 28,) generates a new circle, $HH'H''$, which *lies in the same plane* with CCC, and has the same centre O with it. The two circles are parallel, as in Art. 29. But they are likewise evidently parallel, or equidistant, in another point of view, viz. by counting distance along the plane surface. [For we have not yet shown, what however is the truth, that the shortest path from H to C lies along the plane.] Thus O is more distant from the outmost boundary, $CC'C''$, than is any other point H, which is within the circular area.

82. *No second point* (besides the centre) within that area, *can be equidistant from all the points of the outline* $CC'C''$. For by Art. 63, no point can be equidistant from the circle, unless it be in the axis; and the axis has no point in common with the plane, except the centre.

83. In fact, if the circle $CC'C''$, and therefore its plane, be given, (*Fig.* 28,) we may thus approximate towards one determinate centre. Within the circle, take any point H, and through H pass a line upon the plane, parallel to $CC'C''$. There can be but one such parallel through H, and of course it will be a circle, concentric with $CC'C''$. Within it again take K, and by it determine another circle $KK'K''$, also

parallel and concentric with the former. This process may be repeated as often as we please, and the central area be diminished as much as we please. Each new circle is determinate, if one point in it be given; and hence they converge towards a single determinate point, interior to them all, which of course is the centre.

84. Lastly, the Plane may be looked on as the LOCUS *of Circles, which have the same Axis and Centre.* For there are no circles having the same axis and centre with *C C C*, except those which lie on the plane; and there is no point in the plane which is not likewise a point in one of these circles.

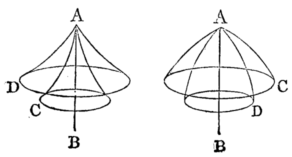
We here conceive of the plane, as *indefinitely extended,* as it evidently may be, by prolonging the line *O C* which generates it. And in this case it suffices to speak of "A Plane," without adding "Plane of *Circle;*" since the circle is but accidentally connected with it.

CURVED LINES.

85. The nature of Straight Lines will be yet better understood, after putting them in contrast with Curved or Bent Lines.

Let *A C* be some stiff line, (*Fig.* 29,) united to a point *B* exterior to it. *A* and *B* remaining fixed, let *A C* generate round *A* and *B* a self-rejoining surface, like an umbrella round its stick. It will be a continuous surface of revolution.

Fig. 29.

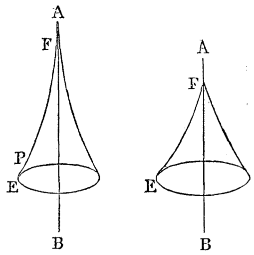


Again: if at *A* be a joint, and *A C* be set in a new direction *A D,* so as no longer to lie on that same surface; and be fixed in the new position; then it may generate a new surface of revolution round *A B.*

Of the two surfaces thus produced, one is interior to the other in the immediate neighbourhood of *A*, and will therefore be justly said to be *sharper* at *A* than the other. We may also say that this surface, or indeed ordinarily each surface, has a *Peak* at *A*.

If to the possible sharpness of the peak, produced by altering the direction of *A C*, there is any limit, let *A E* (*Fig.* 30) be that position of *A C* which makes the peak sharpest. If there be no limit attainable without making a part of the surface disappear at *A*, this is equivalent to saying that a part *A F* of the line *A C* is susceptible of lying evenly with *B*, so as to project out beyond the surface of revolution. Then *A F* is a part of the axis *A B*, and is absolutely Straight.

Fig. 30.



In any other case it is manifest that no portion (as *A F*), however small, can be cut from *A E* such as to be straight; or no portion of *A E*, counted from *A*, coincides with the straight line *A B*. Then *A E* is properly called *Curved* at *A*. The point *A* is a Peak of the surface* of rotation; and there is an evident propriety in calling *A E* *more* or *less* curved at *A*, according as it deviates more or less from the straight line *A B*; that is, according as the peak of the surface at *A* is *blunter* or *sharper*.

86. *Every line may be divided into finite portions that are Straight, and finite portions that are Curved;* if it be not curved throughout, nor straight throughout. For instance, if *A E*, which is curved at *A*, is not curved throughout, there must be some definite point *P*, along *A E*, at which it first *begins* to be straight. Then the finite portion *A P* is wholly curved. Next, setting out from *P*, we may cut off a definite portion which is wholly straight. And so on alternately.

87. *Tangent*, or Osculating Straight Line. *A E* having been brought into such a position that the peak of the

* If a name be needed for this surface, it is obvious to call it a *Bell*.

surface at A is as sharp as possible; we are justified in using the phraseology, that AE lies "as near as possible to the straight line AB at A." This is to use *Distance* in a new sense, yet in a sense perfectly intelligible, and not at all repugnant to its former use.

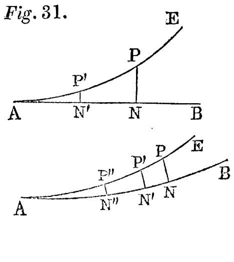
It is immaterial whether AE, turning about A, be brought towards AB, or AB, turning about A, be brought towards AE, as far as proximity of the two lines at A is concerned. In either case, AB and AE are made to take, as nearly as possible, the same direction at A; that is, as nearly as the nature of the curvature at A, and the nature of the straight line AB, allow.

Hence, (if the position of AE be given,) of all possible straight lines that can be drawn from A, none takes so nearly the direction of AE at A, as does the straight line AB; in other words, none *lies* so *close* to AE as does AB. This line AB is therefore said to be the Tangent, (or Rectilinear Tangent,) to the curve AE at A.

In a popular sense, any two lines *touch* one another, when they *meet* one another; and this is a defect in the name Tangent. In consequence of this ambiguity we shall need to be much on our guard; for instance, two solids might "touch one another," and yet *not* be "in contact with one another." It is to be regretted that the word Osculator has not been used in preference to Tangent; and Osculation for Contact. But in the higher Mathematics, Osculation has unfortunately been appropriated to a yet more intimate sort of Contact.

88. It is well to remark on the *visual* characteristic of lines in contact; which is the same, whether one line, as

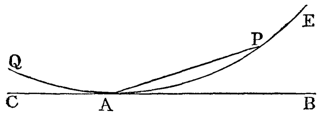
Fig. 31.



AB, be straight, or both lines be curved, (*Fig.* 31.) Let P be any point in the curve AE, and PN the shortest path connecting P to the line AB. Then, when AP diminishes, (by taking P nearer to A,) it is presumed that PN and AN likewise diminish; at least* after AP is less than a certain limit. Now if PN diminish at last far more rapidly than AP, PN attains a size too small to be discerned at all by the eye, while AP is still distinctly visible. When PN is invisible, the two lines appear actually to coincide through the part AP, AN; and this apparent coincidence is the Visual Peculiarity of lines in contact. But the pure science of Geometry recognizes the existence of PN, so long as hypothesis alleges its existence, whether it be visible, or no: hence these considerations do not affect our argument at all.

89. Any *continuous* portion, AP, of a curved line, is called an ARC; and the straight line AP, joining the extremities, its CHORD.

Fig. 32.



90. If A be, not the extremity of a curve, but some intermediate point in a curve PAQ, it has two arcs, AP, AQ, on opposite sides of it, to each of which we may suppose a tangent to be drawn, viz. AB and AC. Now three cases may happen: (I.) as in *Fig.* 32, the two tangents AB, AC, may be opposite branches of the same straight line BAC: (II.) as in *Fig.* 33, AB and AC may be two different straight lines: (III.) They may, as in *Fig.* 34, lie along the same branch of the same straight line AB. In the first case, the curvature is said to be *Continuous* on each side of A. In the two latter cases, the curvature is

Fig. 33.

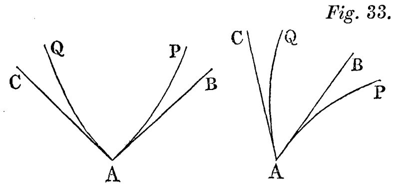
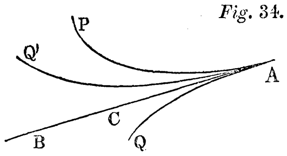
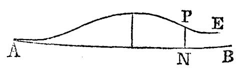


Fig. 34.



* For after AP attains a certain length, it is conceivable that a farther increase of AP might cause a diminution in PN, as in this diagram.



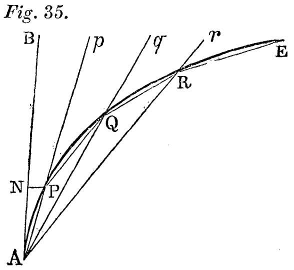
Broken or *Discontinuous* at A, and the curve has a PEAK OF CUSP at A.

It is evident that, in the second case, the curve at A makes an abrupt deviation from straightness, immensely greater than in the first case; while in the last the curve at A turns right back in just the opposite direction.

91. If, in the last case, we look on AP, AQ, as two *separate* curves, then, because they have the common tangent AB, they are said themselves to be *in contact*. It is indeed evident that in no other position could they lie so close together in the neighbourhood of A.

92. If AE be a curve (*Fig.* 35,) divided at P, Q, R.... and the chords AP, PQ, QR .... be all drawn, it appears by Arts. 14—17, that by increasing perpetually the points of division in all parts of the curve, the sum of the chords tends towards the sum of the arcs, (or, towards the whole length AE,) as its limit. Also: that each particular chord, as AP, tends to become equal in length, and coincident in position, with its arc, so as to be entirely confounded with it.

Fig. 35.



It immediately follows, that if the chords AR, AQ, AP, are prolonged to r, q, p, the straight lines Ar, Aq, Ap, tend more and more to coincide in direction as nearly as possible with the arc AE at A. Now the tangent AB, of all straight lines, coincides in direction most nearly with the curve at A. Hence, if AP be perpetually diminished in length, the straight line APp tends towards the position of the tangent, as its *limit*.

93. The review of Arts. 14—17 shows farther, that though a true geometrical curve is not made up of little straight lines, it may be looked on as a limit to which we pass by considering a path made up of straight lines, which become shorter and shorter, and bend oftener and oftener.

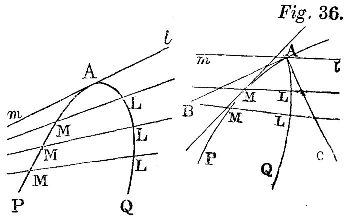
A very small arc nearly coincides with its chord; and by making an arc as small as we please, we may make it coincide, in direction and length, as nearly as we please, with its chord. (Art. 14, 15.) Now, if $P N$ be as short a path as possible from P to the tangent $A B$, it is manifest, since $A p$ tends to assume the direction $A B$ as its limiting position, that the two lines $A P$, $A N$, tend to confound themselves entirely, when the arc $A P$ perpetually diminishes.

If we call the *length* $A N$ by the name of Tangent, in reference to the arc $A P$, we are now warranted to pronounce, that the Tangent, Arc, and Chord, all tend to confound their directions, when the arc perpetually diminishes; and that the *limit* of the two ratios (Tang. : Arc) and (Chord : Arc,) is, the ratio (1 : 1).

94. That the distance $P N$ vanishes when the length of the arc $A P$ vanishes, is, of course, obvious. But this fact *alone* will not suffice to account for $A P$ and $A N$ tending to assume the same direction. It is farther necessary that $P N$ should diminish much faster than $A P$, so that the ratio $(P N : A P)$ must be perpetually getting less, while $A P$ diminishes. In fact, this ratio must be susceptible of indefinite diminution, by lessening $A P$; but the full *proof* of this must be reserved until the subject of Proportional triangles has been discussed.

95. If $P A Q$ be any curve, *having no peak* (*Fig.* 36), and $M L$ be any two points in it, let a straight line of indefinite length be drawn through M and L; then suppose the points $M L$ to move up towards each other, carrying the line with them. If A be the intermediate point in which they tend to concur, and $m A l$ be the limiting position towards which $M L$ tends, then "$m l$ is a Tangent to the curve at A."

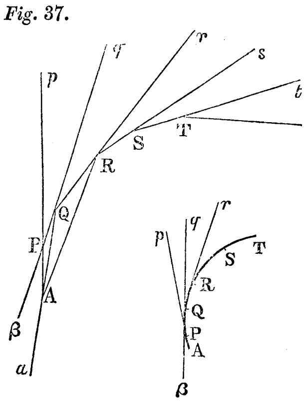
Fig. 36.



For as the arc MAL tends perpetually to confound itself with the chord ML, the chord ML with the tangent at M, and the tangent at M with the tangent at A; the conclusion is evident. It fails only when there is a peak at A, which is excluded by the present hypothesis. In that case the two tangents at M and L do not tend towards the *same* tangent at A, nor does the arc MAL tend to confound itself with the chord ML.

96. All the above will enable the reader to appreciate the statement, that the Rectilinear Tangent "is drawn through *two consecutive* points of a curve." This implies that a chord is first drawn through two neighbouring points, and is prolonged each way; and next, that the two points move together, carrying the chord with them. It then tends to become a Tangent, which is the limit. But if the presence of a Peak be possible, then one of the two points must be stationary, and the other must move towards it. Thus any curve may be approximately represented by portions of its tangents, or of lines which tend to the tangents as their limits. If APp, PQq, QRr, RSs, STt, &c. (*Fig.* 37,) be straight lines, the bent line $APQRS\ldots$ is a rude representation of a curve; and how far it differs from a curve, depends on the lengths of AP, PQ, &c. and their relative directions.

Fig. 37.



Suppose A, P, Q, $R\ldots$ to have been originally taken in some particular curve, and to fix ideas, let the arcs AP, PQ, QR, ... be all equal. Then, as in Art. 92, if the arcs are perpetually lessened, the lines Ap, Pq, Qr, ... will tend to become tangents. We may call them, "ultimately tangents to consecutive points in the curve," and we perceive that the tangents to two consecutive points *inter-*

sect, as $A\,p$, and $P\,q$ in the point P. Thus in passing from A to any other point T along the curve, we find the tangent turn about successive points in its own length, and so deviate into a new position.

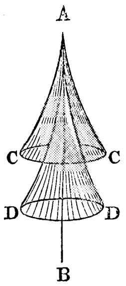
If $Q\,A$, $Q\,P\ldots$ be prolonged to α, $\beta\ldots$ the lines $Q\,\alpha$, $Q\,\beta$ also tend to confound themselves with the tangent at A, when the arcs are perpetually diminished; for $A\,Q$ tends to $A\,p$, as was seen in Art. 92, and so does $P\,q$, or $Q\,\beta$. The same must be true of the prolongations of $P\,R$, $P\,S$, $\ldots Q\,R$, $Q\,S$, $Q\,T\ldots$ since *all* the points $P, Q, R, S\ldots$ tend to merge themselves in A.

97. It now readily follows, that in a limited arc the number of Peaks must be limited; or, what is the same thing, that no two consecutive points of a curve can be Peaks. For instance, if A be a peak, Q, indefinitely near to it, cannot also be a peak. (*Fig.* 37.) For if we take P and R on opposite sides of Q, and draw $\beta\,P\,Q\,q$, $Q\,R\,r$, straight lines, and indefinitely diminish $P\,Q$, $Q\,R$, as also $A\,Q$; the two lines $Q\beta$, Qr, whose limits are the tangents at Q, do both at once approximate towards Ap, so that the opposite tangents $Q\beta$, Qr, at length become a single straight line. Peaks or Cusps are on this account called *Singular* points; because a finite arc, while it contains an infinity of points which are not peaks, has but a finite number which are; and every two consecutive peaks are separated by a finite distance. This is as obvious, as that on a knife edge not every part can be a point or peak.

98. *Deviation of the Tangent.* While a point M, as in *Fig.* 36, traverses the curve from P to Q, let its tangent move with it. Then, by Art. 96, the tangent deviates continually into new positions. But by the same article it appears that (except at a peak) the deviation is gradual, depending on the length of the arc through which M passes, and capable of being perpetually lessened and caused to vanish, by reducing the length of the arc, and causing it to vanish.

Only at a peak is the deviation of a tangent abrupt, and "finite through an infinitely small arc." To explain what may seem an absurd phraseology, consider the curve in *Fig.* 36, in which there is a peak at A. As M moves up from P towards A, the tangent at M tends more and more to assume the position AB, and when M actually reaches A, the tangent attains the position AB. But M cannot move farther along AQ, through any arc, *however small*, without the tangent abruptly passing over into the position AC, or a position indefinitely near to AC. Thus an "infinitely small" motion of M, through A, produces a finite transference of position in the tangent.

Fig. 38.



99. *Deviations of Curves from their Tangents.*—If now we suppose two curves to be placed together, so as to have a common tangent at a common point, as AC, AD have the same tangent AB at A, (*Fig.* 38,) a ready test presents itself, as to "which curve deviates the more from the tangent." For if we suppose them simultaneously to generate bells around AB, the bell whose peak at A is exterior to the other evidently deviates the more.

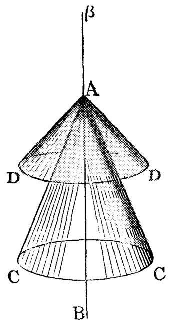
But this does not enable us to pronounce anything concerning the *ratio* of the two deviations. It does not even suggest under what circumstances one curve might be said to deviate *twice* or *three* times, &c. as much as another.

Since it is manifest, by Art. 98, that the principal deviations of curves are at their peaks, at which the tangent itself deviates abruptly, this suggests the propriety of treating on the deviations of *straight* lines, before considering any further those of curves.

RECTILINEAR ANGLES.

100. When two straight lines proceed from one point, the deviation of each from the direction of the other is called a Rectilinear Angle, or more simply, an *Angle*. This Latin term may at first seem to mean the same thing, as its English representative, *Corner;* yet in Geometry they do not mean the same. For the Corner is the bare point in which the lines meet, while the Angle (as we said) is the "deviation," being a relation between the *direction* of the one line, and the *direction* of the other.

Fig. 39.



Depending thus solely on the direction of the lines, the Angle remains the same, whether they be ever so long, or ever so short. It is usual to denote the angle made by AB and AC, by saying, "the angle BAC," or, "the angle CAB," putting the letter which is at the corner *between* the other two.

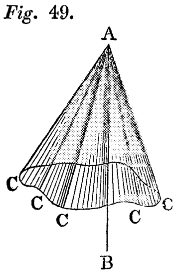
101. A method perfectly similar to that of Art. 99, enables us to decide which of two angles is to be called the *greater*. For if AC, AD be two straight lines, each deviating from the third line AB, we may suppose the whole system to be inflexible, and each of them to generate a surface of revolution round the axis AB; then if (as in *Fig.* 39) the peak of AD is exterior to the peak of AC, we pronounce that AD deviates more from AB, than does AC.

Since AC, AD cannot meet in any second point, the surface of AC forms an entire covering, wholly separating AD from AB; nor is it possible to pass from a point in AD to a point in AB, without piercing the surface. AD is then called *exterior*, because we regard AB as *interior*.

But if BA be prolonged to β, then if $A\beta$ be looked on as interior, AC becomes exterior to the surface of AD, and AC deviates more from $A\beta$ than does AD.

102. *Cone.*—If any straight line AC hang loosely from A (*Fig.* 40), and after performing every circuit soever round AB, return to its original position; the surface which it has traced out is called a Cone. In the particular case supposed above (*Fig.* 39), AC was inflexibly attached to AB; so that in every position its deviation was the same, and the motion was a Rotation. Such a Cone is called for distinction, "a Cone of Revolution;" and as it is for the most part the only Cone spoken of in Elementary Geometry, this is generally understood, when "a Cone" is mentioned, and when the contrary is not specified.

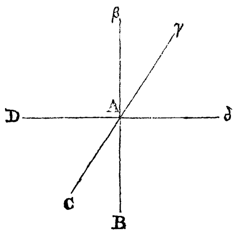
Fig. 49.



103. *Supplement of an Angle.*—Suppose now that in *Fig.* 39, the line AC shifts its place, and occupies that of AD. Hereby the angle which it makes with AB is increased; but the angle which it makes with $A\beta$, the prolongation of BA, is diminished. Thus the two angles BAC, βAC, stand in such a relation, that we cannot increase the former, without diminishing the latter; and of course, conversely. These two are then "Variables," mutually dependent, and they are called Supplements to one another.

104. *Vertical or Opposite Supplements.*—But when a particular angle, BAC, is given, two ways now offer themselves of producing a Supplement to it. For by prolonging BA to β, we get, as before, the supplement βAC; while if (*Fig.* 41) we instead prolong CA to γ, the supplement to BAC is γAB. Before proceeding farther, we must consider whether this involves us in any ambiguity.

Fig. 41.



The following reasoning shows that the opposite supplements are absolutely equal; or are, what we called in Art.

19, *identical* magnitudes. Suppose the whole system of lines (remaining inflexible) to be entirely removed; and then replaced so that the angle BAC shall occupy the same place as before, but with its lines interchanged, AC being where AB was, and AB where AC was. Then the two prolongations $A\beta$, $A\gamma$, will also have exchanged places, and, consequently, the angle $CA\beta$ has precisely exchanged with $BA\gamma$. These angles are then coinciding magnitudes, every way equal; and it is indifferent in which of the two ways the Supplement to BAC is estimated.

105. *Right Angles.*—Let us now imagine that the line AC originally very nearly coincided with AB; in which position the angle BAC was very small, and, consequently, its supplement βAC was large. Suppose then, that BAC gradually opens, and as it increases, its supplement will diminish. If it continue to increase, the supplement will at last become very small, until it all but vanishes.

In such a progress, the angle must have passed *one*, and *only one* intermediate position, in which it is EQUAL to its supplement. Thus, let AD be such as to make the angle $BAD =$ its supplement βAD. In this position, each angle is said to be Right.

106. Moreover, if DA be prolonged to δ, the opposite angles $BA\delta$, $\beta A\delta$, which are the other supplements, are equal to these by Art. 104. Thus the two intersecting lines $BA\beta$, $DA\delta$, produce *four* right angles. Each of them is said "to make right angles," "to be at right angles," or "to be *Perpendicular*" to the other.

107. We said, there is *only one* intermediate position, in which the angle is equal to its supplement. Although no one will question this, it may not be clear to some, how we know it. To remove any doubt on this head, let x be an angle, and y its supplement, and let it be remembered, that the greater x is, the less y is. Hence, if x' is another angle, and y' its supplement, and x' is greater than x, then y' is less than y.

It follows, that should $x = y$, x' is greater than y, and much more greater than y'. Thus in no way can we have simultaneously, $x = y$, and $x' = y'$; unless x and x' were absolutely identical. Or; there is *but one* intermediate angle, between the least and greatest, equal to its supplement.

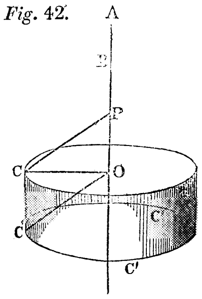
This is generally expressed, by saying that "All Right Angles are equal."

108. *Obliquity.*—Every other straight line, as $A\,C$, which is not perpendicular to $B\,A\,\beta$, is said to be Oblique to it: and of the two angles which it makes, the less (as $C\,A\,B$) is called Acute; while the greater (as $C\,A\,\beta$) is called Obtuse.

109. *Erecting a Perpendicular from a Straight Line.*—It is now evident, that if A be any given point in a given straight line $B\,A\,\beta$; a perpendicular, as $A\,D$, may be erected from A. Nevertheless, "to erect a perpendicular," is not a determinate problem: for an infinity of perpendiculars can be drawn, the LOCUS of all which is the surface of revolution generated by $A\,D$ about the axis $B\,A\,\beta$.

110. On comparing Articles 79, 80, it is very manifest that the surface generated by $A\,D$ is a *Plane*, and $B\,A\,\beta$ the Plane's Axis. Since the Axis is thus perpendicular to the generatrix of the Plane in all its positions, the Axis is said to be "Perpendicular to the Plane."

Fig. 42.



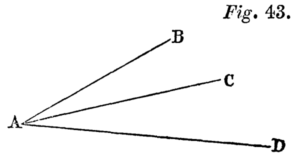
111. *Dropping a Perpendicular on a Straight line.*—Let $A\,B$ (*Fig.* 42) be the straight line, which must be supposed susceptible of indefinite prolongation; and let C be a given point *without* it. Suppose C to generate a circle round the axis $A\,B$; this circle must have a determinate centre O; which is a point in the axis. Join $C\,O$, and it will evidently be perpendicular to $A\,B$. It is said to be "dropt," or "let fall," from C on to $A\,B$.

"To drop a perpendicular" is a wholly determinate problem. There can be no perpendicular as $C P$, P being some *other* point in the axis. For if the system be inflexible, and slide together along the axis, so that the circle may generate a cylinder, when P takes the place which O had before, it appears by Articles 70, 78, that the circle must needs have some new position, as $C' C' C'$. Thus the line $P C$ will have come into the position $O C'$; and, consequently, the angle $C P A$ being equal to $C' O A$, is *not* equal to $C O A$; and is *not* a Right Angle.

112. Thus far we have succeeded in establishing between different angles the relations of Greater, Equal, and Less. But nothing has appeared as yet, by which a *numerical measure* of angles may be attained. It is not possible to affix any sense to the statement that one angle is *double* or *triple* of another, until we can fix on a method by which any number of angles can be added together; and conversely, by which an angle can be resolved into any number of parts, whose sum shall constitute the whole angle. Until this shall have been done, an Angle, if entitled to be called a Quantity, is yet incapable of being measured, or appreciated numerically.

Now that a method of addition may not be illusory, it is requisite and sufficient, (1) that the result may be unaffected by the *order* in which the parts are combined; (2) that no number of resolutions and recompositions may affect it. Yet of various devices by which an unambiguous *Sum* of several angles may be obtained, not all are equally natural and proper, though all may be equally logical. Moreover, if $A B$, $A C$, $A D$, (*Fig.* 43,) are three straight lines proceeding from A, and making three angles, $B A D$, $B A C$, $C A D$, it is by no means justifiable to say that the greatest of the angles is the *Sum* of the other two. For in fact, if the magnitude of the two smaller be given, this does not

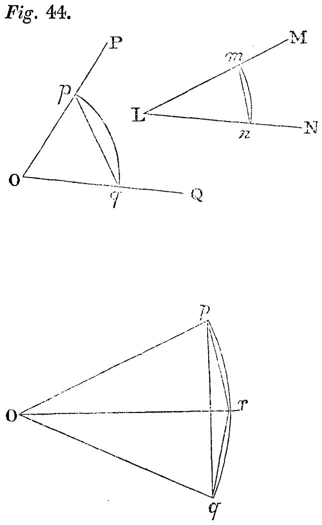
Fig. 43.



suffice to determine a single value for the third. For instance, if the angles BAC, CAD, be given, and BA revolve round CA so as to generate a Cone, we have no right to assert (what indeed is obviously untrue) that the angle BAD remains constantly the same.

113. One method which recommends itself as at once unambiguous and natural, is, to inquire in what position of BA, the angle BAD attains its *maximum:* and to consider this as the genuine *sum* of the two constituent angles. But in our present stage we cannot have recourse to this. We must be satisfied with ultimately proving that the course which we have taken produces this very result.

Fig. 44.



114. When we consider that the quantity to be measured, is, the deviation of the direction of one line from the direction of another line, the thought will instantly arise, whether a comparison of two angles cannot be made, by measuring them from leg to leg. For instance, if POQ, MLN, are two angles, to fix ideas, (*Fig.* 44,) measure off from their legs the lengths $Op = Oq = Lm = Ln$, = one yard; and draw straight lines pq, mn. Then if pq prove to be double of mn, it might at first seem that the angle O must be double of the angle L.

An objection to this presently discovers itself. If pOr, rOq, were angles having a common leg Or, and $Op = Or = Oq$, the line pq, which we have assumed as measuring the angle pOr, is not *made up* of the two lines pr, rq, which are supposed to measure the smaller angles. But if pOq is to be regarded as a whole, made up of parts pOr,

$r\,O\,q$, then the measure of the whole ought to be made up of the measure of the parts.

115. This remark readily leads to a mode of obviating the difficulty. Let a plane be laid upon the lines $O\,P$, $O\,Q$, so that the plane's centre may fall on O, and $O\,P$, $O\,Q$ become two generatrices of the plane: then one circle of the plane (Art. 84) will pass through p and q.* Similarly, through m and n, pass a circle whose centre is L. And let us assume the lengths of the *circular arcs* $p\,q$, $m\,n$, as the measures of the angles $P\,O\,Q$, $M\,L\,N$.

The former objection will not now apply; for if $p\,O\,r$, $r\,O\,q$, be laid down on one plane, whose centre is O, one circle of the same whose centre is O, will pass † through p, r, q; and as the whole arc $p\,q$ = sum of the arcs $p\,r$, $r\,q$, it is congruous that the angle $p\,O\,q$ measured by the arc $p\,q$, should be regarded as the sum of $p\,O\,r$, and $r\,O\,q$, measured by the smaller arcs.

Thus far then, we are led to the principle, that angles which are to be added together should be laid down side by side on a plane, with the plane's centre for their common corner.

116. But a new difficulty may be started, which must be removed before we can acquiesce in this method of measuring the amount of deviation; namely, that an arbitrary quantity has been introduced, in the length of the radius $O\,p$. Now if a change in this length will give different results in our measurement, the method must be abandoned as useless.

To take a simple case: If (*Fig.* 45) $P\,R\,Q$ is a circular arc of centre O, in which $R\,Q$ is double of $P\,R$; whence we infer that the angle $R\,O\,Q$ may, without impropriety, be called double of the angle $P\,O\,R$; let us inquire, whether the propriety of it will be overturned by a change in the

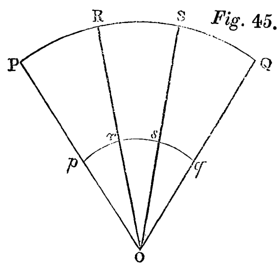


Fig. 45.

* In future this may be expressed, "With centre O, and radius $O\,p$, describe a circular arc $p\,q$."

† Art. 84.

radius OP. Take some *other* length Op along the line OP, and from centre O describe a circular arc prq, cutting OR in r, OQ in q; which is possible, by what has preceded. It is now very manifest, that if RQ be bisected in S, and SO be joined, dividing the arc rq in s, we shall have $rs = sq$. For if the system ROS be applied on the equal and identical system SOQ, so as to coincide, r and s will take the places of s and q. Thus also, PR, RS, SQ, being all equal, pr and rs and sq are likewise equal; which gives, rq double of pr. We find, then, the same ratio as before, between the angles ROQ, POR, if we measure them by pq, and pr, instead of PQ and PR.

But it is at once clear that this may be generalized. For remembering that an axis of rotation passes through O, about which the circles of PQ and pq are described, we may regard PR, pr, as two Variables, depending on each other, as in Articles 33, 41; and in Art. 41, it was shown that they vary *proportionally*. Thus the ratio of PR to RQ, is always equal to that of pr to rq, whatever is the size of the angles at O, and whatever the lengths of the radii OP, Op.

117. Not only, then, does this second objection fall to the ground, but we perceive that we entirely succeed thus in measuring the deviation, (or width of opening between the lines which form the angle,) in the only linear method.* For no other sort of line but a circular arc would give to every elementary equal angle into which we might resolve the whole, an equal measure.

As, however, by Art. 42, it appears that the Areas (or *Circular Sectors*) POR, pOr, are proportional variables, these sectors also might be assumed as measures of the angles. For as the sector PRO is proportional to the arc

* That is: "the only linear method attainable *on a plane*." We might suppose the angles laid down on the surface of a cone, with their common corner at the cone's vertex. But this has a double objection; (I.) It is arbitrary, what sort of cone to choose, or with how large a rotatory angle; (II.) That some angles will be so large as not to lie on the cone at all.

PR, and this arc to the angle POR; it follows that the sector is proportional to the angle. Thus:

Sector PRO : Sector ORQ = Angle POR : Angle ROQ.

118. If, as an arbitrary *unit* for angular measurement, four right angles were assumed; which are measured by the entire circumference of the circle; then since, Any Angle : Four Right Angles, (or 1,) = Arc : Circumference; we get:

$$\text{Angle} = \left(\frac{\text{Arc}}{\text{Circumference}}\right)$$

as a numerical valuation.

By similar reasoning, we get:

$$\text{Angle} = \left(\frac{\text{Sector}}{\text{Whole Circular Area}}\right).$$

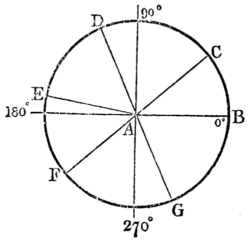
The preceding articles show, that the value of these ratios or fractions is not at all affected by a change of the radius.

119. It is usual to divide the circumference into 360 equal parts, called Degrees; so that every Right Angle contains 90 degrees. But it is needless to enlarge on that which is fully explained in so many other books. It is sufficient here to remark, that, Any angle + its supplement = 180°.

120. *Periodic Magnitude.* We have established that angles are not only Magnitudes, but are Magnitudes resolvable into parts all homogeneous to each other and to the whole, so as to allow of numerical valuation. They have, however, a great peculiarity, distinguishing them from the other magnitudes which we have hitherto met, in their not being susceptible of indefinite increase. There is a maximum value for the angle, which it cannot pass, namely, 180°. On attaining this, the angle vanishes; and if the arc which measured the angle increase yet farther, the angle begins again to increase from nothing.

We may, however, with propriety, extend the limits of angles from 0° to 360°, in order to distinguish between the direction of a line $A B$ from A to B, and the direction of the same from B to A. In *Fig.* 34, we may say that the tangent at A, (which turns back upon itself,) deviates through an angle of 180°. Generally, if A be the centre of a circle $B C D E F G B$, (*Fig.* 46,) and $A C$, $A D$, $A E$, $A F$, $A G$, lines issuing from A in all directions, we may estimate all of these *with reference to the single direction* $A B$, by means of the arcs $B C$, $B C D$, $B C E$, $B D F$, $B E G$, all counted round *in the same direction*, and some of them, perhaps, greater than 180°, or than 270°. Moreover, if $C A F$ be a straight line, and $B C$ (for example) $= 45°$, so that $B D F = 225°$, by assigning these two different arcs, we distinguish between the opposite directions, $A C$, $A F$; a matter which is often of importance in the higher mathematics. Thus in Mechanics, two *opposite* forces might act along $A C$ and $A F$.

Fig. 46.

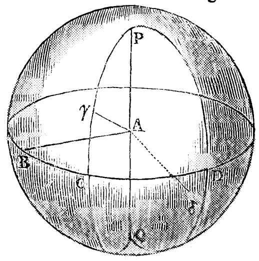


But while we may thus justify the extension of angular magnitude as far as 360°, it is evident that beyond this limit the angle does not increase with the arc. If to any arc, as $B C$, we add 360°, the *direction* determined for the line $A C$ is the very same as before; and by the "angle" we explained that only "relative direction" was meant. Herein, then, consists the *periodicity* of angular magnitude. If the arc by which the angle is determined begin from 0, and increase till it attain the length of 1, 2, 3, 4, . . . circumferences, the angle at the completion of each circumference, vanishes, and then goes through the same series of magnitudes as before.

121. But it may be proper here concisely to point out the method of determining directions universally, whether they do or do not fall on one plane. Suppose a sphere, (*Fig.* 47,) whose centre is A, and $A\gamma$, $A\delta$, to be two straight lines, whose directions, relatively to $A B$, are to be described

or noted down. Let a plane whose centre is A, and axis $P\,A\,Q$, pass through $A\,B$, and cut the sphere along $B\,C\,D$, which, of course, is a circle. Pass a like plane through $A\,P$ and $A\,\gamma$, and a third through $A\,P$ and $A\,\delta$, cutting the sphere along new semicircles $P\,\gamma\,C\,Q$, $P\,D\,\delta\,Q$. Then the direction $A\,\gamma$, with reference to $A\,B$, is fixed by the two arcs $B\,C$, $P\,\gamma$; also the direction of $A\,\delta$ is similarly fixed by the arcs $B\,D$, $P\,\delta$. And so with any other radii.

Fig. 47.



In geography, $B\,C$ or $B\,D$ would be called *Longitude*, and $P\,\gamma$ or $P\,\delta$ *North Polar Distance.*

SCHOLIUM.

122. Very eminent modern geometers,* considering the periodicity which characterizes Angles, have thought themselves justified in pronouncing that the angle has "a natural unit;" and assuming this to be true, have, by a very few steps evolved conclusions, generally supposed to be attainable only by long processes of reasoning, and by help of the properties of Parallel Straight Lines.

In Art. 118, we assumed "four right angles" as an angular unit, and called it "arbitrary." Arbitrary it is; for any other angle might, with equal logical propriety, be assumed: yet at a glance we see that it does not stand on a *like* arbitrary footing with the assumption of a foot, a yard, or a mile, for the *linear* unit; inasmuch as it is the maximum value of angles, while lengths have no maximum. Yet while we are thus led to remark a difference in the two cases, it remains rather vague and uncertain what inferences may be drawn.

* See Dr. Brewster's Translation of Legendre's Elements of Geometry.

But, observe, that if instead of writing; angle $= \frac{\text{arc}}{\text{circumf}}$, we had been able to show that the angle is proportional to $\left(\frac{\text{arc}}{\text{rad.}}\right)$; we should have demonstrated all that Legendre desired. For it is clear that when the arc and the radius are given, the angle is hereby determined, without knowing any angular unit; and the above proportion would show that to give their actual length comes to the same thing as to give their numerical representatives, and conversely. Hence an angle could be determined from knowing the mere ratio of two lines, and without having previously settled on any angular unit. Angles, consequently, *do not need an artificial unit at all*, which circumstance was naturally, and, as the writer believes, truly accounted for, by saying that they had a natural unit in the entire circumference.

123. But can we establish in the present stage, that the angle is proportional to $\left(\frac{\text{arc}}{\text{rad.}}\right)$? This obviously depends on our ability to prove that the circumference and the radius vary proportionally, which must rest on the following train of reasoning.

Let R be radius of a circle, and C its circumference. Then, if R be given, C is geometrically determined, no other element whatever affecting the value of C. We are led to infer, that a mind perfect in intelligence, could deduce by some process of reasoning, the arithmetical *length* of C from the arithmetical *length* of R; and this would imply, that so long as the numerical value of R remained the same, the numerical value of C would likewise be unchanged, namely, that whether the R meant R yards, or R miles, or R furlongs, &c., accordingly, the result would be C yards, C miles, C furlongs, the linear unit being unimportant to the calculation. And this amounts to saying that the circumference must needs bear in all cases the same ratio to the radius. If any one refuse to admit the inference, it must be by alleging, that for the computation of C it might be insufficient to know the numerical length of R, without knowing farther whether it was in *yards* or *miles*, &c. But this seems opposed to the very nature of calculation; as appearing to imply, that help from the *senses* is needed; by which alone a mile can be distinguished to be a mile.

124. Whatever cogency this reasoning may possess, is certainly not due to our secret knowledge that the same result has been attained by other processes, according to the received methods of geometry; for it rests on a far *wider* principle, applicable alike to other sciences, and known as the Law of Homogeneity. In Mechanics, for instance, if homogeneous quantities be mutually dependent on one another, it is considered to be a sort of axiom, that the relation between them, or rather, the equation which expresses it, can only involve their *ratios;* insomuch that such an equation is called "Homogeneous in respect to them." And one might think that every geometer must be conscious, that his mind seizes with a kind of intuition on certain truths which depend solely on this principle, so as to prove fully that we do not need to deduce them by the ordinary steps, of which many are less strikingly obvious than the conclusion. Such truths are those involved in the doctrine of Similar Figures:—that if the three dimensions of a figure vary proportionally, all the linear measurements vary in the very same ratio;—the areas in the duplicate of it, (or as the squares of the lines,)—the volumes in the triplicate of it, (or as the cubes of the lines.)

125. The reasonings of Art. 123, are not really needed as *a part* of Legendre's argument; but they are more or less available for answering objections to it. Yet no geometer thinks it logically incumbent on him to answer objections, which, *if* his demonstration be perfect, must spring from ignorance. It is a condescension on his part to try to help the objector out of a difficulty into which he has plunged himself.

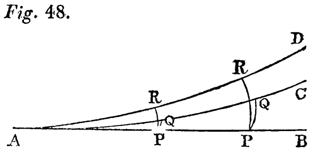
At the same time it is hard to deny that the reasoning (whether as laid down above, or as by Legendre,) involves an assumption which we would not willingly make, while on the threshold of a science which aims at perfect demonstration, namely, *the possibility that the science of Geometry should exist.* If it be conceded as possible, that by a mental process the circumference can be deduced from the radius, the demonstration appears complete. But if a stiff objector * protest, that for anything which has been yet proved to him, geometry cannot in the nature of things become a science of calculation, it may be very hard to answer him.

CURVILINEAR ANGLES.

126. We may now return to the question which we left in Art. 99, and consider how the deviation of *curves*, from one another or from straight lines, is to be measured. And we are naturally led to the following process by what has been already laid down concerning the deviations of straight lines.

Let AB be a common tangent to two curves, AC, AD, (*Fig.* 48,) just as in *Fig.* 38; and suppose a sphere of centre A, small enough to meet AB in P, AC in Q, AD in R. Then with centre A, and radius of the sphere for radius, circular arcs PQ, PR may be described. The lengths of these arcs, when the radius is very short, give a rough or approximate measure of the two curvilinear angles BAC,

Fig. 48.



* This is probably the meaning of Col. Perronet Thompson, who asserts that Legendre has confounded the *determination* of one quantity by others, with its *calculability:* an assertion, the full force of which I did not estimate, when with undue decisiveness I contested it in a Review of his work on Geometry without Axioms.—(*West of England Journal*, 1835.)

BAD. For, in that case, the arcs AC, AD, do not differ greatly from their chords.

If the chord AQ be drawn, the rectilinear angle BAQ is accurately measured by $\left(\frac{PQ}{\text{circumf.}}\right)$; and if the radius of the sphere be perpetually diminished, the chord and arc AQ tend more and more to coincidence. Thus the limit of the rectilinear angle BAQ is the curvilinear angle BAQ. Wherefore this last angle is measured by the *limit* of $\left(\frac{PQ}{\text{circumf.}}\right)$. But this is of no direct utility to us; for as the tangent lies closer to the curve than does any other straight line, it is evident that the peak of the surface generated by AC round AB is sharper than any conical peak; or no conical peak can be introduced at A *interior* to the peak of AC. Hence the curvilinear angle BAC is *sharper than any possible rectilinear angle.* (Which, it will be observed, is thus proved generally of the angle between any curve and its tangent.) It follows that $\left(\frac{PQ}{\text{circumf.}}\right)$ is a ratio which can have no finite limit, but must vanish more and more, and tend perpetually towards zero as the radius of the sphere diminishes.

127. But while the above ratio does not help us to compare the curvilinear with the rectilinear angle, (because the former is indefinitely less,) we may probably in many cases compare *two* curvilinear angles with each other. For, drawing the chord AR, we have:

Rect. angle BAQ : Rect. angle $BAR = PQ : PR$,

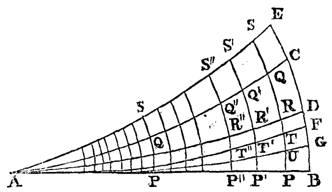
the radius being here quite immaterial, while it is the same for both arcs. Let the radius perpetually diminish; in which case the rect. angles tend to confound themselves with the curvilinear ones; so that we get:

Angle BAC : angle BAD = limit of $\{PQ : PR\}$.

Now *à priori* it is impossible to foresee what will be the limit of the last ratio, which must differ exceedingly in different curves. It is not difficult, however, to invent curves in which the limit may be (1 : 1) or (2 : 1) or any other finite limit: or again, in which it shall be zero, that is, in which the ratio shall diminish *below* all limit.

128. To exemplify this, it will somewhat simplify the matter to confine ourselves to curves drawn on a plane. Let A be the centre of a plane, upon which are drawn an indefinite number of circles, whose common centre is A, as in Art. 81. Let AB be a generatrix of the plane, and AC any curve drawn upon the plane, so as to be touched by AB in A; and

Fig. 49.



of course AC, like AB, will cross the circles. Let $P\,P'\,P'' \ldots$ be points in which AB cuts them, and $Q\,Q'\,Q'' \ldots$ the corresponding points of intersection for AC. We will first show, that a new curve is conceivable, which shall have its curvilinear angle (or *curvature*) at A just one half that of AC.

Bisect the arc BC in D, the arc PQ in R, the arc $P'Q'$ in R', $P''Q''$ in R''; and so on; then the series of points $D, R, R', R'' \ldots$ lie between the curve AC, and tangent AB. And they are indefinite in number, as are the circles which lie on the plane. By increasing the number perpetually, the points $R\,R'\,R'' \ldots$ approach nearer and nearer to one another, and tend to form a continuous line. To use another form of speech; if the radius AP is arbitrary, and the arc $PR = \frac{1}{2}\,PQ$, the LOCUS of R is a certain curve line ARD, which lies between PC and AB. And since $(PR : PQ) = 1 : 2$, a ratio which remains constant however small AP becomes, it follows that $\angle\, BAD : \angle\, BAC = 1 : 2$; or that the curvature of AD at A is half that of AB.

129. It is evident that in like manner a curve is conceivable, whose curvature at A shall bear *any required ratio* to that of AC.

But as it is only the bending in the immediate neighbourhood of A which affects the curvilinear angle at A, and the farther parts of the curve RD may be bent aside without affecting that angle; it is by no means requisite that *every arc* PR should be to its fellow PQ in that required ratio. All that is needed, is that *the limit* of (PR to PQ) should = the ratio proposed.

130. To illustrate this simply, we will devise a new curve wholly distinct from AC,—having no portion, however short, in common with it at A,—and yet having equal curvature with it at A. For this, call the length $PQ = a$, where a is some fractional number, referred to some linear unit, suppose an inch. Then a^2 represents the second power of a, according to algebraic notation; and is less than a, while a is less than 1. Thus if $a = \frac{1}{2}$, $a^2 = \frac{1}{4}$; if $a = \frac{1}{3}$, $a^2 = \frac{1}{9}$; if $a = \frac{1}{4}$, $a^2 = \frac{1}{16}$: if $a = \cdot 1$, $a^2 = \cdot 01$; if $a = \cdot 05$, $a^2 = \cdot 0025$, &c. Now whatever may be the length of PQ or a, which continues to diminish, always take QS, (in the prolongation of the arc PQ,) $= a^2$; and hereby we determine a series of points $S, S', S'' \ldots$ whose locus is a curve ASE. It can never fall *upon* the curve AQC; for QS by hypothesis has always some length. Yet this length (a^2) bears to QP a perpetually decreasing ratio: indeed $SQ : QP = a^2 : a = a : 1$, a ratio which becomes less than any limit, as P approaches A. Thus the approach of ES to CQ in the neighbourhood of A, is indefinitely closer than that of CQ to BP.

It is immediately evident that ASE and AQC have equal curvature at A. For $PS = a + a^2$; $PQ = a$; $\therefore PS : PQ = a + a^2 : a = 1 + a : 1$. But $\angle BAE : \angle BAC =$ limit of $(PS : PQ) =$ limit of $(1 + a : 1)$; which limit is barely $(1 : 1)$; since a is evanescent. Hence $\angle BAE = \angle BAC$.

This teaches us that two curves which have unequal *deviations*, (estimated after the manner of Art. 99,) may nevertheless have equal *curvatures*. The respective deviations through two equal finite arcs, however short, may be unequal; and yet they may approach towards equality as the limiting state, if the arcs be perpetually shortened.

131. Instead of measuring a^2 along the prolongation of PQ, cut off from PQ itself a length $PT = a^2$; or what is the same, $PT = QS$; $P'T' = Q'S'$; $P''T'' = Q''S''$, &c. . . . and let the locus of the points $T\,T'\,T'' \ldots$ be a curve ATF, lying of course between AB and AQC. It is then evident, that as FT approaches A, it lies indefinitely nearer to BP than does CQ. For $PT : PQ = a^2 : a = a : 1$, which is evanescent with a. Hence the curvature BAF is *indefinitely less* than BAC.

132. *Orders of Curvature.* We are led on to remark, that yet a new curve is devisable, whose curvature at A shall be indefinitely less than that of AF, which was itself indefinitely less than that of AQ. For we may suppose PU always cut from PQ, such as to be $= a^3$, the *third* power of a; (thus, if $a = \frac{1}{2}$, $a^3 = \frac{1}{8}$; and if $a = \cdot 1$, $a^3 = \cdot 001$;)

Then $PU : PT = a^3 : a^2 = a : 1 \therefore \angle BAU : \angle BAT$ = limit of $a : 1$; which is evanescent with AP, so that $\angle BAU$ is indefinitely less than $\angle BAT$.

This process may be carried farther and farther by means of the powers a^4, a^5, &c. so that an endless series of curves is devisable, passing through A, and touched by AB, each having its curvature indefinitely less than that preceding it. There is nothing paradoxical or mysterious in this. It is only one form in which *the infinite divisibility of space* (in conception) is set forth. If we can suppose a perpetual and indefinite division of the line PQ as it moves towards A, we can of course equally conceive of curves tending towards AB more and more closely in their approach to A.

The student who is familiarized to algebraic conceptions, will at once perceive that between the series of curves just now supposed, we can at pleasure interpolate others having a like relation to the series. Thus between the curve of a, and the curve of a^2, we may interpolate a curve of $a^{\frac{3}{2}}$ or $\sqrt{a^3}$. Now $a^2 : a^{\frac{3}{2}} = \sqrt{a} : 1$, which is evanescent: also $a^{\frac{3}{2}} : a = \sqrt{a} : 1$, which again is evanescent. Hence the curvature of a^2 is indefinitely less than that of $a^{\frac{3}{2}}$; while the latter again is indefinitely less than that of a.

Thus the geometrical doctrine of the Orders of Contact, is identical with the algebraic doctrine of the Orders of Infinitesimals. See Cauchy's Cours d'Analyse.

133. The most natural inference from the above, is, that Curvatures differ so enormously, and form so many new series of magnitudes *not homogeneous with each other,* as to render hopeless the thought of ordinarily comparing them. Yet the inference is mistaken. It may be interesting to the student, even in this stage, to be informed, that *except at singular points*, the curvature of any two curves soever is of the same order. Thus *A C* and *A F* in *Fig.* 49, although at *A* their curvature is so different, yet at *Q* and *T* probably, (or indeed at every other point but *A*,) have curvatures readily admitting of comparison. And in every curve of limited length, the number of points which have any other than ordinary curvature, is limited.

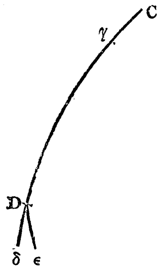
CURVATURE OF CIRCLES AND SPHERES.

134. We have pursued the subject of Curvature into details not logically essential to the argument immediately before us, yet, perhaps, useful in helping the student to distinct ideas on the matter concerning which we are reasoning. We now resume the consideration of the particular curve, which is to us in the present stage most important; taking up the subject in reality from Art. 111.

A Circle is everywhere curved: that is, no circular arc, however short, can be a straight line. For since a circle can slide along itself, the curvature or noncurvature at every point is the same. If then any, however small a part, were perfectly straight, the whole would be straight; and a straight line would rejoin itself.

135. *Prolongation of a Circular Arc.* Any circular arc, however short, admits of sliding along only a single determinate path. For let CD be any arc (*Fig.* 50,) and let it slide into the position $\gamma D \delta$, the part $D\delta$ being its prolongation. We say then, there is no *second* prolongation as $D\varepsilon$, such that it might equally slide into the position $\gamma D \varepsilon$. For, if this were possible, then the system $\gamma D \delta$ might revolve on γD, as an axis, until $D\delta$ came into the position $D\varepsilon$; which would imply that the part γD is a straight line. But we have shown that no portion of a circular arc can be straight. Hence $D\delta$ is the *only* prolongation of CD.

Fig. 50.

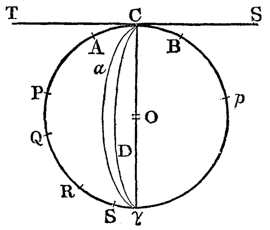


136. We infer that the sliding of CD along itself takes place by a *constrained motion;* and that if the prolongation be continued on and on, the arc will at length complete the whole circumference, and rejoin itself at C.

Moreover; Any arc of a circle, however small, is thus proved sufficient to determine the whole circle. Wherefore, *two circles cannot have any small arc in common.*

137. *No Peak in a Circle.* The uniformity of curvature all round in a circle, proves that there can be no peak anywhere. For if one point were a peak, so would all be, which is contrary to Art. 97.

Fig. 51.



138. *The Tangent is Perpendicular to the Radius.* Let a circle have centre O, (*Fig.* 51,) diameter $CO\gamma$, and let CT, CS, be tangents to arcs CA, CB, on opposite sides of C. It appears by the last Article, that TCS must be a straight line. But besides, since by Art. 75, the circle may be doubled about

its diameter $C\gamma$, so as to make the opposite halves $CA\gamma$, $CB\gamma$, coincide; it follows that CS and CT would coincide, and the angles OCS, OCT are therefore equal. Consequently, each is a right angle.

139. *Three points in a Circle cannot be in the same straight line.* Thus C, A, P cannot lie evenly together. Else, if the circle were doubled about itself, till P and A changed places, and C fell upon Q, Q also would lie evenly with A, P, and C; where $PQ = AC$. Similarly, if $QR = PA$, it would follow that R lay evenly with Q, P, A, C. Again, take $RS = PQ$, and S will be likewise in the straight line. Continue measuring off parts, alternately equal to AP and AC, and we shall at last come round either to C exactly, or to a point beyond C. If we never light on any point twice, however often we go round the circumference; we shall determine an infinite number of points, (as lying evenly with C, A, P,) whose locus is the circumference itself. This would prove the whole circumference to be straight. But if we light again on some point, as C, then a straight line may rejoin itself. Either result is absurd.

140. *The Tangent does not meet the Circle again.* That no small arc of the circle at C coincides with the tangent, appears from Art. 134. Hence at C, the tangent and circle part; the tendency of the circle, even on starting at C, being towards γ, in which CA and CB will at last meet; the tangent having no tendency towards any point on one side of it more than on another side. Now the arcs CA, CB, having once quitted the tangent, by reason of their tendency towards γ, can never again return towards the tangent; but as their path is prolonged, must bend perpetually more and more away from it, since their curvature is all one way. Hence the tangent and circle meet in no point but C, the point of contact.

Otherwise: If the tangent met the curve again as at P, take $Cp = CP$, at the opposite side of C; and it must evidently meet the curve likewise at p. Then P, C, p would be three points in the circle, lying in one straight line; which we have just proved cannot be. For this cannot be evaded by saying that P and p might be but one point, namely γ, (as $C P \gamma = C p \gamma$): for if γ were a point in the tangent, then the diameter and the tangent would coincide; contrary to Art. 138.

141. *Curvature of the Sphere.* If the semicircle $C A \gamma$ revolve about the diameter $C O \gamma$, the arc will generate a sphere; since the surface thus generated is evidently equidistant from O. (See also Art. 64.) Let $C a \gamma$ be any position of the generating semicircle. Then since the curvature of $C A$ and of $C a$, estimated at the point C, is identical, we are justified in saying, that the curvature of the spherical surface at C *in every direction round* C is equal.

Again; since the sphere may slide on its own ground in every direction, until any point C assumes the place of any other point D; in which case the surface immediately round C would occupy the place before held by the surface immediately round D; it follows that the curvature at C = that at D.

It is manifest also, that the curvature of a sphere is measured by that of its generating circle.

142. *Straight line lying upon a Sphere.* The line $T C$, which is tangent to the generating circle, is obviously perpendicular to the sphere's radius, which is the circle's radius. It is besides wholly *without* the sphere. For it cannot be wholly within; for no infinite straight line can be shut up within a limited solid, (Art. 68.) Nor can it pierce nor again meet the surface; for similar reasons to those urged in Art. 140. Hence it lies wholly without the sphere, and *has in common with the surface only the isolated point* C.

143. It hence follows that if $C T$ be a given straight line of indefinite length, and O a point without it, the perpendicular $O C$ being dropped determines C as nearer to O than is any other point in the line.

144. While the semicircle $C A \gamma$ revolves about the diameter $C \gamma$, and generates the spherical surface, let the

tangent CT revolve with it. Then the locus of CT is a Plane, whose axis is $C\gamma$. This is fitly called the *Tangent Plane* to the Sphere at C, because every one of its generatrices is a tangent to a generating circle of the sphere at C. But for a fuller understanding of this, we must take up the matter on more general grounds.

145. If P be a point on any curved surface whatever, and lines PQ, PQ', PQ'' . . . be drawn from it along the surface, to which PT, PT', PT'' . . . . are tangents, the locus of all these tangents is a *single* sheet, forming a Cone whose vertex is P. For if by varying the nature of the curves PQ, (which are subject only to the condition of being drawn from P along the surface,) we could produce *two* or more sheets, then the sheet which lay closer to the given surface would entirely separate the other from it, so that PT on the outer could not lie in contact with the surface at all, nor with any curve drawn on the surface.

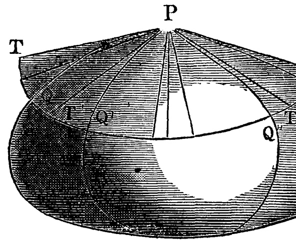
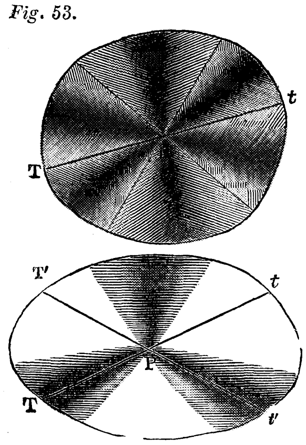


Fig. 52.

Such a cone is called the *Tangent Cone* at P. But the Cone is susceptible of several varieties. (1.) It is possible that the generatrix PT may move round in such a way as to be always perpendicular to some axis drawn through P. In this case the locus becomes a Plane, which is really only a variety of the Cone; though from the absence of any *peak* in the plane at P, we are not used to denominate a plane a sort of Cone. The axis to the Tangent Plane is then called *the Normal* at P. (2.) The Cone, although not a Plane, may be such, that every line PT has its prolongation Pt *also* lying on the Cone's surface. In this case we may conceive the cone to be generated by the half revolution of the line TPt, which is fixed on a pivot P, and vibrates above and below a certain plane, while performing its motion (*Fig.* 53.) No particular name has been given to this variety. (3.) The number of lines $P\,T$, $P\,T'$, . . . which are such that their prolongations lie upon the cone, may be finite; in which case, these form so many *Ridges* crossing in P. Or there may be but one ridge $T\,P\,t$, as along the top of a bank. (4.) No line $P\,T$ may have its prolongation $P\,t$ lie upon the cone; and then there is a true Peak at P; this is what we generally understand by a Cone.

Fig. 53.



146. It would lead us into too long a digression to attempt to prove, what the reader will readily convince himself of,—that except at "Singular Points," a curved surface always admits one Tangent Plane and Normal. To express this otherwise: "Consecutive points on a curved surface cannot be Peaks, and consecutive lines cannot be Ridges." This will be taken up afterwards.

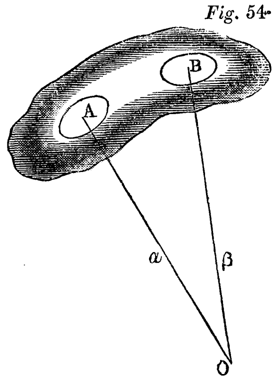
147. Returning to the Sphere, we now see that the plane generated by the revolution of $C\,T$ round $C\gamma$ (*Fig.* 51,) is fitly called tangent to the *sphere*, inasmuch as it contains the tangent at C not only to the circular arcs $C\,A$, $C\,a$, &c. . . . but to every possible line that can be drawn from C along the spherical surface. Also it is evident that no straight line can be drawn from C, between the plane and the sphere. For indeed this is contrary to the very nature of a tangent plane. And the sphere and plane have but one point in common, the point of contact.

Consequently, a straight line $T\,C\,S$, which in a popular sense *touches* the sphere externally, at a point C, does also in a mathematical sense touch (or *osculate*) it; and meets it in that one point C only. It is also perpendicular to the sphere's radius $O\,C$, which meets it in the point of contact.

148. It is included in the above, that a Sphere does not admit of Peaks or Ridges; for we have proved that it has but one Tangent Plane at every point of the surface.

149. *Any portion of a sphere's surface is sufficient to determine the whole sphere.* For in the given portion assume a point *A*, and upon the surface take a distance less than the least distance of *A* from the boundary line. With this distance, determine upon the surface a circle whose pole is *A*. The circle has one determinate axis, as *A a*, which must also be an axis of the sphere. Assuming a second point *B*, we similarly determine *B* β, a second axis of the sphere. But *A a*, *B* β cannot have more than one intersection, *O*, which is thus the single determinate centre to the spherical surface.

Fig. 54.

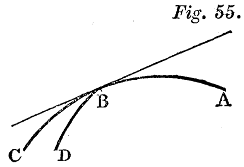


But when the centre *O* is settled, and a point *B* in the surface, the whole sphere is determined. Hence there is *but one* spherical surface, of which the given area can form a portion.

This is equivalent to saying, that "two spheres cannot have any area of their surface, however small, in common.'

150. The given area may of course slide along the sphere's surface, which is its prolongation (or extension) on all sides. But, moreover, it must slide by a certain constraint, so as never to be able to deviate from this one surface; that is, "it has one determinate extension." For if two sheets were imagined, into either of which it might slip, as *B C* and *B D* in *Fig.* 55, then since these must have a common tangent plane at *B*, one must have less curvature at *B* than has the other. Yet unless both had everywhere curvature equal to that of *A B*, and

Fig. 55.

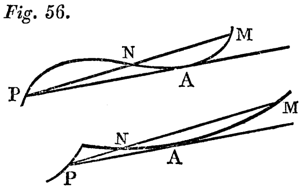


therefore mutually equal, it is evident that AB could not glide along them.

151. It is evident, farther, if in Art. 95, (*Fig.* 36,) we suppose PAQ to be no longer a curve line, but a spherical surface, and M, L, two points which run together, the straight line ML being prolonged will tend towards a tangent (mAl) to the sphere, as its limit. Else, the sphere would have a peak at A; which has been proved impossible.

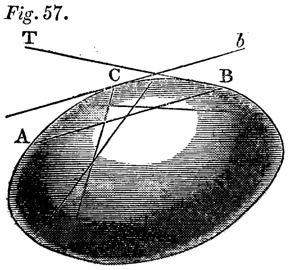
152. Hence we infer, that on a spherical surface, *not more than two points can lie in a straight line.* For if M, N, P (*Fig.* 56,) were three points on a sphere that lay evenly, we might suppose the line PNM to revolve about P, so that M and N might run together into a point A: in which case PA would touch (or osculate) the sphere in E, although it likewise meets the sphere in P; which is contrary to Art. 142.

Fig. 56.



153. *Convexity and Concavity.* If any solid be enclosed by a surface (*Fig.* 57,) such that the straight line (AB) joining any two points (A, B) on the surface lies wholly *within* the solid, it is evident that every tangent plane, or tangent cone to the surface lies *outside*. For if the chord BA revolve about the point B, so that A may move up towards B along any curve ACB drawn on the surface, the prolongation of BA lies entirely without the solid, and, consequently, the tangent BT to which BA tends, will also lie wholly without.

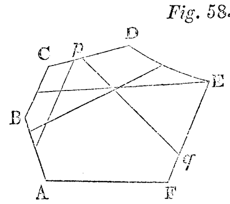
Fig. 57.



Hence the curvature is everywhere turned away from the part exterior to the solid. The outer side is called *Convex*, (protuberant, bulging); the inner, *Concave* (hollow).

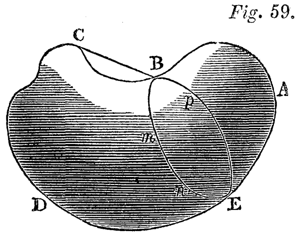
The same names are popularly used, (and may be used with much propriety,) in the case of solids enclosed by *planes,* and which therefore possess no curvature, at least, as far as has yet appeared concerning the plane. Thus if *A B C D E F* (*Fig.* 58,) be a solid, fulfilling the condition that no straight line *p q* joining two points *p*, *q*, on its surface, has any point in it *exterior* to the solid, then the outer surface is called Convex, and the same looked at from within, Concave.

Fig. 58.



A *part* of a surface may be called Convex on one side, even when the whole is not. Thus, let *A B C D E* (*Fig.* 59,) be a solid, which is *not* wholly convex externally, inasmuch as the straight line *C B* lies without it. Nevertheless another portion, as *B A E*, may be externally convex, tried by the following test; that if a line *B m n E p B* be drawn on the surface, cutting off a certain area around *A*, a new surface is conceivable, which shall fill up the line *B m n E p*; and shall, with the opposite area round *A*, form a solid outwardly convex. If so, we are justified in calling the area *A B m n E p B A* by this name.

Fig. 59.



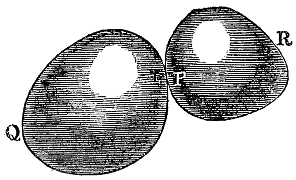
[154. If a solid is not only outwardly convex, but also free from peaks and ridges, and is everywhere curved, so that no straight line upon it can touch it in more than one point at the part of contact; it is often called *oval*, or Egg Shaped (*see Fig.* 57): although this term is sometimes confined to such solids as have a peculiar symmetry, especially those of Revolution.

It is evident from the above, that " Spheres are a species of Ovals," according to this definition. Also, Spheres are externally convex.

155. *Ovals touch one another externally in but one point.* Let the two ovals, *P Q*, *P R*, be in external contact at *P*.

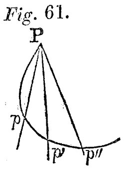
Then, the curvatures at P being in opposite directions, the surfaces bend away from each other on all sides round P. If the two surfaces coincided near the point of contact over any small *area*, this must be because the curvature of one or both was there crushed by pressure. For such area could not possibly be convex on each of its sides, since convexity is the reverse of concavity.

Fig. 60.



Nor can the ovals have any *line* in common, in the neighbourhood of P: for they must have a common tangent plane at P, which entirely separates them; and such line (if it existed) must lie along that plane. Yet if it were a generatrix of the plane, then it would be straight, and would prove the solids not to be convex in this part: or, if it were curved, (*Fig.* 61,) then it would be met by generatrices (Pp, Pp', &c.) of the plane in two or more points; and of such generatrices many must be external to the solids, otherwise the solids would have an *area* (Ppp'') in contact. But if one such generatrix existed, it would prove the solids to be there externally concave; since the straight line (Pp) joining points to (P, p) in their surface, is exterior to them.

Fig. 61.

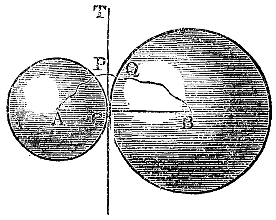


Lastly, it having appeared that in the immediate neighbourhood of P, the ovals have no point in common but the isolated point P itself, it is farther plain that the surfaces meet no more. For as they continue to bend away in opposite directions they can not approach each other again.

156. It is included in the above that Spheres touch each other externally in but one point, and have no second point of the circumference in common. *But as the reasoning in the last Article will to some appear lax, to others difficult to follow, no use has been made of it in the sequel.*]

CONTACT AND INTERSECTION OF SPHERES.

Fig. 62.



157. "A straight line is the *shortest path between two given points.*" Let the given points be A, B, (*Fig.* 62,) and let C be any point in the straight line $A\,C\,B$. We will prove that no path joining A and B can be as short as it might be, unless it passes through C.

Draw $C\,T$ at right angles to $A\,C\,B$, and let it generate a plane round the axis $A\,C\,B$. Also, with centres A, B, and radii $A\,C, B\,C$, describe two spheres. Then since the sphere of centre A is on the same side of the plane that its centre is, and the sphere of centre B is on the same side as is its centre; and the centres are on opposite sides; therefore the plane wholly divides the spheres (Art. 147,) which have thus only the point C in common. Hence any path $A\,P\,Q\,B$ which does *not* pass through C, must pierce the spheres in separate points, as P and Q. Thus a needless length $P\,Q$ is incurred; for the paths $A\,P$ and $B\,Q$ might be otherwise directed, (without change of form,) so as to join the points A and C, B and C; which would save the distance $P\,Q$.

Thus any path, to be as short as possible, must pass through C. But C is *any* point in the straight line $A\,B$. Therefore no path can be as short as it might be, unless it run along the whole straight line $A\,B$. Cor. Hence the straight line is the *measure of distance* between two points in Space.

158. *Addition of Distances.* If now $A, B, C, D \ldots$ are points whose distances, two and two, are given, (A from B, B from C, C from D, &c.) or, what is the same thing, if the line $A\,B\,C\,D \ldots$ is elsewhere inflexible, but has joints at $B, C, D, \ldots$ then the distance of the

first from the last is greatest, when the points $A, B, \ldots E$ are ranged in order along a straight line.

Fig. 63.

For, first, if C be *not* in the straight line AB, the straight line AC is shorter than the sum of the straight lines AB, BC; so that the distance AC is not so great as it might be, namely, by bringing C into the prolongation of AB; and thus the distance AC attains its *maximum.* Next, the same reasoning shows that AD does not attain its maximum, until D is in the prolongation of AC. And so on continually.

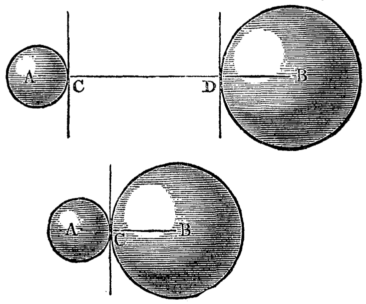
159. *Strained Thread.* If then AE be a thread, which is drawn tight, so as to pull its extremities as far apart as possible, (the length being supposed invariable,) all the points in AE will dispose themselves in a straight line.

This is an experiment made inadvertently by every human creature; so simple and convincing, that it might justly be made an Experimental Law of Geometry. And if any of the reasoning above used is at all questionable as to accuracy, this would be the most preferable mode of obviating all objection. It would also greatly shorten the process of attaining our farther results; but this, in writing an entire scientific treatise, is not always an advantage; for every step which we make is perhaps of intrinsic value, and if omitted in one part, must be introduced in another.

160. *External Contact of Spheres.* Let two spheres be placed at a distance, and two points C, D, on their surfaces, (*Fig.* 64,) which we design to bring into contact, be placed in the line of their centres A, B. Then let A and B, C and D, slide with the spheres along this line, till C coincides with D. Since the tangent planes at C and D lie outside the spheres, and do not meet one another till C and D unite, (for else, contrary to Art. 111, from their concourse would be dropt *two* perpendiculars to the line AB,) it follows that the spheres are wholly separated

until C runs into D, and the two tangent planes merge into one. In this position no second point is common to the surfaces of both spheres; (because this is the only point on which either meets the tangent plane, Art. 147;) and *the point of contact lies in the straight line of the centres.*

Fig. 64.



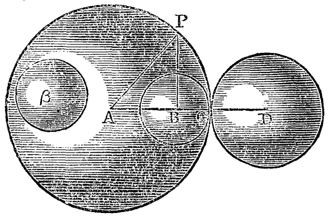
Now I say, this is the only mode in which C and D can be in contact. For if they could remain united while B received ever so small a displacement, by the motion of the sphere B about the fixed point C, let AB be joined, and it would be shorter than $AC + CB$, by Art. 157, or the distance between the centres less than the sum of the radii; which would imply that the curvature of one sphere or other had been crushed in by the motion: which is not contemplated by our hypothesis.

We see, therefore, that though ordinarily one body must be fixed to another immovable body in "at least three points which are not in the same straight line," in order to retain the latter immovably; yet in this case it suffices to fasten the sphere B to the immovable sphere A (or conversely,) by a *single* point C. For the curvatures, though they are not in linear or superficial contact, preclude angular motion as effectually as if they were; the curvilinear angle being less than any rectilinear angle.

161. *Internal Contact of Spheres.* Of two given spheres, let that whose centre is A be the larger, (*Fig.* 65,) and AC a radius. From CA cut off CB, equal to the radius of the smaller; and join AP, BP, where P is any point in the surface of the larger. Then since AP is less than $AB + BP$, while $AP = AC = AB + BC$; therefore BC is less than BP. Hence P is beyond the sphere whose centre is B, and radius BC. If, then, the smaller

sphere be placed with its centre at B, it will touch the other internally at C, *and in no other point.* Also, *the point of contact is now in the line of the centres* prolonged.

Fig. 65.

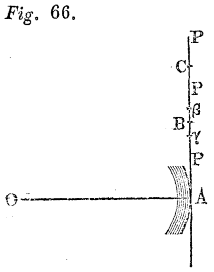


Now I say, this is the only position in which the spheres can have internal contact at C. For if a third sphere, whose centre is D, be applied in external contact at C with the greater of the two, it is, *à fortiori*, in contact with the less, at C. Hence by the last Article, A, C, and D, are in a straight line; and so are B, C, and D, in a straight line; which could not be unless A, B, and C lay evenly with one another.

162. The above is in harmony with what appeared in Art. 149, that it is impossible for two different spheres to have a portion of their surface, however small, in common. Also, the greater sphere has the less curvature.

163. *Intersection of Sphere and Plane.* Let O be any point exterior to the straight line $A B C$, (*Fig.* 66,) and $O A$ be dropt perpendicular to $A C$. We have shown that $O A$ is the shortest path from O to the line, (Arts. 143 and 157). If then, P traverses the line $A P C$, the distance $O P$ increases at first, when P starts from A. Now we farther assert that $O P$ *always continues* to increase, as the distance ($A P$) of P from A increases.

Fig. 66.

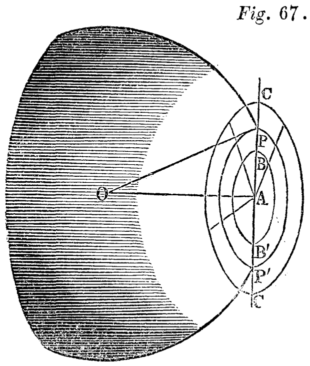


For if not, we must make one of two suppositions. Either, "there is some portion ($B C$) of the line, such, that while P traverses it, the distance $O P$ is invariable." Now this would imply that an entire *ring* of the plane, generated by $B C$ around the axis $O A$, is a portion of the spherical surface whose centre is O, and radius $O\,C$ or $O\,B$: thus, in every position, $B\,C$, as it revolved, would be a straight line touching the spherical surface along its entire length. But this is impossible, by Arts. 142, 152. Again: ELSE, "there is a point B, up to which $O\,P$ keeps increasing, and at which $O\,P$ reaches a maximum, and afterwards decreases." If so, then β, γ, being two points at opposite sides of B, in the line $A\,B\,C$, and ever so near to B, the distances $O\,\beta$, $O\,\gamma$ are less than $O\,B$. Consequently β and γ would be points *within* the sphere of centre O, and radius $O\,B$; and the straight line $\beta\,B\,\gamma$ would be a tangent to the sphere at B, and yet be inside the sphere. which again is contrary to Art. 142.

Since then the length $O\,P$ begins by increasing, when P starts from A, and afterwards, it never remains constant, and never diminishes, while $A\,P$ continues to increase; it follows that $A\,P$ and $O\,P$ perpetually increase together.

164. If now the line $A\,C$ generate a plane round the axis $O\,A$, (*Fig.* 67,) the points B, P, C will each generate a circle of this plane. Let $B\,A\,B'$, $P\,A\,P'$, $C\,A\,C'$, be diameters of these circles. Of course, then, $O\,B = O\,B'$, $O\,P = O\,P'$, $O\,C = O\,C'$; or the points equidistant from A along the line $C\,A\,C'$, are also equidistant from O; as B and B', P and P', C and C', &c... But farther, if a sphere be described with centre O and radius $O\,P$, it is clear that the circle of P is a circle on the sphere. Also, by the last Article, the circle of B lies *within* this sphere, since $O\,B$ is *less* than the radius $O\,P$. This being true for every point B between A and P, it follows that the whole plane area enclosed by the circle of P lies within the sphere. On the other hand, since $O\,C$ is greater than the radius $O\,P$, the

Fig. 67.



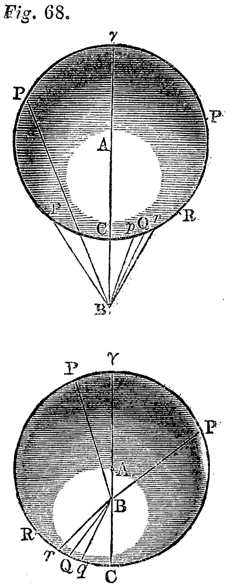
circle of C is outside of the sphere: and this being true, so long as $A\,C$ is greater than $A\,P$, it follows that all that part of the plane which is exterior to the circle of P, is outside the sphere.

Hence the plane *crosses* or *cuts* the sphere in the circle of P; and only this circle is common to the surface of the sphere and the plane.

165. As it will soon be proved that all parts of a plane are like all other parts, the above shows generally, that "the intersection of a plane and sphere is a circle, whose axis is the common axis of the sphere and plane."

166. *Intersection of Spheres.* Let B, (*Fig.* 68,) be within or without a sphere of centre A; and let $B\,A$ cut the sphere in the poles C and γ, of which C is nearer to B than is γ. It is then evident from what was proved about the contact of spheres, (Arts. 160, 161,) that if P is a point traversing the surface, $B\,P$ is least when P is at C, and is greatest when P is at γ. Suppose then that P moves along the surface from C to γ by as short a path as possible. On its starting from C, the distance $B\,P$ begins to increase: but we now farther assert, that $B\,P$ *perpetually* increases with $C\,P$.

Fig. 68.



For, First, there is no small portion ($Q\,R$) of P's path, such, that while P traverses it, the distance $B\,P$ can remain invariable: else the sphere of centre B and radius $B\,Q$ or $B\,R$ would have a *band* of surface, of which $Q\,R$ is the breadth, in common with the given sphere; namely, the surface generated by $Q\,R$ round the axis $C\gamma$. But this, by Arts. 160, 161, is impossible. Next; neither can $B\,P$ attain, as at Q, a maximum value and then again decrease.

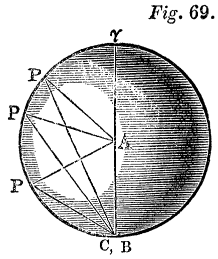
For if q, r, are two points in P's path, very close to Q, on opposite sides, then $B\,q$ and $B\,r$ would be both less than $B\,Q$, however short the distances $Q\,q$, $Q\,r$. Consequently, the sphere of centre B and radius $B\,Q$, would be touched internally by the given sphere along the whole circle which Q generates round $C\gamma$: for the band of the given sphere generated by $q\,Q\,r$ would all lie inside the other sphere. This again is obviously opposed to Arts. 160, 161.

As, therefore, during the increase of the distance $C\,P$, the other distance $B\,P$ never remains constant and never diminishes, it follows that $B\,P$ ever increases with $C\,P$; that is, until $C\,P$ reaches its maximum; which must be when P arrives at the other pole γ.

Hence, if two spheres having centres A and B, are placed so near that the distance of their centres is less than the sum of their radii, they intersect in a circle, (as in the circle of Q,) whose axis is the line of the centres. For it is manifest from the above, that a portion of the surface of sphere A, which is intercepted by the circle of Q round the pole C, is *interior* to the sphere of centre B and radius $B\,Q$; while the rest of the surface is *exterior* to that sphere.

167. *The distance of points on the sphere increases with their absolute distance.* For the above reasoning holds equally, if B (*Fig.* 69,) coincides with C. Then it appears that $C\,P$ *in space* increases with $C\,P$ *on the surface.*

Fig. 69.

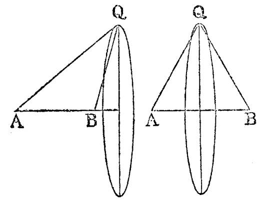


168. Moreover, (*Fig.* 68,) the angle $C\,B\,P$ increases with the distance $C\,P$ along the sphere's surface. Wherefore the distance $B\,P$ in a straight line increases with the angle $C\,B\,P$.

When B coincides with C, (*Fig.* 69,) we can say that "the *chord* of $C\,P$ increases with the *arc* $C\,P$," and consequently, "increases with the *angle* $C\,A\,P$." Thus the diameter $C\,\gamma$ is evidently the longest chord in a sphere.

169. *Triangle of Distances.* When $A\ B\ Q$ is an *inflexible* system, (*Fig.* 70,) it was an experimental Law of Geometry that the point Q revolving round the fixed points A and B, generates a single self-rejoining line, which we call a Circle. But we are now able to say more; namely, that if the system be *not* otherwise known as inflexible at A, B, and Q, these three points *will* be inflexibly connected, if the distances $A\,B$, $A\ Q$, $B\,Q$ are three assigned invariable *lengths*.

Fig. 70.



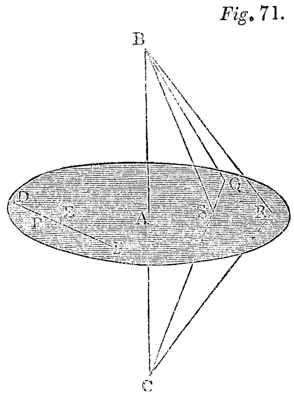
We may state the matter thus. Let A and B be points fixed and known: let Q be a point whose distance from A is known, and whose distance from B is also known. Then Q is on the spherical surface of centre A and radius $A\ Q$; and also on that of centre B and radius $B\ Q$. Hence its locus is in the circle which is the intersection of those spheres. But this is the very circle to which it would be restricted if the system were *by hypothesis* inflexible. Consequently Q is laid under the very same restrictions by assigning its distances from A and B, as by connecting it inflexibly with those points.

Thus if $A\,B$, $B\ Q$, $Q\,A$, are three given distances, the shape of the system is determined, and (if these lines are straight,) the three angles of A, B, Q are determined.

ON THE PLANE.

170. *Evenness of the Plane.* Let the axis $B A C$ (*Fig.* 71,) have $A R$ perpendicular to it, and let $A R$ generate a plane round it. We have to prove that all parts of the plane are Even; that is, "there is no curvature at any part towards either side." This will have been proved, if we have shown that a straight line pressed against any part of the plane lies close against it along the whole length; or, (what amounts to the same,) that if D, E be *any two* points *soever* on the plane, every point P in the straight line $D E$, or in its prolongation, is likewise on the plane.

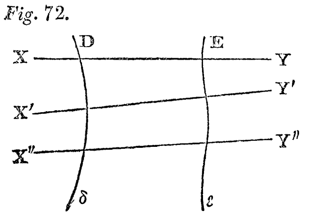
Fig. 71.



Now observe; First; that every point R in the plane, lies at the same distance from B as from C, if $A B = A C$. This is clear by inverting the plane and its axis, so that B and C may exchange places, while A and R remain as before; which is possible by Arts. 80, 81. Next; a point Q on the same side of the plane as B, is nearer to B than to C. For if $Q C$ be joined, it must cut the plane as at S: join $B Q$, $B S$. Then $B S = C S$, as before, S being on the plane, like R: also $B S + S Q$ is longer than $B Q$, (Art. 157); $\therefore$ $C S + S Q$ or $C Q$ is longer than $B Q$. Thirdly; let B and C be fixed points, and R a movable point whose position is assigned by the single condition that it shall be as far from B as from C: and the above justifies us in affirming, that the *locus* of R will be the plane.

Now D, E being two points on the plane, each is as far from B as from C; so that the three distances $D E$, $E B$, $B D$ are identical with the three, $D E$, $E C$, $C D$. Consequently B and C are in a circle whose axis is $D E$, (Art. 169). Now P being in this same axis, is equidistant from all points in the circle; hence $PB = PC$. It follows that P also is on the plane. And P being any point in DE, the whole line DE is on the plane; as was alleged.

Fig. 72.



171. *New mode of generating the Plane.* It is now manifest that a plane can be generated from any two given lines in it, as $D\delta$, $E\varepsilon$ (*Fig.* 72). For if a straight ruler XY press always against both lines, the locus of XY is the plane, if no other restriction is added to its motion.

172. It thus appears also that one and only one plane can pass through two lines, or through a line and a point, or through three points; provided in each case that not all the data lie in one and the same straight line.

173. *Sliding of a Plane.* Supposing $D\delta$, $E\varepsilon$ to be at rest, the plane may be so transferred as to keep it always pressing close against $D\delta$, $E\varepsilon$. But by such transference, the plane (as an indefinitely extended whole,) would always occupy the very same position. Thus it slides on its own ground.

174. *Axes of the Plane.* During the sliding, the axis may be brought to pass through any point required of the plane: so that the plane may be said to have an axis through every point of it. Originally, we supposed the plane to be generated from some one particular axis and centre: but it now appears that there is nothing peculiar to distinguish this axis from others.

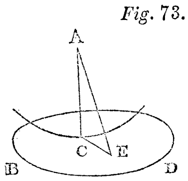
175. *Perpendicular to a Plane.* Since an axis is perpendicular to every generatrix of the plane, it is justly called Perpendicular to the plane itself, as was noted in Art. 110.

From a given point in the plane there can be *but one* axis to the plane, because there can be but one axis to every circle on the plane. Hence but one perpendicular to a plane can be drawn from a given point in it; for a second perpendicular might evidently be made a second axis.

176. Moreover, "from the intersection of two straight lines, can be drawn one and only one line perpendicular to them both:" namely, an axis to the plane which passes through them. For if a second common perpendicular could exist, it might be made axis to a second plane that should pass through both.

177. *Perpendicular dropt on a Plane.* If A be a point (*Fig.* 73,) which is not on a plane $B\,C\,D$, there is some shortest path $A\,C$, from A to the plane. Then the sphere of centre A and radius $A\,C$ cannot intersect and go beyond the plane; therefore, neither can it coincide with the plane in more than one point, as appears by Arts. 142, 147. Hence, too, we ascertain that $A\,C$ is an axis to the plane, and perpendicular to it. But no *second* perpendicular can be dropt, as $A\,E$; else, joining $C\,E$, we should have $A\,C$ and $A\,E$ two perpendiculars dropped on the same straight line $C\,E$.

Fig. 73.



178. *Extension of the Plane.* Any small plane may be prolonged or extended indefinitely, but in a single determinate sheet. This is evident from the generation of the plane in Art. 171, or by its sliding along itself. It follows also that any small plane within a given solid, however large, may be extended so as to cut the surface in a self-rejoining line, and then pass out.

179. *Intersection of Planes.* Moreover, two planes which have two points in common, have in common likewise the straight line joining these two points. But they can have no other point in common without becoming one and the same. Either plane may revolve on this line, until it coincides with the other plane: and from the nature of rotation we deduce that the planes here *cross* each other. The common line is called their Intersection. The angle between them is called Dihedral, and may be readily measured, as in Art. 32. It is clear that if two planes have *one* point in common, they must intersect, viz. in a straight

line: for did they not cross each other, one or other would have a peak or curvature at the common point.

180. *Parameters.* Since any one plane (of indefinite extent,) can be made to coincide with any other, planes do not differ at all. Having everywhere *no* curvature, any two such surfaces are in quality perfectly alike. The same is true of Straight Lines. But Circle differs from Circle, Sphere from Sphere, Cylinder from Cylinder, by reason of the difference of radii, which occasions a difference of curvature in them. When, however, the length of the radius is given, the shape and size of the figures (though not their position,) is completely determined. With reference to this property, their radii are called *Parameters* to them: but Planes and Straight lines are said to have *no* Parameter.

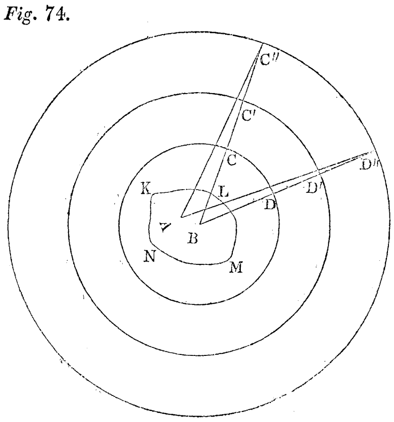
PART II.

ON PARALLEL STRAIGHT LINES.

181. THE term *Parallax* is well known in Astronomy, to indicate "the change in the apparent position of objects, caused by a change of position in the observer:" and more especially, the difference produced by the fact that the observer is on the earth's surface, instead of being at its centre.

Although the name is not particularly needed in Geometry, we meet with the thing. We may regard it as an *error arising from Excentricity*, in the computation of angles, when the corner of the angle is regarded (for simplicity,) as though it were in the centre of a circle, although this should not be accurately true.

Fig. 74.



182. Let CD be the arc of a circle (*Fig.* 74,) of which A is the real centre, and B the *supposed* centre. This false supposition involves the notion that the angle CBD is measured by the number of degrees in the arcs CD: which we know to be true concerning the angle CAD. But as B is

not the centre, it is most probably erroneous to imagine those two angles equal: and the difference between them is the *error* which results from assuming the arc CD as the measure of $\angle CBD$; or, (as we may also put it,) the error arising from the excentricity of B.

183. There are two ways of diminishing the error. The more obvious is, by seeking to diminish the distance of the point B from the true centre A. But another method equally effectual, is, by increasing the radius of the circle, and supposing BC, BD prolonged, so as to meet the new circle in C' and D'; or (if the radius be again increased,) in C'' and D''; then, I say, the angle B will at last, when the radius is of enormous magnitude, be measured with far less error by the degrees in the arc $C'D'$, or $C''D''$. To illustrate the meaning, and at the same time bring conviction of its truth, let B be *one foot* distant from the centre A, and the original radius $AC =$ *two yards*. The error of taking the arc CD to measure the angle B, may be looked on as gross. But take AC', a new radius, $=$ *a thousand miles:* and it is clear that *one foot* is so insignificant in comparison, that the error of confounding B with the centre A, must be very small indeed. If, however, we wish to make it still smaller, take $AC'' =$ *a million miles:* and so on, till it be as small as we choose.

The same method would hold, if B were a mile, or were a thousand miles, distant from A: for we might then suppose $AC' =$ a billion miles, $AC'' =$ a trillion miles; and so on. In short, if $KLMN$ be *any finite area* enclosing the centre A, we may suppose a radius AC'' so vast, that this area may be but a speck in comparison with the circle: and be the error what it may, of confounding any point in this area with the centre, that error may be reduced as small as any one chooses to demand of us, if we may increase the dimensions of the circle at pleasure.

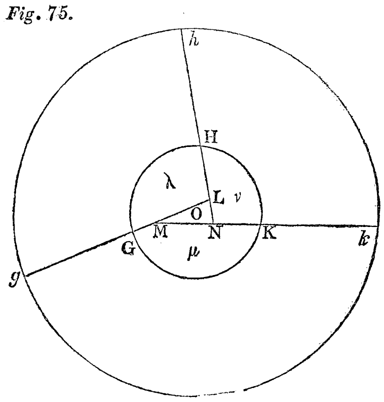
To express ourselves in the phraseology of the higher mathematics: we do not yet know how to estimate the error, when the radius is given; but so much we know *à priori* "it is such a function of the radius, as to vanish when the radius is infinite." And if it be asked how we assure ourselves of this, the only reply is, that by the very nature of Quantity, anything that is finite, (as the amount of excentricity,) becomes insignificant and evanescent in comparison with that which is susceptible of increasing indefinitely: and if any difficulty attach itself to the subject, it is not a purely Geometrical one, but is equally found in the doctrine of Quantity and Number.

184. Like reasonings apply to the *sector* BCD, as well as to the *arc* CD. Were B the true centre, it was shown (Art. 117,) that if we assume the whole area of the circle as the measure of four right angles, the sector BCD would measure the angle BCD. This consequently will, when B is excentric, deviate from the truth only by an error which is evanescent when the radius is indefinitely increased.

185. If the above be conceded as valid argument, all difficulty on this subject is broken down: for it is now easy to prove that "the three external angles of any triangle are together equal to four right angles."

Let LMN be any triangle (*Fig.* 75,) within which take a centre O; and describe a circle containing the triangle. Prolong LM, MN, NL, to meet the circumference in G, K, H. Let L°, M°, N°, be the angles HLG, GMK, KNH, in degrees, and let λ, μ, ν, be the circular areas intercepted by the same angles: also let a = whole area of the circle, τ = area of the triangle.

Fig. 75.



Now if L were the true centre, and not O, we should have $\frac{\lambda}{a} = \frac{L^\circ}{360^\circ}$: but this most probably is erroneous.* Let δ be the error, so that we have accurately:

$$\frac{\lambda}{a} = \frac{L^\circ}{360^\circ} + \delta;$$

where δ is either (possibly) zero, or else some number positive or negative. One thing only is known about δ, by the preceding article, that if the radius of the circle is indefinitely increased, δ diminishes beyond all limit. We similarly have:

$$\frac{\mu}{a} = \frac{M^\circ}{360^\circ} + \delta' \text{ and } \frac{\nu}{a} = \frac{N^\circ}{360^\circ} + \delta'';$$

Consequently:

$$\frac{\lambda + \mu + \nu}{a} = \frac{L^\circ + M^\circ + N^\circ}{360^\circ} + (\delta + \delta' + \delta'').$$

Now $\lambda + \mu + \nu =$ area of circle, *minus* area of triangle; or, $= a - \tau$; hence $\frac{\lambda + \mu + \nu}{a} = \frac{a - \tau}{a} = 1 - \frac{\tau}{a}$. Also, when the circle perpetually increases, a increases *ad infin.*, while τ remains finite. Consequently, $\frac{\tau}{a}$ is evanescent, and the limit of $\frac{\lambda + \mu + \nu}{a}$ is $\frac{1}{1}$. But L°, M°, N° do not vary with the variation of the radius; and the limit of $(\delta + \delta' + \delta'')$ is *zero*, because each separately tends to zero. Hence we obtain:

$$\frac{1}{1} = \frac{L^\circ + M^\circ + N^\circ}{360^\circ},$$

Or, $L^\circ + M^\circ + N^\circ = 360^\circ$; which was to be proved.

Cor. 1. Therefore the three *internal* angles, together $= 180^\circ$. Cor. 2. The four internal angles of a plane Quadrilateral, together $= 360^\circ$.

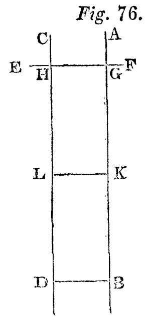
186. This is the proposition, from which Playfair in his

* By using the phrase *ultimately* in its well known acceptation, the details of this argument will be shortened. Thus: $\frac{\lambda}{a}$ *ultimately* $= \frac{L}{360}$ and so on.

Geometry deduces the whole theory of Parallel Straight Lines. But as I have introduced a new definition of the word Parallel, (as equivalent to *Equidistant,*) it is desirable to pursue the subject a little farther.

First, it must be proved that: "If two straight lines (AB, CD,) in the same plane, are both perpendicular to a third straight line, (EF,) they are Parallel." (*Fig.* 76). Let AB, CD intersect EF in G and H. Then since the angles at G and H by hypothesis are right, GH is the distance of the point H from the line AB, and of the point G from the line CD: (Arts. 143, 157.) Let B be any other point in AB, and drop BD perpendicular to CHD; then BD is the distance of B from the line CD. Also: *since the quadrilateral* BGHD *has its four angles together equal to four right angles*, and of these we know three to be right, viz. G, H, and D; it follows that B, the fourth, is also right. Therefore DB is the distance of D from the line GB. And the point B was arbitrarily chosen. It appears then, that to prove the parallelism, or equidistance, of AB and CD, it is only requisite to show that $GH = BD$.

Fig. 76.



Let K be the middle point in GB, and drop KL perpendicular to CD; then, as we showed above that the angles at B were right, so can we show that those at K are right. Now let the figure $KBDL$ turn round KL through half a revolution, till KB, LD have come into the directions KG, LH. Then since $KB = KG$, B will fall on G. But BD being in the plane of KC, and $\angle KBD = \angle KGH$, it follows that BD will have the direction of GH. But LD has simultaneously taken the direction of LH. Therefore D, the intersection of BD and LD, falls upon H, the intersection of GH and LH. Thus GH, BD, coincide, and are equal. Which was to be proved.

187. "Through a given point (G) can be drawn *one* and

only one line in a plane, parallel to a given straight line CD;" and it will be itself straight.

For if we drop GH, perpendicular to CD, we have only to suppose GH to move in the plane GHD, so as to remain always perpendicular to CD, while H traces out the line CD. Then G traces out the *only* line on the plane which can lie at the distance GH from CD; and it is evident from the last Article that the locus of G will be none other than the straight AGB, perpendicular to EF.

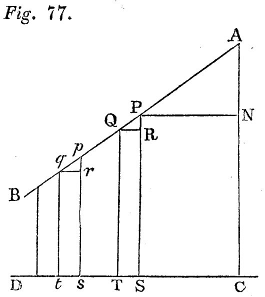
188. "A straight line (GH) perpendicular to another (CHD) is also perpendicular to every line (AGB), which cuts GH, and which, being in the plane GHD, is parallel to the other (CHD)."

For if AGB were *not* perpendicular to GH, the line which should be drawn through G in the same plane and perpendicular to GH, would be also parallel to CD. Thus in the same plane through the same point G pass two lines parallel (or equidistant) to CD; which is obviously absurd.

189. "If a point (P) moves along a sloping path (AB) towards a horizontal line (CD) in the same plane, the *vertical* approach of P towards this line is proportional to the length of the slope which it has traversed." (*Fig.* 77.)

Let P, Q be any two points in the path, and PS, QT be perpendicular to CD: drop QR perpendicular to PS. Then, first, PS, QT are parallel (by Art. 186,) and next, so are QR, TS: whence $QT = RS$. Therefore in moving from P to Q, the point has come nearer to the line CD by the distance PR.—Take $pq = PQ$, in another part of the same slope; and similarly construct the system of lines $pqtsr$: then it is easy to show that the triangles QPR, qpr, are every way equal; and $\therefore pr = PR$. Thus, if the length ($PQ = pq$) along the slope be in two cases the same, then the vertical descent

Fig. 77.



$(PR = pr)$ is also in each case the same, be the previous distance AP what it may. Thus the sloping descent (AP), and the vertical descent (AN), begin together, and increase uniformly; and consequently, (by Articles 38, 39,) the one varies proportionally to the other. Which was to be proved.

190. "Straight lines in the same plane, which are *not* parallel (or equidistant,) may be prolonged so far as to meet." Or, what is the same thing: "Those which, being in the same plane, will never meet, cannot but be parallel; or everywhere equidistant."

For if APB be not parallel to CD, let AN be the vertical approach made towards CD, in the descent AP; (regarding CD as horizontal, to fix ideas.) Also let ANC be the entire perpendicular from A to CD. Since then a multiple of AN can be found, so great as to exceed AC, the same multiple of AP would assign a prolongation of AP sufficient to carry it across CD.

This proposition establishes the identity of Parallelism (or equidistance,) with the notion of the same as given in Euclid, and in most other geometrical treatises: and here, therefore, we stop.

191. But as a matter of curiosity, it may be worth while to go back to Art. 185, and offer an additional thought concerning the argument there employed. We proceeded upon the concession, or established truth, that δ, δ', δ'', were all evanescent when the radius increased indefinitely. Yet it does not appear that the knowledge of this is absolutely essential to the conclusion at which we are driving. It would be sufficient to admit, that "*if* δ have a limit other than zero, yet *that limit does not depend on* L; and so neither the limits of δ' and δ'' on M and N. In this case, if ε = limit of $(\delta + \delta' + \delta'')$, we know that ε does not depend on L, M, N; and we get

$$1 = \frac{L^\circ + M^\circ + N^\circ}{360^\circ} + \varepsilon;$$

consequently $(L^\circ + M^\circ + N^\circ)$ has *a constant value in every* *triangle;* and from this, Legendre shows that we can easily deduce the same results as before.

It may at first appear that this is lower and safer ground: yet in fact we gain nothing by it. For if we cannot infer that δ is *evanescent* for infinite values of the radius, we have nothing at all to convince us, that the limit of δ is not dependent on L; in which case ε would vary in different triangles. Hence the latter mode of stating the argument is unserviceable.

PART III.

ON SOLID ANGLES.

192. *Trihedral and Polyhedral Angles.* If several straight lines in different planes, as OA, OB, OC, &c. meet in one point O; (*Figs.* 78, 79,) and planes pass through each contiguous pair, when these are taken in a certain order; there is formed at O that which is called a *Solid Angle*, by reason of its analogy to plane angles. It is named, according to the number of the planes which form it, Trihedral, Tetrahedral, Polyhedral, for three, four, or more planes.

Fig. 78.

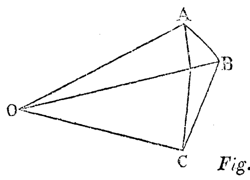
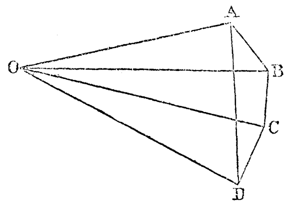


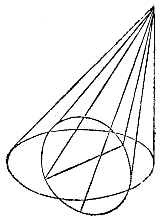
Fig. 79.



193. But we may also conceive of a solid angle formed without any *planes* at all; as at the vertex of a Cone. Such a solid angle bears to a polyhedral angle the same relation as a curved line bears to a straight line, which is bent in many places. We now encounter a difficulty in part like that which was met in discussing plane angles, viz. an inability to estimate their relative *magnitudes*. But plane angles readily admitted of a direct comparison as to greater and less, which is not the case with many solid angles. For instance, if two cones be constituted (*Fig.* 80,) with the same vertex, upon two oval curves, as their

directrices, which intersect each other; no immediate supraposition of the vertical angles avails to establish the relation of greater and less.

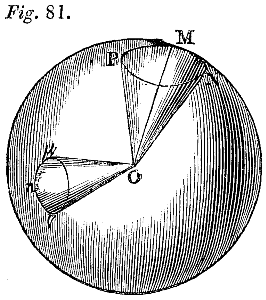
Fig. 80.



194. In another view, however, the Solid Angle is more manageable than the plane angle: viz. that it is very obvious under what circumstances one solid angle is justly said to be divided into two others. In fact, if any plane pass through the vertex of the angle, (as in one of the Cones just supposed,) and divide the base into two parts; this plane also divides the solid angle into two parts, the *sum* of which makes up the *whole* solid angle.

195. It is, then, easy to show the homogeneity of any two solid angles, by a proceeding similar to that which we used in the case of any two solids of limited extent. For, any very small part of a solid angle is homogeneous to the whole. To convince ourselves of this, we have only to consider that by repeatedly taking from the whole a very small part, we may as nearly as possible exhaust the whole. This shows that we may institute a numerical comparison between the whole and such a part; which indeed may be an exact submultiple of the whole. By subdividing this part continually, we may make it differ as little as we please from being a submultiple of any *second* proposed solid angle. Wherefore any two proposed solid angles admit of numerical comparison.

Fig. 81.



196. But as the comparison of plane angles is facilitated, by proving that they are proportional to the arcs of a circle, when the vertex of the angles is at the centre; so the comparison of solid angles is more vividly apprehended, by a like use of the spherical surfaces on which they stand.

Let two solid angles have O for their common vertex. With centre O, suppose any sphere to be described; and on the surface, let the solid angles determine the areas or bases MNP, $\mu\nu\pi$ (*Fig.* 81.) *The two solid angles shall be proportional to these their spherical bases.*

Call the two solid angles A and α, and the areas of their bases B and β; we have then to prove, that A is to α as B is to β.

197. Now, FIRST, it shall be shown that in every case in which the base B = the base β, it is also true that the angle A = the angle α.

The possible cases of equality between B and β are three: (I.) B and β may have the same *shape* as well as size; or be absolutely identical. In this case the sphere may slide on its own ground, while the centre is unmoved, until B actually coincides with β; and then A precisely coincides with α; or $A = \alpha$. (II.) B and β may be divisible into an equal finite number of parts, as $R_1\ R_2\ R_3 \ldots\ R_n$, and $\rho_1\ \rho_2\ \rho_3 \ldots\ \rho_n$, such that every R is identical with a ρ that corresponds. For this gives

$$\left.\begin{array}{l} B = R_1 + R_2 + \ldots + R_n \\ \beta = \rho_1 + \rho_2 + \ldots + \rho_n \end{array}\right\} \quad \text{Also, } \left.\begin{array}{l} R_1 = \rho_1 \\ R_2 = \rho_2 \end{array}\right\}$$

$$\text{whence } B = \beta. \qquad \&c.\ \&c.$$

Now in this case, the division of the base B into (n) such parts, furnishes a corresponding division of the angle A into (n) parts, which may be called $S_1\ S_2\ S_3 \ldots S_n$; so that

$$A = S_1 + S_2 + S_3 + \ldots + S_n.$$

$$\text{Similarly } \alpha = \sigma_1 + \sigma_2 + \sigma_3 + \ldots + \sigma_n;$$

since the division of the base β equally gives rise to a division of the other angle α into (n) parts. Moreover, since R_1 is identical with ρ_1, in shape and size, therefore $S_1 = \sigma_1$, (by Case I. just treated,) if these are the angles which have the bases R_1 and ρ_1. Similarly we have $S = \sigma_2$, $S_3 = \sigma_3$, and so on. Wherefore $A = \alpha$.

(III.) B and β may be called equal, on yet a third ground; viz. when each is divisible into parts which form a converging infinite series, say,

$$\left.\begin{aligned} B &= \text{limit of } R_1 + R_2 + R_3 + R_4 + \text{\&c. } ad\ infin. \\ \beta &= \text{limit of } \rho_1 + \rho_2 + \rho_3 + \rho_4 + \text{\&c. } ad\ infin. \end{aligned}\right\}$$

such that $R_1 = \rho_1$, $R_2 = \rho_2$, $R_3 = \rho_3$, &c. *ad infin.*

where these last equalities imply areas which are either identical, or may become so by a mere redistribution of parts. This falls under Cases I. or II. just treated. Hence, retaining the same notation as before, we infer by those Cases, that $S_1 = \sigma_1$, $= S_2 = \sigma_2$, $S_3 = \sigma_3$, &c. *ad infin.* At the same time we get:

$$\left.\begin{aligned} A &= \text{limit of } S_1 + S_2 + S_3 + \text{\&c. } ad\ infin. \\ \alpha &= \text{limit of } \sigma_1 + \sigma_2 + \sigma_3 + \text{\&c. } ad\ infin. \end{aligned}\right\}$$

whence it follows that $A = \alpha$.

The fourth case of conceivable equality, mentioned in Art. 22, need not here be treated, because the bases B and β are surfaces of like curvature everywhere, so that every part may coincide by supraposition with every other part.

Generally then, it has been proved that if $B = \beta$, $A = \alpha$; or, what is the same, that the magnitude of the angle (A) is determined, when the area of the base (B) is given.

198. We have further to prove, that *if the area* (B) *varies at all, the angle* (A) *varies in the same ratio,* the sphere being unaltered.

They are magnitudes which vary *together,* by what has already been established. Moreover, when B becomes very small so as to be ready to vanish into nothing, A likewise is ready to vanish, and may be made as small as we please by diminishing B. Contrariwise then, if we regard A and B as increasing, we may state that *they begin together from zero,* and increase together.

Suppose, then, any increment ΔB to be assigned to B; then a corresponding increment ΔA at once accrues to A.

Also the magnitude of ΔA is determined by the magnitude of ΔB, and by that alone; be the magnitude of B what it may. Hence it immediately follows, (by the doctrine of Proportion delivered in Art. 40,) that *A varies proportionably to B.*

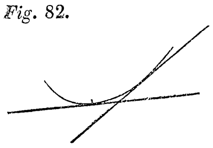
199. We have thus established generally, that on the surface of a given sphere, the area inclosed by a solid angle whose vertex is at the centre, is a proper MEASURE of the Solid Angle.

PART IV.

CERTAIN ELEMENTARY PROPERTIES OF PLANE CURVES.

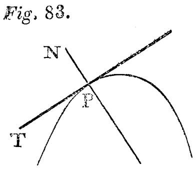
200. From the *Evenness* of the Plane, it follows that all the Chords of a plane curve lie in its plane: so therefore do all its Tangents, since they are limits to the chords.

Fig. 82.



201. That side of a plane curve on which the tangent falls, (*Fig.* 82,) is suitably called *Convex*, and the opposite side *Concave;* in conformity with the language used in Art. 153.

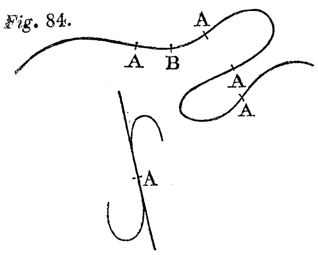
Fig. 83.



202. *Normal.* By this word is understood a perpendicular to the tangent, drawn through the point of contact, and in the plane of the curve. (*Fig.* 83.) Hence the radii of a circle are all Normals.

203. *Undulation.* If, at any point (A) in a plane curve, (*Fig.* 84,) the tangent changes its side, so that the curve, from being convex at one side, becomes presently concave, or *vice versâ;* the curve is said to undulate at this point, if there is here no breach of the continuity; that is, if there is here no peak. The tangent *cuts* the curve at a point of undulation.

Fig. 84.

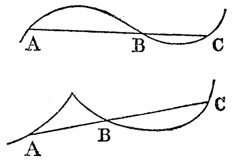


This may lead us to remark the incorrectness of laying down that a Tangent is a straight line which "meets a curve without cutting it;" a coarse definition, suited only to a very crude state of geometrical knowledge.

204. Points of Undulation, like peaks, are *Singular* points; that is, are of finite number in a finite arc. If we try to conceive them occurring consecutively, all idea both of convexity and of concavity is destroyed. Wherefore from a given curve at a given point A, there can always be cut an arc $A B$ so short as to have neither peak nor undulation. (*Fig.* 84.)

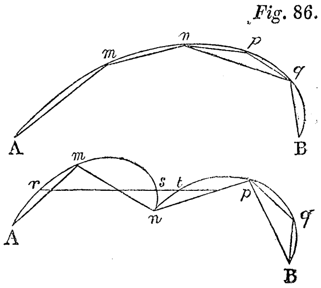
205. If a connected curve have *three* points (A, B, C) in a straight line, this implies that the curvature has not been all towards one side continuously: hence there must be between the extreme points either a peak or an undulation. (*Fig.* 85.)

Fig. 85.



206. If the plane curve ($A B$) (*Fig.* 86,) be destitute of singular points, and $m, n, p, q \ldots$ are taken in it, between A and B; and chords $A m$, $m n$, $\ldots q B$ are drawn; it is evident that all the angles $A m n$, $m n p$, $\ldots p q B$, are pointed towards the side on which the curve is convex. For if two of these angles, as $A m n$, $m n p$, were turned opposite ways, a straight line could be drawn from a point in the arc $A m$, cutting the chords mn, np; and consequently, cutting the arcs of those chords. It would then meet the curve in as many as three points, (r, s, t.) But this is not possible, if the curve have neither peak nor undulation.

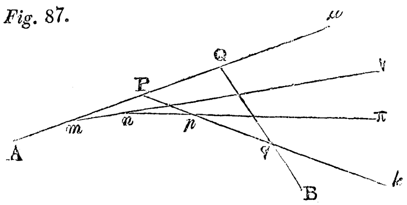
Fig. 86.



The Rectilinear path $A m n p q B$ may be called Convex on the side towards which the angles present themselves.

207. *Deviation.* In such a path, each straight line leaves the direction of that immediately preceding it, and deviates into a new direction: but all deviate towards the same side. If we prolong Am to μ, mn to ν, np to π, pq to κ (*Fig.* 87,) the successive deviations at m, n, p, q, are measured by the angles $\mu m n$, $\nu n p$, $\pi p q$, $\kappa q B$; and the sum of all these together might be called the *Total* deviation.

Fig. 87.



But if again we prolong qp, Bq backward, so as to meet $A\mu$ in P and Q, between m and μ, the angles μPq, μQB, measure the deviations of pq and qB from the original direction Am. We may call $\angle \mu QB$ the *ultimate* deviation attained by the path qB.

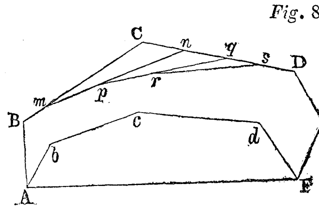
208. That the *ultimate* is equal to the *total* deviation, readily appears from the doctrine of Parallel Straight Lines, and the propositions connected with it. In fact, considering the triangle PqQ, of which μQB is an *external* angle, and PqQ an internal angle $= \kappa q B$, we have at once: $\angle \mu QB = \angle \mu Pq + \angle \kappa q B$: which expresses, that the ultimate deviation of qB, exceeds that of pq, by the amount of the deflection at q. Thus each successive deviation is *added* to that which before existed.

On the other hand, it is so simple a principle as to bring conviction to the mind by a direct process; that when all the deviations are *in one plane* and *towards the same side,* the Ultimate deviation must be equal to the Sum of the separate deviations. It may, therefore, deserve the consideration of geometers, whether this might not be proposed in such a form, as to make it the foundation on which the doctrine of Parallels might rest.

209. Once more, suppose such a path, $ABCDEF$, with the angles all turned one way. (*Fig.* 88.) Select one corner, as C, and cut it off by the line mn. By taking either m or n

as near as we please to C, we may evidently make the difference of length between $(m\,C + C\,n)$ and $m\,n$, as small as we please: yet $(m\,C + C\,n)$ is always longer than $m\,n$, while a triangle $m\,C\,n$ exists. Hence $A\,B\,m\,n\,D\,E\,F$ is a new path like the former, only shortened, and that, as little as we please.

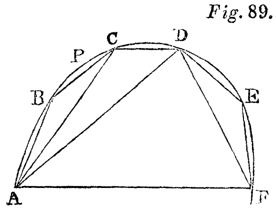
Fig. 88.



But again, we may cut off the corner n by a straight line $p\,q$; and next, cut off the corner q by a straight line $r\,s$; and so on continually. The resulting interior path which connects A to F, is always shorter than the exterior. Hence if $A\,b\,c\,d\,F$ be any such interior path, (the angles at b, c, d . . . being all turned the same way as B, C, D . . .) it is shorter than $A\,B\,C\,D\,E\,F$.

210. Let now $A\,P\,F$ be any arc of a plane curve (*Fig.* 89,) concave towards the chord $A\,F$, and *without peak:* and by perpetual division of the arc let chords be inscribed. Thus, first, take D on the arc, and join $A\,D$, $D\,F$. Next, on the arc $A\,D$ take C, and on $D\,F$ take E; join $A\,C$, $C\,D$; $D\,E$, $E\,F$. Next; on $A\,C$ take B; and so on.

Fig. 89.



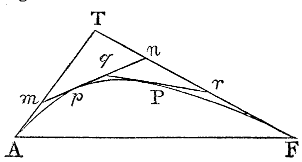
Then the chord $A\,F$ is shorter than $A\,D\,F$; $A\,D\,F$ than $A\,C\,D\,E\,F$; this last than $A\,B\,C\,D\,E\,F$: and so on. Or *the sum of the chords*, as they thus increase in number and diminish in length, *is a perpetually increasing quantity.*

But the limit towards which their sum tends, is, the length of the arc $A\,P\,F$ itself. (Art. 17.) Since, then, they *increase towards it*, it is always "greater than their sum," which also is manifest from the circumstance that each small arc is longer than its chord, by Art. 157.

211. Again: let the tangents at A and F meet in T, on the side of the convexity $A\,P\,F$; and we may now show

that* *the sum of the tangents* ($AT + TF$) *is longer than the arc* APF; the limitation being continued as before, that APF has no Singular point; that is, neither undulation nor peak. (*Fig.* 90.)

Fig. 90.

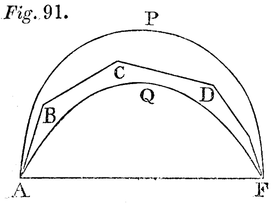


For, a series of paths is conceivable, of which ATF is the first in order; that shall continually diminish in length, and shall tend towards the arc APF as their limit. If this can be proved, then the arc will manifestly be shorter than any of them, and of course shorter than the longest of them, namely, than ATF.

To exhibit the truth of the above: first suppose the angle T cut off by a straight line mpn, which touches the curve in p. Then since mn is shorter than $(mT + Tn)$, the path $AmnF$ is shorter than ATF.—Next, let the corner n be cut off by a straight line qPr, touching the curve in P; then qr being shorter than $(qn + nr)$, the path $AmqrF$ is shorter than $AmnF$. By proceeding thus, always cutting off the angles of the path by tangents, it is evident that each new path which is produced is shorter than that from which it was formed. It is likewise manifest that the curve APF itself is the limit towards which we tend: and this is what we undertook to prove.

212. Let us now imagine two curves APF, AQF, both concave towards the common chord AF, and on the same side of it. If one contains the other, that which is the interior (as AQF) is the shorter.—For between the two we can draw a rectilinear path $ABC \ldots F$,

Fig. 91.

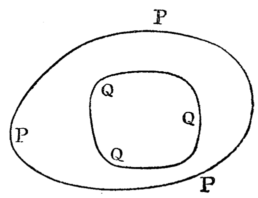


* Many Geometers assume this as an *Axiom*. But as all our definitions are complete, we have no pretence for any such assumption; but if it be true, it can be, and ought to be, deduced from the definitions. The proposition, moreover, makes a very cumbrous axiom, because of its being embarrassed by the limitation that the arc must be free from peaks and undulations.

having all its corners presented towards $A\,P\,F$: and by the preceding articles it is evident that this path will be shorter than $A\,P\,F$, but longer than $A\,Q\,F$ (*Fig.* 91.)

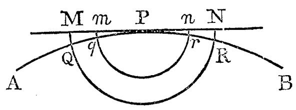
213. The reader will with great ease infer, that of *two plane ovals* that which may be contained within the other, (as $Q\,Q\,Q$ within $P\,P\,P$,) has the shorter outline. (*Fig.* 92.)

Fig. 92.



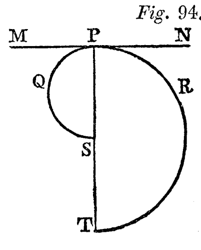
214. Besides Peaks and Undulations, there are other singular points to be noticed, depending on an irregularity in the curvature. It was explained above, (Art. 127,) how two different curvatures may be arithmetically compared: and this same method is evidently applicable to compare the curvatures *at the opposite sides of the same point in one and the same curve.* Thus, if $A\,P\,B$ is a curve, (*Fig.* 93,) we have to consider at P two different curvatures, that along $P\,A$, and that along $P\,B$. With centre P and any radius $P\,M = P\,N$, describe a semicircle, cutting the tangent at P in M and N, and the curve in Q and R. Then the ratio $\{M\,Q : N\,R\}$ will roughly show the ratio of the curvatures $P\,Q$ and $P\,R$ to each other, if $P\,M$ is small: and if $P\,M$ is perpetually diminished, the *limit* of the ratio $\{M\,Q : N\,R\}$ expresses the ratio of the two curvatures.

Fig. 93.



215. If now the limit of $\{M\,Q : N\,R\}$ is the ratio $\{1 : 1\}$, the opposite curvatures are exactly equal. But no one is able to predict concerning any curve soever that this must needs be the case. In fact, it is easy to construct a curve in which it shall be otherwise. Thus, if two unequal semicircles $P\,Q\,S$, $P\,R\,T$ be applied on opposite sides of a straight line $P\,S\,T$, which is perpendicular to $M\,P\,N$, it is at once clear that the curve

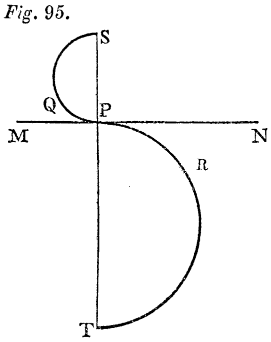
Fig. 94.



$SQPRT$ has on opposite sides of the point P unequal curvatures. (For although we have as yet given no rigorous demonstration that unequal circles have unequal curvatures, it is allowable to assume it, when we are only aiming at illustration.)

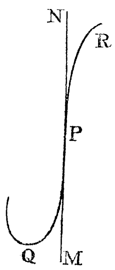
216. A similarly abrupt change of curvature might happen at a point of undulation; as will be seen by reversing one of the semicircles so as to produce an undulation at P. Then, if for illustration we suppose the curvature of SQP to be represented by 1, and that of PRT by 3, we may with propriety say that at the point P the curvature changes suddenly from $+1$ to -3; or from -1 to $+3$; since it changes in direction as well as in amount.

Fig. 95.



217. This leads us to remark that a point of undulation will always imply an abrupt and finite change of curvature, unless the curvature becomes actually *zero* on each side of the point: in which case we may say that, in changing from positive to negative, it passes through zero. There is then no discontinuity. Should this be the case, the curve in the immediate neighbourhood of the undulation appears almost straight; and although the tangent cuts it, the contact is infinitely closer than under ordinary circumstances. (*Fig.* 96.)

Fig. 96.



218. *Measure of Circular Curvature.* Let us assume, in accordance with Art. 208, that the ultimate deviation of a bending path is equal to the sum of all the separate deviations. It immediately follows, that, if AB be a circular arc, (*Fig.* 97,) and $AQ\mu$, BQ tangents, the angle $BQ\mu$, which is the ultimate deviation of the arc, between A and B, is also the sum of the deviations, (or, as we may now say, the sum of the curvatures) of the whole

arc. Hence the angle $B\,Q\,\mu$, *divided by the length $A\,B$*, is the proper measure or quantum of curvature at every point of the circular arc.

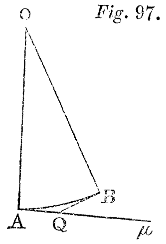


Fig. 97.

This expression admits of a very simple transformation. Let $O\,A$, $O\,B$ be radii of the circle. Then the angles at A and B being right, in the quadrilateral $O\,A\,Q\,B$, the other two angles at O and Q in that quadrilateral must be mutually supplements; (Art. 185, Cor. 2): and therefore, $\angle B\,Q\,\mu = \angle O$. Next; the angle O is to four right angles, as the arc $A\,B$ is to the circumference. Hence the fraction $\left\{\frac{\angle B\,Q\,\mu}{\text{arc } AB}\right\}$ which measures the curvature $= \left\{\frac{4 \text{ right angles}}{\text{circumf.}}\right\} = \frac{1^*}{\text{radius}}$, if we suppose a right angle to be measured by one quarter of the circumference whose radius = 1.—Also, when rad. = 1, curvature = 1.

Making then the standard "unit of curvature," the curvature of the circle whose radius is 1, the curvature of any other circle is measured by $\left(\frac{1}{\text{radius}}\right)$.

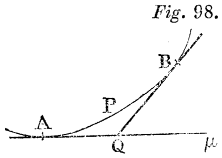


Fig. 98.

219. *Finite Curvature.*—Let $A\,B$ be any plane curve soever, which is not a circle, (*Fig.* 98,) and having its curvature turned all one way. Suppose the extreme tangents $A\,\mu$, $B\,Q$ to meet in Q, so that $\angle B\,Q\,\mu$ as before expresses the ultimate and total deviation through $A\,B$, or the sum of all the curvatures; which may be conceived of as angles infinite in number, but each infinitely small. Then the fraction $\left\{\frac{\angle B\,Q\,\mu}{\text{arc } A\,B}\right\}$ expresses the *average* curvature in the course of the arc $A\,B$: and as each term of this ratio or

* We here borrow the proposition, that the circumferences of circles are proportional to their radii.

fraction is finite, a finite line R must exist, such that $\left(\frac{1}{R}\right)$ is equal to the fraction. Then the circle whose radius is R, has a curvature which is the *average* of that in the arc $A B$.

It is then impossible that this arc should have, at *every* point of it, curvature infinitely less than that of a circle, or, what is the same, LESS than the curvature of any circle however great: for in that case, its average curvature would also be less than that of any circle; which, we have just seen, cannot be. And this holds, however short the arc $A B$ be supposed, so long as it is finite.

In like maner it appears, that the curvature of $A B$ cannot be, at every point, GREATER than that of any circle however small.

In the above, we have supposed the curvature of $A B$ to be turned all the same way: but every finite plane curve can be divided into a finite number of portions, alternately concave and convex. We can then pronounce generally, that the points of curves at which the curvature is infinitely greater or infinitely less than that of a circle, are isolated or Singular; their number is finite in a finite arc, and every adjoining pair of such points is separated by a finite distance. The ordinary curvature may thus be measured by that of a Circle; the cases in which this happens being, in a finite arc, infinite in number, but the exceptions finite in number. Hence the curvature of circles is the ordinary standard, and is named *Finite* Curvature.

220. *The* CHANGES *of Curvature are ordinarily gradual:* or the anomaly remarked on in Art. 215, can occur only at Singular points.

To prove this, we must first consider how to determine for any point of a curve, the circle which shall have equal curvature with it. That some such circle exists, except at singular points, appears by the last article. If then $A C P B$ be any curve, (*Fig.* 99,) take k, l, points in the curve near to C, and through C, k, l, suppose a circle to pass, which is ordinarily possible. Next, let k, l, move up towards C, and the circle change its form and position with them. The limiting circle, if any, to which the variable circle will tend, may be said to pass through *three contiguous points* of the curve; and since no circle can be made to pass through more than three given points, no circle can be imagined that shall more nearly coincide with the curve at C than this does. It will therefore be the equicurve circle itself; and this proves, that, unless C be a singular point, such a limiting circle exists.

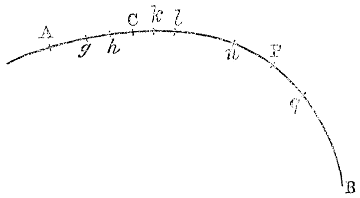


Fig. 99.

Take h, g, on the opposite sides of C; and by passing a circle through C, h, g, we may form the idea of a second equicurve circle osculating the curve along $C\,h\,g$. Our immediate business is to prove, that, except at singular points, the *same* circle osculates along $C\,k\,l$, and along $C\,h\,g$.

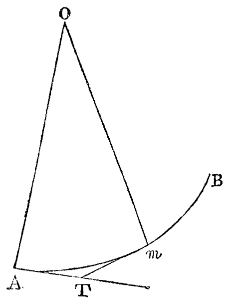
Let A be a singular point, having *un*equal curvatures on its opposite sides. Then, I say, C, *a contiguous point*, cannot have a like property. This means, that if C constantly move up towards A, the two circles which osculate along $C\,k\,l$ and along $C\,h\,g$, must, at least at length, tend to become one and the same; and this, nearer than any assignable difference. For: when C moves up towards A, it simultaneously moves up towards g, which is always between A and C. Therefore the circle through g, h, C, and that through C, k, l, tend to confound themselves in one; since indeed we may regard the former as either osculating at C, in the direction $C\,g$, or (equally well) as osculating at g, in the direction $g\,C$.

221. We do but put the same under a different aspect, in saying, that if P is not a singular point, we may measure off $P\,n$, $P\,q$, finite distances, on each side of it, such, that within $n\,P\,q$ the curvature receives no abrupt increment

(like that noticed in Art. 215); but, that if n, q move together, meet where they will within nPq, they bring equal curvatures with them, counted along nA, qB, opposite ways. Thus we may announce generally, that "*Except* at Singular points, the change of curvature in an indefinitely small arc is indefinitely small."

222. Supposing AB to be any curve soever, having finite curvature at A, (*Fig.* 100,) a part Am may be cut off so small, that "the change of curvature in Am may be less than any assigned amount;" as is manifest from the last article. This is equivalent to saying, that "the arc Am shall differ from a circular arc as little as may be required."

Fig. 100.



Hence if tangents be drawn from A and m to meet each other in T, the parts AT, mT are nearly equal, if Am be very small; and when Am is perpetually diminished, the ratio $\{AT : Tm\}$ tends to the limit $\{1 : 1\}$. Thus also the angles (whether curved or rectilinear) TAm, TmA, are said to be *ultimately equal.*

Moreover, if from A and m normals to the curve are drawn, meeting in O, it follows that these likewise are ultimately equal; that is, the limit of $\{AO : mO\}$ is $\{1 : 1\}$, if the arc Am is perpetually diminished. On this is founded the simplest method of determining the circle which shall have equal curvature with the curve at A; viz. Draw normals from *contiguous* points A, m, and suppose them to intersect in O; then O is the centre, and OA or Om' the radius of the circle required. Also, as in Art. 218, we have $\frac{1}{\text{rad.}} = \frac{\text{angle } O}{\text{arc } Am}$; a formula much used in arithmetic geometry.

In strictness, if Am has a *small finite* length, this formula does but determine a radius which expresses the *average* curvature of the arc $A\,m$; yet as the variation of curvature is very small, this is nearly the same as the curvature at A. But we must suppose the length $A\,m$ perpetually diminished, and so pass to the limit of the ratio, in order to determine a radius which rigorously corresponds to the curvature *at the very point* A. This is called the Radius of Curvature to that point; and the centre O which corresponds, the Centre of Curvature. The Circle is also said to *osculate* the curve. From Articles 219, 220, we infer, that except at Singular points, the radius of curvature has a finite and only a single value.

Fig. 101.

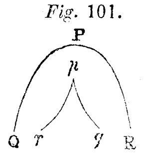
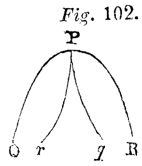


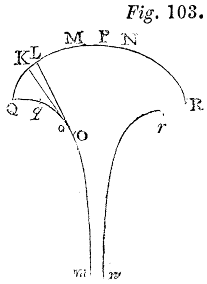
Fig. 102.



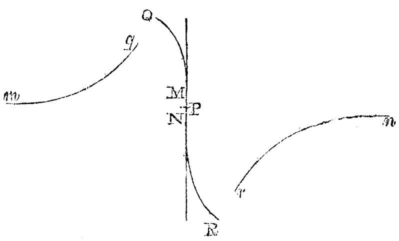
223. In concluding this subject, it may be well to point out how the results at which we have arrived affect the nature of the curve which is *the Locus of the centres of curvature.*

Let $Q\,P\,R$ be any curve, and suppose the centre of curvature corresponding to every point in it to be found. The assemblage of these centres will make a new curve, called the *Evolute* to the former; or to speak more accurately, the *locus* of the centres is the Evolute. In *Fig.* 101, the portion $P\,Q$ gives rise to the evolute $p\,q$, and the opposite portion $P\,R$ to the evolute $p\,r$. In this figure it is supposed that the curvature is always finite, increasing from Q to P, and decreasing from P to R.

Fig. 103.



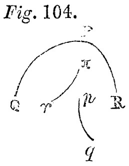
But if $Q\,P\,R$ contains a singular point P, at which the curvature is *infinitely great*, the radius of curvature is here infinitely small, or p runs up into P, as in *Fig.*



102. Then the evolute and original curve have the point P in common.

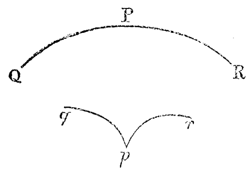
If the curvature at P is *infinitely small*, as in each of the figures 103, then the radius is here infinitely great. Consequently the evolute has two infinite branches $r\,n$, $q\,m$. Should there be also a point of undulation at P, as in the second *Fig.* 103, then $q\,m$ and $r\,n$ are at opposite sides of $Q\,P\,R$, and the infinite branches run off in contrary directions.

Fig. 104.



If the singularity of P consist in this, that the curvature at opposite sides is unequal, there are here *two* centres of curvature, as π and p in *Fig.* 104. Then the evolute consists of two finite portions, as $p\,q$, $\pi\,r$, which are broken apart at π and p. A yet more entire disruption of the evolute would happen, if there were a *peak* at P.

Fig. 105.



Finally, if the curvature come to a maximum at P, as in *Fig.* 101, or to a minimum, as in *Fig.* 105, without becoming infinitely small or infinitely great, the evolute turns directly back on itself at the corresponding point p, producing there a sharp peak.

224. It is also very obvious that any radius of curvature to the original curve, as $O\,L$ in *Fig.* 103, is a tangent to the evolute at O. In fact, if K, L are consecutive points in the curve, and o, O the corresponding centres of curvature, which are therefore consecutive points of the evolute, it has already appeared that O is likewise a point in the normal $K\,o$, since it is the intersection of the contiguous normals $K\,O$, $L\,O$. Wherefore $K\,o\,O$ passes through the consecutive points o, O of the evolute; and is consequently a tangent.

On this is grounded the property which has given rise to the name Evolute; viz. that the original curve may be looked on as produced by the unwinding of a string KO, which is ever kept stretched, and pressing against the evolute.

225. Hence also it follows, that the length of any part of the evolute, as Oq, is equal to the difference of the radii OL and qQ, which belong to the points O and q.

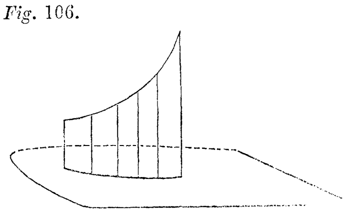
PART V.

DOUBLE CURVATURE.

226. If we suppose a curve to be drawn on a piece of flat paper, and next that this paper is curled up in some way, as by rolling it upon any cylinder; the curve assumes a new shape, in which it may be said to have *two* curvatures. The one is its own, such as it had while yet on the plane, the other is the curvature which the plane itself has received, or the curvature of the cylinder, which is identical with that of the cylinder's base.

Upon regarding the matter thus, we are led to inquire, conversely, whether, if any curve which is not a plane be given, we can resolve its curvature into that of two plane curves. And the above suggests the method of *projecting* the curve by perpendiculars on to a plane; so as to produce a cylindrical* surface (*Fig.* 106.) The uncurling of this surface will exhibit upon it a plane curve, the curvature of which, joined to the curvature of the cylinder's base, constitutes that of the given curve.

Fig. 106.

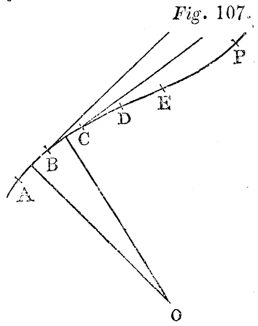


227. On farther consideration it appears that this introduces an arbitrary element,—the position of the plane, —which might be such as to produce results needlessly

* The word Cylinder is here used in a larger sense than in Art. 69.

complicated. Nay, if the given curve were actually plane, yet by projecting it thus on another plane, it would seem to be a curve of double curvature. It is therefore to be desired, that we estimate the curvature of the proposed line without introducing anything arbitrary.

Fig. 107.



228. Along any curve take points A, B, C, D, E . . . which we call *consecutive;* (that is, we intend at a later stage to introduce the supposition, that they approximate towards one another without limit:) and to fix ideas, let them be at equal distances, two and two, counting along the curve.

Draw chords AB, BC, CD, &c. . . . Then since the curve is by hypothesis not plane, we presume that the plane ABC is not the same as the plane BCD, except in singular positions of A, B, &c. Yet as the points ABC can be in one straight line only in singular positions, a plane ABC is always determined by them, when the distances AB, BC perpetually diminish. Hence the plane which is the limit towards which ABC tends, may be said *to pass through three consecutive points* in the curve; and is called the Osculating Plane.

Thus each different point, as A and P, (if these are separated by a finite distance,) has its different osculating plane; and when a pencil traces out the curve, we may suppose an osculating plane to accompany its motion, turning about into such and such successive positions. Consecutive points A, B, C . . . yield consecutive osculating planes.

229. If the chords AB, BC be prolonged, these ultimately confound themselves with tangents. Thus the limit to the plane ABC is the same as the limit to the plane of the tangents to two consecutive points at A, B. Hence we may also describe the Osculating Plane as *passing through two consecutive tangents.*

230. Since the arcs AB, BC may be made to differ as

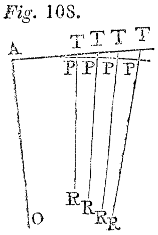
little as we please from their chords, by diminishing the distances AB, BC; the whole arc ABC may be made to differ as little as may be desired from a plane curve, by shortening it as much as we please. Bisecting, therefore, the chords AB, BC, by perpendiculars drawn in the plane of BAC, we obtain two consecutive normals, which intersect in O. This will of course be the centre of the Osculating Circle.

231. Every point, as A, admits evidently an infinity of normals, the locus of which is the plane passing through A, perpendicular to the tangent A. This plane is called *the Normal Plane*. But of all the normals, that which lies in the osculating plane is the most important; and is named the *Principal Normal*.

By the last Article it appears that two consecutive principal normals of necessity intersect each other; viz. in the centre of curvature; wherever the curvature is finite.

232. Another way of looking at the principal normal is sometimes convenient. Let AT be tangent to the curve AP; and take equal lengths AT, AP along them. Join TP and prolong it to R; then let the distance AT be perpetually diminished; and TPR will tend more and more towards some position AO, as its limit. I say, this is the Principal Normal.

Fig. 108.



For since the amount of curvature in AP becomes indefinitely small, when AP is perpetually diminished, APT tends more and more to become an isosceles triangle. But the angle TAP being infinitely small, TPR is ultimately perpendicular to AT; or becomes a normal at A. Yet this normal (AO) is in the plane of the tangent AT, and of a consecutive point P; that is, it is in the osculating plane. Hence it is the Principal Normal.

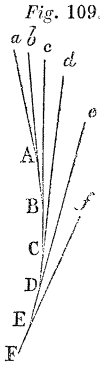
233. It must be already manifest, that the two sorts of curvature which of necessity meet us in any curve which is not plane, are, *first*, the curvature as measured by the oscu-

lating circle in the osculating plane; *secondly*, the curvature or revolving which is to be discovered in the osculating plane itself. The former differs not at all from plane curvature, and needs no farther remark here, except as it leads us to estimate the latter.

If AB (*Fig.* 107,) $= a$, and the angle O between two consecutive principal normals $= \omega$, it appears by Art. 222, that $\frac{1}{\text{Rad.}} = \text{limit of } \left(\frac{\omega}{a}\right)$; which determines the first curvature. Suppose, then, that while the principal normal thus turns through angle $= \omega$, the osculating plane (revolving about that normal) turns through an angle $= \psi$. Take a length ρ, such that $\frac{1}{\rho} = \text{limit of } \frac{\psi}{a}$. Then it is clear that the circle whose radius is ρ will exhibit the *second* curvature at A; or the proportionate rate at which the osculating plane is revolving.

234. The locus of all the centres of curvature produces an evolute, exactly as in plane curves; but besides, the locus of the *radii* of curvature is a surface worth attention. Generated by the motion of this radius, it has the property that any two consecutive generatrices meet each other, (viz. in the evolute,) and of course, as in plane curves, these are tangents to the evolute, but principal normals to the original curve. But, in consequence of the mode of generation, it is a surface of such a nature as to be susceptible of being unfolded and spread out on a plane: for which reason it is named *Developable*.

Fig. 109.

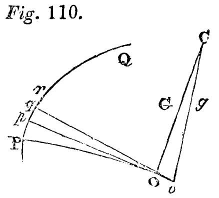


To see the truth of this statement, suppose a series of straight lines aA, bAB, cBC, dCD, eDE, fEF . . . to be drawn, (*Fig.* 109,) intersecting one another successively in A, B, C, D, E, . . . but in any planes soever. If, then, we suppose the system capable of revolving about any of these straight lines, so as to change its shape in any manner; provided only that the angles aAb, bBc, cCd, &c. . . .

retain the same magnitude, and the lengths AB, BC, CD, &c. . . . remain unaltered; it is evident that we may unfold each angle in succession so as to bring all into the plane of the first, without any tearing of the system. This, being true however small the angles aAb, bBc, &c. . . . become, and however short the distances AB, BC, CD, &c. will be true also when the lines aA, bB, &c. are the tangents to a curve $AB \ldots F$. Wherefore the surface which is the locus of the tangents of any curve of double curvature, is Developable: and this will apply to the osculating surface above spoken of, by considering that its generatrices are tangents to the evolute. It is manifest that the plane of two consecutive generatrices, as dCD, eD, is ultimately a Tangent Plane to the developable surface.

235. Let PO, po be consecutive radii of curvature of the curve PQ (*Fig.* 110); and from O, o, draw two perpendiculars OG, og to the successive osculating planes. Then whatever is the angle of inclination between those planes, the same must be the angle between the perpendiculars, if they meet. Suppose them to meet in C; then the angle OCo = the angle between successive osculating planes, or may be used to measure the second curvature of PQ at P.

Fig. 110.



Since every point in OG may be made centre of a circle passing through three consecutive points of the curve, say P, p, q; and similarly every point in og centre of a circle passing through p, q, r, the three next points: it follows that C, the point of meeting, is equidistant from four consecutive points P, p, q, r; and may thus be regarded as centre of an Osculating Sphere. It is the point of concourse of three successive normal planes.

236. But it may be inquired whether OG, og are ultimately in one plane, so as to have *any* ultimate point of concourse C. The reply is, that except at Singular Points,

the deflections of the curve $P\,p\,q\,r$ follow the law of continuity, and so therefore does the change of position in the normal plane. Wherefore the locus of $O\,G$ is a surface, whose curvature does not change abruptly except at Singular Positions of $O\,G$. It follows, that ordinarily the pair of lines $O\,G$, $o\,g$ determine its tangent plane. There is, then, no incongruity in supposing them to meet at C.

It is true that $O\,G$, $o\,g$, may be parallel at Singular Points. But if through a finite arc $P\,Q$, the series of lines $O\,G$, $O'G'$, $O''G''$ . . . were *always* parallel to each other, their locus would be a Cylinder, and the arc $P\,Q$ would be Plane. Hence in a curve of double curvature, the point C ordinarily exists.

237. The locus of the centre C is a curve, to which the perpendicular $O\,C$ is a tangent: that is, the system of tangents to this curve forms the developable surface, which has for its tangent planes the normal planes of our given curve. This surface is named by French geometers, *La Surface Reglée;* (ruled surface ?)

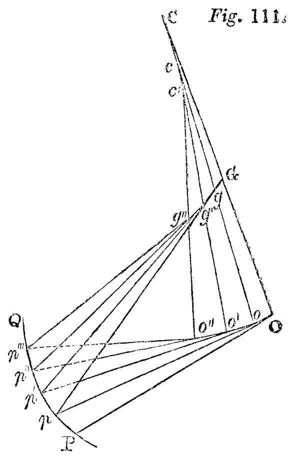


Fig. 111.

238. It is now not difficult to show, that, (besides the *principal* Evolute, which is the locus O, as in plane curves,) an infinity of other evolutes exist, all lying on the ruled surface: each having the property which gave rise to the name Evolute, viz. that the original curve can be generated from it by the unwinding of a string. Thus if $O\,C$, $o\,C$, $o'\,c$, $o''\,c'$ are successive generatrices of the Ruled Surface, (*Fig.* 111,) meeting one another in C, c, c'; while O, o, o', o'', are the centres of curvature determined by the intersections of five consecutive principal normals, at P, p, p', p'', p'''; take G arbitrarily in $O\,C$; join $G\,p$, and

let it cut $o\,C$ in g. Next, join $g\,p'$, and let it cut $o''\,c$ in g'. Join $g'\,p''$, and let it cut $o''\,c'$ in g''; and so on. Then $G\,g\,g'\,g''$ is a part of an Evolute; and a string $G\,g\,g'\,g''\,p'''$ by unwinding will generate $p''\,P$, for the string is always perpendicular to $P\,Q$, and the circle whose radius is $G\,p, g\,p'' \ldots$ passes through three consecutive points of $P\,Q$. Because of this property, the Ruled Surface might be termed the *Surface of Evolutes*.

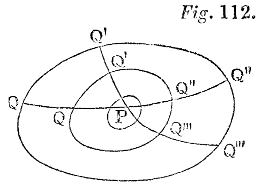
239. Since the curve which is the locus of the centres $C, c, c' \ldots$ cannot have two consecutive Cusps; since also its successive deflections are equal to the second curvature of the given curve, the second curvature is subject to the same laws of continuity as is the first curvature. Therefore also it is only at Singular Points that a curve can have *two* Osculating Planes, any more than two tangents. In fact, since the successive angles $O\,C\,o$ may be all laid down on a plane, by uncurling the Surface of Evolutes, we can thus produce a plane curve, whose deflections shall accurately represent the second curvature of our original curve.

PART VI.

CURVED SURFACES.

240. If P be any point on a curved surface, (*Fig.* 112,) and lines $P\,Q$, $P\,Q''$, $P\,Q'$ . . . be drawn in all directions round P along the surface; it appears by Art. 238, that if we take all of these lines as short as we please, we may make every one differ as little as we please from a straight line.

Fig. 112.



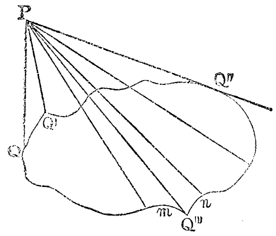
Round P draw on the surface a a line $Q\,Q'\,Q''\,Q'''\,Q$ rejoining itself; then the above may be otherwise stated thus: If we take the area round P as small as we please, we may attain a portion of surface differing as little as we please from a Cone, generated round P by the motion of a straight line: (See Art. 102.) Such an area then, differs to an indefinitely small amount from an area marked out on the Tangent Cone round P, (See Art. 145).

Now the cases in which the tangent cone does not merge itself into a tangent plane, may be spoken of under two heads: first, when at P is a *peak*, as in ordinary cones; secondly, when P is (what we may call) *a centre of undulation*, such as is described in Art. 145, under numbers (2) and (3).

241. Consider next the case of a Cone, (*Fig.* 113,) (using the word in its most extended sense,) which has P for vertex,

and $Q Q' Q'' Q''' Q$ for the directrix by which it is generated. If this directrix have any peaks, as at Q' and Q''', the lines $P Q'$, $P Q'''$ will probably be *Ridges* along the surface. For if m, n be taken in the directrix, at opposite sides of Q''', and the distances $Q m$, $Q n$, be diminished indefinitely; each of the planes $P Q''' m$, $P Q''' n$, tend to become tangent planes to the cone along $P Q'''$. But by reason of the peak at Q''' in the directrix, it is possible that the two planes may not tend to one and the same plane: in which case $P Q'''$ will be a Ridge.

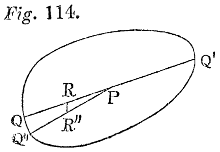
Fig. 113.



But two consecutive generatrices $P m$, $P Q'''$, could not be ridges; otherwise the consecutive points m, Q''', must needs be both peaks in the directrix; which is impossible.

It is obvious also, that no point but P in the line $P Q$ or $P Q'$, &c. can be a peak to the surface; for these lines are straight. Hence both the peak P and any ridges that proceed from P, are Singular; so that in a finite curvilinear area both the peaks and the ridges are finite in number.

Fig. 114.



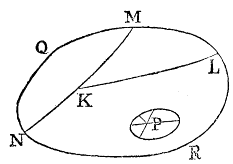
The same is true of such points as we have called centres of undulation. If P be such a one, (*Fig.* 114,) suppose R to be a second point indefinitely near to P, and in the generatrix $P Q$; still considering $P Q$, $P Q'$, &c. as straight lines. Take $P Q''$ another generatrix, indefinitely near to $P R Q$; then the plane $Q P Q''$ is a tangent plane to the surface along the line $P R Q$, and consequently the point R admits a tangent plane, which contains the tangent to every curve $R R''$ drawn from R to meet $P Q''$. Every point R in $Q P Q'$, except the point P, (which admits of no such lines as $R R''$,) has thus a tangent plane. Wherefore P is altogether isolated and unique.

242. What has been proved of the Cone, applies to the

surface in *Fig.* 112, which by perpetual diminution might be made to differ as little as we pleased from the Cone. Thus any given curved surface of finite dimensions,—*except* at a limited number of isolated points, and along a limited number of straight lines,—admits at every point of it *one*, and *only one*, Tangent Plane.

243. *Every curved surface may consequently be distributed into portions so small, that each may differ from a plane surface as little as may be required.*—To fix ideas, let *Fig.* 115 represent any curved surface, having an isolated peak P, and ridges NKM, KL. Round P draw a line, cutting off an area as small as we please; and we may make it differ as little as we please from a cone. But a cone is a developable surface, and may be laid out on a plane; hence this portion, (which might indeed be neglected as evanescent,) is comparable to a plane surface. Divide the rest into three parts, $NQMK$, MKL, and $NKLR$, (omitting the system of P;) then neither of these parts has any singular point such as have here been discussed. Consequently, each is resolvable into small parts, differing from the *tangent planes* as little as we please.

Fig. 115.



244. A curved surface is then always* comparable in respect to magnitude with a plane surface; or, Any curved surface being given, a plane surface is conceivable, (say a circle or a square,) equal to it in area. See Art. 12.

245. Analogy might seem to require that we follow the subject of curvature in surfaces; and establish the very remarkable and elegant properties, which, except at Singular Points, they are known to possess. But the

* In the Diff. Calc. this is assumed as an Axiom. Thus, if u = area of an ellipsoid, whose semi-axes are a, b, c, it is at once assumed that u is a *function* of a, b, c, *of two dimensions.*

writer has not succeeded in finding how to do this by any simple * and satisfactory method; and, in fact, there is little or no occasion for it, inasmuch as every step has now been made good, which was a prerequisite to the application of the Differential Calculus to curved lines or surfaces.

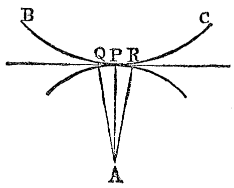
* Monge has attempted it in his Descriptive Geometry. He appears to me to fall short of demonstrative reasoning on this subject, which yet is well worthy of being studied by those who do not aim at high mathematical attainments.

PART VII.

SHORTEST PATH ON A SPHERE.

246. In investigating the question, "What is the shortest path along a sphere from one point to another?" it is found convenient to establish several subordinate propositions.

Fig. 116.

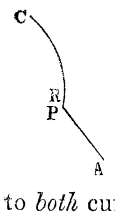


First: "The shortest path AP (in space) (*Fig.* 116,) from a point A to a curve BPC that has no peak, is at right angles to the curve."

For if Q, R are points near to P, and at opposite sides of it, the distances AQ, AR are never less than AP, however near Q and R are to P; because AP is a shortest path from A to the curve. Hence if a sphere be conceived, of centre A and radius AP, Q and R can never fall within the sphere. And since we suppose that there is no peak in the curve at P, the straight line QR being prolonged each way, tends to become a tangent to the curve at P, when the distances PQ, PR, are perpetually diminished, (Art. 95.) But it also tends to become a tangent to the sphere at P; for if it entered the sphere at P, either Q or R must lie inside the sphere; which is not the case. Thus the curve and sphere have a common tangent at P. This tangent being perpendicular to the radius AP, it remains that the shortest path AP is justly called perpendicular to the *curve* at P.

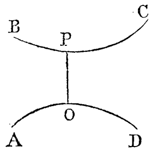
Observe: If P were an extreme point of the curve (*Fig.* 117,) so as not to allow opposite points Q, R; this would be equivalent to the case of P being a peak: for the curvature would be there suddenly arrested.

Fig. 117.



247. Cor. If $B\,C$, $A\,D$ are curves without peaks, no path $P\,O$ to join them can be as short as might be, unless it is perpendicular to *both* curves. (*Fig.* 118.)

Fig. 118.



248. Secondly: "The same things are true, if the path is constrained to a particular surface, on which the curves $B\,P\,C$, $A\,O\,D$, lie." (*Fig.* 119.)

When from a *given* point A, the path $A\,P$ is drawn along the surface to $B\,C$, let $\alpha, \beta, \gamma \ldots$ be points in the path. Then $A\,P$ cannot be as short as possible, if αP could be shortened without shifting α; and the like may be said of βP, γP, &c. however near $\alpha, \beta, \gamma \ldots$ may be to P. But when these distances become very small, the *arcs* αP, βP, $\gamma P, \ldots$ tend to confound themselves with the *chords;* and by the theorem just proved, the chords are perpendicular to $B\,C$, when they are the shortest paths (in space). Here, therefore, the chords tend more and more to perpendicularity, as they diminish: consequently, the tangent $P\,T$ to the curve $P\,A$, is actually perpendicular to $A\,P\,C$; that is, the two curves $A\,P$, $B\,P\,C$, are perpendicular to each other at P.

Fig. 119.

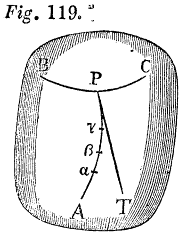
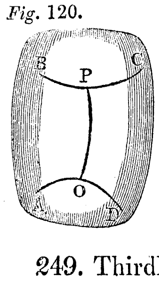


Fig. 120.



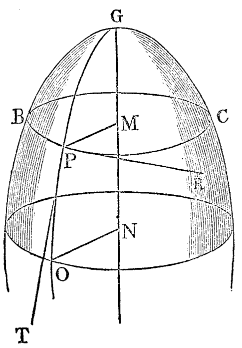
When the point A is not given, but two curves $B\,C$, $A\,D$, are given; it follows exactly as in the Cor. to the former article, that $O\,P$ must be perpendicular to both curves, if it is as short a path as possible from one to the other. *Fig.* 120.

249. Thirdly: "On a Surface of Revolution, the shortest

possible path to connect two points in its plane generatrix lies along the generatrix itself." (*Fig.* 121.)

Let $G P O$ be any portion of a plane curve, which, by revolving round the axis $G M N$ in its own plane, generates a surface of revolution. It is first easy to see, that *every circle generated is perpendicular to the generating curve.* For if P generate the circle $B P C$, whose centre is M, and $P R$ be a tangent to the circle, and $P T$ a tangent to the curve $P O$; then since the axis $N M$ is perpendicular to the plane of the circle, so also is the plane $N M P$ that passes through the axis. Now the intersection of these rectangular planes is $M P$; to which is drawn in one of them the perpendicular $P R$. Wherefore $P R$ is perpendicular to the other plane $N M P$; and consequently, since $P T$ is in this last plane, the angle $R P T$ is right. That is to say, $O P$ is perpendicular to $B P C$ at the point P.

Fig. 121.



Next: a path on the surface to join O and G, will not be as short as possible, unless the path from O to the circle $B P C$ be also a minimum: for all points in that circle lie at the same distance from G. Hence by the last Article, the shortest path from O to G, must cut the circle $B P C$ (and every parallel circle) at right angles.

But no path joining O and G, except the plane generatrix $O P G$, can cut *all* the parallel circles at right angles. That this path does, we have shown; but any deviation from this must instantly produce a path more or less oblique to the circles. Wherefore no other line drawn on the surface from O to G, is so short as the plane generatrix $O P G$.

Finally; if O and P be given points in the same plane generatrix, this line $P O$ is as short as any other that can be drawn on the surface to connect O and P. For $O P G$

cannot be a minimum, unless its part $O P$ is also a minimum. Which was to be proved.

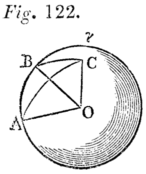
250. A Sphere is a particular sort of Solid of Revolution, its plane generatrix being a semicircle. By the theorem just proved, it appears that the shortest line which can join two points on a sphere, is, one arc of the generating semicircle which passes through them. To determine this arc, we have only to pass a plane through the centre of the sphere, and the two given points: the intersection of the plane and sphere affords the path sought for.

SCHOLIUM.

251. To the establishment of Art. 249, some propositions concerning the intersections of planes are needed, which are found in the XIth book of Euclid. In all this, however, there is nothing which absolutely demands that we shall have settled any principles concerning the addition and subtraction of rectilinear angles. The comparison of angles, as regards *greater* and *less*, is effected as in Art. 101; and we may proceed from thence as far as Art. 111, and then skipping to Art. 134, continue to the end of the 1st Part. If any one choose next to treat of the Inclination of Planes, so far as to establish the simple theorems needed in Art. 249, (which is easy, especially by help of the Sphere,) he might then lay down principles for effecting the addition and subtraction of angles, in some respects more satisfactory than those of Articles 113—117.

252. For if $A O B$, $B O C$ are any two angles to be added together, (*Fig.* 122,) we must inquire how we can make the angle $A O C$ greatest; and we must count this to be their *Sum*. Suppose then, that we cut off $OA = OB = OC$, and regard A, B, C as on the surface of a sphere whose centre is O; and that $A B$,

Fig. 122.



$B\,C$, $A\,C$, are arcs of great circles. Then by Art. 250, $A\,C$ is less than $A\,B + B\,C$, except when C is in the prolongation of $A\,B$; as at γ. Thus the distance $A\,C$, (and consequently, the angle $A\,O\,C$,) is greatest, when A, B, C are all in one plane with O, and A, B, C stand in order round the circle. And this requires that the two angles $A\,O\,B$, $B\,O\,C$, be put side by side *upon a plane*.

253. In composing a continuous treatise of Elementary Geometry, the writer is (on the whole) of opinion that it might be advisable to follow this order. Articles 113—133, will then be omitted from their place above; and after establishing the Evenness of the Plane, &c., as in Articles 170—180, Dihedral Angles must be treated, and a certain part of what is commonly found in Spherical Trigonometry: after which, the doctrine of the Addition and Measurement of Angles upon the basis just suggested would follow.

The nearer we come to the true and natural method of a science, the more immediately do we find ourselves able to deduce our conclusions out of first principles. This necessarily draws after it a new difficulty, viz. an uncertainty as to which arrangement of subjects is most to be preferred: and if there be also a doubt, which of several Experimental Laws deserves to be made the basis of the science, there is still more room for hesitating as to the most advisable Order of a treatise. This has evidently been much felt by writers on Statics; and I am strongly conscious of the same, in regard to Geometry. But there is here some danger of a fantastic desire of an artificial consecutiveness. For however specious may be the system, which would make Geometry as nearly as possible a chain of propositions, linked each to the one before it, so as to admit of no dislocation; it may be questioned whether this can ever be an arrangement *to be aimed at*. On the contrary, if we desire a deep knowledge on the student's part as to what he is about, the more our proofs are drawn from first principles, the better. Yet this will give to a treatise the appearance of being ill-connected.

254. The doctrine of Parallel Straight Lines may be regarded as dividing Geometry into two parts. Before this doctrine is established, the whole theory of the mensuration of angles, plane, spherical, curved, and solid, can be treated satisfactorily; but we can scarcely touch the theory of linear, superficial, and solid measurements without it. As this is the critical step by which we pass to the *calculation* of lengths, surfaces, and volumes, so also it was shown in Art. 123, that we can prove the doctrine itself by barely assuming that Geometry is a science of calculation.

It may appear strange to some, that we do not propose to prove by *experiment* some fundamental law on which the doctrine of Parallels may rest, since this proceeding has been vindicated above, as the basis of the science. The only reply which the writer can give, is, that while he has an internal consciousness that his conviction of the Laws of Rotation is of a mechanical origin, he has just the opposite inward persuasion concerning the relations of Parallel Straight Lines. *Here* the appeal seems to him to be made to the pure reason, and not to outward trial of any sort; and the remark made in Art. 124, concerning the doctrine of Homogeneity, as common to other sciences, confirms him in this view.

255. Yet, as the Evenness of the Plane has been shown to result out of the principle that Peaks are Singular points, there is much speciousness (to his mind) in the thought, that the doctrine of Parallels ought to be elicited out of the theorem that "curvature is finite," (or comparable with that of a circle) "except at singular points." He once thought that he had succeeded in demonstrating this; but at last it appeared, that there was a concealed assumption that "the total deviation is not infinitely greater than the ultimate deviation." But in case any should be disposed to pursue this investigation, it may be remarked, (1) that the doctrine of Parallels is easily established, if it can be shown that the Cylinder, defined as in Art. 69, has *no* longitudinal curvature. (2) Since a longitudinal

section of the cylinder is clearly *not* a circle, it must either be a straight line, or an unknown curve which has at every point equal curvature. (3) The latter alternative is instantly disproved, if it be conceded that ordinary curvature is finite. For the (supposed) curve under examination can nowhere have its curvature so great as that of any circle: which is contrary to the concession.

It may finally be noticed, that by the method of Articles 127, 214, the conception of finite curvature can be distinctly formed, without any premature assumptions concerning the circle of curvature.

APPENDIX.

No I.

After the foregoing pages were in the printer's hands, my attention was arrested by Lieut.-Colonel Perronet Thompson's small book, on the "Equiangular Spiral," used as the foundation of the Theory of Parallels. Having before found him to supply instructive suggestions, I turned with interest to the examination of his new proof.

It appears quite safe to assert, that if the Second Corollary to his Proposition A is sound, his method is logically perfect. But I am sorry that in that one most important step his proof is quite deficient, in my judgment. He is essaying to demonstrate, that if (r, ω) are coordinates to the Spiral, we may make (ω) as great as we please, by taking (r) as great as we please; or, what is the same, that if (r) increases without limit, (ω) cannot approximate to a finite limit; or that while (ω) is finite, (r) cannot be infinite. But there is a clear *petitio principii*, in his proof, that (r) cannot be infinite; and moreover, it proves too much. His Second Corollary is not limited to the *Equiangular* Spiral, but to a spiral which fulfills the condition named in his first Corollary. This is, virtually, the equation, $d\,r \propto \phi\,(r)\,.\,d\,\omega$, where $\phi\,(r) =$ circumference of a circle, whose radius is (r); a function about which he has established nothing, except that it is finite, while (r) is finite. It is easy to prove farther, that it increases with (r). But the conclusion from such premises, in his Second Corollary, is undoubtedly too wide: for instance, if $\phi\,(r) = e^r$, it is readily proved false. Or, again, if $\phi\,(r) = a\,r^{1+\mu}$, the conclusion is false as long as (μ) has a real positive value. It is then essential to Col. Thompson's method, first to establish that the circumference does *not* increase in a *higher* ratio than as the radius directly.

On a superficial view it may seem that all that is needed, is to prove that the Equiangular Spiral cannot have an asymptote *through its pole;* for which an easy geometrical proof is devisable. But this is not really enough; nor can I complete the proof without assuming the following