EUCLID, BOOK V,

PROVED ALGEBRAICALLY.

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PROVED ALGEBRAICALLY

SO FAR AS IT RELATES TO

COMMENSURABLE MAGNITUDES.

TO WHICH IS PREFIXED

A SUMMARY

OF ALL THE NECESSARY ALGEBRAICAL OPERATIONS,

ARRANGED IN ORDER OF DIFFICULTY.

BY

CHARLES L. DODGSON, M.A.,

SENIOR STUDENT AND MATHEMATICAL LECTURER

OF CH. CH., OXFORD.

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PREFACE.

The student is recommended to go through this treatise in the following order:—

First, to master the 'Preliminary Algebra,' and not to go further until he finds that, when covering up the right-hand column and setting himself any question in the left-hand column, he can at once *work out* (not merely supply from memory) the required answer.

Secondly, taking the Algebraical Enunciation which stands at the top of the right-hand column in each Proposition, to learn to supply the proof which follows it. To do this, he should cover the rest of the right-hand column, and try to work out the proof for himself, with the help of the directions in the left-hand column. As every step of the work has been already done in the 'Preliminary Algebra,' this ought to be possible without any reference to the right-hand column: but if any difficulty *should* occur, there will usually be found a marginal reference to the 'Preliminary Algebra,' and it will be better to turn back to the section referred to, and so refresh the memory, than to look at the right-hand column, which should only be uncovered, when the proof has been written out, as a test of the correctness of the work.

Thirdly, to practise himself in working out the same proofs, without the help of the directions in the left-hand column, from the Algebraical Enunciations only. These are given by themselves at p. 49.

Fourthly, taking the Enunciation printed in *small* type in each Proposition, and covering up all below it, to learn to express it algebraically, as given in the first sentence of the right-hand column.

Fifthly, taking the Enunciation printed in *large* type in each Proposition, and covering up all below it, to learn to repeat it with the addition of algebraical symbols for the magnitudes, as given in the small-type Enunciation.

Sixthly, to learn Euclid's Definitions and Axioms, given at p. 53.

CONTENTS.

PRELIMINARY ALGEBRA.

1.

1. Given a;	
multiply it by m	ma.
divide it by n, by a, and by ab	$\frac{a}{n}$, 1, $\frac{1}{b}$.
divide 1 by it	$\frac{1}{a}$.
divide x by it	$\frac{x}{a}$.
2. Multiply by m, both with and without a bracket,	
$a+b$	$m.(a+b)$, $ma+mb$.
$a-b$	$m.(a-b)$, $ma-mb$.
$a+1$	$m.(a+1)$, $ma+m$.
$a-1$	$m.(a-1)$, $ma-m$.
$a+b+c+$ &c. . . .	$m.(a+b+c+$ &c.$)$, $ma+mb+mc+$ &c.
3. Divide by n, both with and without a vinculum,	
$a+b$	$\frac{a+b}{n}$, $\frac{a}{n}+\frac{b}{n}$.
$a-b$	$\frac{a-b}{n}$, $\frac{a}{n}-\frac{b}{n}$.
$a+b+c+$ &c. . . .	$\frac{a+b+c+\text{\&c.}}{n}$, $\frac{a}{n}+\frac{b}{n}+\frac{c}{n}+$ &c.
4. Divide by b, both with and without a vinculum,	
$a+b$	$\frac{a+b}{b}$, $\frac{a}{b}+1$.
$a-b$	$\frac{a-b}{b}$, $\frac{a}{b}-1$.

5. Resolve into factors,	
$ma+mb$	$m.(a+b)$.
$ma-mb$	$m.(a-b)$.
$ma+na$	$(m+n).a$.
$ma-na$	$(m-n).a$.
$ma+a$	$(m+1).a$.
$ma-a$	$(m-1).a$.
$ma+mb+mc+$ &c. . .	$m.(a+b+c+$ &c.$)$.
$ma+na+ra+$ &c. . .	$(m+n+r+$ &c.$).a$.
6. Reduce 1 to a fraction with denominator b	$\frac{b}{b}$.

2.

Given a; what process will convert it into	
ma	multiply by m.
$\frac{a}{n}$	divide by n.
1	divide by a.
$\frac{1}{b}$	divide by ab.
$\frac{1}{a}$	divide 1 by it.
$\frac{x}{a}$	divide x by it.

3.

1. Given $\frac{a}{b}$;	
multiply by m, and by b .	$\frac{ma}{b}$, a.
divide by n, and by a . .	$\frac{a}{nb}$, $\frac{1}{b}$.
multiply by $\frac{m}{n}$, and by $\frac{b}{c}$	$\frac{ma}{nb}$, $\frac{a}{c}$.
add, and subtract, $\frac{x}{b}$ . .	$\frac{a+x}{b}$, $\frac{a-x}{b}$.
add, and subtract, 1 . .	$\frac{a+b}{b}$, $\frac{a-b}{b}$.
divide 1 by it	$\frac{b}{a}$.
2. Given $\frac{a+b}{b}$; subtract 1 . .	$\frac{a}{b}$.
3. Multiply together $\frac{a}{b}$, $\frac{b}{c}$, &c., $\frac{x}{y}$, $\frac{y}{z}$ . . .	$\frac{a}{z}$.
4. Reduce to their simplest forms—	
$\frac{a}{a}$, $\frac{ma}{a}$, $\frac{a}{na}$, $\frac{ma}{mb}$, $\frac{ma}{na}$.	1, m, $\frac{1}{n}$, $\frac{a}{b}$, $\frac{m}{n}$.
$\frac{ma+a}{a}$, $\frac{ma-a}{a}$ . . .	$m+1$, $m-1$.
$\frac{ma+mb+mc+\&c.}{a+b+c+\&c.}$ . .	m.
$\frac{ma}{ma-a}$, $\frac{ma-mb}{a-b}$ . . .	$\frac{m}{m-1}$, m.

4.

1. Given $\frac{a}{b}$; what process will convert it into	
$\frac{ma}{b}$	multiply by m.
a	multiply by b.
$\frac{a}{nb}$	divide by n.
$\frac{1}{b}$	divide by a.
$\frac{ma}{nb}$	multiply by $\frac{m}{n}$.
$\frac{a}{c}$	multiply by $\frac{b}{c}$.
$\frac{a+b}{b}$, $\frac{a-b}{b}$.	add 1, subtract 1.
$\frac{b}{a}$	divide 1 by it.
2. Given $\frac{a+b}{b}$; what process will convert it into $\frac{a}{b}$ . .	subtract 1.

5.

1. Given '$a=b$';	
add x	$a+x=b+x$.
subtract x	$a-x=b-x$.
multiply by x	$xa=xb$.
divide by x	$\frac{a}{x}=\frac{b}{x}$.
divide 1 by each side . .	$\frac{1}{a}=\frac{1}{b}$.
divide x by each side . .	$\frac{x}{a}=\frac{x}{b}$.

2. Given '$a>b$';	
add x	$a+x>b+x$.
subtract x	$a-x>b-x$.
multiply by x	$xa>xb$.
divide by x, by b, and by ab	$\frac{a}{x}>\frac{b}{x}$, $\frac{a}{b}>1$, $\frac{1}{b}>\frac{1}{a}$.
divide by ab and multiply by x	$\frac{x}{b}>\frac{x}{a}$.
3. Given '$a-b>c-d$'; add $(b+d)$	$a+d>b+c$.

6.

1. Given '$a=b$'; what process will prove	
$\frac{a}{x}=\frac{b}{x}$ . . .	divide by x.
$\frac{1}{a}=\frac{1}{b}$ . . .	divide 1 by each side.
$\frac{x}{a}=\frac{x}{b}$ . . .	divide x by each side.
2. Given '$a>b$'; what process will prove	
$\frac{a}{x}>\frac{b}{x}$ . . .	divide by x.
$\frac{a}{b}>1$ . . .	divide by b.
$\frac{1}{b}>\frac{1}{a}$ . . .	divide by ab.
$\frac{x}{b}>\frac{x}{a}$ . . .	divide by ab and multiply by x.
3. Given '$a-b>c-d$'; what process will prove	
$a+d>b+c$	add $(b+d)$.

7.

1. Given '$\frac{a}{b}=1$'; multiply by m, and by b	$\frac{ma}{b}=m,\ a=b.$
2. Given '$\frac{a}{b}=k$';	
multiply by m, and by b .	$\frac{ma}{b}=mk,\ a=bk.$
add 1	$\frac{a+b}{b}=k+1.$
subtract 1	$\frac{a-b}{b}=k-1.$
divide 1 by each side . .	$\frac{b}{a}=\frac{1}{k}.$
3. Given '$\frac{a}{b}=\frac{c}{d}=k$'; clear of fractions (by multiplying both sides of '$\frac{a}{b}=k$' by b, and of '$\frac{c}{d}=k$' by d) . .	$a=bk,\ c=dk.$
4. Given '$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\&c.=k$'; clear of fractions	$a=bk,\ c=dk,\ e=fk,\ \&c.$
5. Given '$\frac{a}{x}=\frac{b}{x}$'; multiply by x	$a=b.$
6. Given '$\frac{1}{a}=\frac{1}{b}$'; multiply by ab	$b=a.$
7. Given '$\frac{x}{a}=\frac{x}{b}$';	
divide by x	$\frac{1}{a}=\frac{1}{b}.$
divide by x and multiply by ab	$b=a.$

8. Given '$\frac{a}{b} > 1$'; multiply by m, and by b	$\frac{ma}{b} > m$, $a > b$.
9. Given '$\frac{a}{x} > \frac{b}{x}$'; multiply by x	$a > b$.
10. Given '$\frac{1}{a} > \frac{1}{b}$'; multiply by ab	$b > a$.
11. Given '$\frac{x}{a} > \frac{x}{b}$'; divide by x	$\frac{1}{a} > \frac{1}{b}$.
divide by x and multiply by ab	$b > a$.

8.

1. Given '$\frac{a}{x} = \frac{b}{x}$'; what process will prove '$a = b$'	multiply by x.
2. Given '$\frac{x}{a} = \frac{x}{b}$'; what process will prove '$b = a$'	divide by x and multiply by ab.
3. Given '$\frac{a}{b} > 1$'; what process will prove '$a > b$'	multiply by b.
4. Given '$\frac{a}{x} > \frac{b}{x}$'; what process will prove '$a > b$'	multiply by x.
5. Given '$\frac{1}{a} > \frac{1}{b}$'; what process will prove '$b > a$'	multiply by ab.
6. Given '$\frac{x}{a} > \frac{x}{b}$'; what process will prove '$b > a$'	divide by x and multiply by ab.

9.

1. Given '$\frac{a}{b}=\frac{c}{d}$';

multiply by m	$\frac{ma}{b}=\frac{mc}{d}$.
divide by n	$\frac{a}{nb}=\frac{c}{nd}$.
multiply by $\frac{m}{n}$, and by $\frac{b}{c}$	$\frac{ma}{nb}=\frac{mc}{nd}$, $\frac{a}{c}=\frac{b}{d}$.
add 1	$\frac{a+b}{b}=\frac{c+d}{d}$.
subtract 1	$\frac{a-b}{b}=\frac{c-d}{d}$.
divide 1 by each side . .	$\frac{b}{a}=\frac{d}{c}$.

2. Given '$\frac{a+b}{b}=\frac{c+d}{d}$';

subtract 1	$\frac{a}{b}=\frac{c}{d}$.

3. Given '$\frac{A}{B}=\frac{a}{b}$, $\frac{B}{C}=\frac{b}{c}$';

multiply corresponding sides together	$\frac{A}{C}=\frac{a}{c}$.

4. Given '$\frac{A}{B}=\frac{b}{c}$, $\frac{B}{C}=\frac{a}{b}$';

multiply corresponding sides together	$\frac{A}{C}=\frac{a}{c}$.

5. Given $\frac{A}{B}=\frac{a}{b}$,

$\frac{B}{C}=\frac{b}{c}$,

&c.

$\frac{X}{Y}=\frac{x}{y}$,

$\frac{Y}{Z}=\frac{y}{z}$;

multiply corresponding sides together . . . . $\frac{A}{Z}=\frac{a}{z}$.

6. Given $\frac{A}{B}=\frac{y}{z}$,

$\frac{B}{C}=\frac{x}{y}$,

&c.

$\frac{X}{Y}=\frac{b}{c}$,

$\frac{Y}{Z}=\frac{a}{b}$;

multiply corresponding sides together . . . . $\frac{A}{Z}=\frac{a}{z}$.

10.

1. Given '$\frac{a}{b}=\frac{c}{d}$'; what process will prove	
$\frac{ma}{b}=\frac{mc}{d}$.	multiply by m.
$\frac{a}{nb}=\frac{c}{nd}$ . .	divide by n.
$\frac{ma}{nb}=\frac{mc}{nd}$.	multiply by $\frac{m}{n}$.
$\frac{a}{c}=\frac{b}{d}$ . .	multiply by $\frac{b}{c}$.
$\frac{a+b}{b}=\frac{c+d}{d}$.	add 1.
$\frac{a-b}{b}=\frac{c-d}{d}$.	subtract 1.
$\frac{b}{a}=\frac{d}{c}$ . .	divide 1 by each side.
2. Given '$\frac{a+b}{b}=\frac{c+d}{d}$'; what process will prove '$\frac{a}{b}=\frac{c}{d}$'	subtract 1.
3. Given '$\frac{A}{B}=\frac{a}{b}$, $\frac{B}{C}=\frac{b}{c}$'; what process will prove '$\frac{A}{C}=\frac{a}{c}$'	multiply corresponding sides together.
4. Given '$\frac{A}{B}=\frac{b}{c}$, $\frac{B}{C}=\frac{a}{b}$'; what process will prove '$\frac{A}{C}=\frac{a}{c}$'	the same.

5. Given '$\frac{A}{B}=\frac{a}{b}$, $\frac{B}{C}=\frac{b}{c}$, &c., $\frac{X}{Y}=\frac{x}{y}$, $\frac{Y}{Z}=\frac{y}{z}$'; what process will prove '$\frac{A}{Z}=\frac{a}{z}$'

multiply corresponding sides together.

6. Given '$\frac{A}{B}=\frac{y}{z}$, $\frac{B}{C}=\frac{x}{y}$, &c., $\frac{X}{Y}=\frac{b}{c}$, $\frac{Y}{Z}=\frac{a}{b}$'; what process will prove '$\frac{A}{Z}=\frac{a}{z}$'

the same.

11.

1. Given '$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$ &c.;

to prove $\frac{a+c+e+\&c.}{b+d+f+\&c.}=\frac{a}{b}$.

Equate to k, and clear of fractions . . . . .

Let each $=k$;

$\therefore$, clearing of fractions, $a=bk$, $c=dk$, $e=fk$, &c.;

simplify $\frac{a+c+e+\&c.}{b+d+f+\&c.}$

$$\therefore\ \frac{a+c+e+\&c.}{b+d+f+\&c.}=\frac{bk+dk+fk+\&c.}{b+d+f+\&c.}$$
$$=\frac{k.(b+d+f+\&c.)}{b+d+f+\&c.}$$
$$=k;$$

thence deduce conclusion .

$$\therefore\ \frac{a+c+e+\&c.}{b+d+f+\&c.}=\frac{a}{b}.$$

11 (*continued*).

2. Given '$\frac{a}{b}=\frac{c}{d}$';	
to prove $\frac{a-c}{b-d}=\frac{a}{b}$.	
Equate to k, and clear of fractions	Let each $= k$;
	$\therefore$, clearing of fractions, $a=bk$, $c=dk$;
simplify $\frac{a-c}{b-d}$	$\therefore \frac{a-c}{b-d}=\frac{bk-dk}{b-d}=\frac{k(b-d)}{b-d}=k$;
thence deduce conclusion .	$\therefore \frac{a-c}{b-d}=\frac{a}{b}$.
3. Given '$\frac{a}{b}=\frac{c}{d}$';	
to prove $\frac{a}{a-b}=\frac{c}{c-d}$.	
Equate to k, and clear of fractions	Let each $= k$;
	$\therefore$, clearing of fractions, $a=bk$, $c=dk$;
simplify $\frac{a}{a-b}$	$\therefore \frac{a}{a-b}=\frac{bk}{bk-b}=\frac{k}{k-1}$;
and $\frac{c}{c-d}$	and $\frac{c}{c-d}=\frac{dk}{dk-d}=\frac{k}{k-1}$;
thence deduce conclusion .	$\therefore \frac{a}{a-b}=\frac{c}{c-d}$.
4. Given '$\frac{a}{b}=\frac{c}{d}$, and $a>b$';	
to prove $c>d$.	
6. 2) Prove $\frac{a}{b}>1$	$\because a>b$,
	$\therefore$, $\div^{ng}$ by b, $\frac{a}{b}>1$;
thence prove $\frac{c}{d}>1$.	$\therefore$, substituting, $\frac{c}{d}>1$;
8. 3) thence prove conclusion .	$\therefore$, $\times^{ng}$ by d; $c>d$.

5. Given $\frac{a}{b} = \frac{c}{d}$, and $a > c$; to prove $b > d$.	
6. 2) Prove $\frac{a}{b} > \frac{c}{b}$	$\because$ $a > c$,
	$\therefore$, $\div^{ng}$ by b, $\frac{a}{b} > \frac{c}{b}$;
thence prove $\frac{c}{d} > \frac{c}{b}$ . .	$\therefore$, substituting, $\frac{c}{d} > \frac{c}{b}$;
8. 6) thence prove conclusion .	$\therefore$, $\div^{ng}$ by c and $\times^{ng}$ by bd, $b > d$.

12.

1. When is a less magnitude said to be a **part**, or **measure**, of a greater?	A less magnitude is said to be a **part**, or **measure**, of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.
2. When is a greater magnitude said to be a **multiple** of a less?	A greater magnitude is said to be a **multiple** of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

3. If a magnitude be multiplied, what is the new magnitude called with regard to the old, in the following cases—	
(α) When the multiplier is a whole number . .	(α) A multiple.
(β) When it is a fraction whose numerator is 1	(β) A part, or measure.
(γ) When it is any other fraction	(γ) A fraction.
4. If two or more magnitudes be multiplied by the same whole number, what are the new magnitudes called with regard to the old?	**Equimultiples.**

13.

Express algebraically

1. 'A is five times a'	$A=5a$.
2. 'A is a multiple of a' . . .	$A=ma$.
3. 'A is a multiple of a, and B is a multiple of b' . . . , .	$A=ma$, $B=nb$.
4. 'A is the same multiple of a as B is of b'.	$A=ma$, $B=mb$.
5. 'A, B are equimultiples of a, b'	the same.
6. 'A, B, C, &c. are equimultiples of a, b, c, &c.'	$A=ma$, $B=mb$, $C=mc$, &c.
7. 'A, B, C, &c., are multiples of X; and a, b, c, &c., are the same multiples of x' . . .	$A=mX$, $B=nX$, $C=rX$, &c.; $a=mx$, $b=nx$, $c=rx$, &c.

8. 'a is the same multiple of b as c is of d; and e is the same multiple of b as f is of d' .	$a=mb$, $c=md$; $e=nb$, $f=nd$.
9. 'a, c are equimultiples of b, d; and e, f are also equimultiples of b, d'	the same.

14.

1. When is a magnitude said to be **common measure** of two or more others? . . .	When it measures each of them.
2. When are magnitudes said to be **commensurable**? . .	When they have a common measure.
3. If a magnitude be said to be **represented** by a number, what is to be understood? .	First, that a measure of it has been agreed on; secondly, that the number denotes what multiple it is of that measure.
4. If two or more magnitudes be said to be **represented** by numbers, what is to be understood?	First, that they are commensurable; secondly, that a common measure of them has been agreed on; thirdly, that the numbers denote what multiples they are of that common measure.
5. Give the algebraical definition of **ratio**	The **ratio** of one magnitude to another is the number denoting what multiple, part, or fraction, the one is of the other.

6. When are four magnitudes called **proportionals**?	Four magnitudes are called **proportionals**, when the first has the same ratio to the second as the third has to the fourth.
7. When are any number of magnitudes called **proportionals**?	Any number of magnitudes are called **proportionals**, when the ratio of the first to the second, of the third to the fourth, of the fifth to the sixth, and so on, are all the same.

15.

Express algebraically

1. '*a* has the same ratio to *b* as *c* has to *d*'	$a : b :: c : d$.
2. '*a*, *c* have the same ratio to *b*, *d*'	the same.
3. '*a* is to *b* as *c* is to *d*' . . .	the same.
4. '*a*, *b*, *c*, *d* are proportionals' .	the same.
5. '*a*, *b*, *c*, *d*, *e*, *f*, &c. are proportionals'	$a : b :: c : d :: e : f ::$ &c.
6. '*a* has a greater ratio to *b* than *c* has to *d*'	$a : b > c : d$.
7. '*A*, *B*, *C* and *a*, *b*, *c* have the same ratio, taken two and two'	$A : B :: a : b$, and $B : C :: b : c$.
8. '*A*, *B*, *C* and *a*, *b*, *c* have the same ratio, taken two and two in a cross order' . . .	$A : B :: b : c$, and $B : C :: a : b$.

9. '*A, B, C, &c., X, Y, Z*, and *a, b, c, &c., x, y, z*, have the same ratio, taken two and two' . . . . $A : B :: a : b$, $B : C :: b : c$, &c., $X : Y :: x : y$, $Y : Z :: y : z$.

10. '*A, B, C, &c., X, Y, Z*, and *a, b, c, &c., x, y, z*, have the same ratio, taken two and two in a cross order' . . . $A : B :: y : z$, $B : C :: x : y$, &c. $X : Y :: b : c$, $Y : Z :: a : b$.

11. 'Two ranks of magnitudes, *A, B, C, &c.*, and *a, b, c, &c.*, are such that each of the first rank has to *X* the same ratio as the corresponding one of the second rank has to *x*' . $A : X :: a : x$, $B : X :: b : x$, $C : X :: c : x$, &c.

16.

1. If two commensurable magnitudes be represented by the numbers *a*, *b*, what is the ratio of the first to the second? . . . . $\frac{a}{b}$.

2. Prove your answer . . . . The common measure is $\frac{1}{b}$ of the second. Hence the first, being *a* times the common measure, is $\frac{a}{b}$ of the second; that is, the ratio of the first to the second is $\frac{a}{b}$.

3. Given $a : b :: c : d$;	
express in words . . .	a is to b as c is to d.
express more fully, by giving meaning of phrase 'is to' . . .	a has the same ratio to b as c has to d.
express yet more fully, by giving meaning of 'ratio'	a is the same multiple, part, or fraction of b, as c is of d.
what are a, b, c, d, called?	**Proportionals.**
4. Given '$a : b :: c : d$'; deduce an equation . .	$\frac{a}{b} = \frac{c}{d}$.
5. Given '$a : b :: c : d :: e : f ::$ &c.' deduce equations . . .	$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ &c.
6. Given '$a : b > c : d$'; deduce an inequality . .	$\frac{a}{b} > \frac{c}{d}$.
7. Given '$\frac{a}{b} = \frac{c}{d}$'; deduce a proportion . .	$a : b :: c : d$.
8. Given '$\frac{a}{b} > \frac{c}{d}$'; deduce a disproportion .	$a : b > c : d$.

THE PROPOSITIONS OF EUCLID, BOOK V.

[*The Student is recommended to take the Propositions in the following order.*

§ **1.** 1, 5, 3, 2 (*without Corollary*), 6, *Corollary of* 2.
§ **2.** *C*, *D*, 15.
§ **3.** 11, 13.
§ **4.** 7, 9, 8, 10.
§ **5.** 16, *B*, 18, 17, 17*, *E*, 12, 19, 24, 22, 23.
§ **6.** 4.
§ **7.** *A*, 14, 20, 21, 25.]

PROP. I.

If any number of magnitudes be equimultiples of as many: whatever multiple any one of them is of its part, the same multiple are all the first magnitudes of all the others.

[If any number of magnitudes (A, B, C, *&c.*) be equimultiples of as many (a, b, c, *&c.*), each of each; then, whatever multiple any one of them (A) is of its part (a), the same multiple are all the first magnitudes ($A+B+C+$ *&c.*) of all the others ($a+b+c+$ *&c.*)].

Express enunciation.	Let $A=ma$, $B=mb$, $C=mc$, &c.; then shall $(A+B+C+\&c.)=m.(a+b+c+\&c.)$.
Simplify terms of conclusion	Now $A+B+C+\&c.=ma+mb+mc+\&c.$ $=m.(a+b+c+\&c.)$.
	Q. E. D.

Prop. II.

If the first magnitude be the same multiple of the second as the third is of the fourth, and if the fifth be the same multiple of the second as the sixth is of the fourth; the first and fifth together are the same multiple of the second as the third and sixth together are of the fourth.

[If the first magnitude (a) be the same multiple of the second (b) as the third (c) is of the fourth (d), and if the fifth (e) be the same multiple of the second (b) as the sixth (f) is of the fourth (d); then the first and fifth together ($a+e$) are the same multiple of the second (b) as the third and sixth together ($c+f$) are of the fourth (d).]

Express enunciation.	Let $a=mb$, and $c=md$, $e=nb$, and $f=nd$; then shall $(a+e)$, $(c+f)$ be equimultiples of b, d.
Simplify terms of conclusion	Now $a+e=mb+nb=(m+n).b$, $c+f=md+nd=(m+n).d$. Q. E. D.

Corollary.

The Proposition holds true of two ranks of magnitudes, of which each of the first rank is the same multiple of a single magnitude as each of the second rank is of another single magnitude.

[*This Corollary is usually stated as follows.*]

If any number of magnitudes (A, B, C, *&c.*) be multiples of another (X), and as many others (a, b, c, *&c.*)

the same multiples of (x): all the first $(A+B+C+\&c.)$ are the same multiple of (X) as all the others $(a+b+c+\&c.)$ are of (x).

Express enunciation .	Let $A=mX$, and $a=mx$, $B=nX$, and $b=nx$, $C=rX$, and $c=rx$, &c., then conclusion shall follow.
Simplify terms of conclusion	Now $A+B+C+\&c.=mX+nX+rX+\&c.$ $=(m+n+r+\&c.).X$, $a+b+c+\&c.=mx+nx+rx+\&c.$ $=(m+n+r+\&c.).x$. Q. E. D.

Prop. III.

If the first be the same multiple of the second as the third is of the fourth, and if of the first and the third there be taken equimultiples: these are equimultiples of the second and the fourth.

[If the first (a) be the same multiple of the second (b) as the third (c) is of the fourth (d), and if of the first (a) and the third (c) there be taken equimultiples (A, C): these are equimultiples of the second (b) and the fourth (d).]

Express enunciation.	Let $a=mb$, and $c=md$, $A=na$, and $C=nc$; then shall A, C, be equimultiples of b, d.
Simplify terms of conclusion	Now $A=na=nmb$, $C=nc=nmd$.

Q. E. D.

Prop. IV.

If the first have to the second the same ratio as the third has to the fourth: any equimultiples of the first and third have the same ratio to any equimultiples of the second and fourth, viz. "the multiple of the first has to the multiple of the second the same ratio as the multiple of the third has to the multiple of the fourth."

[If the first (a) have to the second (b) the same ratio as the third (c) has to the fourth (d): any equimultiples (A, C) of the first and third have the same ratio to any equimultiples (B, D) of the second and fourth, viz., "the multiple (A) of the first has to the multiple (B) of the second the same ratio as the multiple (C) of the third has to the multiple (D) of the fourth."]

Express enunciation.	Let $a : b :: c : d$, and let $A=ma$, and $C=mc$, $B=nb$, and $D=nd$; then shall $A : B :: C : D$.
Simplify terms of equation required for conclusion . .	Now $\frac{A}{B}=\frac{ma}{nb}$, $\frac{C}{D}=\frac{mc}{nd}$.
Taking given proportion	$\because a : b :: c : d$,
deduce equation .	$\therefore \frac{a}{b}=\frac{c}{d}$;
thence deduce $\frac{ma}{nb}=\frac{mc}{nd}$ . .	$\therefore$, $\times^{ng}$ by $\frac{m}{n}$, $\frac{ma}{nb}=\frac{mc}{nd}$;
thence deduce equation required for conclusion . .	$\therefore$, substituting, $\frac{A}{B}=\frac{C}{D}$;
thence deduce conclusion . . .	$\therefore A : B :: C : D$.

Q. E. D.

COROLLARY 1.

With the same *data:* the multiple of the first has the same ratio to the second as the multiple of the third has to the fourth.

[With the same *data:* the multiple (A) of the first has the same ratio to the second (b) as the multiple (C) of the third has to the fourth (d).]

Express conclusion .	— then shall $A : b :: C : d$.
Simplify terms of equation required for it	Now $\frac{A}{b} = \frac{ma}{b}$, $\frac{C}{d} = \frac{mc}{d}$.
Taking given proportion	$\because$ $a : b :: c : d$,
deduce equation .	$\therefore$ $\frac{a}{b} = \frac{c}{d}$;
thence deduce $\frac{ma}{b} = \frac{mc}{d}$ . .	$\therefore$, $\times^{ng}$ by m, $\frac{ma}{b} = \frac{mc}{d}$;
thence deduce equation required for conclusion . .	$\therefore$, substituting, $\frac{A}{b} = \frac{C}{d}$;
thence deduce conclusion . . .	$\therefore$ $A : b :: C : d$.

Q. E. D.

Corollary 2.

With the same *data*: the first has the same ratio to the multiple of the second as the third has to the multiple of the fourth.

[With the same *data*: the first (a) has the same ratio to the multiple (B) of the second as the third (c) has to the multiple (D) of the fourth.]

Express conclusion	— then shall $a : B :: c : D$.
Simplify terms of equation required for it	Now $\frac{a}{B} = \frac{a}{nb}$, $\frac{c}{D} = \frac{c}{nd}$.
Taking given proportion	$\because$ $a : b :: c : d$,
deduce equation .	$\therefore$ $\frac{a}{b} = \frac{c}{d}$;
thence deduce $\frac{a}{nb} = \frac{c}{nd}$ . .	$\therefore$, $\div^{ng}$ by n, $\frac{a}{nb} = \frac{c}{nd}$;
thence deduce equation required for conclusion . .	$\therefore$, substituting, $\frac{a}{B} = \frac{c}{D}$;
thence deduce conclusion . . .	$\therefore$ $a : B :: c : D$.

Q. E. D.

Prop. V.

If one magnitude be the same multiple of another, as a magnitude taken from the first is of a magnitude taken from the other: the remainder is the same multiple of the remainder as the whole is of the whole.

[If one magnitude (A) be the same multiple of another (a), as a magnitude (B) taken from the first is of a magnitude (b) taken from the other: the remainder $(A-B)$ is the same multiple of the remainder $(a-b)$ as the whole (A) is of the whole (a).]

Express enunciation	Let $A=ma$, and $B=mb$; then shall $(A-B)=m.(a-b)$.
Simplify terms of conclusion	Now $A-B=ma-mb$ $=m.(a-b)$.

Q. E. D.

Prop. VI.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two: the remainders are either equal to these others or equimultiples of them.

[If two magnitudes (a, b) be equimultiples of two others (c, d), and if (e, f), equimultiples of these (c, d), be taken from the first two (a, b), the remainders $(a-e, b-f)$ are either equal to these others (c, d) or equimultiples of them.]

Express enunciation	Let $a=mb$, and $c=md$, $e=nb$, and $f=nd$; then shall $(a-e)$, $(c-f)$ be either equal to, or equimultiples of b and d.
Simplify terms of conclusion	Now $a-e=mb-nb=(m-n).b$, $c-f=md-nd=(m-n).d$.
Thence deduce conclusion	then, if $(m-n)=1$, $a-e=b$, and $c-f=d$; if not, they are equimultiples of b, d.

Q. E. D.

[N.B. *This Proposition is identical with Prop. II, excepting that where one has the sign* $+$, *the other has the sign* $-$.]

Prop. A.

If the first have to the second the same ratio as the third has to the fourth: then, if the first be greater than the second, the third is greater than the fourth; if equal, equal; if less, less.

[If the first (a) have to the second (b) the same ratio as the third (c) has to the fourth (d): then, if the first be greater than the second, the third is greater than the fourth; if equal, equal; if less, less.]

	Express enunciation, with first hypothesis	Let $a : b :: c : d$, and let $a > b$; then shall $c > d$.
	Taking given proportion	$\because\ a : b :: c : d$,
16. 4)	*deduce equation* .	$\therefore\ \frac{a}{b} = \frac{c}{d}$.
	Taking given inequality	$\because\ a > b$,
11. 4)	*deduce conclusion* .	$\therefore$, $\div^{ng}$ by b, $\frac{a}{b} > 1$;
		$\therefore$, substituting, $\frac{c}{d} > 1$;
		$\therefore$, $\times^{ng}$ by d, $c > d$.
		Similarly, if $a = b$, $c = d$; and if $a < b$, $c < d$.

Q. E. D.

PROP. B. (*Invertendo.*)

If four magnitudes be proportionals: they are also proportionals when taken inversely.

[If four magnitudes (a, b, c, d) be proportionals: they are also proportionals when taken inversely (b, a, d, c).]

Express enunciation	Let $a : b :: c : d$; then shall $b : a :: d : c$.
Taking given proportion	$\because\ a : b :: c : d$,
deduce equation .	$\therefore\ \frac{a}{b} = \frac{c}{d}$;
10. 1) *thence deduce equation required for conclusion* . .	$\therefore$, $\div^{ng}$ 1 by each, $\frac{b}{a} = \frac{d}{c}$;
thence deduce conclusion . . .	$\therefore\ b : a :: d : c$.

Q. E. D.

PROP. C.

If the first be the same multiple or part of the second as the third is of the fourth: the first is to the second as the third to the fourth.

Repeat enunciation, with full algebraical meaning . .	If the first be the same multiple or part of the second as the third is of the fourth; then the first is the same multiple, part, or fraction, of the second, as the third is of the fourth.

Quod est tautologicum.

Prop. D.

If the first be to the second as the third to the fourth: then, if the first be a multiple or part of the second, the third is the same multiple or part of the fourth.

Repeat enunciation, with full algebraical meaning.	If the first be the same multiple, part, or fraction, of the second, as the third is of the fourth, and if the first be a multiple or part of the second; then the third is the same multiple or part of the fourth. Quod est tautologicum.

Prop. VII.

Equal magnitudes have the same ratio to any other. And any other has the same ratio to them.

[Equal magnitudes (a, b) have the same ratio to any other (x). And any other (x) has the same ratio to them.]

Part 1.

	Express first enunciation	Let $a=b$; then shall $a : x :: b : x$.
	Taking given equation	$\because$ $a=b$,
6. 1)	*deduce equation required for conclusion*	$\therefore$, $\div^{ng}$ by x, $\frac{a}{x}=\frac{b}{x}$;
	thence deduce conclusion . . .	$\therefore$ $a : x :: b : x$. Q. E. D.

Part 2.

	Express second enunciation	Let $a=b$; then shall $x : a :: x : b$.
	Taking given equation	$\because$ $a=b$,
6. 1)	*deduce equation required for conclusion*	$\therefore$, $\div^{ng}$ x by each, $\frac{x}{a}=\frac{x}{b}$;
	thence deduce conclusion . . .	$\therefore$ $x : a :: x : b$. Q. E. D.

Prop. VIII.

Of two unequal magnitudes the greater has a greater ratio to any other than the lesser has. And any other has a greater ratio to the lesser than it has to the greater.

[Of two unequal magnitudes (a, b) the greater (a) has a greater ratio to any other (x) than the lesser (b) has. And any other (x) has a greater ratio to the lesser (b) than it has to the greater (a).]

Part 1.

	Express first enunciation	Let $a > b$; then shall $a : x > b : x$.
	Taking given inequality	$\because\ a > b$,
6. 2)	*deduce equation required for conclusion* . . .	$\therefore$, $\div^{ng}$ by x, $\frac{a}{x} > \frac{b}{x}$;
	thence deduce conclusion . . .	$\therefore\ a : x > b : x$.

Q. E. D.

Part 2.

	Express second enunciation	Let $a > b$; then shall $x : b > x : a$.
	Taking given inequality	$\because\ a > b$,
6. 2)	*deduce equation required for conclusion* . . .	$\therefore$, $\div^{ng}$ by ab and $\times^{ng}$ by x, $\frac{x}{b} > \frac{x}{a}$;
	thence deduce conclusion . . .	$\therefore\ x : b > x : a$.

Q. E. D.

Prop. IX.

Magnitudes, which have the same ratio to any other, are equal. And magnitudes, to which any other has the same ratio, are equal.

[Magnitudes (a, b), which have the same ratio to any other (x), are equal. And magnitudes (a, b), to which any other (x) has the same ratio, are equal.]

Part 1.

	Express first enunciation	Let $a : x :: b : x$; then shall $a = b$.
	Taking given proportion	$\because \quad a : x :: b : x$,
	deduce equation .	$\therefore \quad \frac{a}{x} = \frac{b}{x}$;
8. 1)	*thence deduce conclusion* . . .	$\therefore$, $\times^{ng}$ by x, $a = b$.

Q. E. D.

Part 2.

	Express second enunciation	Again, let $x : a :: x : b$; then shall $a = b$.
	Taking given proportion	$\because \quad x : a :: x : b$,
	deduce equation .	$\therefore \quad \frac{x}{a} = \frac{x}{b}$;
8. 2)	*thence deduce conclusion* . . .	$\therefore$, $\div^{ng}$ by x and $\times^{ng}$ by ab, $b = a$.

Q. E. D.

Prop. X.

Of magnitudes, which have unequal ratios to any other, that one, which has the greater ratio, is the greater. And of magnitudes, to which any other has unequal ratios, that one, to which it has the greater ratio, is the lesser.

[Of magnitudes (a, b), which have unequal ratios to any other (x), that one (a), which has the greater ratio, is the greater. And of magnitudes (a, b), to which any other (x) has unequal ratios, that one (b), to which it has the greater ratio, is the lesser.]

PART 1.

	Express first enunciation	Let $a : x > b : x$; then shall $a > b$,
	Taking given disproportion	$\because\ a : x > b : x$,
	deduce inequality .	$\therefore\ \frac{a}{x} > \frac{b}{x}$;
8. 4)	*thence deduce conclusion* . . .	$\therefore$, $\times^{\text{ng}}$ by x, $a > b$. Q. E. D.

PART 2.

	Express second enunciation	Again, let $x : b > x : a$; then shall $b < a$.
	Taking given disproportion	$\because\ x : b > x : a$,
	deduce inequality .	$\therefore\ \frac{x}{b} > \frac{x}{a}$;
8. 6)	*thence deduce conclusion* . . .	$\therefore$, $\div^{\text{ng}}$ by x and $\times^{\text{ng}}$ by ba, $a > b$. Q. E. D.

PROP. XI.

Ratios, that are equal to the same ratio, are equal to one another.

Show that this needs no demonstration .	This is an instance of the Axiom "Things, that are equal to the same, are equal to one another."

Prop. XII.

If any number of magnitudes be proportionals: as one of the antecedents is to its consequent, so are all the antecedents to all the consequents.

[If any number of magnitudes (a, b, c, d, e, f, &c.) be proportionals: as one of the antecedents (a) is to its consequent (b), so are all the antecedents ($a+c+e+$&c.) to all the consequents ($b+d+f+$&c.).]

Express enunciation. Let $a : b :: c : d :: e : f ::$ &c.; then shall
$(a+c+e+\&c.) : (b+d+f+\&c.) :: a : b.$

Taking given proportion . . . . . $\because\ a : b :: c : d :: e : f ::$ &c.,

deduce equations . $\therefore\ \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$ &c.;

(11. 1) *thence deduce equation required for conclusion* . . let each $=k$;

$\therefore$, clearing of fractions, $a=bk$, $c=dk$, &c.;

$$\therefore\ \frac{a+c+e+\&c.}{b+d+f+\&c.}=\frac{bk+dk+fk+\&c.}{b+d+f+\&c.}$$
$$=\frac{k.(b+d+f+\&c.)}{b+d+f+\&c.}$$
$$=k;$$

$$\therefore\ \frac{a+c+e+\&c.}{b+d+f+\&c.}=\frac{a}{b};$$

thence deduce conclusion . . . $\therefore\ a+c+e+\&c. : b+d+f+\&c. :: a : b.$

Q. E. D.

Prop. XIII.

If the first have to the second the same ratio as the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth: then the first has to the second a greater ratio than the fifth has to the sixth.

Show that this needs no demonstration .	This is an instance of the general statement "If the first of three things be equal to the second, but the second greater than the third; the first is greater than the third:" which is axiomatic.

Corollary.

If the first have to the second a greater ratio than the third to the fourth, but the third to the fourth the same ratio as the fifth has to the sixth: the same conclusion follows.

Show that this needs no demonstration .	This is an instance of the general statement "If the first of three things be greater than the second, but the second equal to the third; the first is greater than the third:" which is axiomatic.

PROP. XIV.

If the first have to the second the same ratio as the third has to the fourth: then, if the first be greater than the third, the second is greater than the fourth; if equal, equal; if less, less.

[If the first (a) have to the second (b) the same ratio as the third (c) has to the fourth (d): then, if the first (a) be greater than the third (c), the second (b) is greater than the fourth (d); if equal, equal; if less, less.]

	Express enunciation with first hypothesis	Let $a : b :: c : d$, and $a > c$; then shall $b > d$.
	Taking given proportion	$\because \; a : b :: c : d$,
	deduce equation .	$\therefore \; \frac{a}{b} = \frac{c}{d}$.
	Taking given inequality	Again, $\because \; a > c$,
11. 5)	*deduce conclusion* .	$\therefore$, ÷ng by b, $\frac{a}{b} > \frac{c}{b}$;
		$\therefore$, substituting, $\frac{c}{d} > \frac{c}{b}$;
		$\therefore$, ÷ng by c and ×ng by bd, $b > d$.
		Similarly, if $a = c$, $b = d$; and if $a < c$, $b < d$.
		Q. E. D.

PROP. XV.

Magnitudes have to each other the same ratio as their equimultiples have.

[Magnitudes (a, b) have to each other the same ratio as their equimultiples (A, B) have.]

Express enunciation .	Let $A = ma$, and $B = mb$; then shall $a : b :: A : B$.
Simplify terms of equation required for conclusion .	Now $\frac{A}{B} = \frac{ma}{mb} = \frac{a}{b}$;
thence deduce conclusion . . .	$\therefore\ a : b :: A : B$.

Q. E. D.

PROP. XVI. (*Alternando.*)

If four magnitudes *of the same kind* be proportionals: they are also proportionals when taken alternately.

[If four magnitudes *of the same kind* (a, b, c, d) be proportionals: they are also proportionals when taken alternately (a, c, b, d).]

Express enunciation .	Let $a : b :: c : d$; then shall $a : c :: b : d$.
Taking given proportion	$\because\ a : b :: c : d$,
deduce equation .	$\therefore\ \frac{a}{b} = \frac{c}{d}$;
10. 1) *thence deduce equation required for conclusion* . .	$\therefore$, $\times^{ng}$ by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d}$;
thence deduce conclusion . . .	$\therefore\ a : c :: b : d$. Q. E. D.

Prop. XVII.

If four magnitudes, taken jointly, be proportionals: they are also proportionals when taken separately. That is—If two magnitudes together have to one of them the same ratio as two others together have to one of them: the remaining one of the first two has to the other the same ratio as the remaining one of the last two has to the other.

[If two magnitudes together $(a+b)$ have to one of them (b) the same ratio as two others together $(c+d)$ have to one of them (d): the remaining one of the first two (a) has to the other (b) the same ratio as the remaining one of the last two (c) has to the other (d).]

	Express enunciation .	Let $a+b : b :: c+d : d$; then shall $a : b :: c : d$.
	Taking given proportion	$\because\ a+b : b :: c+d : d$,
	deduce equation .	$\therefore\ \frac{a+b}{b} = \frac{c+d}{d}$;
10. 2)	*thence deduce equation required for conclusion* . .	$\therefore$, subtracting 1, $\frac{a}{b} = \frac{c}{d}$;
	thence deduce conclusion . . .	$\therefore\ a : b :: c : d$.

Q. E. D.

PROP. XVII.* (*Dividendo.*)

(*Stated according to Definition XVI.*)

If four magnitudes be proportionals: the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.

[If four magnitudes (a, b, c, d) be proportionals: the excess of the first above the second ($a-b$) is to the second (b) as the excess of the third above the fourth ($c-d$) is to the fourth (d).]

	Express enunciation .	Let $a : b :: c : d$; then shall $a-b : b :: c-d : d$.
	Taking given proportion	$\because\ a : b :: c : d$,
	deduce equation .	$\therefore\ \frac{a}{b} = \frac{c}{d}$;
10. 1)	*thence deduce equation required for conclusion* . .	$\therefore$, subtracting 1, $\frac{a-b}{b} = \frac{c-d}{d}$;
	thence deduce conclusion . . .	$\therefore\ a-b : b :: c-d : d$.

Q.E.D.

[N.B. *This Proposition is identical with Prop. XVIII, excepting that where one has the sign* +, *the other has the sign* −.]

PROP. XVIII. (*Componendo.*)

If four magnitudes, taken separately, be proportionals: they are also proportionals when taken jointly. That is—If the first be to the second as the third is to the fourth: the first and second together are to the second as the third and fourth together are to the fourth.

[If the first (a) be to the second (b) as the third (c) is to the fourth (d): the first and second together ($a+b$) are to the second (b) as the third and fourth together ($c+d$) are to the fourth (d).]

Express enunciation .	Let $a : b :: c : d$; then shall $a+b : b :: c+d : d$.
Taking given proportion	$\because\ a : b :: c : d$,
deduce equation .	$\therefore\ \frac{a}{b} = \frac{c}{d}$;
10. 1) *thence deduce equation required for conclusion* . .	$\therefore$, adding 1, $\frac{a+b}{b} = \frac{c+d}{d}$;
thence deduce conclusion . . .	$\therefore\ a+b : b :: c+d : d$.

PROP. XIX.

If one magnitude be to another as a magnitude taken from the first is to a magnitude taken from the other: the remainder is to the remainder as the whole is to the whole.

[If one magnitude (a) be to another (b) as a magnitude (c) taken from the first is to a magnitude (d) taken from the other: the remainder ($a-c$) is to the remainder ($b-d$) as the whole (a) is to the whole (b).]

Express enunciation .	Let $a : b :: c : d$; then shall $a-c : b-d :: a : b$.
Taking given proportion	$\because \quad a : b :: c : d,$
deduce equation .	$\therefore \quad \frac{a}{b} = \frac{c}{d};$
11. 2) *thence deduce equation required for conclusion* . .	let each $= k$;
	$\therefore$, clearing of fractions, $a = bk$, $c = dk$;
	$\therefore \quad \frac{a-c}{b-d} = \frac{bk-dk}{b-d} = \frac{k.(b-d)}{b-d} = k;$
	$\therefore \quad \frac{a-c}{b-d} = \frac{a}{b};$
thence deduce conclusion . . .	$\therefore \quad a-c : b-d :: a : b.$

Q. E. D.

Corollary.

With the same *data:* the remainder is to the remainder as the magnitude taken from the first is to the magnitude taken from the other.

[With the same *data:* the remainder ($a-c$) is to the remainder ($b-d$) as the magnitude (c) taken from the first is to the magnitude (d) taken from the other.]

The Demonstration is the same as in the Proposition.

Prop. E. (*Convertendo.*)

If four magnitudes be proportionals: they are also proportionals by conversion; that is, the first is to its excess above the second as the third is to its excess above the fourth.

[If four magnitudes (a, b, c, d) be proportionals: they are also proportionals by conversion (a, $a-b$, c, $c-d$); that is, the first (a) is to its excess above the second ($a-b$) as the third (c) is to its excess above the fourth ($c-d$).]

	Express enunciation .	Let $a : b :: c : d$; then shall $a : a-b :: c : c-d$.
	Taking given proportion	$\because \; a : b :: c : d$,
	deduce equation .	$\therefore \; \frac{a}{b} = \frac{c}{d}$;
11. 3)	*thence deduce equation required for conclusion* . .	let each $= k$,
		$\therefore$, clearing of fractions, $a = bk$, $c = dk$;
		$\therefore \; \frac{a}{a-b} = \frac{bk}{bk-b} = \frac{k}{k-1}$,
		and $\frac{c}{c-d} = \frac{dk}{dk-d} = \frac{k}{k-1}$;
		$\therefore \; \frac{a}{a-b} = \frac{c}{c-d}$;
	thence deduce conclusion . . .	$\therefore \; a : a-b :: c : c-d$.

Q. E. D.

Prop. XX.

If there be three magnitudes and other three, which, taken two and two, have the same ratio: then, if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; if less, less.

[If there be three magnitudes (A, B, C) and other three (a, b, c), which, taken two and two, have the same ratio: then, if the first (A) be greater than the third (C), the fourth (a) is greater than the sixth (c); if equal, equal; if less, less.]

	Express enunciation with first hypothesis	Let $A : B :: a : b$, $B : C :: b : c$; and let $A>C$; then shall $a>c$.
	Taking given proportions	$\because$ $A : B :: a : b$, and $B : C :: b : c$,
	deduce equations .	$\therefore$ $\frac{A}{B}=\frac{a}{b}$, and $\frac{B}{C}=\frac{b}{c}$;
10. 3)	*thence deduce* $\frac{A}{C}=\frac{a}{c}$ . .	$\therefore$, $\times^{ng}$ corresponding sides together, $\frac{A}{C}=\frac{a}{c}$.
	Taking given inequality	Now, $\because A>C$,
11. 4)	*deduce conclusion* .	$\therefore$, $\div^{ng}$ by C, $\frac{A}{C}>1$;
		$\therefore$, substituting, $\frac{a}{c}>1$;
		$\therefore$, $\times^{ng}$ by c, $a>c$.
		Similarly, if $A=C$, $a=c$; and if $A<C$, $a<c$.

Q.E.D.

Prop. XXI.

If there be three magnitudes and other three, which, taken two and two in a cross order, have the same ratio: then, if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; if less, less.

[If there be three magnitudes (A, B, C) and other three (a, b, c), which, taken two and two in a cross order, have the same ratio: then, if the first (A) be greater than the third (C), the fourth (a) is greater than the sixth (c); if equal, equal; if less, less.]

Express enunciation with first hypothesis	Let $A : B :: b : c$, $B : C :: a : b$; and let $A > C$; then shall $a > c$.
Taking given proportions	$\because$ $A : B :: b : c$, and $B : C :: a : b$,
deduce equations .	$\therefore$ $\frac{A}{B} = \frac{b}{c}$, and $\frac{B}{C} = \frac{a}{b}$;
10. 4) *thence deduce* $\frac{A}{C} = \frac{a}{c}$ . . .	$\therefore$, $\times^{ng}$ corresponding sides together, $\frac{A}{C} = \frac{a}{c}$.
Taking given inequality	Now, $\because$ $A > C$,
11. 4) *deduce conclusion* .	$\therefore$, $\div^{ng}$ by C, $\frac{A}{C} > 1$; $\therefore$, substituting, $\frac{a}{c} > 1$; $\therefore$, $\times^{ng}$ by c, $a > c$. Similarly, if $A = C$, $a = c$; and if $A < C$, $a < c$.

Q. E. D.

PROP. XXII.

If there be any number of magnitudes, and as many others which, taken two and two, have the same ratio: the first of the first rank shall have to the last the same ratio as the first of the second rank has to the last.

[If there be any number of magnitudes (A, B, C, &c., X, Y, Z), and as many others (a, b, c, &c., x, y, z) which, taken two and two, have the same ratio: the first of the first rank (A) shall have to the last (Z) the same ratio as the first of the second rank (a) has to the last (z).]

Express enunciation .	Let $A : B :: a : b$, $B : C :: b : c$, &c., $X : Y :: x : y$, $Y : Z :: y : z$; then shall $A : Z :: a : z$.
Taking given proportions	$\because$ $A : B :: a : b$, &c.,
deduce equations .	$\therefore$ $\frac{A}{B} = \frac{a}{b}$, $\frac{B}{C} = \frac{b}{c}$, &c., $\frac{X}{Y} = \frac{x}{y}$, $\frac{Y}{Z} = \frac{y}{z}$;
10. 5) *thence deduce* $\frac{A}{Z} = \frac{a}{z}$ . .	$\therefore$, $\times^{ng}$ corresponding sides together, $\frac{A}{Z} = \frac{a}{z}$;
thence deduce conclusion . . .	$\therefore$ $A : Z :: a : z$.

Q. E. D.

Prop. XXIII.

If there be any number of magnitudes, and as many others which, taken two and two in a cross order, have the same ratio: the first of the first rank shall have to the last the same ratio as the first of the second rank has to the last.

[If there be any number of magnitudes (*A*, *B*, *C*, *&c.*, *X*, *Y*, *Z*), and as many others (*a*, *b*, *c*, *&c.*, *x*, *y*, *z*) which, taken two and two in a cross order, have the same ratio: the first of the first rank (*A*) shall have to the last (*Z*) the same ratio as the first of the second rank (*a*) has to the last (*z*).]

	Express enunciation .	Let $A : B :: y : z$, $B : C :: x : y$, *&c.*, $X : Y :: b : c$, $Y : Z :: a : b$; then shall $A : Z :: a : z$.
	Taking given proportions	$\because\ A : B :: y : z$, *&c.*,
	deduce equations .	$\therefore\ \frac{A}{B} = \frac{y}{z},\ \frac{B}{C} = \frac{x}{y}$, *&c.*, $\frac{X}{Y} = \frac{b}{c},\ \frac{Y}{Z} = \frac{a}{b}$;
10. 6)	*thence deduce* $\frac{A}{Z} = \frac{a}{z}$ . .	$\therefore$, $\times^{ng}$ corresponding sides together, $\frac{A}{Z} = \frac{a}{z}$;
	thence deduce conclusion . . .	$\therefore\ A : Z :: a : z$.

Q. E. D.

Prop. XXIV.

If the first have to the second the same ratio as the third has to the fourth; and the fifth the same ratio to the second as the sixth has to the fourth: then the first and fifth together shall have the same ratio to the second as the third and sixth together have to the fourth.

[If the first (a) have to the second (b) the same ratio as the third (c) has to the fourth (d); and the fifth (e) the same ratio to the second (b) as the sixth (f) has to the fourth (d): then the first and fifth together ($a+e$) shall have the same ratio to the second (b) as the third and sixth together ($c+f$) have to the fourth (d).]

Express enunciation .	Let $a : b :: c : d,$ $e : b :: f : d;$ then shall $a+e : b :: c+f : d.$
Taking given proportions	$\because\ a : b :: c : d$, and $e : b :: f : d$,
deduce equations .	$\therefore\ \frac{a}{b} = \frac{c}{d}$, and $\frac{e}{b} = \frac{f}{d}$;
thence deduce equation required for conclusion . .	$\therefore$, adding corresponding sides, $\frac{a+e}{b} = \frac{c+f}{d}$;
thence deduce conclusion . . .	$\therefore\ a+e : b :: c+f : d.$

Q. E. D.

Corollary 1.

With the same data; the excess of the first above the fifth shall have the same ratio to the second as the excess of the third above the sixth has to the fourth.

[With the same data: the excess of the first above the fifth $(a-e)$ shall have the same ratio to the second (b) as the excess of the third above the sixth $(c-f)$ has to the fourth (d).]

Express enunciation .	Let $a : b :: c : d$, $e : b :: f : d$; then shall $a-e : b :: c-f : d$.
Taking given proportions	$\because\ a : b :: c : d$, and $e : b :: f : d$,
deduce equations .	$\therefore\ \frac{a}{b} = \frac{c}{d}$, and $\frac{e}{b} = \frac{f}{d}$;
thence deduce equation required for conclusion . .	$\therefore$, subtracting corresponding sides, $\frac{a-e}{b} = \frac{c-f}{d}$;
thence deduce conclusion . . .	$\therefore\ a-e : b :: c-f : d$.

Q. E. D.

[N.B. *This is identical with the Proposition, excepting that where one has the sign* $+$, *the other has the sign* $-$.]

COROLLARY 2.

The Proposition holds true of two ranks of magnitudes, of which each of the first rank has to a single magnitude the same ratio as each of the second rank has to another single magnitude.

[The Proposition holds true of two ranks of magnitudes (A, B, C, &c., and a, b, c, &c.), of which each of the first rank has to a single magnitude (X) the same ratio as each of the second rank has to another single magnitude (x).

Express enunciation .	Let $A : X :: a : x$, $B : X :: b : x$, $C : X :: c : x$, &c.; then shall $(A+B+C+\&c.) : X :: (a+b+c+\&c.) : x$.
Taking given proportions	$\because$ $A : X :: a : x$, &c.,
deduce equations .	$\therefore$ $\frac{A}{X}=\frac{a}{x}$, $\frac{B}{X}=\frac{b}{x}$, $\frac{C}{X}=\frac{c}{x}$, &c.;
thence deduce equation required for conclusion . .	$\therefore$, adding corresponding sides, $\frac{A+B+C+\&c.}{X}=\frac{a+b+c+\&c.}{x}$;
thence deduce conclusion . . .	$\therefore (A+B+C+\&c.) : X :: (a+b+c+\&c.) : x$. Q. E. D.

Prop. XXV.

If four magnitudes of the same kind be proportionals: the greatest and least together are greater than the other two together.

Obtain a proportion in which the greatest stands first . . .	Taking that ratio first which contains the greatest magnitude, and inverting if necessary, we obtain a proportion in which the greatest stands first.	
Name magnitudes, pointing out which is greatest . . .	Let magnitudes, so arranged, be named a, b, c, d, of which a is greatest.	
Express data . . .	Let $a : b :: c : d$;	
Prove that 'd' is least	$\because\ a>b\ \therefore\ c>d$;	[V. *A*.
	and $\because\ a>c\ \therefore\ b>d$;	[V. 14.
	$\therefore\ d$ is least.	
Express conclusion .	Then shall $a+d > b+c$.	
Taking given proportion	$\because\ a : b :: c : d$,	
deduce $a : a-b :: c : c-d$	$\therefore$, *convertendo*, $a : a-b :: c : c-d$;	[V. *E*.
thence deduce $a-b>c-d$ . .	then, $\because\ a>c$, $\therefore\ a-b>c-d$;	[V. 14.
thence deduce conclusion . . .	$\therefore$, adding $(b+d)$, $a+d>b+c$.	

Q. E. D.

THE ENUNCIATIONS,

SYSTEMATICALLY ARRANGED.

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§ 1. *From given Equimultiples to prove others.*

DATA.	QUÆSITA.
1. $A=ma,\ B=mb,\ C=mc$, *&c.*	$(A+B+C+\&c.)=$ $m.(a+b+c+\&c.)$.
5. $A=ma,\ B=mb$	$A-B=m.(a-b)$.
3. $\left.\begin{array}{l} a=mb, \text{ and } c=md \\ A=na, \text{ and } c=nc \end{array}\right\}$ . . .	A, C are equimultiples of b, d.
2. $\left.\begin{array}{l} a=mb, \text{ and } c=md \\ e=nb, \text{ and } f=nd \end{array}\right\}$ . . .	$(a+e)$, $(e+f)$ are equimultiples of b, d.
6. *Same data*	$(a-e)$, $(e-f)$ are equal to, or equimultiples of, b, d.
2. Cor. $\left.\begin{array}{l} A=mX, \text{ and } a=mx \\ B=nX, \text{ and } b=nx \\ C=rX, \text{ and } c=rx \\ \&c. \end{array}\right\}$.	$(A+B+C+\&c.)$, $(a+b+c+\&c.)$ are equimultiples of X, x.

§ 2. *From given Equimultiples to prove Proportions: and* vice versâ.

DATA.	QUÆSITA.
C. First is same multiple or part of second as third is of fourth	first is to second as third to fourth.
D. First is to second as third to fourth, and first is multiple or part of second. .	third is same multiple or part of fourth.
15. $A = ma$, and $B = mb$ . .	$a : b :: A : B$.

§ 3. *From given Proportions (and Disproportions) to prove others involving the same Ratios.*

11. Ratios equal to the same ratio	are equal to one another.
13. Ratio of first to second equals ratio of third to fourth, but ratio of third to fourth is greater than ratio of fifth to sixth .	ratio of first to second is greater than ratio of fifth to sixth.
Cor. Ratio of first to second is greater than ratio of third to fourth, but ratio of third to fourth equals ratio of fifth to sixth .	*same conclusion.*

§ 4. *From given Equations (or Inequalities) to prove Proportions (or Disproportions): and* vice versâ.

DATA.	QUÆSITA.
7. (1) $a=b$	$a : x :: b : x$.
(2) do.	$x : a :: x : b$.
9. (1) $a : x :: b : x$	$a=b$.
(2) $x : a :: x : b$	do.
8. (1) $a>b$.	$a : x>b : x$.
(2) do.	$x : b>x : a$.
10. (1) $a : x>b : x$	$a>b$.
(2) $x : b>x : a$	do.

§ 5. *From given Proportions to prove others involving new Ratios.*

16. $a : b :: c : d$	$a : c :: b : d$ (*alternando*).
B. do.	$b : a :: d : c$ (*invertendo*).
18. do.	$a+b : b :: c+d : d$ (*componendo*).
17. $a+b : b :: c+d : d$	$a : b :: c : d$ (*dividendo*).
17*. $a : b :: c : d$	$a-b : b :: c-d : d$ (*ditto*).
E. do.	$a : a-b :: c : c-d$ (*convertendo*).
12. $a : b :: c : d :: e : f ::$ &c.	$(a+c+e+\&c.) : (b+d+f+\&c.) :: a : b$.
19. $a : b :: c : d$	$a-c : b-d :: a : b$.
Cor. do.	$a-c : b-d :: c : d$.
24. $a : b :: c : d$ and $e : b :: f : d$	$a+e : b :: c+f : d$.
Cor. (1) do.	$a-e : b :: c-f=d$.
Cor. (2) $A : X :: a : x$ $B : X :: b : x$ $C : X :: c : x$ &c.	$(A+B+C+\&c.) : X :: (a+b+c+\&c.) : x$.
22. $A : B :: a : b$ $B : C :: b : c$ &c. $X : Y :: x : y$ $Y : Z :: y : z$	$A : Z :: a : z$.

DATA.	QUÆSITA.
23. $\left.\begin{array}{l} A : B :: y : z \\ B : C :: x : y \\ \quad \&c. \\ X : Y :: b : c \\ Y : Z :: a : b \end{array}\right\}$ . . .	$A : Z :: a : z.$

§ 6. *From a given Proportion, combined with Equimultiples, to prove Proportions involving new Ratios.*

4. $\left.\begin{array}{l} a : b :: c : d \\ A = ma, \text{ and } C = mc \\ B = nb, \text{ and } D = nd \end{array}\right\}$ . .	$A : B :: C : D.$
Cor. (1) do. . .	$A : b :: C : d.$
Cor. (2) do. . .	$a : B :: c : D.$

§ 7. *From given Proportions, combined with Equations (or Inequalities), to prove Equations (or Inequalities).*

A. $\left.\begin{array}{l} a : b :: c : d \\ \text{and } a >= \text{ or } < b \end{array}\right\}$ . . .	$c >= \text{ or } < d.$
14. $\left.\begin{array}{l} a : b :: c : d \\ \text{and } a >= \text{ or } < c \end{array}\right\}$ . . .	$b >= \text{ or } < d.$
20. $\left.\begin{array}{l} A : B :: a : b \\ B : C :: b : c \\ A >= \text{ or } < C \end{array}\right\}$	$a >= \text{ or } < c.$
21. $\left.\begin{array}{l} A : B :: b : c \\ B : C :: a : b \\ A >= \text{ or } < C \end{array}\right\}$	*same conclusion.*
25. That four magnitudes are proportionals, and that one is greatest and one least:	greatest and least together are greater than other two.

THE DEFINITIONS.

I.

A less magnitude is said to be a **part** of a greater when the less measures the greater; that is, 'when the less is contained a certain number of times exactly in the greater.'

II.

A greater magnitude is said to be a **multiple** of a less, when the greater is measured by the less, that is 'when the greater contains the less a certain number of times exactly.'

Euclid always means, by 'a certain number of times,' '*more than once*': but, in Algebra, 'one' is considered as a number, so that a magnitude is a multiple of itself, and also a part of itself.

III.

'**Ratio** is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.'

This is too vague to be of any practical use. Euclid explains more clearly what he means by 'ratio,' when he comes to define 'identity of ratio' in Def. V. That Definition is applicable to incommensurable, as well as to commensurable, magnitudes: whereas the Algebraical Definition of 'ratio,' and so of 'identity of ratio,' is applicable to commensurable magnitudes only.

IV.

Magnitudes are said **to have a ratio to one another,** when the less can be multiplied so as to exceed the other.

Euclid probably meant, by this Definition, to exclude *infinite* magnitudes.

V.

The first of four magnitudes is said **to have the same ratio to the second, which the third has to the fourth,** when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

The Algebraical Definition answering to this would be 'The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when the first is the same multiple, part, or fraction of the second which the third is of the fourth': but such a Definition would be quite superfluous, as it may be easily deduced from the Algebraical Definition of 'ratio.'

This Definition is discussed in the Appendix.

VI.

Magnitudes which have the same ratio are called **proportionals.**

This Definition is also true in Algebra.

VII.

(*To be omitted.*)

When of the equimultiples of four magnitudes, taken as in the fifth definition, the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater than the multiple of the fourth, then the first is said to have to the second a greater ratio than the third has to the fourth; and the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

'**Analogy,** or **proportion,** is the similitude of ratio.'

This might be expressed, equally well, as 'the equality of ratio,' or 'the identity of ratio.'

IX.

Proportion consists in three terms at least.

This is an Axiom.

X.

When three magnitudes are proportionals, the first is said to have to the third, **the duplicate ratio** of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth, **the triplicate ratio** of that which it has to the second, and so on, quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

A.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them **the ratio compounded** of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

XII.

In proportionals, the antecedent terms are called **homologous** to one another, as also the consequents.

XIII.

Alternando. This word is used when there are four proportionals, and it is inferred that the first is to the third as the second to the fourth. (Prop. XVI.)

XIV.

Invertendo; when there are four proportionals, an it is inferred, that the second is to the first, as th fourth to the third. (Prop. *B.*)

XV.

Componendo; when there are four proportionals, and it is inferred that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. (Prop. XVIII.)

XVI.

Dividendo; when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. (Prop. XVII.)

XVII.

Convertendo; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. (Prop. *E*.)

XVIII.

Ex æquali; when there is any number of magnitudes more than two, and as many others such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: 'Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two.'

XIX.

Ex æquali in proportione directa; this term is used when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and the second is to the third of the first rank, as the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition. (PROP. XXII.)

XX.

Ex æquali in proportione perturbata seu inordinata; this term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and the second is to the third of the first rank, as the last but two to the last but one of the second rank: and the third is to the fourth of the first rank, as the last but three to the last but two of the second rank; and so on in a cross order: and the inference is as in the 18th definition. (PROP. XXIII.)

THE AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

APPENDIX.

EUCLID'S Definition of Proportion is not used in the Fifth Book, when proved algebraically, since this method of Proof applies to commensurable Magnitudes only, for which a much simpler Definition is sufficient: but it is required for Prop. I of the Sixth Book, on which the rest of that Book depends, so that some explanation of it may fitly be given here.

The Student should pay particular attention to the word "*whatsoever.*" Four magnitudes are said to be Proportionals, not merely when "*any* equimultiples" fulfil certain conditions (in which case *one* successful instance would be enough to justify the use of the name, in spite of many *un*successful instances being found), but when "*any whatsoever*" fulfil them (in which case *all* instances must be successful to justify the use of the name, and a *single* unsuccessful instance would be enough to destroy our right to use it). To take an illustration from Chemistry, a drug might fairly be called "dangerous to life," if *any* instance could be found of a person having died from swallowing it, but it could not be called "*certainly* fatal to life," unless it could be shown that *any person whatsoever*, who swallowed it, *must* die in consequence; and a *single* proved case of a person having survived it would destroy our right to use the name.

In other words, before we have a right to call certain Magnitudes "Proportionals" in Euclid's sense of the word, we must first have proved what is called in Logic "a Universal Proposition" (whose typical form is "All A are B"); that is, we must have proved that *all* equimultiples of these Magnitudes fulfil certain conditions.

Now a Universal Proposition may be proved by two totally distinct methods. One may be defined as "the enumeration of all instances," the other as "the establishment of a general law." Before explaining these phrases as applied to Euclid's Definition, let us illustrate them by examples from another subject.

Suppose we wish to establish the Universal Proposition that "All English Queens who have reigned since the Conquest have had the letter 'A' in their names." This may be proved true by enumerating all the names, and pointing out that *each* fulfils the condition stated. But, inasmuch as the circumstance is merely an *accident* as to each name, no "general law" can be shown to exist. In this case, then, we are restricted to the *first* method of proof.

Next, suppose the Proposition to be "All English Queens who have reigned since the Conquest have been of royal descent." In this case we have a choice of methods: we may either enumerate all the names, giving the genealogy of each: or we may show that, by the principles of our Constitution, such royal descent is essential to a Queen. In many cases of this kind, it would be quite a matter for consideration, *which* method to employ: sometimes the one would be found the more convenient, sometimes the other.

Thirdly, suppose the Proposition to be "All English Queens, who have reigned since the Conquest, or who ever will reign, have had, or will have, weight." In this case, since some of the instances referred to do not yet exist, so that no evidence, concerning them individually, can be given, we are restricted to the use of the *second* method: that is, we can only prove the Proposition by showing that all English Queens, past, present, and future, are necessarily human beings; and that human beings, by a law of Nature, have weight.

Now which of these two methods does Euclid mean us to employ, when he tells us, before we can use the name "Proportionals" of certain Magnitudes, to prove the Universal Proposition that "*all* equimultiples" fulfil certain conditions?

The number of equimultiples we may take of the magnitudes is infinite: hence "the enumeration of all instances" is *impossible* in this case, and the only method left us is "the establishment of a general law."

To take the actual instance in which the Definition is used by Euclid—Prop. I of the Sixth Book. Euclid wishes to show that if two Triangles be of the same altitude, the two bases and the two

Triangles constitute four "Proportionals." To do this, out of the infinite number of *possible* equimultiples of the first and third, he chooses a single instance; and out of the infinite number of *possible* equimultiples of the second and fourth, he also chooses a single instance. He does not assume it to be enough for his purpose to show that, in this particular pair of instances, the test of "if greater, greater; if equal, equal; if less, less" is fulfilled. The essence of his argument consists in showing that what is true in the particular instances chosen would *also*, from the nature of the case, be true in *every other* conceivable instance which could be taken.

This part of the proof depends on two theorems—first, Prop. XXXVII of the First Book, namely "Triangles on equal bases, and between the same parallels, are equal"; secondly, on an easy deduction from this (which is not explicitly stated by Euclid, and so is often overlooked), namely, "Triangles between the same parallels, but on unequal bases, are unequal, that which is on the greater base being the greater." Both Prop. XXXVII and this deduction from it are clearly of *universal* applicability. Hence Euclid's proof of the Universal Proposition, that "*all* equimultiples" fulfil the requisite conditions, is complete, and his conclusion, that the original four Magnitudes are "Proportionals," is legitimate.

THE END.