A

KEY TO THE EXERCISES

IN THE

FIRST SIX BOOKS

OF

CASEY'S ELEMENTS OF EUCLID.

THIRD EDITION

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BY

JOSEPH B CASEY,

TUTOR, UNIVERSITY COLLEGE, DUBLIN,

EDITED BY

PATRICK A E DOWLING, B A, R.U I,

EXAMINER TO THE BOARD OF INTERMEDIATE EDUCATION, IRELAND,

AND

PROFESSOR OF MATHEMATICS, UNIVERSITY COLLEGE, DUBLIN

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EDITOR'S PREFACE

IN this edition a few exercises, omitted in former editions, have been inserted; and in No XXXIV, Miscellaneous Exercises, Book VI, an alternative demonstration of the converse of Ptolemy's Theorem has been added

Though the proof-sheets have been very carefully read throughout, some misprints have probably escaped notice, and the Editor will be grateful for a list of any that may be found in the present edition

P A E D

4, UXBRIDGE-TERRACE,
LEESON PARK, DUBLIN,
Jan 20th, 1893

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EXERCISES ON EUCLID.

BOOK I

PROPOSITION I

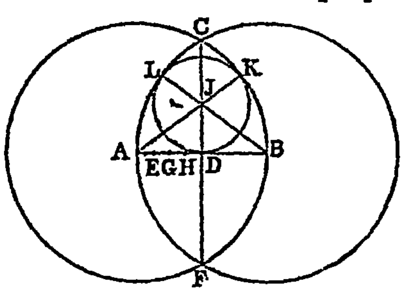
1 **Dem.**—The four lines AC, AF, BC, BF are each = AB, and = to each other Hence ACBF is a lozenge

2 **Dem** —Because AC = BC, and CF common, and the base AF = BF, (VIII) the ∠ ACF = BCF, ∴ ACF is $\frac{1}{2}$ an ∠ of an equilateral Δ Again, the ∠ CAB = ACD + ADC (XXXII), but ACD = ADC, ∴ CAB = 2ACD, ∴ ACD is $\frac{1}{2}$ an ∠ of an equilateral Δ, and ACF is $\frac{1}{2}$ an ∠ of an equilateral Δ, ∴ DCF is an ∠ of an equilateral Δ Similarly DFC is an ∠ of an equilateral Δ Hence the Δ CDF is equilateral

3 **Dem.**—Join AF Because AG = AF, the ∠ AGF = AFG, and because AF = AC, the ∠ ACF = AFC, ∴ the ∠ GFC = FGC + FCG, and is ∴ (XXXII *Cor* 7) a right ∠ In like manner HFC is a right ∠ Hence (XIV) G, F, H are collinear

4 **Dem** —$GC^2 = GF^2 + FC^2$ (XLVII), and $GC^2 = 4AG^2$, ∴ $GF^2 + FC^2 = 4AG^2$, but GF = AG Therefore $FC^2 = 3AG^2 = 3AB^2$

5 **Sol** —Join CF Divide AD into four equal parts in E, G, H



From DC cut off DJ = ED J is the centre of the required ○

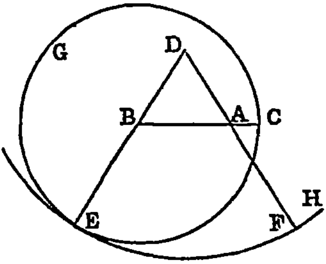
Dem —Join AJ, BJ, and produce them to meet the ○s in K, L

7V

Because the ∠ ADJ is right, $AJ^2 = JD^2 + DA^2 = 3^2 + 4^2 = 5^2$, ∴ AJ is = 5 of the parts into which AD is divided, but AK = AB, ∴ JK = 3 of the parts, ∴ JK = JD Again, AD = DB, and DJ common, and the ∠ ADJ equal BDJ, ∴ (IV) AJ = BJ, but AK = BL, ∴ JK = JL Hence the lines JD, JK, JL are equal, and the ○, with J as centre and JD as radius, will pass through the points K, L

PROPOSITION II

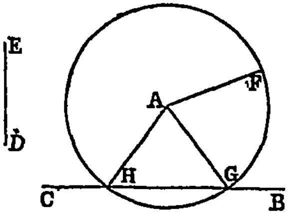
1 **Sol** —On AB describe the equilateral △ ABD With B as centre and BC as radius, describe the ○ CEG, and produce DB



to meet it in E With D as centre and DE as radius, describe the ○ EFH, and produce DA to meet it in F AF is the required line

Dem —Because D is the centre of the ○ EFH, ∴ DE = DF, but DB = DA, ∴ BE = AF, and BE = BC, ∴ AF = BC

2 **Sol** —Let A be the given point, and BC the given line



It is required from the point A to inflect to BC a line equal to a given line DE From A draw AF = DE [II] With A as centre, and AF as radius, describe a ○ cutting BC in G, H Join AG, AH AG, AH are the required lines

Dem.—Because AF = AG, and AF = DE, AG = DE In like manner AH = DE Hence there are two solutions

PROPOSITION IV

1 Let AD bisect the vertical ∠ of the isosceles Δ ABC It is required to prove that it bisects the base BC perpendicularly

Dem —AB = AC, and AD common, and the ∠ BAD = CAD, (IV) the ∠ ADB = ADC, and the side BD = CD Hence BC is bisected, and (Def XIV) AD is ⊥ to BC

2 Dem —Let ABCD be the quadrilateral, and BD its diagonal Because AB = CB, and BD common, and the ∠ ABD = CBD, (IV) the base AD = CD

3 Let the lines AB, CD, bisect each other in E

Dem —Take any point F in ED Join AF, BF Because AE = BE, and EF common, and the ∠ AEF = BEF, . the base AF = BF

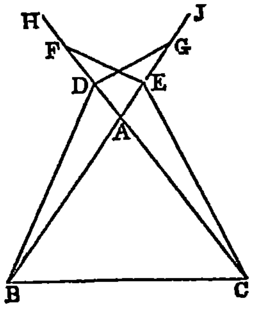
4 Let ABC be the Δ On the sides AB, AC, describe equilateral Δ• ABD, ACE Join CD, BE It is required to prove that CD = BE

Dem —Because the ∠ DAB = CAE, to each add the ∠ BAC, then the ∠ DAC = BAE, and since DA = BA, and CA = EA, the sides DA, AC = BA, AE, and we have shown that the ∠ DAC = BAE, (IV) the bases CD, BE, are equal.

PROPOSITION V

1 (1) Dem —Take any point D in AB, and from AC cut off AE = AD (III) Join BE, CD, DE Because AB = AC, and AE = AD, BA and AE = CA and AD, and the ∠ A is common, BE = CD, and the ∠ ABE = ACD Again, because BE = CD, and BD = CE, BD and BE = CE and CD, and the ∠ DBE = ECD, (IV) the ∠ BDE = CED, and the ∠ BED = CDE, hence the remainders, the ∠• BDC, BEC, are equal Again, BD = CE, and DC = EB, BD and DC = CE and EB, and the contained ∠• BDC, CEB, have been shown to be equal, (IV) the ∠• DBC, ECB, are equal

(2) Dem —Produce BA, CA, to J, H, in AJ take any points E, G, and from AH cut off AD = AE, and AF = AG Join DG,

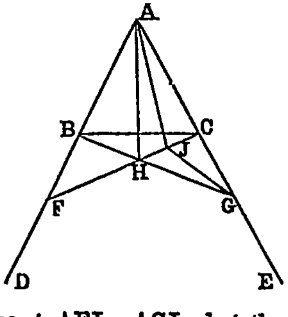


DB, EC, EF Because AF = AG, and AE = AD, ∴ AF and AE = AG and AD, and the ∠ FAG common, the base FE = DG, and the ∠ AFE = AGD, and the ∠ FEA = GDA

Again, because BG = CF, and GD = FE, BG and GD = CF and FE, and the ∠ DGB = EFC, the base DB = EC, and the ∠ GDB = FEC, but the ∠ GDA = FEA ∴ the remainders, the ∠s BDC, BEC, are equal

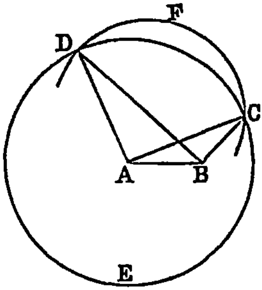
Now, since BD = CE, and DC = EB, BD and DC = CE and EB, and the ∠ BDC = CEB, the ∠ DCB = EBC

2 Dem —If AH be not an axis of symmetry, let AJ be one Join JG Because AF = AG, and AJ common, and the ∠ FAJ



GAJ (hyp), the ∠ AFJ = AGJ, but the ∠ AFC = AGB, the ∠ AGJ = AGB, a part = to the whole, which is absurd, AH must be an axis of symmetry

○ FCD, BC = BD, but this is contrary to Prop VII Hence



the ○s cannot intersect in more than one point on the same side of the line AB Hence two ○s cannot intersect in more than two points, which must be situated on opposite sides of the line joining the centres of the ○s

PROPOSITION IX

1 **Dem** —Because AD = AE, the ∠ ADE = AED, and because FD = FE, the ∠ FDE = FED Now we have two Δs ADF, AEF, having two sides AD, DF, and the contained ∠ ADF respectively = to the two sides AE, EF, and the contained ∠ AEF, (IV) the ∠ DAF = EAF

2 **Dem** —Let G be the point where AF meets DE Because AD = AE, and AG common, and the ∠ DAG = EAG, the ∠ AGD = AGE Hence (Def XIV) AF is ⊥ to DE

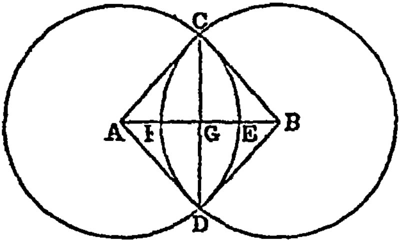
3 See Ex 3, Prop IV

4 **Dem** —Take any point P in AF, and from P let fall the ⊥ PH on AB From AC cut off AJ = AH, and join PJ Because AH = AJ, and AP common, and the ∠ HAP = JAP, (IV) the ∠ AJP = AHP Hence the ∠ AJP is right, and the base PH = PJ

PROPOSITION X.

1 **Sol** —Let AB be the given line Take a part AE greater than half AB With A as centre and AE as radius, describe the ○ CED Take BF = AE With B as centre and BF as radius, describe the ○ CFD, cutting the ○ CED in C, D Join CD, cutting AB in G AB is bisected in G

Dem —Join AC, BC, AD, BD Because AC = BC, and CD common, and the base AD = BD, (VIII) the ∠ ACD = BCD



Again, since AC = BC, and CG common, and the ∠ ACG = BCG, (IV) AG = BG

2 Dem —Take any point H equally distant from A, B Join AH, BH, CH Because AC = BC, and CH common, and the base AH = BH, (VIII) the ∠ ACH = BCH Hence any point equally distant from A, B, is in the bisector of the ∠ ACB

PROPOSITION XI

1 Dem —Let the diagonals AD, BC, of the lozenge ABDC, intersect in E Because AB = AC, and AD common, and the base BD = CD, (VIII) the ∠ BAE = CAE Again, AB = AC, AE common, and the ∠ BAE = CAE, (IV) BE = CE, and the ∠ AEB = AEC Hence AD bisects BC perpendicularly

2 Dem —Because DF = EF, the ∠ FED = FDE (V), and CD = CE, (IV) the △ DCF = ECF, the ∠ DCF = ECF, and (Def XIV) each of them is a right ∠

3 Sol.—Let AB be the given line At the point A draw AC, making an ∠ with AB In AC take AD = AB At D erect DE ⊥ to AC Bisect the ∠ BAC by AE, meeting DE in E Join BE BE is ⊥ to AB

Dem —AD = AB, AE common, and the ∠ DAE = BAE, (IV) the ∠ ADE = ABE, but ADE is a right ∠ (const), hence ABE is a right ∠

4 Sol —Let AB be the given line, and C, D, the points Join CD, bisect CD in E Draw EF ⊥ to CD, meeting AB in F F is the required point

Dem —Join CF, DF Because (IV) the △ CEF = DEF, ∴ FC = FD Hence the point F is equally distant from C and D

5 Sol —Let AB be the given line, and C, D, the points From C let fall a ⊥ CG on AB, and produce it to E, so that GE will be equal to CG Join ED, and produce it to meet AB in F F is the required point

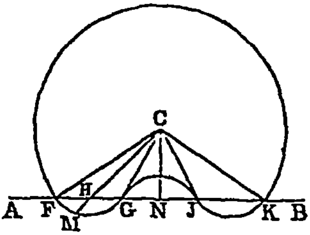
Dem —Join CF Because CG = EG, and GF common, and the ∠ CGF = EGF, ∴ (IV) the ∠ CFG = EFG Hence the ∠ CFD is bisected by the line AB

6 Sol —Let A, B, C, be the three given points Join AB, BC Bisect AB at D, and erect DF ⊥ to AB Bisect BC at E, and erect EF ⊥ to BC F is the required point

Dem —Join AF, BF, CF Because AD = BD, and DF common, and the ∠ ADF = BDF, ∴ (IV) AF = BF In like manner BF = CF Hence the three lines AF, BF, CF, are equal

PROPOSITION XII

1 Dem —If possible let FGJK be a ○ meeting AB in the points F, G, J, K Bisect FG in H Join CH, and produce it to



M Join CF, CG Bisect GJ in N Join CN, CJ, CK Because FH = GH, and HC common, and the base FC = CG, ∴ the ∠ FHC = GHC, and (Def XIV) each of them is a right angle

Again, since GN = JN, and CN common, and the base CG = CJ, ∴ the ∠ CNG = CNJ, and each is a right angle Hence the ∠ CNH = CHN, ∴ CH = CN, but CN is greater than CK, because the point N is outside the ○, ∴ CH is greater than CK, and CM = CK, ∴ CH is greater than CM, which is absurd. Hence the ○ cannot meet AB in more than two points

2 Dem —Let ABC be the Δ, having the ∠ BAC equal to the sum of the ∠s ABC, ACB Bisect AB in D, and erect DE ⊥ to AB, meeting BC in E Join AE

Because AD = BD, DE common, and the ∠ ADE = BDE, ∴ (IV) the ∠ DAE = DBE, but the ∠ BAC = ABC + ACB, hence the ∠ EAC = ECA, ∴ each of the Δs ABE, ACE, is isosceles, and since AE = BE = CE, ∴ BC = 2AE

PROPOSITION XVII

Dem —Let ABC be the Δ Take any point D in BC Join AD The ∠ ADC is greater than ABC (XVI), and the ∠ ADB is greater than ACB, but ADC and ADB equal two right ∠s, ∴ ABC and ACB are less than two right ∠s

PROPOSITION XVIII

1 Dem—Let ABC be the Δ, of which AC is greater than AB From AC cut off AD = AB With A as centre, and AB as radius, describe the circle DBE, cutting CB produced in E Join AE Now the ∠ ABC is greater than AEB, but AEB = ABE, ∴ ABC is greater than ABE, and ABE is greater than ACB (XVI) Hence ABC is greater than ACB

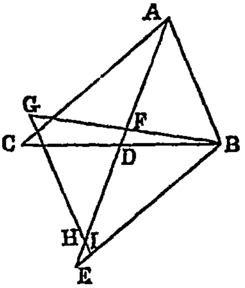
2 Dem —Produce AB to D, so that AD = AC Join CD Now the ∠ ABC is greater than ADC (XVI), but ADC = ACD, ∴ ABC is greater than ACD Much more is ABC greater than ACB

3 Dem —Let ABCD be a quadrilateral, whose sides AB, CD, are the greatest and least It is required to prove that the ∠ ADC is greater than ABC Join BD Because BC is greater than DC, the ∠ BDC is greater than DBC (XVIII) Similarly the ∠ ADB is greater than ABD Hence the ∠ ADC is greater than ABC

4 Dem —Let ABC be a Δ, whose side BC is not less than AB or AC From A let fall a ⊥ AD on BC Because BC is not less than AB, the ∠ BAC is not less than BCA, ∴ BCA must be acute In like manner CBA must be acute Hence AD must fall within the Δ ABC

PROPOSITION XIX

1 Dem —Bisect BC in D Join AD, produce it to E, so that DE = AD Join BE Now the Δˢ BDE, ADC, have the sides BD, DE, of one respectively equal to CD, DA, of the other, and the contained ∠ˢ equal (xv), (iv) BE = AC, and the



∠ DBE = DCA, but the ∠ ABD is greater than DCA (hyp), ABD is greater than EBD, hence the line BF which bisects the ∠ ABE falls above BC Produce BF to G, and make GF = BF Now, since ED = AD, EF is greater than AF Cut off FH = AF Join GH, and produce it to meet BE in I Now we have in the Δˢ AFB, GFH, two sides AF, FB, in one equal HF, FG, in the other, and the contained ∠ˢ equal, hence AB = GH, and the ∠ ABF = HGF, but ABF = FBI (const), BGI = GBI, and (vi) IB = IG, but EB is greater that IB, and IG greater than HG, EB is greater than GH, and we have proved BE = AC, and GH = AB Hence AC is greater than AB

2 Dem —Take any point D in the base BC of an isosceles Δ ABC Join AD Now the ∠ ADC is greater than ABD (xvi), and greater than ACD Hence (xix) AC is greater than AD

If we take the point D in the base produced, we have the ∠ ACB, that is, ABC greater than ADC, AD is greater than AB

3 Dem —This follows from the last exercise For when we took the point in the base, and joined it to the vertex, the joining line was less than either side of the triangle, and when the point was in the base produced, the joining line was greater

4 (1) Dem.—Let A be the given point, and EF the given line From A let fall a ⊥ AB, and draw any other line AC to EF The ∠ ACB is less than ABC (xvii), (xix) AC is greater than AB

(2) Dem.—Take another point D in EF Join AD Now the ∠ ACD is greater than ABC, and therefore obtuse, hence ADC must be acute, AD is greater than AC

5 Dem.—Because AB is greater than AC, the ∠ ACB is greater than ABC (xviii) Much more is the ∠ BCF greater than CBF Hence (xix) BF is greater than CF Again (hyp), AB is greater than BC, but AB = CF (iv), CF is greater than BC, (xviii) the ∠ CBF is greater than CFB, that is, than ABE Hence ABE or CFB is less than half ABC

PROPOSITION XX.

1 Dem.—Let ABC be a △ It is required to prove that the difference between two sides AB, AC, is less than BC From AC cut off AD = AB, and join BD Now AB and BC are greater than AD and DC, but AB = AD, BC is greater than DC, that is, greater than the difference between AB and AC

2 Dem.—Let D be any point within a △ ABC Join AD, BD, CD Now (xx) DA + DB > AB, DB + DC > BC, DC + DA > AC Adding, we get 2 (DA + DB + DC) > (AB + BC + CA), (DA + DB + DC) > ½ (AB + BC + CA)

3 Dem.—Let AD be the bisector of the ∠ BAC Take any point E in AD Join BE, CE From AB cut off AF = AC, and join EF Because AF = AC, and AE common, and the ∠ EAF = EAC, (iv) the base EF = EC Again, since EF = EC, the difference between BE and EC is equal to the difference between BE and EF, but BE − EF is less than BF (Ex 1), BE − EC is less than BF, but BF is the difference between BA and AC Hence the difference between BE and EC is less than the difference between BA and AC

4 Dem.—Produce BA to F, so that AF = AC Take any point E in the external bisector AD Join EB, EE, EF Now (iv) EF = EC To each add EB, and we have EF and EB = EC and EB, but EF and EB are greater than FB, that is, greater than AB and AC Hence EB and EC are greater than AB and AC

5 Dem —Let ABCD be the polygon Join BD Now (xx) AB + AD > BD, and BC + BD > CD, hence AB + AD + BC > CD

6 Dem —Let the Δ DEF be inscribed in ABC Now (xx) AD + AE > DE, EC + CF > EF, FB + BD > FD Adding, we get (AB + BC + CA) > (DE + EF + FD)

7 Dem —Let the polygon FGHJK be inscribed in the polygon ABCDE Now (xx) AF + AG > FG, BG + BH > GH, CH + CJ > HJ, DJ + DK > JK, EK + EF > KF Adding, we get the perimeter of ABCDE greater than that of FGHJK

8 Dem —Let ABCD be a quadrilateral, AC, BD, its diagonals Now, if AC, BD, are not equal, one of them must be the greater Let BD be the greater, then we have the sum of the sides AB, BC, CD, DA, greater than 2BD, and greater than AC and BD

9 Dem —Let ABC be the Δ, AD one of its medians Produce AD to E, so that ED = AD Join EC Now (iv) EC = AB, and (xx) AC and CE, that is, AC and AB, are greater than AE, that is, greater than 2AD Similarly BC and CA are greater than 2CG, and AB and BC are greater than 2BF, (AB + BC + CA) > (AD + BF + CG)

10 Dem —Let the diagonals AC, BD, of the quadrilateral ABCD intersect in E Take any other point F in the quadrilateral Join AF, BF, CF, DF Now (xx) BF + FD > BD, and AF + FC > AC Adding, we get (AF + BF + CF + DF) > (AC + BD)

PROPOSITION XXI

1 Dem —Let ABC be the Δ, and O any point within it Join OA, OB, OC Now, AB + AC > OB + OC (xxi), AC + BC > OA + OB, and AB + BC > OA + OC Adding, we get 2(AB + BC + CA) > 2(OA + OB + OC), (OA + OB + OC) < (AB + BC + CA)

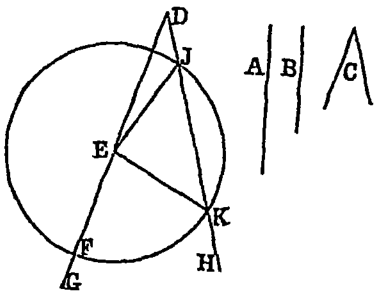
2 Dem —Produce BC both ways to meet AM, DN, in E, F Now (xx) AE + EB > AB, and DF + FC > DC To each add BC, and we have AE + EF + FD > AB + BC + CD Again, EM + MN + NF > EF (Ex 5, xx) To each add AE and DF, and we get AM + MN + ND > AE + EF + FD, but we have shown that AE + EF + FD > AB + BC + CD, AM + MN + ND > AB + BC + CD

PROPOSITION XXIII.

1 Sol.—Let A, B, be the given sides, and C the ∠ between them Draw any line DG, and from DG cut off DE = A At the point D in DG draw DH, making the ∠ GDH = C (xxiii) In DH take DF = B, and join EF DEF is the Δ required

2 Sol —Let AB be the given side, and D, E, the given ∠s. At the point A in AB make the ∠ BAC = D, and at the point B in AB make the ∠ ABC = E ABC is the Δ required

3 Sol —Let A, B, be the given sides, and C the given angle Draw any line DG, and in it make DE = A, and EF = B At the point D in DG make the ∠ GDH = C With E as centre,



and EF as radius, describe a ○, cutting DH in J, K Join EK, EJ Then evidently either of the Δs DEJ, DEK, will fulfil the given conditions

4 (1) Sol.—Let AB be the base, C the given ∠, and S the sum of the sides At the point A in AB make the ∠ BAF = C, and in AF take AE = S Join BE At the point B in BE make the ∠ EBG = BEG ABG is the Δ required

Dem —Because the ∠ EBG = BEG, (vi) EG = BG To each add AG, and we have AG + GB = AE, but AE = S (const), AG + GB = S

(2) Sol —Let AB be the base, C the given ∠, and D the difference of the sides At the point A in AB make the ∠ BAG = C, and let AG = D Produce AG to E Join BG, and at the point B in BG make the ∠ GBE = EGB AEB is the Δ required

Dem —Because the ∠ GBE = EGB, (vi) EG = EB, but AE − GE = AG, AE − BE = AG = D Hence the difference between AE and BE is D

5 (1) Let A, B, be two points, one of which, B, is in the given line GF It is required to find another point C in GF, such that CB + CA may be equal to a given line D

Sol —In GF take a part BE = D Join AE, and at the point A in AE make the ∠ CAE = CEA, then C is the required point

Dem —Because the ∠ CAE = CEA, CA = CE (vi) To each add CB, then CA + CB = BE, but BE = D, CA + CB = D. Hence C is the required point

(2) Let A, B, be the points, GF the given line

Sol —In GF take a part BG = D Join AG, and at the point A in AG make the ∠ GAE = AGE E is the required point

Dem —Because the ∠ GAE = AGE, GE = AE, AE − EB = GE − EB, but GE − EB = GB, that is, equal to D Hence AE − EB = D Since a part = D can be measured from B in either direction, there are evidently two solutions in each case

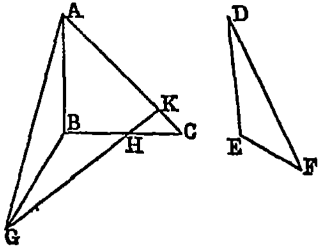
PROPOSITION XXIV

1 Dem —At the point A, in AB, make the ∠ BAH = EDF, and make AH = AC or DF Join BH Now (iv) BH = EF And because the ∠ BAC is greater than EDF, the bisector of the ∠ HAC must fall to the right of AB Let AG be the bisector Join HG Now since AH = AC, and AG common, and the ∠ HAG = CAG, (iv) GH = GC To each add BG, and we have BC = HG + GB, (xx) BC is greater than BH, that is, greater than EF

2 Dem (Diagram to Ex 1) —The ∠ AHG = ACG, but AHG is greater than AHB, ACG is greater than AHB, that is, greater than EFD

PROPOSITION XXV

Dem.—From BC cut off BH = EF On BH describe the



Δ BGH = DEF, that is, having BG = DE, and GH = DF Join

AG Because BA = DE, and BG = DE, BA = BG, (vi) the ∠ BGA = BAG Produce GH to meet AC in K Now since AC = DF, and GH = DF, AC = GH, GK is > AK, (xviii) the ∠ GAK is > AGK, but BAG = BGA, BAC is > BGH, that is, > EDF

PROPOSITION XXVI

1 Let ABC be the △

Dem —Let fall the ⊥ AD on BC Now (xxvi) the △s ADB, ADC, are equal, DB = DC Take any point E in AD Join BE, CE Now (iv) the △s BDE, CDE, are equal, BE = CE. Hence the point E is equally distant from the points B, C

2 Let AD bisect the vertical ∠ BAC, and also the base BC

Dem —Produce AD to E, so that DE = AD Join EC Now (iv) the △s ADB, EDC, are equal, AB = CE, and the ∠ BAD = CED, but BAD = CAD (hyp), CAD = CED, hence (vi) CE = CA, but CE = BA, CA = BA. Hence the △ BAC is isosceles

3 Let AB, AC, be two fixed lines, and D a point equally distant from them

Dem.—Let fall ⊥s DE, DF, on AB, AC Join EF, AD. Because DE = DF, the ∠ DFE = DEF, but the ∠ DFA = DEA; the ∠ AFE = AEF, and AE = AF Now AE = AF, AD common, and the base DE = DF, the ∠ EAD = FAD, the bisector of the ∠ BAC is the locus of the point D In like manner, if we produce BA to G, the locus of a point equally distant from AC, AG, will be the bisector of the ∠ CAG

4 Let AB be the given right line, and CD, EF, the other lines

Sol —Let CD, EF, intersect in G, and meet AB in H, J Bisect the ∠ HGJ by GK, meeting AB in K K is the point required

Dem —Let fall ⊥s KM, KN, on CD, EF Because the ∠ NGK = MGK, and GNK = GMK, and GK common, (xxvi) KN = KM There are evidently two solutions

5 Let ABC, DEF, be two △s, right-angled at A and D, having the base BC = EF, and the acute ∠ ABC = DEF

Dem —The △s ABC, DEF, have the ∠s BAC, ABC, equal to the ∠s EDF, DEF, and the side BC = EF, (xxvi) they are equal in every respect

6 Let the right-angled Δ^s ABC, DEF, have the sides AB, DE, equal, and also their hypotenuses BC, EF equal. It is required to prove that the Δ^s are equal in every respect

Dem —At the point B in BC make, on the side remote from A, the $\angle$ GBC = DEF (xxiii), and make BG = DE or AB Join CG, AG

Now the Δ^s GBC, DEF, have the sides GB, BC = DE, EF, and the $\angle$ GBC = DEF , (iv) CG = DF, and the $\angle$ BGC = EDF , but EDF is a right $\angle$, BGC is right, and = BAC Now BG = AB , the $\angle$ BAG = BGA , but BAC = BGC , CAG = CGA , hence CG = CA , but CG = DF , AC = DF Hence the Δ^s ABC, DEF, are equal in every respect

7 Let ABC be the Δ, and let the bisectors of the $\angle^s$ ABC, ACB, meet in O Join OA. It is required to prove that OA bisects the $\angle$ BAC

Dem —From O let fall $\perp^s$ OD, OE, OF, on AB, BC, CA Join DF The Δ^s OBD, OBE, are equal (xxvi) , OD = OE Similarly OE = OF , OF = OD, and (v) the $\angle$ ODF = OFD , but the $\angle$ ODA = OFA (const) , the $\angle$ ADF = AFD , (vi) AF = AD Now AF = AD, AO common, and the base OF = OD , hence (viii) the $\angle$ OAF = OAD Therefore AO is the bisector of the $\angle$ BAC

8 Let ABC be the Δ, and let BO, CO, bisecting the two external $\angle^s$ meet in O Join OA It is required to prove that OA bisects the $\angle$ BAC

Dem —From O let fall $\perp^s$ OD, OE, OF, on AB, BC, CA Join DF Now, as in the last exercise, OD = OF , the $\angle$ OFD = ODF , but the $\angle$ OFA = ODA , AFD = ADF, and AD = AF Now AD = AF, AO common, and the base OD = OF , the $\angle$ OAD = OAF Therefore AO bisects the $\angle$ BAC

9 Let A, B, C, be the given points It is required to draw a line through C, such that the $\perp^s$ on it from A, B, may be equal

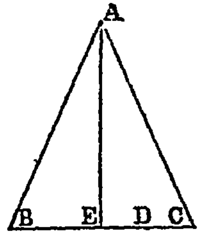
Sol —Join AB , bisect it in O Join CO, and produce it to D From A, B, let fall the $\perp^s$ AE, BF, on CD

Dem —Because AO = BO, and the $\angle^s$ AEO, AOE = BFO, BOF , (xxvi) AE = BF

10 Let AB, AC, be the given lines, and D the given point

Sol —Bisect the $\angle$ BAC by AE From D let fall a $\perp$ DE on AE, and produce it both ways to meet AB, AC, in B, C

Dem —The Δ^s ABE, ACE, have the $\angle^s$ AEB, EAB, equal to the ∠ˢ AEC, EAC, and the side AE common, ∴ the ∠ ABE = ACE Hence the Δ ABC is isoceles There are two solutions For if we produce BA to F, bisect the ∠ CAF by AG,



and from D let fall the ⊥ DH on AG, and produce it to meet AF in F, we will have another isosceles Δ

PROPOSITION XXIX

1 (1) Dem.—If AB, CD, are not ∥, let them meet in K Then we have the exterior ∠ EGK of the Δ GKH equal to the interior ∠ GHK, but this is impossible (xvi) Therefore AB, CD, must be ∥

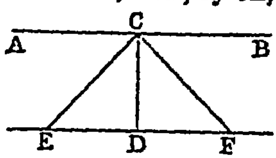
(2) If AB, CD, are not ∥, let them meet in K Then we have the ∠ˢ KGH, GHK, of the Δ GHK, equal to two right ∠ˢ, which is impossible (xvii) Hence AB, CD, must be ∥

2 Let AB, CD, be the ∥ lines, and AC, BD, the ⊥ˢ intercepted between them.

Dem.—Join AD Now, the ∠ ACD is right (hyp), and ABD, CDB, together equal two right ∠ˢ (xxix), but CDB is right, ∴ ABD is right, and hence = ACD, and the ∠ BAD = ADC (xxix) Therefore the Δˢ ABD, ACD, have two ∠ˢ of one equal to two ∠ˢ of the other, and the side AD common Hence (xxvi) BD = AC

3 Let EF be ∥ to AB

Dem —Bisect the ∠ˢ ACD, BCD, by CE, CF Now (xxix)



the ∠ ACE = DEC, but ACE = DCE, ∴ DEC = DCE, and ∴ DC = DE In like manner DC = DF Therefore DE = DF.

4 Let EF be the line whose middle point is O, and terminated by the ||ˢ AB, CD

Dem —Through O draw a line GH, meeting AB, CD in G, H The ∠ GOE = HOF (xv), and the ∠ GEO = OFH (xxix), and OE = OF (hyp), therefore (xxvi) OG = OH

5 Let AB, CD, be the ||ˢ, and O the point equidistant from them

Dem —Through O draw EF, meeting AB, CD, in E, F, and draw GH, JK, meeting them in G, H, J, K Because EF is bisected in O, (4) GH, JK, are bisected in O, then the Δˢ GOJ, HOK, have two sides GO, OJ, and the ∠ GOJ in one equal to the sides HO, OK, and the ∠ HOK in the other Hence (iv) GJ = HK

6 Let AEFD be the ▭ formed by drawing || lines FD, FE from a point F in BC to the sides AB, AC, of the equilateral Δ ABC

Dem —The ∠ EFB = ACB (xxix), EFB is an ∠ of an equilateral Δ, and EBF is an ∠ of an equilateral Δ (hyp), EBF is an equilateral Δ, EF = BF, but EF = AD, EF + AD = 2BF In like manner, AE + DF = 2CF Hence AE + AD + FE + FD = 2BC

7 Let ABCDEF be the hexagon, and let its diagonals AD, BE intersect in O Join CO, FO It is required to prove that CO, FO are in one straight line

Dem —The ∠ ABO = DEO (xxix), and the ∠ AOB = DOE (xv), and the side AB = DE (hyp), (xxvi) BO = EO Again (xxix) the ∠ CBO = FEO, and CB = EF (hyp), and we have shown that BO = EO, (iv) the ∠ BOC = EOF, to each add the ∠ FOB, and we have BOC + FOB = EOF + FOB, but EOF + FOB = two right ∠ˢ (xiii), BOC + FOB = two right ∠ˢ, and (xiv) CO, OF are in one straight line

PROPOSITION XXXI

1 Let A, B, be the given ∠ˢ, and H the altitude

Sol —Draw any line CD, and make the ∠ DCE = A, and the ∠ CDE = B, let fall a ⊥ EF on CD If EF = H, the Δ is constructed If not, produce it, and cut off EG = H Through G draw JK || to CD, and produce EC, ED, to meet it in J, K

Dem —The ∠ EJK = ECD (xxix) = A In like manner EKJ = B, and EG = H Therefore EJK is the Δ required.

2 Let AB be the given line, C the given point, and M the given ∠

Sol.—Through C draw CE ∥ to AB (xxx) At the point C in CE make the ∠ ECD = M The ∠ ECD = CDA (xxix) CDA = M

3 Dem —The ∠ CAD = ADE (xxix), but CAD = EAD (const), ADE = EAD, and EA = ED In like manner FB = FD Again, the ∠ CAB = DEF (xxix), but CAB is an ∠ of an equilateral Δ, DEF is an ∠ of an equilateral Δ Similarly DFE is an ∠ of an equilateral Δ, hence DEF is an equilateral Δ, DE = EF, but DE = AE, AE = EF In like manner BF = EF Hence AB is trisected.

4 Let ABC be the equilateral Δ

Sol —Let fall a ⊥ AD on BC Bisect the ∠ BAD by AE, meeting BC in E Through E draw EF ∥ to AD, meeting AB in F Through F draw FG ∥ to BC, and complete the ▭ EFGH EFGH is a square

Dem —The ∠ FEA = EAD (xxix), = FAE, FA = FE, but FAG is an equilateral Δ, because FG is ∥ to BC, AF = FG, but AF = EF, EF = GF, and EF = GH, and GF = EH, the four sides are equal, and (xxix) the ∠ GFE = BEF, but BEF is a right ∠, GFE is right. Hence EFGH is a square

5 (1) Let ABC be the Δ

Sol —Produce AB to G Bisect the ∠ GBC by BF, meeting AC produced in F Through F draw FG ∥ to BC

Dem.—The ∠ CBF = BFG (xxix), but CBF = GBF (const), GBF = BFG, and FG = BG If we bisect the ∠s BCF, ABC, or ACB we get in each case another solution

(2) Sol —Produce AB, AC to E, F Bisect the ∠s CBE, BCF, and through D, where the bisectors meet, draw EF ∥ to BC, meeting AE, AF, in E, F

Dem —The ∠ CBD = EDB (xxix), but CBD = EBD (const.), EDB = EBD, and (vi) EB = ED Similarly, FC = FD Hence EB + FC = EF

If we bisect the ∠s ABC, ACB, we have another solution.

(3) Sol.—Produce the base BC to G Bisect the ∠s ABC, ACG, by BD, CD Through D draw DF ∥ to BC, meeting AB, AC in F, E

Dem —The ∠ FDB = CBD (xxix), but CBD = FBD (const), FBD = FDB, and therefore FB = FD In like manner CE = DE Hence BF − CE = FD − DE = FE If we produce CB to H, and bisect the ∠s ACB, ABH, we will have another solution

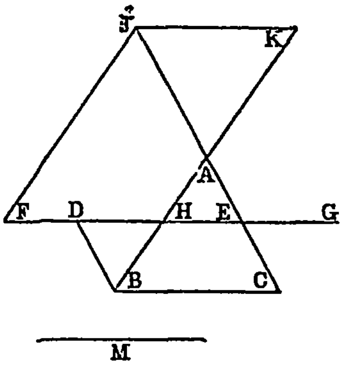
6 Let AB, DC be the ∥ lines, and B, D the given points

Sol.—Join BD, bisect it in E At E erect EA ⊥ to BD produce it to meet CD in C Join AD, BC

Dem —Because EB = ED, EA common, and the ∠ AEB = AED, (iv) AB = AD In like manner BC = DC, and the four sides are equal to each other Hence (Def xxix) ABCD is a lozenge

7 Let AB, AC be the lines given in position, M the line of given length, and FG the line to which the required line is to be ∥

Sol —(1) In FG take a part ED = M, through D draw DB ∥ to AC, and through B draw BC ∥ to DE BC fulfils the required conditions



Dem —Because DBCE is a ▭, BC = DE, but DE = M, BC = M.

Sol —(2) In FG take FH = M Through F draw FJ ∥ to AB, meeting CA produced in J, and through J draw JK ∥ to FH JK fulfils the required conditions

Dem —Because FJKH is a ▭, FH = JK, but FH = M, JK = M

PROPOSITION XXXII

1 Let ABC be the right ∠

Sol.—Make the ∠ ABD equal an ∠ of an equilateral Δ (XXIII), and draw BE bisecting it.

Dem —Because the ∠ ABD is an ∠ of an equilateral Δ, it is two-thirds of a right ∠, CDB is one-third, and half ABD is one-third Hence ABC is trisected

2 (1) Let ABC be the Δ

Dem —Draw the median AD Now if BD be greater than AD, the ∠ BAD will be greater than ABD (XVIII) Similarly the ∠ CAD will be greater than ACD Hence the ∠ BAC will be greater than ABC + BCA, and will be obtuse, when the side BC is greater than 2AD

(2) **Dem** —If BD = AD, the ∠ BAD = ABD , and if CB = AD, the ∠ CAD = ACD Hence the ∠ BAC is = ABC + BCA, and is right when BC = 2AD

(3) In like manner it can be shown that the ∠ BAC is acute, when BC is less than 2AD

3 Let ABCDE be the polygon.

Dem —Produce AB, DC to meet in A′, BC, ED to meet in B , &c

Now the sum of the ∠s of the Δ BA′C is two right ∠s, similarly the sum of the ∠s of each of the external Δs is two right ∠s Hence if there be n external Δs, the sum of their ∠s will be $2n$ right ∠s, but the sum of the exterior ∠s BCA′, CDB′, &c , is four right ∠s, and the sum of the exterior ∠s CBA , DCB′, &c , is four right ∠s Hence the sum of the remaining ∠s must be $(2n - 8)$ right ∠s, that is, $2(n - 4)$ right ∠s

4 Let BAC be the Δ

Dem —Produce BA to D, and bisect the ∠ CAD by the line AE ∥ to BC

The ∠ EAC = ACB (XXIX.), but EAC = EAD, and EAD = ABC , ACB = ABC And hence AB = AC

5 Let E be the point where CD cuts AB

Dem.—Bisect AB in F Join CF, DF Now the lines AF, BF, CF, DF are equal (XII , Ex. 2) And because FD = FB, the ∠ FBD = FDB = FDE + EDB , to each add the ∠ EDB , then the ∠s EBD + EDB = FDE + 2EDB , but the ∠ CEB = EBD + EDB (XXXII) , CEB = FDE + 2EDB, but CEB = FCE + CFE, and FCD = FDE , CFE = 2EDB Again, CFE = ACF + CAF, but ACF = CAF (v) CFE = 2CAF, 2CAF = 2EDB And hence CAF = EDB

6 Let ABC be the △

From B, C draw ⊥ˢ BD, CE to the sides AC, AB, and let them meet in G, join AG, and produce it to meet BC in F It is required to prove that AF is ⊥ to BC

Dem —Join DE. Now we have two right-angled △ˢ BEC, BDC, and we have joined their vertices E, D, hence (5) the ∠ EDB = ECB Similarly from the △ˢ AEG, ADG, the ∠ EAG = EDG (5), EAG = FCG, and AGE = CGF (xv), hence (Cor 2) the ∠ AEG = GFC, but AEG is a right ∠, CFG is right, and hence AF is ⊥ to BC

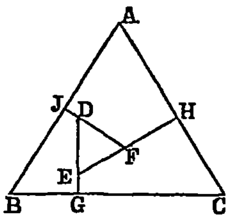
7 Let ABCD be the ▭, and BE, CE the bisectors of the adjacent ∠ˢ B, C It is required to prove that the ∠ BEC is right

Dem —The ∠ˢ ABC, DCB equal two right ∠ˢ (xxix), EBC + ECB equal a right ∠, and hence the ∠ BEC is right

8 Let ABCD be the quad Bisect the external ∠ˢ A, B, C, D, let the bisectors meet in E, F, G, H It is required to prove that the ∠ˢ EHG, EFG, of the quad EFGH, are together equal to two right ∠ˢ

Dem —Produce BA, CD to J, K Now the ∠ˢ ADC, ADK, DAB, DAJ equal four right ∠ˢ, and the ∠ˢ DHA, HAD, ADH equal two right ∠ˢ, the ∠ˢ of the △ HAD equal half sum of the ∠ˢ ADC, ADK, DAB, DAJ, but the ∠ˢ HAD, ADH are the halves of JAD, ADK, hence the ∠ DHA is half sum of BAD, ADC, in like manner BFC is half sum of ABC, BCD Hence the sum of the ∠ˢ DHA, BFC is half sum of the four ∠ˢ of the quad ABCD, and equal to two right ∠ˢ

9 Let the sides of the △ DEF be ⊥ to the sides of ABC It



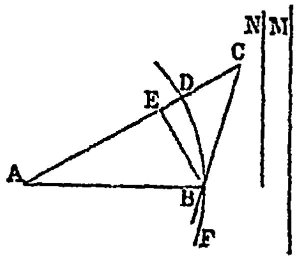
required to prove that the △ˢ DEF, ABC are equiangular

Dem —Since the ∠s CHE, EGC are right, the sum of the ∠s HCG + HEG = two right ∠s (Cor 3), and HED + HEG = two right ∠s Reject the common ∠ HEG, and we have the ∠ HCG = DEF, that is, the ∠ ACB = DEF In like manner the ∠ BAC = EFD, and ABC = EDF

10 (1) Let M equal sum of sides, and N the hypotenuse

Sol —Draw any line AC, and make it equal to M In AC take a part AD = N At the point C in AC make the ∠ ACB equal half a right ∠ With A as centre, and AD as radius, describe the ○ DBF, cutting CB in B Join AB, and at the point B in BC make the ∠ EBC = ACB AEB is the required Δ

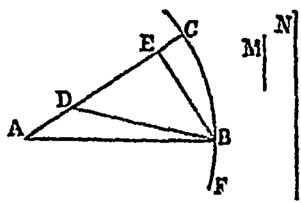
Dem —Because the ∠ EBC = ACB, EC = EB (vi) To each add AE, and we have AC = AE + EB, but AC = M (const),



AE + EB = M Again, the ∠ AEB = EBC + ECB (xxxii), but EBC = ECB, AEB = 2ECB, and is a right ∠

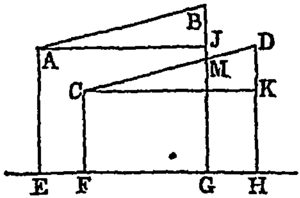
(2) Let M equal difference of sides, and N the hypotenuse

Sol —Draw any line AC = N In AC take AD = M At the point D in AC make the ∠ CDB = half a right ∠ With A as centre, and AC as radius, describe the ○ CBF, cutting DB in B From B let fall the ⊥ BE on AC Join AB AEB is the required Δ.



Dem —Because the ∠ AEB is right, and EDB half right, EBD is half right, and (vi) ED = EB Hence AD is the

Dem —Through A, C draw AJ, CK ‖ to EF



Now, because AJ, CK are each ‖ to EF, they are ‖ to one another, and AB is ‖ to CD , hence (xxix , Ex 8) the ∠ BAJ = DCK, also the ∠ AJB = CKD, because each is right, and the side AB = CD , (xxvi) AJ = CK , but AJ = EG, and CK = FH Hence EG = FH

(2) As in (1) the ∠s BAJ, AJB are respectively equal to the ∠s *DCK, CKD, and the side AJ = CK Hence AB = CD*

3 Dem —Since AB=CD, and AJ=CK, and the ∠ AJB = CKD, each being right , (xxvi , Ex 6) the △s ABJ, CDK, are equal in every respect , hence the ∠ ABJ = CDK , but CDK = CMG (xxix) , ABJ = CMG Hence AB is ‖ to CD

4 Let AB, CD be the equal and ‖ lines Join AD, BC, intersecting in E It is required to prove that AD, BC bisect each other in E

Dem —The ∠s ABE, BAE are respectively equal to the ∠s DCE, EDC, and the side AB = CD (hyp) Hence (xxvi) BE = CE, and AE = DE

PROPOSITION XXXIV

1 See last exercise to Prop xxxiii

2 Let ABCD be the ▭, AC, BD its diagonals, which are equal It is required to prove that the ∠s of ABCD are right ∠s

Dem —Because AD = BC, and AB common, and the bases BD, AC equal, (viii) the ∠ BAD = ABC , but (xxix) BAD + ABC equal two right ∠s , hence each is right, and (xxxiv) the ∠ BAD = BCD, and ABC = ADC Therefore all the ∠s are right ∠s

3 See "Sequel to Euclid," Prop xv , p 11, 6th Edition

4 Let AB, CD be two ∥ lines, of which AB is the greater Join AC, BD It is required to prove that AC, BD produced will meet

Dem —From BA cut off EB = CD Join EC Because EB is equal and ∥ to CD, (xxxIII) EC is equal and ∥ to BD, and (xxix) the ∠ AEC = ABD To each add the ∠ CAE, then CAE + AEC = CAE + ABD, but CAE and AEC are less than two right ∠s (xvII), hence CAE and ABD are less than two right ∠s And AC, BD, if produced, will meet

5 Let ABCD be a quad, having AB, CD ∥, but not equal, and AC, BD equal, but not ∥ It is required to prove that the ∠s CAB, CBD are supplemental

Dem —In CD take CE = AB Join BE Now (xxxIII) AC is = and ∥ to BE, but AC = BD (hyp), BE = BD, and (v) the ∠ BDE = EED, and (xxxIV) the ∠ CAB = CEB, hence the ∠s CAB + BDE = CEB + BED But CEB and BED are supplemental, hence CAB and BDE are supplemental

6 Let A, B, C be the middle points of the sides

Sol —Join AB, BC, CA, and through the points A, B, C draw DE, EF, FD ∥ to BC, AC, AB DEF is the required △

Dem —AB = CD (xxxIV), and AB = CF, hence CD = CF In like manner AD = AE, and BF = BE

7 Let ABCD be a quad, whose diagonals are AC, BD Through B, D, draw FG, EH ∥ to AC, and through C, A, draw GH, EF ∥ to BD Join FH It is required to prove that the area of the △ EFH is equal to the area of ABCD

Dem —The area of the △ EFH is half the area of the ▭ EFGH (xxxIV), and the area of ABCD is half the area of EFGH, EFH = ABCD, and the sides EF, EH are equal to BD, AC, and the ∠ FEH = AJD, which is the ∠ between AC, BD

PROPOSITION XXXVI

Dem —Produce AB, EF to meet in J Through J draw JK ∥ to AH or BG, and produce DC to meet it in K Join KG Now JK = BC (xxxIV), but BC = FG (hyp), JK = FG, and it is ∥ to it, hence JFGK is a ▭, JF is ∥ to KG, but JE is ∥ to GH Hence KG, GH are in one straight line, JEHK is a ▭ and it is equal to JADK (xxxv), but JBCK = JFGK Hence ABCD = EFGH

PROPOSITION XXXVII

1 See "Sequel to Euclid," Prop vi, p 4, 6th Edition

2 Let ABCD be a given quad It is required to construct a Δ equal in area to ABCD

Sol —Join AC Produce DC to E, and through B draw BE ‖ to AC Join AE ADE is the Δ required

Dem —The Δs ABC, AEC are equal (xxxvii) To each add the Δ ACD, and we have the Δ ADE equal to the quad ABCD

3 Let the pentagon ABCDE be the given rectilineal figure It is required to construct a Δ equal in area to ABCDE

Sol —Join AC, AD Through B, E draw BF, EG ‖ to AC, AD, and meeting DC produced both ways in F, G Join AF, AG AGF is the Δ required

Dem —The Δs ABC, AFC are equal (xxxvii), to each add ACDE, and we have the pentagon ABCDE equal to the quad AFDE Again (xxxvii), the Δ AGD = AED To each add the Δ ADF, and we have the Δ AGF equal to the quad. AFDE, but AFDE = ABCDE Hence AGF = ABCDE

4 Let ABCD be a given ▭ It is required to construct a lozenge equal to ABCD, and having CD as base.

Sol —If AD = DC, the thing required is done If not, let DC be the greater With D as centre, and DC as radius, describe a ○ ECG, cutting AB in E Join DE Through C draw CF ‖ to DE, meeting AB produced in F DEFC is the required lozenge

Dem —DE = DC, but DC = EF (xxxiv), DE = EF Hence the four sides are equal, DEFC is a lozenge, and (xxxv) is equal to ABCD

5 Let ABC be a Δ, whose base BC is given, and whose area is given. It is required to find the locus of its vertex A

Sol —Through A draw DE ‖ to BC DE is the required locus

Dem —Take any other point F in DE Join BF, CF Now (xxxvii) the Δs ABC, FBC are equal. Hence DE is the locus of the vertex of all Δs having BC as base, and whose area is equal to the area of the Δ ABC

6 Dem —Through E draw EG ‖ to FD, and meeting AD in G Join GF, GC Now (xxxvii) the Δ EFD = GFD, but GFD = GCD, and GCD is less than ACD, EFD is less than ACD, that is, is less than half ABCD

PROPOSITION XXXVIII.

1 Let ABC be the Δ, and AD one of its medians It is required to prove that AD bisects the Δ

Dem —BD = CD (Def Prop xx.), (xxxviii) the Δ ABD = ACD

2 Let ABC, DEF be two Δs, having the sides AB, BC equal to the sides DE, EF, and the contained ∠s supplemental It is required to prove that the Δs are equal

Dem —Produce CB to G, and make BG = BC or EF Join AG Now the ∠s ABC, DEF are supplements (hyp), and ABC, ABG are supplements (xiii.) Reject ABC, and we have ABG = DEF, hence (iv) the Δ ABG = DEF, but ABG = ABC (xxxviii) Hence DEF = ABC

3 **Dem** —Divide the base BC of the Δ ABC into any number, such as four equal parts, in the points D, E, F Join AD, AE, AF It is required to prove that the four Δs into which ABC is divided are equal

The Δ BAD = EAD (xxxviii) Similarly EAD = EAF, and EAF = CAF Hence the four Δs are equal

4 Let ABCD be a ▭ whose diagonals AC, BD intersect in F In BD take a point E Join EA, EC It is required to prove that the Δ ABE = CBE, and that ADE = CDE

Dem —AF = CF (xxxiv , Ex 1), hence (xxxviii) the Δ AFB = CFB, and AFE = CFE, hence AEB = CEB, but ABD = CBD, AED = CED

5 Let ABCD be a quad., and let AC, one of its diagonals, bisect the other, BD in E It is required to prove that AC bisects ABCD

Dem —The Δ AEB = AED (xxxviii), and the Δ CEB = CED Hence ABC = ADC

6 See "Sequel to Euclid," Prop xiii , p 10, 6th Edition

7 See "Sequel to Euclid," Prop xiii , p 10, Cor 1

8 See "Sequel to Euclid," Prop iii , Cor 1, p 2

9 Let ABC be a Δ, D, E the middle points of AB, AC, F any point in BC Join DE, EF, FD It is required to prove that DEF = $\frac{1}{4}$ ABC

Dem —Bisect BC in G Join DG, EG Now (xxxvii) the Δ DEF = DEG, but DEG = $\frac{1}{4}$ ABC (8) Hence DEF = $\frac{1}{4}$ ABC

10 Let ABC be a given Δ, and D a given point in BC It is required to draw a line through D, bisecting the Δ ABC

Sol.—Join AD Bisect BC in E Through E draw EF ‖ to AD, and meeting AB in F Join DF DF is the required line

Dem —Join AE Now (xxxvii) the Δˢ EFD, EFA are equal. To each add the Δ BEF, and we have the Δ BFD = BAE , but BAE = ½ BAC Hence BFD = ½ BAC

11 Let ABC be a given Δ, and D a given point within it It is required to trisect ABC by three lines drawn from D

Sol.—Trisect BC in E, F (xxxiv , Ex 3) Join AD, DE, DF Through A draw AG, AH ‖ to DE, DF Join DG, DH AD, DG, DH trisect ABC

Dem.—Join AE, AF Now (xxxvii) the Δˢ ADG, AEG are equal To each add the Δ AGB, and we have the quad ADGB equal to the Δ AEB , but AEB = ⅓ ABC (3) , hence ADGB = ⅓ ABC In like manner ADHC = ⅓ ABC , the Δ DGH = ⅓ ABC Hence the Δ ABC is trisected by the lines AD, GD, HD

12 Let ABCD be a ▭ whose diagonals AC, BD intersect in E Through E draw any line FG, meeting AB, CD in F, G It is required to prove that FG bisects ABCD

Dem —The ∠ BEF = GED (xv), and the ∠ FBE = GDE (xxix), and the side EB = ED (xxxiv , Ex 1), hence (xxvi , the Δˢ BEF, DEG are equal Similarly, AEF = CEG, and AED = CEB Hence FG bisects ABCD

13 Let ABCD be a trapezium Bisect AD in E Join EB, EC It is required to prove that the Δ BEC = ½ ABCD

Dem —Produce BE, CD to meet in F Now (xxvi) the Δ AEB = DEF, and EB = EF And since AEB = DEF, AEB + CED = CEF , but (xxxviii) CEF = BEC Hence BEC = AEB + CED

PROPOSITION XL

1 Let ABC, DEF be two Δˢ whose bases and altitudes are equal It is required to prove that the Δˢ are equal

Dem —Produce BC , and in BC produced cut off GH = EF or BC, and construct the Δ JGH, having its sides JG, GH, HJ respectively equal to the sides DE, EF, FD of the Δ DEF Join AJ , and from A, J let fall ⊥ˢ AL, JK on BH Because the Δ DEF = JGH, their altitudes are equal , but the altitudes of DEF and ABC are equal (hyp) , hence the altitudes of JGH and ABC are equal, that is, JK = AL, and they are parallel, hence (XXXIII.) AJ, BH are parallel, (XXXVIII.) the Δ ABC = JGH, but JGH = DEF. Hence ABC = DEF.

3. See "Sequel to Euclid," Prop. II., p. 2, 6th Edition.

4. See "Sequel to Euclid," Prop. III., Cor. 1, p. 2.

5. See "Sequel to Euclid," Prop. II., Cor., p. 2.

6. See "Sequel to Euclid," Prop. V., p. 3.

7. Let ABCD be a trapezium, whose opposite sides AD, BC are ∥, E, F the middle points of AB, DC. Join EF. It is required to prove that AD + BC = 2EF.

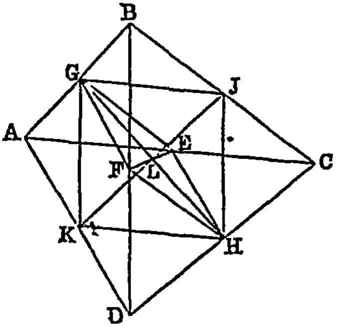
Dem.—Through A draw AH ∥ to DC, meeting EF, BC in G, H.

Now (XXXIV.) AD = GF, and HC = GF, ∴ AD + HC = 2GF, and (5) BH = 2EG. Hence AD + BC = 2EF.

8. See "Sequel to Euclid," Prop. III., Cor. 2, p. 3.

9. Let ABCD be a quad., AC, BD its diagonals. Bisect AC, BD in E, F. Join EF. Bisect AB, CD, BC, AD in G, H, J, K. Join GH, JK. It is required to prove that the lines EF, GH, JK are concurrent.

Dem.—Join EG, EH, FG, FH, GJ, GK, HJ, HK.



Now ((2) and (5)) GF is ∥ to AD, and = ½AD. Similarly, EH is ∥ to AD, and = ½AD, hence GF is = and ∥ to EH, ∴ (XXXIII.) GFHE is a ▭, hence (XXXIV., 1) the diagonal EF bisects GH in L. In like manner GJHK is a ▭, and the diagonal JK bisects GH. Hence the lines EF, GH, JK are concurrent.

PROPOSITION XLV

1 Let A and B be two rectilineal figures It is required to construct a rectangle equal to the sum of A and B

Sol —Construct a rectangular ▭ EFGH equal to A (XLV), and to the straight line GH apply a ▭ GHIK equal to B, and having the ∠ GHI a right ∠ FI is the required rectangle

Dem —The figure FI is equal to the sum of A and B, and it is evidently a rectangle

2 If we apply the rectangular ▭ GHIK to the left of GH, it is evident that EFKI will be the required rectangle

PROPOSITION XLVI

1 (1) Let AB, CD be equal lines Upon AB, CD describe squares ABEF, CDGH It is required to prove that ABEF = CDGH

Dem —Join AE, CG Now AB = BE, and CD = DG, but AB = CD, hence AB and BE = CD and DG, and the ∠ ABE = CDG, (IV) the Δ ABE = CDG, but ABEF = 2ABE, and CDGH = 2CDG Hence ABEF = CDGH

(2) Let ABEF = CDGH It is required to prove that AB = CD

Dem —If not, from AB cut off AJ = CD, and on AJ describe the square AJKL Now since AJ = CD, AJKL = CDGH, but CDGH = ABEF (hyp), AJKL = ABEF, which is absurd Hence AB = CD

2 Let ABCD be a square, and BD one of its diagonals In BD take a point E, and through E draw FG, HJ ∥ to AB, AD It is required to prove that HG, FJ are squares

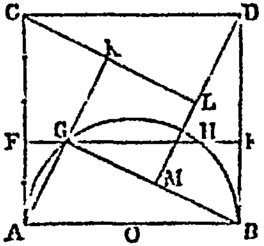
Dem —The ∠ ADB = ABD (V), but ADB = HEB (XXIX); ABD = HEB, hence the side HE = HB, but HB = EG, and HE = BG, HB, HE, GB, EG, are all equal Again, the ∠s EHB, GBH equal two right ∠s but GBH is right, EHB is right, and (XXXIV) the opposite ∠s are equal Hence EGBH is a square In like manner EJDF is a square

3 Let ABCD be a square, and E, F, G, H points in the sides AB, BC, CD, DA respectively equidistant from A, B, C, D Join EF, FG, GH, HJ It is required to prove that EFGH is a square

Dem —The Δ• AHE, BEF are equal in every respect (iv), the side EH = EF Similarly, EF = GF, and EH = GH Hence the four sides are equal Again, the ∠ AHE = BEF To each add the ∠ AEH, and we have the ∠• AHE, AEH equal to the ∠• BEF, AEH but AHE + AEH = a right ∠, since the ∠ at A is right, BEF + AEH = a right ∠ Hence the ∠ FEH is right In like manner the other ∠• are right, EFGH is a square Similar proof for other figures

4 Let ABCD be a square It is required to divide it into five equal parts, namely, four right angled Δ• and a square

Sol —Divide AC into five equal parts, and let AE = $\frac{2}{5}$ AC Through E draw EF ‖ to AB Upon AB describe the semicircle AGHB, cutting EF in the points G, H Join AG, and produce



it From C let fall a ⊥ CK on AK, and produce it Join BG From D let fall DM ⊥ to BG, meeting CK produced in L ABCD is divided into five equal parts

Dem —Join OG Because O is the centre of AGHB, OG = OA, (v) the ∠ OAG = OGA Similarly, the ∠ OBG = OGB Hence (xxxii, Cor 7) the ∠ AGB is right Again, since the ∠ AKC is right, the ∠• KCA, KAC are together equal to a right ∠, and therefore equal to the ∠ CAB, which is right Reject the ∠ KAC, and we have the ∠ KCA = KAB, and the ∠ CKA = AGB, because each is right, and the side AC = AB, hence (xxvi) the Δ AKC = AGB, AK = BG, and CK = AG In like manner it can be shown that the Δ• CLD, BMD are each equal to AGB Hence the four Δ• are equal, and the lines AK, BG, CL, DM are equal, and also the lines AG, BM, CK, DL, hence the remainders GK, GM, LK, LM, are equal Again, the rectangle ABEF is $\frac{2}{5}$ ABCD, and the Δ AGB is $\frac{1}{2}$ ABEF, AGB is $\frac{1}{5}$ ABCD, AKC, CLD, BMD are each $\frac{1}{5}$ ABCD Hence KGML must be $\frac{1}{5}$ ABCD, and it is a square, for we have proved the sides equal, and the ∠• are right ∠•

PROPOSITION XLVII

1 Dem —ACHK=AOLG, but AOLG is the rectangle AG AO, that is, AB AO, and ACHK is AC^2 Hence $AC^2 = AB$ AO Similarly, $BC^2 = AB$ BO

2 Dem —From GA cut off GM = GL, and draw MN ‖ to GL Now the figure AL = AH (XLVII), but $AH = AC^2 = AO^2 + OC^2$, and $GN = MN^2 = AO^2$, hence $OM = CO^2$, but OM = AO OB, since ON = OB Hence $CO^2 = AO$ OB

3 Dem —$AC^2 = AO^2 + OC^2$, and $BC^2 = BO^2 + OC^2$ Subtracting, we get $AC^2 - BC^2 = AO^2 - BO^2$

4 Let AB, CD be the lines whose squares are given It is required to find a line whose square shall be equal to the sum of the squares on AB and CD

Sol —Erect AE ⊥ to AB, and make it equal to CD Join BE Now (XLVII) $BE^2 = AB^2 + AE^2 = AB^2 + CD^2$

5 Let ACB be a △ whose base AB is given, and the difference of the squares of its sides It is required to prove that the locus of C is a right line ⊥ to AB

Dem —From C let fall a ⊥ CO on AB Now (3) $AC^2 - BC^2 = AO^2 - BO^2$, but $AC^2 - BC^2$ is given, $AO^2 - BO^2$ is given, and O is a given point, the line OC is given in position Hence OC is the locus of C

6 Dem —Let P, Q be the points in which AC, GC intersect BK Now (IV) the △s CAG, BAK are equal in every respect, the ∠ ACG = AKB, and the ∠ CPQ = APK (XV), (XXXII, Cor 7) the ∠ CQP = KAP, CQP is a right ∠, and CG is ⊥ to BK

7 See "Sequel to Euclid," Book I, Prop XXIII (3)

8 Dem —Since EB = AH, AB = AE + AH, and AC is the square on AB, ∴ AC is equal to the square on the sum of AE and AH, but AC exceeds EG by four times the △ AEH, and EG is the square on EH, hence the square on the sum of AE and AH exceeds the square on EH by four times the △ AEH

9 Dem —Join PH, QC Now (XXXVII) the △ PCQ=PBQ To each add APQ, and we have the △ ACQ = APB Again, the sum of the △s KAP, HCP equals ½ KC, and the △ KAB = ½ KC (XLI), ∴ KAB = KAP and HCP Reject the △ KAP, and we have the △ APB = HCP, but APB = AQC, hence HCP = AQC, and their bases HC, AC are equal Hence (XL) their altitudes PQ, PC are equal

10 See "Sequel to Euclid," Book I, Prop XXIII (2)

11 Let M, N be two lines It is required to find a line whose square shall be equal to $M^2 - N^2$.

Sol —Draw a line $AB = M$, and in it take $AC = N$ Erect CE ⊥ to AB With A as centre, and AB as radius, describe a ○ cutting CE in D CD is the required line

Dem —Join AD Now $AD^2 = AC^2 + CD^2$, ∴ $CD^2 = AD^2 - AC^2 = AB^2 - AC^2 = M^2 - N^2$

12 Dem —From AC cut off $AD = BC$, then, evidently, CD is the difference between AC and CB. On AB describe a square ABFG, and on CD describe a square CDEH, and produce DE, EH to meet ABFG in G, F (figure similar to that on p 89, "Elements")

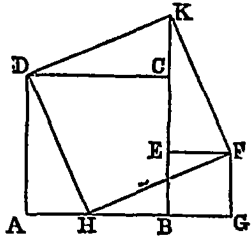
Now CE is less than AF by the sum of the four △s, that is, by four times the △ ABC Hence $CD^2 + 4ABC = AB^2$

13 Dem —Join CF, CG, cutting AE, BK in P, Q Through A draw AM ∥ to GC, cutting BK in R, and meeting LC produced in M Join BM, cutting AE in N Now, because AM is ∥ to GC, and AG to ML, AGCM is a ▭, ∴ AG = CM, but AG = BF, ∴ BF = CM, ∴ FCMB is a ▭, ∴ CF is ∥ to BM, hence (XXIX) the ∠ ANM = APC, but APC is a right ∠ (6), ∴ ANM is right, and AN is ⊥ to BM In like manner BR is ⊥ to AM, and CM being ⊥ to AB, ∴ AN, BR, CM are the ⊥s of the △ AMB, ∴ (XXXII, Ex 6) these lines are concurrent, that is, the lines AE, BK, CL are concurrent

14 Let ABC be an equilateral △ Let fall a ⊥ AD on BC

Dem —$AB^2 = AD^2 + BD^2$ (XLVII), ∴ $4AB^2 = 4AD^2 + 4BD^2$, but $AB^2 = 4BD^2$, since $AB = BC = 2BD$ Subtracting, we get $3AB^2 = 4AD^2$

15 Sol —In AB take AH = BG Join DH, FH These lines



divide the figure into the parts required

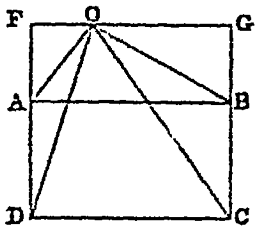
Dem —For if we take the Δ AHD and place it in the position DCK, and place the Δ FHG in the position FKE, the figure HFKD will be equal to the figure AGFECD, and it is evidently a square

16 Let AB be the hypotenuse of the right-angled Δ ACB. Bisect BC, AC in D, E Join AD, BE It is required to prove that $4AD^2 + 4BE = 5AB^2$

Dem —$4\,AD^2 = 4\,AC^2 + 4\,CD^2$; but $BC^2 = 4\,CD^2$. $\therefore$ $4\,AD^2 = 4\,AC^2 + BC^2$ Similarly, $4\,BE^2 = 4BC^2 + AC^2$. Adding, we get $4\,(AD^2 + BE^2) = 5\,(AC^2 + BC^2) = 5\,AB^2$.

17 Let ABC be a Δ, and O a point within it. Through O draw $\perp$s OD, OE, OF to BC, CA, AB It is required to prove that $AF^2 + BD^2 + CE^2 = BF^2 + DC^2 + EA^2$ Now (2) $AF^2 - BF^2 = AO^2 - BO^2$, $BD^2 - CD^2 = BO^2 - CO^2$, and $CE^2 - AE^2 = CO^2 - OA^2$ Adding, we get $AF^2 + BD^2 + CE^2 - (BF^2 + DC^2 + EA^2) = 0$, and hence $AF^2 + BD^2 + CE^2 = BF^2 + DC^2 + EA^2$. Similarly for a figure of any number of sides.

18 Let ABCD be a rectangle and O any point Join OA, OB, OC, OD It is required to prove that $OA^2 + OC^2 = OB^2 + OD^2$



Dem —Produce DA, CB to F, G, and let fall $\perp$s OF, OG on DF, CG

Now, $OD^2 = DF^2 + OF^2$, and $OA^2 = AF^2 + OF^2$, $\therefore$ $OD^2 - OA^2 = DF^2 - AF^2$ Similarly, $OC^2 - OB^2 = CG^2 - GB^2$, but $DF^2 = CG^2$, and $AF^2 = GB^2$, $\therefore$ $OD^2 - OA^2 = OC^2 - OB^2$, and, by transposition, we have $OD^2 + OB^2 = OC^2 + OA^2$.

19 Let AB be the hypotenuse of a right-angled Δ ABC It is required to divide it into two parts, such that the difference of their squares shall equal AC^2.

Sol —Bisect BC in D Join AD, and let fall the $\perp$ DE on AB $AE^2 - BE^2 = AC^2$.

Dem.—$AD^2 - BD^2 = AE^2 - BE^2$ (3) that is, $AC^2 + CD^2 - BD^2 = AE^2 - BE^2$, but $CD^2 = BD^2$ (const), $\therefore$ $AC^2 = AE^2 - BE^2$.

20 Let ABC be the △ From B, C let fall ⊥ˢ BE, CD on AC, AB It is required to prove that AB . BD + AC . CE = BC²

Dem —On BC describe a square BCFG Produce BE, CD to H, J, and through B, C draw BL, CK ∥ to DJ, EH and make BL = AB, and CK = AC Complete the ▭ˢ BLJD, CKHE Draw AM ∥ to CF, meeting GF in M Now it can be shown, as in (XLVII), that BM = BJ, and CM = CH, ∴ BF = BJ + CH, but BF = BC², BJ = AB . BD, and CH = AC . CE Hence BC² = AB . BD + AC . CE

Miscellaneous Exercises on Book I

1 See "Sequel to Euclid," Book I, Prop III, Cor 1.

2 Let DEF be the original △, ABC the △ formed by drawing through each vertex a ∥ to the opposite side Let fall a ⊥ FG on DE It is required to prove that GF bisects BC perpendicularly

Dem —The ∠ CFG = DGF (XXIX), but DGF is right, ∴ CFG is right Again, BF = DE (XXXIV), and CF = DE, ∴ BF = CF Hence GF bisects BC perpendicularly Similarly, the ⊥ˢ from D, E on EF, DF bisect AB, AC perpendicularly

3 Let ABC be a given ∠, and D a given point It is required to draw a line through D, so that the parts DA, DC, intercepted by AB, BC, may be equal

Sol —Through D draw DE ∥ to AB, meeting BC in E, and make EC = BE Join CD, and produce it to meet AB in A

Dem —AC is bisected in D (XL, Ex 3)

4 Let BD, CE, two of the medians of the △ ABC, intersect in H Join AH, and produce it to meet BC in F It is required to prove that AF is the third median

Dem —Produce AF to G, draw BG ∥ to EH, and join GC Now (XL, Ex 3) AG is bisected in H, and in the △ AGC, HD is ∥ to GC (XL, Ex 2), hence BHCG is a ▭, and ∴ (XXXIV, Ex 1) BC is bisected by HG, in F Hence AF is a median of the △ ABC

5 See "Sequel to Euclid," Book I, Prop IV, Cor

6 Let a, b be the two sides, and c the median of the third side It is required to construct a △ having two sides respectively equal to a and b, and the median of the third side equal to c

Sol —Construct the △ ABC, having AB = a, AC = b, and BC = $2c$ Bisect BC in D Join AD, and produce it until DE = AD Join EC ACE is the required △

Dem —The △ˢ ADB, CDE are equal (IV) in every respect;

AB = CE, but AB = a, ∴ CE = a, and AC = b, and BC = $2c$, ∴ CD = c

7 (1) See xx, Ex 9

(2) Let a, b, c be the sides of the Δ, and α, β, γ the medians

Dem —$\frac{2}{3}\beta + \frac{2}{3}\gamma > a$ (Ex 5) In like manner $\frac{2}{3}\gamma + \frac{2}{3}\alpha > b$, and $\frac{2}{3}\alpha + \frac{2}{3}\beta > c$ Adding, we have $\frac{4}{3}(\alpha + \beta + \gamma) > (a + b + c)$, and therefore $(\alpha + \beta + \gamma) > \frac{3}{4}(a + b + c)$

8 Let a be the side, and b, c, the medians It is required to construct a Δ, having a side equal to a, and the medians of the remaining sides equal to b, c

Sol.—Construct a Δ ABC (xxii), having BC (the base) = a, AB = $\frac{2}{3}b$, and AC = $\frac{2}{3}c$ Bisect BC in D Join DA, and produce to E, so that AE = 2AD BEC is the required Δ

Dem —Produce BA, CA to meet CE, BE in F, G Now ED is a median of the Δ EBC (const), ∴ (4) BF, CG are medians, hence (5) BA = $\frac{2}{3}$ BF, but BA = $\frac{2}{3}b$, ∴ BF = b Similarly, CG = c

9 Let a, b, c be the medians of a Δ It is required to construct it

Sol.—Construct a Δ ABC, having AB = $\frac{2}{3}a$, BC = $\frac{2}{3}b$, and CA = $\frac{2}{3}c$ Bisect BC in D Join AD, and produce it to E, so that DE = AD Produce CB to F, and make BF = BC Join AF, EF AFE is the Δ required

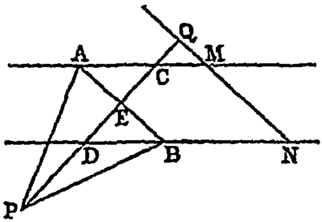
Dem —Join EB, and produce it to meet AF in H Produce AB to meet EF in G Join CE Now since AD = DE, and BD = CD, ABEC is a ▭, ∴ BH is ∥ to AC Hence (xl, Ex 3) AF is bisected in H Similarly, FE is bisected in G, and (const) AE is bisected in D, ∴ (Def) AG, DF, EH are the medians, hence (Ex 5) AB = 2BG but AB = $\frac{2a}{3}$, ∴ AG = a In like manner it can be shown that FD = b, and EH = c

10 Let ABC be the Δ having AC > AB, and from AC cut off AD = AB, and join BD Let fall AE ⊥ to BC, meeting BD in G, and bisect the ∠ BAC by AF meeting BD in F

Dem —The ∠ AFG is right (iv, Ex 1), and GEB is right, and the ∠ AGF = BGE (xv), ∴ the ∠ GAF = GBE, but GBE = $\frac{1}{2}$(ABC − ACB) (xxxii, Ex 13), ∴ GAF = $\frac{1}{2}$(ABC − ACB)

11 Let AM, BN be the two ∥ lines, and P the given point It is required to find in AM, BN two points equidistant from P, and whose line of connexion shall be ∥ to a given line MN

Sol —From P let fall a ⊥ PQ on MN Bisect the part CD



between AM, BN in E Through E draw AB ∥ to MN A, B are the required points

Dem —Join AP, BP Now the ∠ PEB = PQN (xxix), but PQN is a right ∠, PEB is right, and since CD is bisected in E, (xxix, Ex 4) AB is bisected in E Now AE = BE, and EP common, and the ∠ AEP = BEP, (iv) AP = BP

12 Let a be the side, and b, c the two diagonals

Sol —Construct the △ AEB, having AB = a, AE = $\frac{1}{2}b$, and BE = $\frac{1}{2}c$ Produce AE, BE to C, D, so that CE = AE, and DE = BE Join CD, AD, BC ABCD is the required ▭

Dem —The side AB = a, and AC, BD = b, c

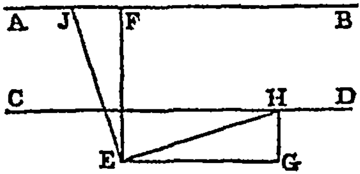
13 Let ABC be a △, having the side AB greater than AC It is required to prove that BE, the median of AC, is greater than CF, the median of AB

Dem —Let BE, CF intersect in G Join AG, and produce it to meet BC in D AD is the median of BC Now because BD = CD, AD common, and the base AB greater than AC, (xxv) the ∠ ADB is greater than ADC Again, BD = CD, GD common, and the ∠ BDG greater than CDG, (xxiv) BG is greater than CG, but BG = $\frac{2}{3}$ BE, and CG = $\frac{2}{3}$ CF (5) Hence BE is greater than CF

14 Let AB, CD be two ∥ lines, and E a given point. It is required to find in AB, CD two points that shall subtend a right angle at E, and be equally distant from it.

Sol —From E let fall a ⊥ EF on AB Draw EG ∥ to AB, and make it equal to EF From G draw GH ⊥ to CD In AB take = GH H, J are the required points

Dem —Join EH, EJ Because EF = EG, and FJ = GH, and the ∠ EFJ = EGH, (IV) EJ = EH, and the ∠ FEJ = GFH. To each add the ∠ FEH, and we have the ∠ JEH = FEG, but FEG is a right ∠ Hence JEH is right



15 Let ABC be an isosceles Δ, and D a point in the base BC From D let fall ⊥s DE, DF on AB, AC From B let fall a ⊥ BG on AC It is required to prove that BG = DE + DF

Dem —From D draw DH ∥ to AC, meeting BG in H Now (XXIX) the ∠ HDB = ACD, but ACD = ABD (hyp), HDB = EBD, and the ∠ BHD = BED, each being right, (XXVI) BH = DE, but HG = DF (XXXIV) Hence BG = DE + DF

16 Let ABC be the Δ At the middle points G, F of AB, AC erect ⊥s to those sides meeting at O Join O to E the middle point of BC It is required to prove that OE is ⊥ to BC BO = OC, since each is = OA (IV), ∴ ∠ OBE = OCE (V), and (IV) ∠ OEB = OEC, and ∴ OE is ⊥ to BC Hence prop is proved For second part see "Elements," 11th Edition, Book IV, Prop V, Ex I

17 Let ABC be the Δ Bisect the ∠ BAC by AD, meeting BC in D From D draw DE, DF ∥ to AB, AC AEDF is an inscribed lozenge

Dem —The ∠ EAD = ADF (XXIX), but EAD = FAD (const), ADF = FAD, and AF = DF Similarly, AE = DE, but (XXXIV) AF = DE, and AE = DF. Hence the four sides AF, DF, AE, DE are equal, AEDF is a lozenge

18 See "*Sequel to Euclid,*" Book I, Prop XIV, 6th Edition

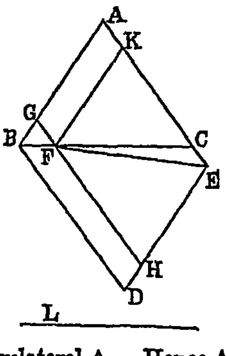
19 (1) Let AB, AC be two fixed lines, and P the point Let fall ⊥s PD, PE on AB, AC, then, being given the sum of PD and PE, it is required to find the locus of P

Dem.—Produce EP to F, and make PF = PD Through F draw GF ∥ to AC, meeting AB in G Join GP, and produce it both ways, GP is the required locus Because PF = PD, to each add PE, and we have EF = PD + PE, EF is given,

·om AC, GF is given in
PFG, PDG is right, PF^2
const), $GF^2 = GD^2$;
nmon, and the base PF
Then, since AB, GF are
between them, GP is

id P any point within it
C, CA, and from A let fall
hat PD + PE + PF = AK

Dem —Through P draw GH ∥ to BC, meeting AB, AC, AK in G, H, L, and from G let fall a ⊥ GJ on AC Now the ∠ AGH = ABC (xxix), . AGH is an ∠ of an equilateral Δ Similarly,



AHG is an ∠ of an equilateral Δ Hence AGH is an equilateral Δ, AL = GJ, but GJ = PD + PF (Ex 15), AL = PD + PF, and PE = LK Hence AK = PD + PE + PF

21 See "Sequel to Euclid," Book I, Prop xi 6th Edition

22 See "Sequel to Euclid," Book I, Prop xi, Cor 1

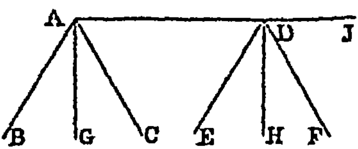
23 Let ABC be a Δ, and L a given length It is required to find a point F in BC, such that if FK, FG, be drawn ∥ to AB, AC, the sum of AG, AK shall be equal to L

Sol —From B draw BD ∥ to AC, and make it = L From D draw DE ∥ to AB, and produce AC to meet it in E Bisect the ∠ AED by EF, meeting BC in F F is the point required

Dem —Through F draw GH ∥ to BD, and FK ∥ to AB Now the ∠ HEF = KEF (const), and (xxix) the ∠ KEF = EFH, ∴ EFH = HEF, and ∴ HE = HF, but HE = FK, ∴ FK = FH To each add FG, and we have FK + FG = GH, that is, AG + AK = GH, but GH = BD = L Hence AG + AK = L

24 (1) Let BAC, EDF be two ∠s whose legs AB, DE, AC, DF are respectively ∥ Bisect BAC, EDF by AG, DH It is required to prove that AG, DH are ∥

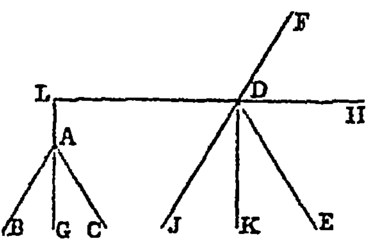
Dem —Join AD, and produce it to J Now (xxix) the ∠ JDE = JAB, and JDF = JAC, ∴ FDE = CAB, hence FDH = CAG



And it has been shown that JDF = JAC, ∴ JDH = JAG Hence (xxviii) DH is parallel to AG

(2) Let BAC, EDF be the ∠s Bisect BAC, EDF by AG, DH Produce GA, HD to meet in L It is required to prove that HL is ⊥ to GL

Dem —Produce FD to J, and bisect the ∠ JDE by DK Now



the ∠ FDH = EDH, and JDK = EDK, hence HDK = half sum of JDE and EDF, but JDE and EDF = two right ∠s, HDK is a right ∠, and HDK = HLG, HLG is right And hence HL is ⊥ to GL

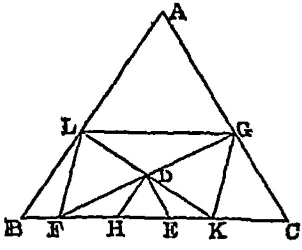
25 Let ABC be the Δ of which A is the vertex, produce BA, CA to D, E Bisect the ∠s CAD, BAE, by the line FAG From B, C let fall ⊥s BG, CF, on GF Bisect the ∠ BAC by AH Join BF, CG It is required to prove that BF, CG meet on AH

Dem —Produce CF to meet AD in D Now the ∠ CAF = DAF, and CFA = DFA, and AF is common, (xxvi) CF = DF, and because the ∠ DFA = HAF, each being right, AH is ∥ to CD Now, since F is the middle point of the base CD of the Δ CBD, and BF joined, and AH ∥ to CD, (xxviii, Ex. 7), BF bisects AH In like manner CG bisects AH Hence BF, CG meet on AH

26 Dem —From the vertices A, B, C, of the Δ ABC, let fall ⊥s AD, BE, CF on the opposite sides, let them intersect in G Join DE, EF, FD It is required to prove that the ⊥s AD, BE, CF bisect the ∠s EDF, DEF, and EFD

Now the ∠ CDE = CGE (xxxii, Ex 5), and BDF = BGF, but (xv) CGE = BGF, CDE = BDF, and CDA = BDA, since each is right, EDA = FDA, hence the ∠ EDF is bisected by AD In like manner the ∠s DEF, EFD are bisected by BE and CF

27 Let ABC be a given Δ, and D a given point within it. It is required to inscribe, in ABC, a ▭ whose diagonals shall intersect in D

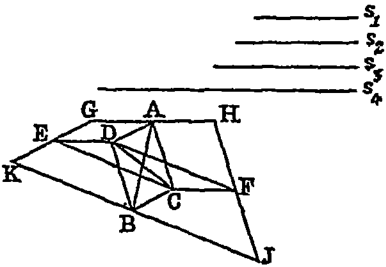


Sol.—Through D draw DE ∥ to AC, and from EB cut off EF = EC Join FD, and produce it to meet AC in G Draw DH ∥ to AB, and from HC cut off HK = BH Join KD, and produce it to meet AB in L Join GL, FL, GK GLFK is the required ▭

Dem —FG is bisected in D (XL, Ex 3) Similarly, KL is bisected in D Hence (XXXIV, Cor 5) GLFK is a ▭

28 Let s_1, s_2, s_3, s_4 be the sides of the quadrilateral, and A, B the middle points of two opposite sides It is required to construct it.

Sol.—Join AB, and on it describe the Δ ACB, having BC $= \frac{1}{2}s_1$, and CA $= \frac{1}{2}s_3$ Complete the ▭ ADBC Join DC, and describe the Δ CDE, having DE $= \frac{1}{2}s_2$, and CE $= \frac{1}{2}s_4$ Complete the



▭ DECF Through A, E, B, F draw HG, GK, KJ, JH ∥ respectively to DE, BC, CE, CA GHJK is the required quadrilateral

Dem —HF = AC (XXXIV), and JF = BD, but AC + BD = 2 AC, hence HJ = s_3 In like manner GH = s_2, GK = s_1, and JK = s_4

29 See "Sequel to Euclid," Book I, Prop VIII

30 Let ABC be the given rectilineal figure, and O the given point. From O let fall ⊥ˢ on BC, CA, AB, and let them be denoted by p, p_1, p_2, then, if $p + p_1 + p_2$ be given, it is required to prove that the locus of O is a right line

Dem —In BC take a part EF, equal to any given line Join OE, OF In AC, AB take GH, JK, each equal to EF Join OG, OH, OJ, OK. Now let EF be denoted by b, and we have $bp = 2$ Δ OEF (II I Cor 1), and, similarly, for the Δˢ OGH, OJK Therefore $b(p + p_1 + p_2)$ is equal to twice the sum of the areas of those Δˢ, but the bases, and sum of the areas, are given Hence (Ex 29) the locus of O is a right line

31 Dem —Through C and B′ draw CD, B D ∥ to BB′ and BC Join DC′, cutting BC in E Now (XXXIV) BB′ = CD, but BB′ = CC′ (hyp), ∴ CD = CC′, and CE is common, and

the ∠ ACB = DCB, because each is equal to ABC, hence (IV) the ∠ CEC′ = CED, each is a right ∠, (XXIX.) B′DE is right, hence B C′D is acute, and (XIX) B C′ is greater than B D, that is, greater than BC

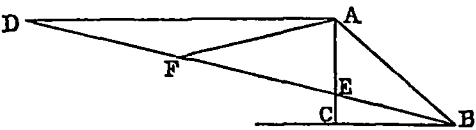
32 (1) Dem —From B let fall a ⊥ BC on L, and produce it to meet AP in Q In L take any other point S Join AS, BS, QS Now, because BCP = QCP, and the ∠ BPC = QPC, and CP common, (XXVI) BP = QP Similarly, BS = QS Hence AS − SQ = AS − SB, but AS − SQ is less than AQ, AS − SB is less than AQ, that is, less than AP − BP

(2) See "Sequel to Euclid," Book I, Prop XXI

33 Let ABCD be a quadrilateral It is required to bisect it by a line drawn from A, one of its angular points

Dem —Join AC Produce DC to E Through B draw BE ∥ to AC Join AE Bisect DE in F Join AF AF bisects ABCD Now the Δ AEC = ABC (XXXVII) To each add the Δ ACD, and we have the Δ AED = the quadrilateral ABCD, but AED = 2 ADF (XXXVIII), ABCD = 2 ADF

34 Dem —Bisect ED in F Join AF Now (XII, Ex 2), the lines EF, AF, DF are equal, hence the ∠ FAD = FDA,



but (XXXII) the ∠ AFE = FAD + FDA, AFE = 2 FDA, and (XXIX) = 2 DBC, but AF = AB, because each is equal to ½ ED, the ∠ ABF = AFB, but AFB = 2 DBC, ABF = 2 DBC Hence DBC = ⅓ ABC

35 Dem —The three ∠s ABC, BCA, CAB are equal to two right ∠s, ABO, BAO, BCO are equal to a right ∠, but BOD = ABO + BAO, BOD and BCO equal a right ∠, and EOC + BCO equal a right ∠, hence BOD + BCO = EOC + BCO, the ∠ BOD = EOC

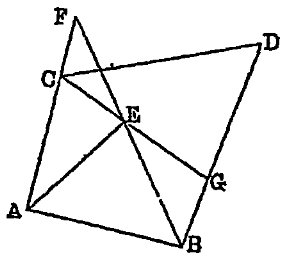
36 The angles of each external Δ are respectively equal to ½ (B + C), ½ (C + A), ½ (A + B) See XXXII, Ex 14 Hence the three external Δs are equiangular

37 (1) Dem —Let ABCD be the quadrilateral Bisect the ∠s BCD, CDA by CE, DE It is required to prove that the ∠ CED = ½ (DAB + ABC)

Now the ∠s DAB, ABC, BCD, CDA are together equal to four right ∠s, and the ∠s CED, EDC, DCE are equal to two right ∠s, hence (CED + EDC + DCE) = $\frac{1}{2}$ (DAB + ABC + BCD + CDA), but EDC = $\frac{1}{2}$ ADC, and DCE = $\frac{1}{2}$ DCB Hence CED = $\frac{1}{2}$ (DAB + ABC)

(2) Bisect the ∠s ABD, ACD by BE, CE Produce BE, CE to meet AC, BD in F, G It is required to prove that the ∠ CEF = $\frac{1}{2}$ (BAC − BDC)

Dem —Join AE Now the ∠s of the figure ABEC are equal to



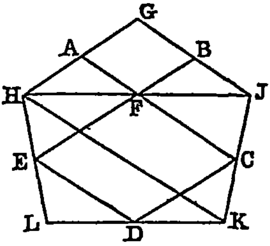
four right ∠s, and the ∠s of the figure BECD are equal to four right ∠s, hence the ∠s (BAC + ABE + BEC + ACE) = (BEG + GEF + FEC + ECD + CDB + DBE), but ABE = DBE, and ACE = ECD, and BEC = GEF Reject these, and we have BAC = CDB + GEB + CEF = CDB + 2 CEF Hence the ∠ BAC exceeds CDB by 2 CEF, that is, CEF = $\frac{1}{2}$ (BAC − CDB)

38 Dem —It has been proved (XLVII, Ex 7) that $EF^2 = AC^2 + 4 BC^2$ Similarly, $KG^2 = BC^2 + 4 AC^2$ Adding, we get $EF^2 + KG^2 = 5 (AC^2 + BC^2) = 5 AB^2$

39 Let A, B, C, D, E be the middle points of the sides of a convex polygon of an odd number of sides It is required to construct it

Sol —Join CD, DE, and through C, E draw CF, EF ∥ to DE, CD, and (XXXIV, Ex 6) construct the Δ GHJ, having A, F, B for the middle points of its sides, GH, HJ, JG Join JC, and produce JC to K, so that CK = CJ Join KD, HE, and produce them to meet in L GHLKJ is the required polygon

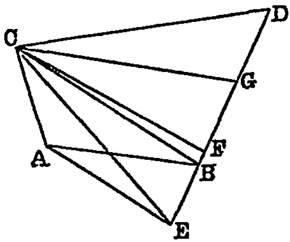
Dem —Join HK Now in the Δ HJK, HJ, JK are bisected in F, C, hence (XL, Exercises 2 and 5) FC is ∥ to HK, and equal to half of it, but FC = ED, ED is ∥ to HK, and



equal to half HK And hence (xl, Ex 3) HL, LK are bisected in E, D

40 Let ABDC be a quadrilateral It is required to trisect it by lines drawn from C, one of its angular points

Sol —Join BC Produce DB to E, and draw AE ∥ to BC Join CE Trisect ED in F, G (xxxiv, Ex 3) Join CF, CG. CF, CG trisect the quadrilateral



Dem —The △ CEB = CAB (xxxvii) To each add CBD, and we have the △ CED = the quadrilateral CABD, but the △ CGD = $\frac{1}{3}$ CED, CGD = $\frac{1}{3}$ CABD In like manner CFG = $\frac{1}{3}$ CABD If F falls between B and E we can (Ex 33), by a line drawn from C, bisect the quad ACGB, each half of which will be $\frac{1}{3}$ CABD

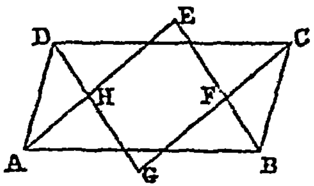
41 Let ABC be a △ whose base BC is given in magnitude and position, and the sum of its sides BA, AC also given. Produce BA to D, and make AD = AC Bisect the ∠ CAD by AE Erect CE ⊥ to AC Join BE, DE, and from E let fall a ⊥ EF on BC produced It is required to prove that the locus of E is the ⊥ EF

Dem.—Because AC = AD, and AE common, and the ∠ CAE

=DAE, ∴ (IV) CE = DE, and the ∠ ACE = ADE, but ACE is a right ∠ (const), ∴ ADE is right, hence (XLVII) $BE^2 - ED^2 = BD^2$, but BD is given, since it is equal to BA + AC, and ED = EC, ∴ $BE^2 - EC^2$ is given, and the base BC is given Hence (XLVII, Ex 5) the locus of E is EF, the ⊥ from E on BC

42 (1) See XXXII, Ex 8

(2) Let ABCD be a ▭ It is required to prove that EFGH is a rectangle



Dem —The ∠s ABC, BAD are together equal to two right ∠s (XXIX), ∴ the ∠s EBA, EAB together make a right ∠, hence the ∠ AEB is right Similarly, the ∠s at F, G, H are right Hence EFGH is a rectangle

(3) Let ABCD be a rectangle It is required to prove that EFGH is a square

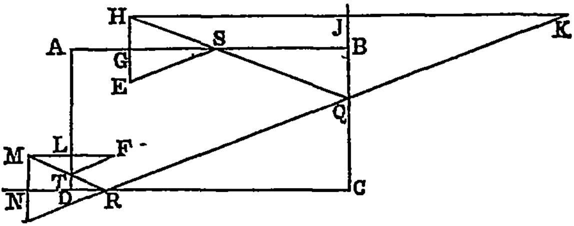
Dem.—Because the ∠ BAD = CDA, the ∠ BAE = CDG In like manner the ∠ ABE = DCG, and the side AB = CD, ∴ (XXVI) AE = DG, but AH = DH, since the ∠ ADH = DAH, ∴ HE = HG In like manner all the sides are equal, and the ∠s are right ∠s Hence EFGH is a square

43 Dem.—Join AE Now (XL, Ex 5) EF = ½ AB = BD, and FG = BD, ∴ EF = FG, and AF = CF (hyp), ∴ CF and FG = AF, FE, and the ∠ CFG = AFE (XV), hence (IV) CG = AE, but AE is a median of the Δ ABC, also CD, a side of the Δ CDG, is one of the medians of ABC, and BF, the remaining median, is equal to DG (XXXIV) Hence the sides of the Δ CDG are equal to the medians of ABC

44 Let ABCD be the billiard table, E the point from which the ball starts, and F the point through which it will pass

Sol —From E let fall a ⊥ EG on AB, produce EG to H, so that GH = EG From H let fall a ⊥ HJ on CB produced; and produce HJ to K, so that JK = HJ From F let fall a ⊥ FL on AD, and produce to M, so that LM = LF, and from M let fall a

⊥ MN on CD produced, and produce to P, so that NP=MN Join KP, intersecting BC in Q and CD in R Join HQ, MR, inter-

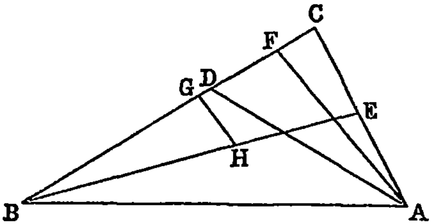


secting AB in S, and AD in T Join ES, FT ESQRTF will be the path of the ball

Dem —Because EG=HG, GS common, and the ∠ EGS=HGS, the ∠ ESG=HSG, but HSG=BSQ (xv), ESG=BSQ, hence the ball will be reflected in the direction SQ In like manner it can be shown that the ∠ HQJ=RQC, and therefore the ball will be reflected from Q in the direction QR Similarly, it will be reflected from R along RT, and from T along TF

45 Let ABC be the △, AD, BE the bisectors of the ∠ˢ A, B It is required to prove, if AD=BE, that the ∠ CAB=ABC

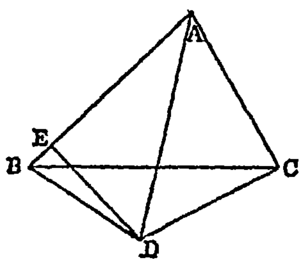
Dem —If the angle CAB be not equal to ABC, let CAB be the greater, then, since the ∠ CAB is greater than ABC, its



half, the ∠ DAC, is greater than EBC, the half of ABC, then make DAF equal to EBC Now, since the ∠ DAB is greater than ABE, the whole ∠ FAB is greater than FBA, the side FB is greater than FA Cut off BG=FA, and draw GH ∥ to FA, then the △ˢ GBH, FAD have evidently two ∠ˢ in one respectively equal to two ∠ˢ in the other, and the side BG=AF Hence BH is equal to AD, but BE is=AD (hyp) Hence BH=BE, which is absurd Hence the ∠ CAB is not unequal to ABC, that is, it is equal to it, and (vi) the △ ABC is isosceles

46 Let ABC be a △, whose base and difference of sides are given Bisect the ∠ BAC by AD Erect CD ⊥ to AC The locus of D is a right line

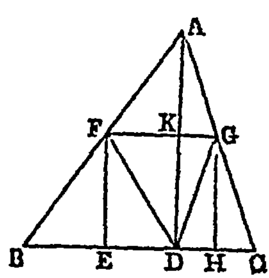
Dem —Let fall a ⊥ DE on AB Join BD Now (I xxvi) the △s ACD, AED are equal in every respect, ∴ DC=DE, and AC=AE, ∴ AB − AC=BE, but AB−AC is given, ∴ BE is given Again, $BD^2 - DE^2 = BE^2$, that is, $BD^2 - CD^2 = BE^2$,



hence $BD^2 - CD^2$ is given, and the base BC is given Now we are given the base, and the difference of the squares of the sides of the △ BCD Hence (xlvii , Ex 5) the locus of the vertex D is a right line ⊥ to BC

47 Let EFGH be a square inscribed in the △ ABC It is required to prove that (BC + AD) s=2 △ ABC, where s denotes the side of the square

Dem —Let fall a ⊥ AD on BC Join DF, DG Now BD EF =2 △ BFD (II i , Cor 1), that is, BD s =2 △ BFD Similarly, DC s = 2 △ DGC, ∴ BC . s = 2 △ BFD + 2 △ DGC

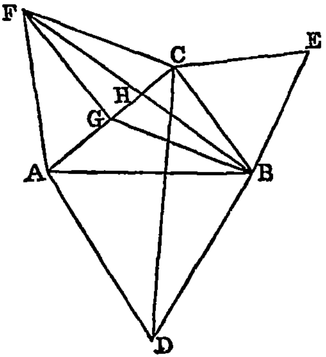


Again, AD FK=2 △ AFD, and AD GK=2 △ AGD, ∴ AD . s =2 AFDG Adding, we get (BC + AD) s=2 △ ABC

48 Dem —Let fall a ⊥ CE on AB Now (XLVII, Ex 20) BC^2 = AB BE + AC CD, but (XXVI) the △ˢ BEC, BDC are equal, since the △ ABC is isosceles, BE = DC, and AB = AC Hence BC^2 = 2 AC CD

49 Let ABC be a right-angled △, and let equilateral △ˢ be described on its three sides It is required to prove that the △ ABD is equal to the sum of the △ˢ ACF, BCE

Dem —Bisect AC in G Join FG, BG, FB, CD Now the ∠ CAF = BAD, to each add CAB, and we have the ∠ FAB = CAD, and AF = AC, and AB = AD, (IV) the △ˢ AFB, ACD are equal Again, because each of the ∠ˢ FGC, ACB is right, BC, FG are ∥, (XXXVII) the △ FGC = FGB To



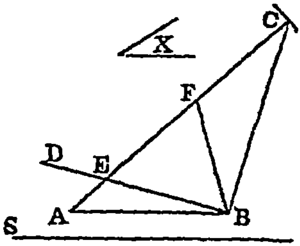
each add the △ FGA, and we have AFC = to the quadrilateral AFBG Again, to each add the △ AGB, which is ½ ACB, and we have AFC + ½ ACB = AFB Hence ACD = AFC + ½ ACB Similarly BCD = BEC + ½ ACB Add, and we have ACBD = AFC + ACB + BEC Reject the right-angled △ ACB, which is common, and the △ ABD = AFC + BEC

50 (1) Let AB be the base, X the difference of the base ∠ˢ, and S the sum of the sides It is required to construct the △

Sol —Draw BD, making the ∠ ABD = ½ X, and draw BC ⊥ to BD With A as centre, and a radius equal to S, describe a ○, cutting BC in C Join AC, cutting BD in E Bisect CE in F Join BF AFB is the required △

Dem —The lines BF, CF, EF are equal (XII, Ex 2), FE

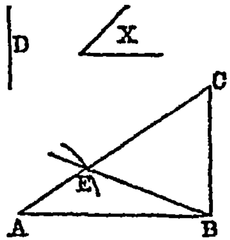
= FB, the ∠ FBE = FEB, but FEB = FAB + ABE (xxxii), ∴ FBE = FAB + ABE, hence the ∠ FBA = FAB + 2 ABE,



and hence the ∠ ABE is half the difference of the base ∠s, but ABE = ½ X Hence the difference of the base ∠s = X, and since FB = FC, AF + FB = AC = S, the sum of the sides = S

(2) Let AB be the base, X the difference of the base ∠s, and D the difference of the sides

Sol.—Draw BE, making the ∠ ABE = ½ X With A as centre, and a radius equal to D, describe a ○, cutting BE in E Join AE,



and produce it Draw BC, making the ∠ CBE = CEB, and meeting AE produced in C ACB is the required △

Dem —CB = CE (vi), AE = AC − CB, but AE = D, AC − CB = D, and, as before, the difference of the base ∠s = X

51 Sol —Let AB be the base, and M the median that bisects the base To AB apply a ▭ ABCD, whose area is equal to twice the given area (xlv) Bisect AB in E With E as centre, and a radius equal to M, describe a ○, cutting CD in F Join AF, BF AFB is the required △

52 Dem —Join AG, CG, FG The △ CED = CGD + CEG, and the △ EBC = BGC − CEG Subtracting, we get CED − EBC

= 2 CEG Similarly AED − AEB = 2 AEG Subtracting, we have AEB + CED − (AED + EBC) = 2 (CEG − AEG) Again, CEG = CFG + EFG, and AEG = AFG − EFG, CEG − AEG = 2 EFG And hence 4 EFG = AEB + CED − (AED + EBC)

53 (1) Let ACB be the △ Describe squares AH, AF, CE on the sides AC, AB, BC respectively Bisect AC in J Join BJ, EF It is required to prove that EF = 2 BJ

Dem —Produce BJ to M, so that JM = JB, and join MC

Now (IV) the △ˢ MJC, AJB are equal in every respect, MC = AB = BF, and CB = BE, hence MC, CB equal BF, BE And because AC and BM bisect each other in J, MC and AB are ∥, the ∠ˢ MCB and ABC are together equal to two right ∠ˢ, and the ∠ˢ EBF, ABC are equal to two right ∠ˢ, since ABF and CBE are right, the ∠ MCB = EBF, hence (IV) MB = EF, but MB = 2 BJ, EF = 2 BJ

(2) Produce MB to meet EF in N MN is ⊥ to EF

Dem —From the equal △ˢ CMB, BFE we have the ∠ CMB = BFE, but CMB = ABM, BFE = ABM To each add NBF, and we have BFN + NBF = ABM + NBF, but since ABF is right, ABM + NBF equal a right ∠, BFN + NBF equal a right ∠, and hence the ∠ BNF is right

BOOK II

PROPOSITION IV

1 Dem —$AB^2 = AB\ AC + AB\ BC$ (II),

but $AB\ AC = AC^2 + AC\ CB$ (III),

and $AB\ BC = BC^2 + AC\ CB$ (III),

Therefore $AB\ AC + AB\ BC = AC^2 + BC^2 + 2\,AC\ CB$,

that is, $AB^2 = AC^2 + BC^2 + 2\,AC\ CB$

2 Let C be the vertical ∠ of the right-angled △ ABC From C let fall a ⊥ CD on AB It is required to prove that $DC^2 = AD\ DB$

Dem —$AB^2 = AC^2 + CB^2$ (I XLVII), but $AC^2 = AD^2 + DC^2$, and $CB^2 = BD^2 + DC^2$, $AB^2 = AD^2 + BD^2 + 2\,DC^2$ Again, $AB^2 = AD^2 + DB^2 + 2\,AD\ DB$ (IV) Hence $DC^2 = AD\ DB$

3 Let ABC be the right-angled △ In the base AB cut off $AD = AC$, and $BE = BC$ It is required to prove that $ED^2 = 2\,AE\ DB$

Dem —$AB^2 = AC^2 + CB^2$ (I XLVII) $= AD^2 + BE^2$, but $AD^2 = AE^2 + ED^2 + 2\,AE\ ED$ (IV), and $BE^2 = BD^2 + DE^2 + 2\,BD\ DE$, ∴ $AB^2 = AE^2 + ED^2 + 2\,AE\ ED + BD^2 + DE^2 + 2\,BD\ DE$, also $AB^2 = AE^2 + ED^2 + DB^2 + 2\,AE\ ED + 2\,ED\ DB + 2\,AE\ DB$ (IV, Cor 3) Hence $ED^2 = 2\,AE\ DB$

4 Let ABC be the right-angled △, CD the ⊥ from the right angle on the base It is required to prove that $(AB + CD)^2$ exceeds $(AC + CB)^2$ by CD^2

Dem —AC CB is equal to twice the △ ACB, and AB CD is equal to twice the △ ACB, $AC\ CB = AB\ CD$

Now $(AB + CD)^2 = AB^2 + CD^2 + 2\,AB\ CD$,

and $(AC + CB)^2 = AC^2 + CB^2 + 2\,AC\ CB$

Subtracting, we have $(AB + CD)^2 - (AC + CB)^2 = AB^2 - BC^2$

$-CA^2+DC^2$, but $AB^2-BC^2=AC^2$, $\therefore (AB+CD)^2-(AC+CB)^2 = AC^2-AC^2+DC^2=DC^2$

5 Let the sides of the Δ be denoted by a, b, c, c being the hypotenuse It is required to prove that $(a+b+c)^2=2(c+a)(c+b)$

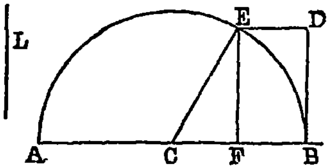
Dem —$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$, but $a^2+b^2=c^2$ (I XLVII), $\therefore a^2+b^2+c^2=2c^2$ Hence $(a+b+c)^2 = 2(c^2+ac+bc+ab)=2(c+a)(c+b)$

PROPOSITION V

1 Let AB be the given straight line Bisect it in C It is required to prove that AC . CB is a maximum

Dem —Take any other point D in AB, then AD . DB + CD^2 = CB (V), but CB^2 = AC . CB, $\therefore$ AC . CB = AD . DB + CD^2, that is, AC . CB is greater than AD . DB by CD^2 Hence, when a line is bisected, the rectangle contained by the parts is a maximum

2 Let AB be the given straight line, and L the line whose square is given It is required to divide AB, so that the rectangle contained by its segments will be equal to L^2



Sol —Bisect AB in C, with C as centre, and CB as radius, describe a semicircle Draw BD $\perp$ to AB, and = to L Through D draw DE $\parallel$ to AB, cutting the semicircle in E, let fall a $\perp$ EF on AB The rectangle AF . FB = L^2

Dem —Join CE Now AF . FB + CF^2 = CB^2 (V) = CE^2 = CF^2+FE^2 (I XLVII) Take away CF^2, which is common, and AF . FB = $FE^2=BD^2=L^2$

3 Let ABC be the Δ From C let fall a $\perp$ CD on AB It is required to prove that $(AC+BC)(AC-BC)=AB(AD-DB)$

Dem —$AC^2 = AD^2 + DC^2$ (I XLVII), and $BC^2 = BD^2 + DC^2$ Subtracting, we get $AC^2 - BC^2 = AD^2 - DB^2$, that is

(AC + BC) (AC − BC) = (AD + DB) (AD − DB) = AB (AD − DB)

4 Dem —(AC + BC) (AC − BC) = AB (AD − DB) (Ex 3), but (AC + BC) is greater than AB (I xx), (AC − BC) is less than (AD − DB)

5 See "Sequel to Euclid," Book II, Prop I, Cor

6 Let ABC be the Δ It is required to prove that AC^2 = (AB + BC) (AB − BC)

Dem —$AC^2 + BC^2 = AB^2$, $AC^2 = AB^2 - BC^2$ = (AB + BC) (AB − BC)

PROPOSITION VI

1 Let AB be the straight line which is bisected in C, and divided externally in D It is required to prove Prop VI by Prop V, by producing the line DA in the opposite direction

Dem —Produce DA to O, and make OA = BD

Now OB BD + $CB^2 = CD^2$ (V), but since OA = BD, OB = AD Therefore AD DB + $CB^2 = CD^2$

2 Let AB be the given line It is required to divide it externally in E, so that AE EB = L^2, L being a given line

Sol —Bisect AB in C Erect BD ⊥ to AB, and make it equal to L Join CD With C as centre, and CD as radius, describe a circle, meeting AB in E E is the point required

Dem —Now AE EB + $CB^2 = CE^2 = CD^2 = CB^2 + BD^2$ Reject CB^2, which is common, and AE EB = $BD^2 = L^2$

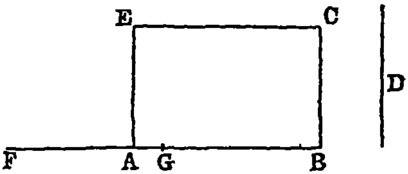
3 In Ex 2 AB = AE − EB, and is given, L^2 = AE EB, we find the point E and AE, EB are then the lines required

4 Let AD, DB be two lines Bisect AB in C

Dem —Because AB is the sum, CB is half sum, and AD = AC + CD, and DB = CB − CD, AD − DB = 2 CD, hence CD is half difference Now AD DB + $CD^2 = CB^2$ (V), AD DB = $CB^2 - CD^2$ = square on half sum − square on half difference

5 Dem —Let AB be the sum, and D^2 the difference of their squares To AB apply the rectangular ▭ ABCE = D^2 Now, since the sum multiplied by the difference is equal to the difference of the squares, and that AB is the sum, therefore AE must be the difference Produce BA to F, and make AF = AE Therefore, since the sum together with the difference is equal to twice the greater, if we bisect BF in G, BG will be the greater, and AG the less

If we take AE equal to the difference, and apply the rectangular ▭ ABCE $= D^2$, we have the second case



6 See "Sequel to Euclid," Book II, Prop I, Cor

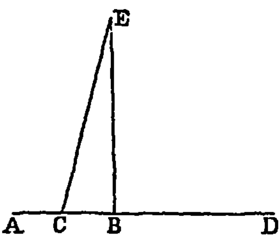
7 The rectangle contained by two straight lines, together with the square described on half their difference, is equal to the square on half their sum

PROPOSITION VIII

1 Dem —By the third proof of Prop VIII $(AB + BO)^2 = 4\,AB\ \ BO + AO^2$, but $AB\ \ BO = BC^2$ (I XLVII, Ex 1), and $AO^2 = AC^2 - CO^2$, $(AB + BO)^2 = 4\,BC^2 + AC^2 - CO^2$, but $4\,BC^2 + AC^2 = EF^2$ (I XLVII, Ex 7), $(AB + BO)^2 = EF^2 - CO^2$

2 Dem —$GK^2 = 4\,AC^2 + BC^2$ (I XLVII, Ex 7), and $EF^2 = 4\,BC^2 + AC^2$, $GK^2 - EF^2 = 3\,AC^2 - 3\,BC^2$, but (I XLVII, Ex 1) $AC^2 = AB\ \ AO$, and $BC^2 = AB\ \ BO$, $GK^2 - EF^2 = 3(AB\ \ AO - AB\ \ BO) = 3\,AB\,(AO - BO)$

3 Sol.—Let AB be the difference of the lines Bisect AB in C, erect BE ⊥ to AB, and make it equal $2\,AB = 2\,R$ Join CE, and produce CB to D Cut off $CD = CE$. AD, DB are the required lines



Dem —$AD\ \ DB + CB^2 = CD^2$ (VI) $= CE^2 = CB^2 + BE^2$

Reject CB^2, which is common, and we have $AD \ DB = BE^2 = 4 R^2$ Hence AD, BD are the required lines, for their difference is AB, that is, R, and their rectangle is equal to $4 R^2$

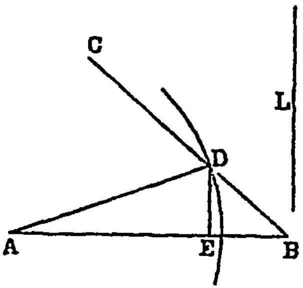
PROPOSITION IX

1 Let AB be the given line Bisect it in C It is required to prove that $AC^2 + CB^2$ is a minimum

Dem —Take any other point D in AB Now $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$ (IX) $= AC^2 + CB^2 + 2 CD^2$, therefore $AC^2 + CB^2$ is less than $AD^2 + DB^2$ by $2 CD^2$ Hence, when a line is bisected, the sum of the squares on its segments is a minimum.

2 Let AB be a given line It is required to divide it internally, so that the sum of the squares on the parts may be equal to L^2

Sol.—Draw BC, making the ∠ ABC half a right ∠ With A as centre, and a radius equal to L, describe a ○, cutting BC in D From D let fall a ⊥ DE on AB E is the point required



Dem —Because the ∠ EBD is half a right ∠, and the ∠ BED right, the ∠ BDE is half a right ∠, $EB = ED$, $EB^2 = ED^2$, $AE^2 + ED^2$, that is, AD^2, that is $L^2 = AE^2 + EB^2$ If the ○ does not meet the line BC, the question is impossible

3 Dem —From AC cut off $AE = DB$ Now $AD^2 + AE^2 = 2 AD \ AE + ED^2$ (VII), that is, $AD^2 + DB^2 = 2 AD \ DB + 4 CD^2$

4 Let ABC be the Δ In AB take any point D Join CD, It is required to prove that $2\,CD^2 = AD^2 + DB^2$ From C let fall a $\perp$ CE on AB Now $AD^2 + DB^2 = 2\,AE^2 + 2\,ED^2$ (IX), but AE = EC Therefore $AD^2 + DB^2 = 2\,EC^2 + 2\,ED^2 = 2\,CD^2$

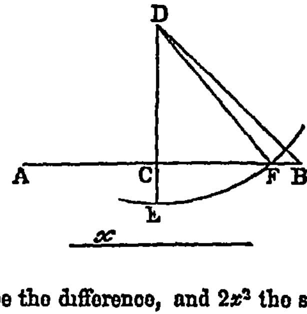
5 See "Sequel to Euclid," Book II, Prop XII

PROPOSITION X

1 (1) Let AB be the sum of the lines, and $2x^2$ the sum of the squares

Sol —Bisect AB in C Erect CD $\perp$ to AB, and make it equal to AC or CB Produce DC to E Cut off DE = x With D as centre and DE as radius, describe a ○, cutting AB in F AF and FB are the required lines

Dem —Join DF, DB Now $AF^2 + FB^2 = 2\,AC^2 + 2\,CF^2$ (IX) $= 2\,DC^2 + 2\,CF^2 = 2\,DF^2 = 2\,DE^2 = 2\,x^2$



(2) Let AB be the difference, and $2x^2$ the sum of the squares

Sol —Bisect AB in C, and erect CD $\perp$ to AB, and make it equal to AC or CB Produce DC to E Cut off DE = x With D as centre, and DE as radius, describe a ○, cutting AB produced in F AF and FB are the required lines

Dem —Join DB, DF Now $AF^2 + FB^2 = 2\,AC^2 + 2\,CF^2 = 2\,DC^2 + 2\,CF^2 = 2\,DF^2 = 2\,DE^2 = 2x^2$

2 Let CE be the median which bisects the base AB It is required to prove that $AC^2 + CB^2 = 2\,AE^2 + 2\,CE^2$

Dem —From C let fall a $\perp$ CD on AB Now $AD^2 + DB^2 = 2\,AE^2 + 2\,ED^2$ (IX), and $CD^2 + CD^2 = 2\,CD^2$ Add, and we get $AC^2 + CB^2 = 2\,AE^2 + 2\,CE^2$ Or apply Props XII. and XIII

3 Let BC be the given base of a △ ABC, the sum of the squares of whose sides AB, AC, is equal to a given square It is required to prove that the locus of the vertex A is a ○

Dem —Bisect BC in D Join AD Now (Ex 2), $BA^2 + AC^2 = 2 BD^2 + 2 DA^2$, but $BA^2 + AC^2$ is given (hyp), ∴ $2 BD^2 + 2 DA^2$ is given, and $2 BD^2$ is given, since BD is half of the given base BC, ∴ $2 DA^2$ is given, ∴ DA is given, and the point D is given Hence the locus of A is a ○, having D as centre, and DA as radius

4 Dem —Bisect AD in E Join BE, CE Now (Ex 2) $AB^2 + BD^2 = 2 AE^2 + 2 BE^2$, and $AC^2 + CD^2 = 2 AE^2 + 2 CE^2$, but $AB^2 + BD^2 = AC^2 + CD^2$ (hyp), hence $2 AE^2 + 2 BE^2 = 2 AE^2 + 2 CE^2$, and therefore $2 BE^2 = 2 CE^2$, ∴ BE = CE

5 See "Sequel to Euclid," Book II, Prop III

PROPOSITION XI

1 Let AB be the line It is required to cut it externally in extreme and mean ratio

Sol —Erect BC ⊥ to and equal to AB Bisect AB in D Join DC. Produce AB to E Cut off DE = DC AB is cut in E in extreme and mean ratio

Dem.—$AE \cdot EB + DB^2 = DE^2$ (VI) $= DC^2 = DB^2 + BC^2$ Reject DB^2, which is common, and $AE \cdot EB = BC^2 = AB^2$

2 Let AB be a line divided in extreme and mean ratio at C It is required to prove that $AC^2 - CB^2 = AC \cdot CB$

Dem —$AB \cdot BC = AC^2$ (hyp), but AB = AC + CB, ∴ $(AC + CB) CB = AC^2$, that is, $AC \cdot CB + CB^2 = AC^2$, and $AC \cdot CB = AC^2 - CB^2$

3 Let ACB be a right-angled △, having $AC^2 = AB \cdot BC$ From C let fall a ⊥ CD on AB It is required to prove that $AB \cdot BD = AD^2$

Dem.—$AC^2 = AB \cdot BC$ (hyp), and $AC^2 = AB \cdot AD$ (I XLVII, Ex 1), ∴ AD = BC, ∴ $AD^2 = BC^2$, but $BC^2 = AB \cdot BD$ (I XLVII, Ex 1) Hence $AB \cdot BD = AD^2$

4 (1) Dem —$AB^2 + BC^2 = 2 AB \cdot BC + AC^2$ (VII), but $AB \cdot BC = AC^2$ (hyp) Hence $AB^2 + BC^2 = 3 AC^2$

(2) Dem —$(AB + BC)^2 = 4 AB \cdot BC + AC^2$ (VIII), but $AB \cdot BC = AC^2$ (hyp) Hence $(AB + BC)^2 = 5 AC^2$

5 * Dem —Join FK, AD Now the square AFGH is double of the Δ AFK (I XLI) And the rectangle HBDK is double of AKD, but AFGH = HBDK (XI), the Δ AFK = AKD, and hence (I XXXIX) AK is ∥ to FD In like manner, by joining BF, GD, it can be shown that GB is ∥ to FD Hence the three lines AK, FD, GB are parallel

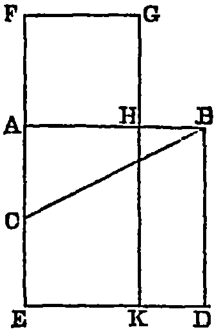
6 Dem —Join BF, and produce CH to meet it in L

Because EB = EF, the ∠ EBF = EFB, and the ∠s at L are right (XI, Ex 7), the ∠ BOL = FCL, but BOL = EOC, EOC = ECO, and EO = EO, but EC = EA, EO = EA, the ∠ EOA = EAO, and EOC = ECO Hence the ∠ AOC = OAC + OCA, and is therefore (I XXXII, Cor 7) a right ∠

7 Let CH be produced to meet BF at L It is required to prove that CL is ⊥ to BF

Dem —The Δs FAB, HAC, are equal (I IV) in every respect, the ∠ FBA = HCA, and the ∠ LHB = AHC (I XV), the ∠ HLB = HAC (I XXXII, Cor 2), but HAC is a right ∠ Hence HLB is right

8 Dem —In AB take AH = BC − AC Produce CA to F, so that AF = AH, then evidently CF = CB Complete the square AFGH Produce AC to E, and make CE = AC, and complete



the square ABDE Produce GH to meet ED in K Now we have the construction as in Prop XI, and AB BH = AH^2 Hence AB is divided in "extreme and mean ratio" at H

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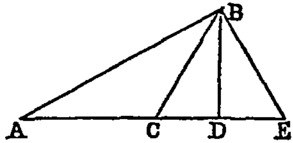
* See diagram in Euclid [II XI] for this and the two following Exercises

PROPOSITION XII

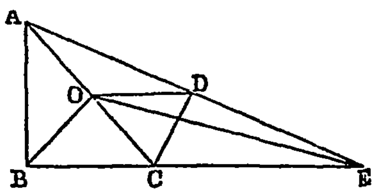
1 Dem —Produce AC, and let fall a ⊥ BD on AC produced Make DE = CD, and join BE Now the Δ^{s} BCD, BED are equal in every respect (I iv), the ∠ BCE = BEC And



since the ∠ ACB is twice an ∠ of an equilateral Δ, each of the ∠^{s} BCE, BEC is an ∠ of an equilateral Δ, hence the Δ BCE is equilateral, BC = CE = 2 CD

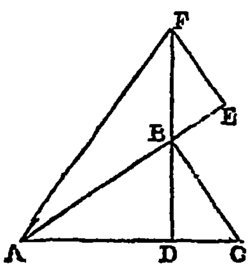
Again, $AB^2 = AC^2 + CB^2 + 2\ AC\ CD$, but we have shown that BC = 2 CD Hence $AB^2 = AC^2 + CB^2 + AC\ CB$

2 Dem —Join AC, bisect it in O Join BO, DO, EO Now the lines AO, BO, CO are equal (I xii, Ex 2), hence OBC is



an isosceles Δ, $OE^2 - OC^2 = BE\ CE$ (vi, Ex 6) In like manner $OE^2 - OD^2 = AE\ DE$, but OC = OD Hence AE DE = BE CE

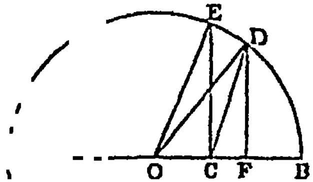
3 Dem —Produce AB, DB Cut off BE = DC, and BF = BC



FBC = BDC + BCD (I xxxii), but
∠s, FBE = BCD, hence (I iv)
equal in every respect, therefore the
∠ BEF is right Now $AF^2 = AB^2$
and $AF^2 = AB^2 + BF^2 + 2 FB \cdot BD$,
t BE = DC, and BF = BC Hence

o AB, and equal BC Join AD, CD
AC CB (xii), and $CD^2 = CB^2 + BD^2$
$+ CD^2 = 2 AC^2$, $AD^2 = 2 AC^2$
CB) = 2 AC AB Again, $AD^2 = AB^2$
$AD^2 = AB^2 + BC^2$ Hence AB^2

From D let fall a ⊥ DF on AB
OC CF $= p^2$ (the given square) C is



B Join OE, OD, CD
2 OC CF (xii) $= OC^2 + CD^2 + p^2$,
$OC^2 + CD^2 + p^2$, that is, $OC^2 + CE^2$
$- CD^2 = p^2$
$)^2 - CB^2$ (vi Ex 6), but $OD^2 = 2 AB^2$
$B^2 - CB^2 = 2 AB^2 - AB^2 = AB^2$

SITION XIII.

tex A let fall a ⊥ AD on BC From
Join AE Now the △ ACD = AED
AC = AE, and the ∠ AEC = ACE,
equilateral △, the △ ACE is
= 2 CD Again, $AB^2 = BC^2 + CA^2$
we have shown that 2 CD = AC
BC AC

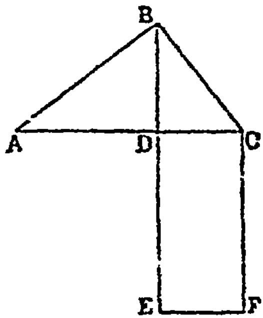
2 See "Sequel to Euclid," Book II, Prop IV.

3 Sol —Erect BD ⊥ to and equal to AB Join AD Produce AB to C Cut off AC = AD C is the point required

Dem.—$AD^2 = AB^2 + BD^2 = 2\ AB^2$, ∴ $AC^2 = 2\ AB^2$ To each add BC^2, and we have $2\ AB^2 + BC^2 = AC^2 + BC^2 = 2\ AC \cdot BC + AB^2$ (VII.), ∴ $AB^2 + BC^2 = 2\ AC \cdot BC$

PROPOSITION XIV

1 Sol —Let a line CD be found (XIV) whose square is equal to the given difference of squares On CD construct a rectangle CE equal to the given rectangle Produce CD to A, so that $CA \cdot AD = DE^2$ (VI, Ex 2) Produce ED From A inflect AB = DE to the line DB, and join BC BC and BD are the required lines



Dem —Because $AB^2 = DE^2 = CA \cdot AD$, the ∠ ABC is right (I XLVII, Ex 1), ∴ $AB \cdot DC = BD \cdot BC$ (XII, Ex. 3), hence the rectangle CE = BD · BC, and CE is equal to the given rectangle Also because the ∠ BDC is right, $BC^2 - BD^2 = DC^2$, which is equal to the given difference of squares

2 See Book II, Ex 6, Miscellaneous

Miscellaneous Exercises on Book II.

1 Let ABCD be a quadrilateral, AC, BD its diagonals, and EF, GH lines joining the middle points of BC, AD, AB, CD It is required to prove that $AC^2 + BD^2 = 2\ EF^2 + 2\ GH^2$

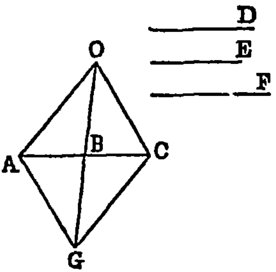
Dem —Join GE, EH, HF, FG Now GEHF is a ▭ (I xl, Ex 6), $2\ GH^2 + 2\ EF^2 = 2\ GE^2 + 2\ EH^2 + 2\ HF^2 + 2\ FG^2$ (x, Ex 5) $= 4\ GE^2 + 4\ EH^2$

Again, $GE = \frac{1}{2} AC$ (I xl, Ex 5), and $EH = \frac{1}{2} BD$, $4\ GE^2 + 4\ EH^2 = AC^2 + BD^2$ Hence $2\ GH^2 + 2\ EF^2 = AC^2 + BD^2$

2 Let AD, BE, CF be the medians

Dem —$AB^2 + AC^2 = 2\ BD^2 + 2\ AD^2$ (x., Ex 2), $2\ AB^2 + 2\ AC^2 = BC^2 + 4\ AD^2$, but $AO = \frac{2}{3} AD$, $AO^2 = \frac{4}{9} AD^2$, $9\ AO^2 = 4\ AD^2$, hence $2\ AB^2 + 2\ AC^2 = BC^2 + 9\ AO^2$ Similarly $2\ AC^2 + 2\ CB^2 = AB^2 + 9\ CO^2$, and $2\ CB^2 + 2\ AB^2 = AC^2 + 9\ BO^2$, $3\ (AB^2 + BC^2 + CA^2) = 9\ (AO^2 + BO^2 + CO^2)$ Hence $AB^2 + BC^2 + CA^2 = 3\ (AO^2 + BO^2 + CO^2)$

3 Sol —Construct the △ OCG, having OC = D, OG = 2 E, and CG = F Bisect OG in B Join CB, and produce it to A Cut off AB = BC Join AO OA, OB, OC are the required lines



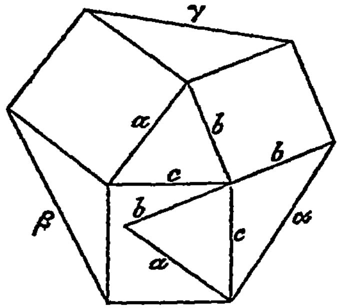
Dem —The △s ABO, CBG are equal in every respect (I iv), AO = CG = F, and OC = D, and OB = E

4 Let ABCD be a quadrilateral, AC, BD its diagonals Bisect AB, CD in E, F Join EF It is required to prove that $AD^2 + BC^2 + AC^2 + BD^2 = AB^2 + DC^2 + 4\ EF^2$

Dem —Join CE, DE Now $AD^2 + BD^2 = 2\ AE^2 + 2\ ED^2$ (x, Ex 2), and $AC^2 + BC^2 = 2\ BE^2 + 2\ CE^2$, $AD^2 + BD^2 + AC^2 + BC^2 = 2\ AE^2 + 2\ BE^2 + 2\ CE^2 + 2\ DE^2$, but $2\ AE^2 + 2\ BE^2 = 4\ AE^2 = AB^2$, and $2\ CE^2 + 2\ DE^2 = 4\ DF^2 + 4\ EF^2 = DC^2 + 4\ EF^2$ Therefore $AD^2 + BD^2 + BC^2 + AC^2 = AB^2 + DC^2 + 4\ EF^2$

5 Let *a*, *b*, *c* be the sides of the triangle On *a*, *b*, *c* describe squares Join the adjacent corners, and let the joining lines be

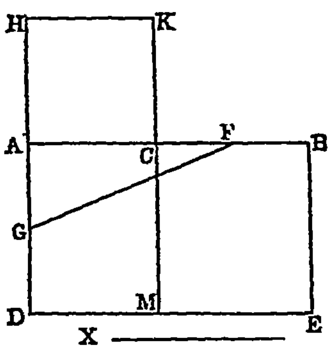
denoted by α, β, γ It is required to prove that $\alpha^2 + \beta^2 + \gamma^2 = 3(a^2 + b^2 + c^2)$



Dem.—Complete the construction, as in I XLVII, Ex 6. Now we have (x, Ex 2) $\alpha^2 + a^2 = 2b^2 + 2c^2$, $\beta^2 + b^2 = 2c^2 + 2a^2$, and $\gamma^2 + c^2 = 2a^2 + 2b^2$ Add together, and we get $\alpha^2 + \beta^2 + \gamma^2 + (a^2 + b^2 + c^2) = 4(a^2 + b^2 + c^2)$, and $\alpha^2 + \beta^2 + \gamma^2 = 3(a^2 + b^2 + c^2)$

6 Let AB be a given line It is required to divide it into two parts at C, so that the rectangle contained by another given line X, and one segment BC, will be equal to AC²

Sol.—Erect AD ⊥ to AB, and equal to X. Complete the rectangular ▭ ABED Construct a square equal to ABED, and let

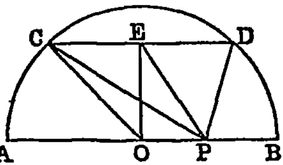


AF be one of its sides Bisect AD in G Join GF Produce DA to H Cut off GH = GF In AB take AC = AH C is the required point

Dem —Complete the square AHKC Produce KC to meet DE in M Now $DH \cdot HA + AG^2 = GH^2$ (VI), but $GH^2 = GF^2 = AG^2 + AF^2$, $\therefore DH \cdot HA = AF^2$, but $AF^2 = ABED$ (const); $\therefore$ the figure $HM = BD$ Reject DC, and $HC = BM$, but BM is the rectangle $BC \cdot BE$, that is, $BC \cdot X$, and HC is AC^2, $\therefore BC \cdot X = AC^2$.

If we put $\frac{AB}{m} = X$, where m is any quantity, we get $AB \cdot BC = m\,AC^2$

7 Dem —Bisect AB in O Erect OE ⊥ to AB, and join OC, EP Now (III, 3) CD is bisected at E, $\therefore$ (X, Ex 2)



$CP^2 + PD^2 = 2\,CE^2 + 2\,EP^2 = 2\,CE^2 + 2\,EO^2 + 2\,OP^2 = 2\,CO^2 + 2\,OP^2 = 2\,AO^2 + 2\,OP^2 = AP^2 + PB^2$ (IX)

8 See "Sequel to Euclid," Book II, Prop VII

9 Let ABCDE be the pentagon, AC, BD, CE, AD, BE its diagonals Bisect the diagonals Let α be the line joining the middle points of AC, BD, β of BD, CE, γ of CE, AD, δ of AD, BE, and ϵ of BE, AC It is required to prove that $3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) = AC^2 + BD^2 + CE^2 + AD^2 + BE^2 + 4\,(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2)$

Dem —From XIII, Ex 2, we have—

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\,\alpha^2$$
$$BC^2 + CD^2 + DE^2 + EB^2 = BD^2 + CE^2 + 4\,\beta^2,$$
$$CD^2 + DE^2 + EA^2 + AC^2 = CE^2 + DA^2 + 4\,\gamma^2,$$
$$DE^2 + EA^2 + AB^2 + BD^2 = DA^2 + EB^2 + 4\,\delta^2,$$
$$EA^2 + AB^2 + BC^2 + CE^2 = EB^2 + AC^2 + 4\,\epsilon^2$$

Add together, and we have

$$3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) = AC^2 + BD^2 + CE^2 + AD^2 + BE^2 + 4\,(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2)$$

10 See "Sequel to Euclid," Book II, Prop V

11 See "Sequel to Euclid," Book II, Prop VIII

12 See "Sequel to Euclid," Book II, Prop IX

13 See "Sequel to Euclid," Book II, Prop IX, Cor

14 (1) Dem —It is proved in Ex 12 that

$$m\,AC^2 + n\,BC^2 = m\,AD^2 + n\,DB^2 + (m+n)\,DC^2,$$

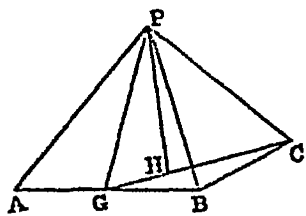
but $m\,AC^2 + n\,BC^2$ is given (hyp), ∴ $m\,AD^2 + n\,DB^2 + (m+n)\,DC^2$ is given, and $m\,AD^2 + n\,DB^2$ is given, ∴ $(m+n)\,DC^2$ is given, but $(m+n)$ is given, ∴ DC^2 is given, ∴ DC is given, and D is a given point Hence the locus of the vertex is a ○, having D as centre, and DC as radius

(2) This case can be proved in a similar manner by using Ex 13

15 Let ABCD be a rectangle, of which AB, AD are adjacent sides On AB, AD describe squares AF, AE Draw the diagonals AF, AE It is required to prove that AF . AE is equal to twice the rectangle AC

Dem —The diagonals AF, AE are evidently in the same right line Let fall a ⊥ BG on AF Now, because the ∠ ABF is right, $AF^2 = AB^2 + BF^2 = 2\,AB^2$ For a similar reason $AE^2 = 2\,AD^2$, hence $AF^2 . AE^2 = 4\,AB^2 . AD^2$, therefore $AF . AE = 2\,AB . AD$, that is, AF . AE is equal to twice the rectangle AC

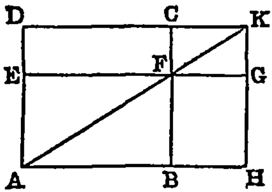
16. Dem —Join AB, BC Bisect AB in G Join PG, CG,



AP, BP, CP Divide GC in H, so that HC = 2 GH Join PH. Now $AP^2 + BP^2 = 2\,AG^2 + 2\,GP^2$ (x, Ex 2), and $2\,PG^2 + PC^2 = 2\,GH^2 + HC^2 + 3\,HP^2$ (Ex 12), ∴ $AP^2 + BP^2 + CP^2 = 2\,AG^2 + 2\,GH^2 + HC^2 + 3\,HP^2$, but $AP^2 + BP^2 + CP^2$ is given (hyp), ∴ $2\,AG^2 + 2\,GH^2 + HC^2 + 3\,HP^2$ is given, but $2\,AG^2$ is given, and $2\,GH^2$, and HC^2, hence $3\,HP^2$ is given, ∴ HP is given, and the point H is given Hence the locus of P is a ○

17 Let ABCD be a square, and AEGH a rectangle of equal area. It is required to prove that the perimeter of ABCD is less than that of AEGH

Dem—ABCD = AEGH (hyp) Take away the common part AEFB, and we have EDCF = BFGH, hence these must be the complements about the diagonal of a ▭, if DC, AF, HG be produced, they are concurrent Let them meet in K Now DK is greater than DA the ∠ DAK is greater than DKA, that



is, CFK is greater than CKF, CK is greater than CF, and therefore greater than DE To each add CD + EA, and we get KD + EA, that is, GE + EA, greater than CD + DA Hence the perimeter of the rectangle is greater than that of the square.

18 Let the tranversal be divided by the lines, so that m AC $= n$ CB, then $\frac{m}{n} = \frac{BC}{AC}$

Dem.—m $AD^2 + n$ $DB^2 = m$ $AC^2 + n$ $BC^2 + (m+n)$ CD^2 (Ex 12),

$$\frac{m}{n} AD^2 + DB^2 = \frac{m}{n} AC^2 + BC^2 + \left(\frac{m}{n} + 1\right) CD^2, \text{ but } \frac{m}{n} = \frac{BC}{AC};$$

$$\frac{BC}{AC} \ AD^2 + DB^2 = \frac{BC}{AC} \ AC^2 + BC^2 + \left(\frac{BC}{AC} + 1\right) CD^2,$$

$$BC \ AD^2 + AC \ DB^2 = BC \ AC^2 + AC \ BC^2 + AB \ CD^2,$$

$$BC \ AD^2 + AC \ DB^2 - AB \ CD^2 = AC \ CB\,(AC + CB),$$

$$\therefore BC \ AD^2 + AC \ DB^2 - AB \ CD^2 = AB \ BC \ CA$$

Lemma—If a ○ be described about an equilateral △ the square of the side of the △ is equal to three times the square of the radius

Dem—Let BC be the side of the equilateral △ ABC, and O the centre of the circumscribing ○ Join BO, and produce it to meet the circumference in D Join DC, OC, OA

The radii BO, OC, OD are equal the ∠ OBC = OCB and the ∠ ODC = OCD (I xxxii Cor 7), the ∠ BCD is right;

$BD^2 = BC^2 + CD^2 = BC^2 + CO^2$ Let BO be denoted by r, then $BD^2 = 4r^2$, and $OC^2 = r^2$, $4r^2 = BC^2 + r^2$ And therefore $BC^2 = 3r^2$

19 Dem.—Join AD, CD, CD′ Now in the Δ DCD′, $DD'^2 = DC^2 + CD'^2 + DC \cdot CD'$ (XII, Ex 1), $6\,DD'^2 = 6\,DC^2 + 6\,CD'^2 + 6\,DC \cdot CD'$

Again, $AC^2 = 3\,CD^2$ (Lemma), and $CB^2 = 3\,CD'^2$, $AC^2 \cdot CB^2 = 9\,CD^2 \cdot CD'^2$, $AC \cdot CB = 3\,CD \cdot CD'$, $2AC \cdot CB = 6\,CD \cdot CD'$, hence we have $6\,DD'^2 = 2\,AC^2 + 2\,CB^2 + 2\,AC \cdot CB = AC^2 + CB^2 + (AC^2 + CB^2 + 2\,AC \cdot CB) = AC^2 + CB^2 + AB^2$

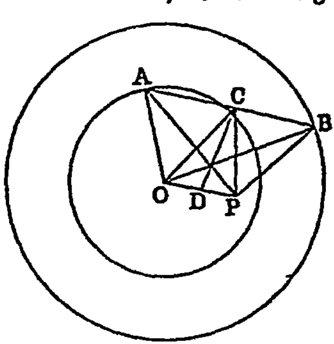
20 —Dem —Let c be the hypotenuse, then $ab = cp$ (I, Cor 1), ∴ $a^2 b^2 = c^2 p^2$, $a^2 b^2 = (a^2 + b^2) p^2 = a^2 p^2 + b^2 p^2$ Divide by $a^2 b^2 p^2$, and $\frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2}$

21 Dem.—Since ABD is an isosceles Δ, $DC^2 - DB^2 = AC \cdot CB$ (VI, Ex 6) $= AB^2$ (hyp) Hence $DC^2 = DB^2 + AB^2 = 2\,AB^2$

22 Let a variable line AB, whose extremities rest on the circumferences of two given concentric ⊙s, subtend a right ∠ at a fixed point P It is required to prove that the locus of its middle point C is a ⊙

Dem —Join OA, OB, OP Bisect OP in D Join CO, CD, CP

Now $AO^2 + OB^2 = 2\,BC^2 + 2\,CO^2$ (X, Ex 2), but AO, OB are given, being radii of the given ⊙s, $2\,BC^2 + 2\,CO^2$ is given, ∴ $BC^2 + CO^2$ is given, but BC = CP (I XII, Ex 2), $CO^2 + CP^2$ is given, that is, $2\,OD^2 + 2\,DC^2$ is given, but $2\,OD^2$ is given, since OP is bisected in D, ∴ $2\,DC^2$ is given, ∴ DC is a



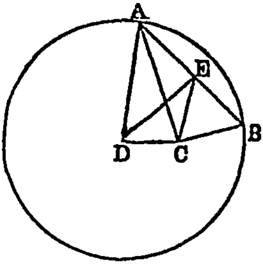
given line, and D is a fixed point. Hence the locus of C is a ⊙, having D as centre, and DC as radius

BOOK III

PROPOSITION XII

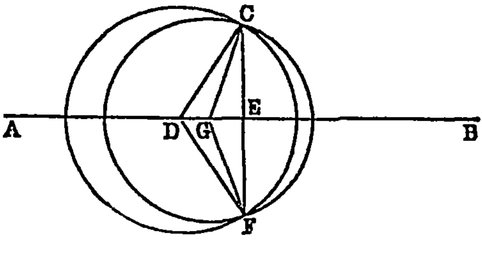
1 Let AB be the chord subtending a right ∠ at the point C. It is required to prove that the locus of the middle point of AB is a ○

Dem —Let D be the centre Draw DE ⊥ to AB, and join CD, AD, CE



Now (III) AB is bisected in E, the lines AE, BE, CE are equal (I xii , Ex 2) Again, $AD^2 = AE^2 + ED^2 = ED^2 + EC^2$, but AD^2 is given, since AD is the radius, $ED^2 + EC^2$ is given, and the base DC is given, (II x , Ex 3), the locus of E is a ○

2 Let AB be the given line, and C the given point Take any

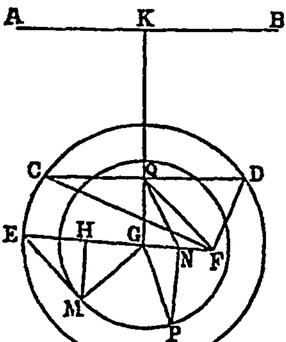


point D in AB Join DC With D as centre, and DC as radius, describe a ⊙ From C let fall a ⊥ CE on AB, and produce it to meet the circumference in F It is required to prove that every ⊙ having its centre in AB, and passing through C, must pass through F

Dem —Take any other point G in AB, Join GC With G as centre, and GC as radius, describe a ⊙ Join FG Now EC = EF (III), and EG common, and the ∠ CEG = FEG , (I IV) CG = FG Hence the second ⊙ must pass through F

3 Let CDE be the given ⊙, AB the given line, and F the given point It is required to draw a chord in CDE which shall subtend a right ∠ at F, and be ∥ to AB

Sol —Let G be the centre of CDE From G let fall a ⊥ GK on AB Join FG, and produce it to meet the ⊙ in E Bisect EG in H Erect HM ⊥ to EG, and make it equal to GH

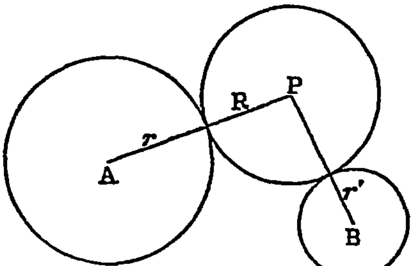


Join GM Bisect FG in N, and erect NP ⊥ to FG With G as centre, and GM as radius, describe a ⊙, meeting NP in P With N as centre, and NP as radius, describe a ⊙, cutting GK in Q Through Q draw CD ∥ to AB CD is the required line

Dem —Join GP, GC, CF, QF, QN, FD Now, since EG = 2 GH, $EG^2 = 4\ GH^2$, but $MG^2 = MH^2 + HG^2 = 2\ GH^2$ Hence $EG^2 = 2\ MG^2 = 2\ GP^2 = 2\ PN^2 + 2\ NG^2 = 2\ GN^2 + 2\ NQ^2$, but $2\ GN^2 + 2\ NQ^2 = QG^2 + QF^2$ (II x , Ex 2), and $EG^2 = GC^2$, ∴ $GC^2 = QG^2 + QF^2$, but $GC^2 = QC^2 + QG^2$, ∴ $QF^2 = QC^2$, and QF = QC, but QC = QD (III.), hence the three lines QC, QF, QD are equal, ∴ (I XII , Ex 2) the ∠ CFD is right.

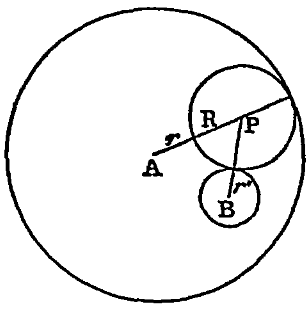
PROPOSITION XIII

1 (1) Dem —Let A, B be the centres of the fixed ⊙s, and P the centre of the variable one Join AP, BP, and let the radii be denoted by R, r, r'. Now AP = R + r, and BP = R + r'; $\therefore$ AP − BP = $r - r'$.



(2) If the contact of the variable ⊙ with the ⊙ whose centre is B be of the second species, we have AP = R + r, and BP = R − r', $\therefore$ AP − BP = $r + r$

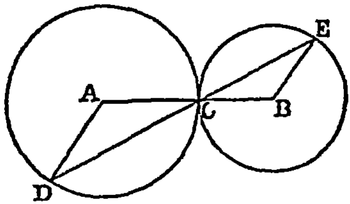
2 (1) Dem —Let the ⊙ whose centre is P touch that whose centre is A internally, and be touched by the one whose centre is B externally, then, denoting the radii as in the last Exercise, we get AP = r − R, BP = r + R, and $\therefore$ AP + BP = $r + r'$



(2) If the ⊙ whose centre is B touches the variable ⊙ internally, we get AP = r − R, and BP = R − r', $\therefore$ AP + BP = $r - r$

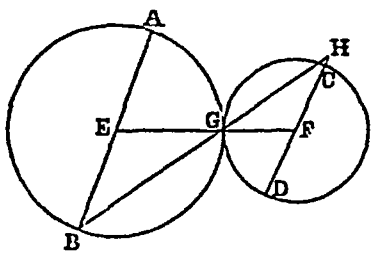
3 Dem —Let A, B be the centres, and C the point of con-

tact Join AB Through C draw DE, meeting the ⊙s in D, E. Join AD, BE



Now the ∠ ADC = ACD, and BCE = BEC, but ACD = BCE (I xv), ∴ ADC = BEC, and hence (I xxvii) AD is ∥ to BE

4 Let AB, CD be the diameters, G the point of contact, and E, F the centres Join BG It is required to prove that BG produced must pass through C



Dem —If possible, let it pass through H Produce DC to meet BH Join GE, GF

Now the ∠EBG = FHG (I xxix), but EBG = EGB = FGH, ∴ FHG = FGH, ∴ FG = FH, but FG = FC, ∴ FC = FH, which is absurd Hence BG produced must pass through C. In like manner DG produced must pass through A

PROPOSITION XIV

(1) Dem —Let ABC be the fixed ⊙, and AB the chord. From the centre D let fall a ⊥ DE on AB Join AD

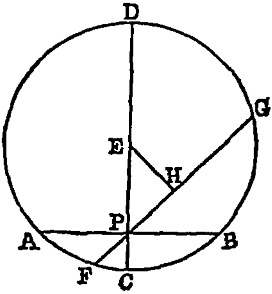
Now AB is bisected in E (III), ∴ AE is a line of given length, and AD is given, since it is the radius, but $AD^2 = AE^2 + DE^2$, ∴ DE is given, and the point D is given Hence the locus of E is a ⊙

(2) Let ABC be the O, AB the chord, and E any fixed point in AB

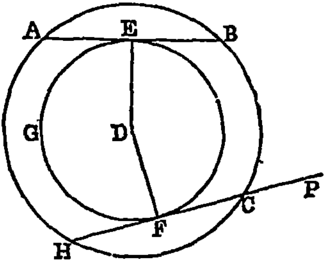
Dem —Let D be the centre Join AD, BD, ED Now, because AB is given, and E is a fixed point in it, AE and EB are each given, AE EB is given, and because ADB is an isosceles Δ, AE EB = $BD^2 - DE^2$ (II v Ex 5, or VI, Ex 6), but AE EB is given, and BD^2 is given, since BD is the radius, DE is given, and the point D is given Hence the locus of E is a O.

PROPOSITION XV

1 Let ABC be the O, and P the point Through P draw a chord AB ⊥ to the diameter CPD It is required to prove that AB is the minimum chord



Dem —Through P draw any other chord FG, and from E, the centre, let fall a ⊥ EH on it. Now the ∠ EHP is right,



EPH is acute, EP is greater than EH, (XV) FG is greater than AB

2 Let ABC be the given O, AB the given chord, and P the

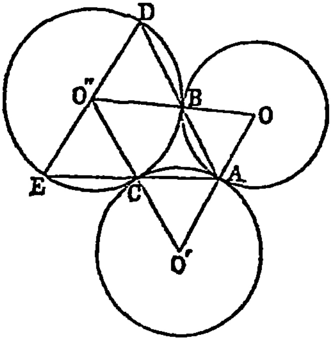
point It is required through the point P, to draw a chord equal in length to AB

Sol.—From the centre D let fall a ⊥ DE on AB With D as centre, and DE as radius, describe a ○ EFG Through P draw PCFH, touching EFG in F, and cutting ABC in C and H CH is the chord required

Dem —Join DF Now because DF = DE, (xiv) CH = AB

3 See "Sequel to Euclid," Book III, Prop xv, 6th Edition

4 Dem —Let O, O, O be the centres Now the lines joining OO′, O O″, O′O must pass through A, C, B (xii)

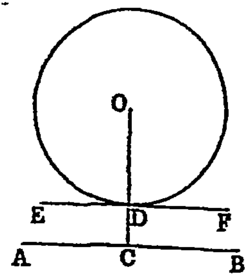


And because OA = OB, the ∠ OBA = OAB Similarly, the ∠ O″BD = O″DB but O″BD = OBA, hence O″DB = OAB, and O″D is ∥ to OA. In like manner O′E is ∥ to OA, and hence O″D, O″E are in the same straight line

PROPOSITION XVI

1 Dem —Let D be the common centre, and AB, CH the chords of the greater which touch the less, then AB = CH (xiv) See diagram to Prop xv, Ex 2

2 Let AB be the given line, and O the centre of the given



○ It is required to draw a ∥ to AB which shall touch the ○.

Sol —Let fall a ⊥ OC on AB, and through D, where OC cuts the ○, draw EF ∥ to AB EF is the required line

Dem —Now the ∠ ODF = OCB (I xxix), ∴ ODF is a right ∠, hence (xvi) EF touches the ○

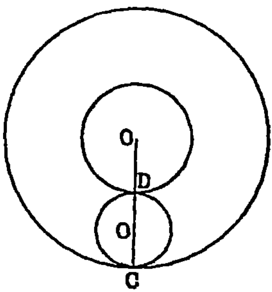
3 Let AB be the given line, and O the centre of the given ○ It is required to draw a ⊥ to AB which shall touch the ○

Sol —From O let fall a ⊥ OC on AB Draw OF ∥ to AB, and from F, where it meets the ○, draw FB ∥ to OC FB is the required line

Dem —The ∠s OCB, FBC are together equal to two right ∠s (I xxix), ∴ the ∠ FBC is right, and FB is ⊥ to AB, and (xvi) FB touches the ○

4 (1) Sol —Let O be the given point, and AB the given line Let fall a ⊥ OC on AB With O as centre, and OC as radius, describe a ○ Hence there is only one solution

(2) Let O be the given point, and O′ the centre of the given ○ It is required to describe a ○ having its centre at O, and touching the ○ whose centre is O′.

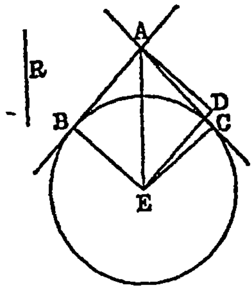


Sol —Join OO′, and produce to meet the circumference of ○ in C, with O as centre, and OC as radius, describe a ○, or, with O as centre, and OD as radius, describe a ○ Hence there are two solutions

5 Let AB, AC be the given lines, and R the given radius. It is required to describe a ○, touching AB, AC, and having a r equal to R

Sol —Erect AD ⊥ to AB, and equal to R Draw DE ∥ to AB.

Bisect the ∠ BAC by AE, meeting the line DE in E E is the centre of the required O.



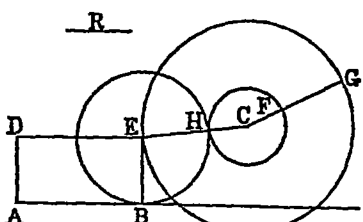
Dem —Draw EB, EC, ⊥ˢ to AB, AC

Now the ∠ BAE = CAE, and the right ∠ˢ ABE, ACE are equal, and AE common, (I xxvi) BE = CE, and the O, with E as centre and BE as radius, will pass through C There are evidently four solutions a O in each of the four ∠ˢ formed by BA, CA

6 Let AB, AC be the given lines, and E the centre of one of the Oˢ which touch AB, AC

Sol —Join AE, and produce it Join E to the points B, C, where the O touches AB, AC Now, since the ∠ˢ at B, C are right (xvi), $AE^2 = AB^2 + BE^2 = AC^2 + CE^2$, but $BE^2 = CE^2$, $AB^2 = AC^2$, AB = AC, AE common, and the base BE = CE, (I viii) the ∠ BAE = CAE, the ∠ between the lines is bisected by the line joining their intersection to the centre of one of the Oˢ Hence the locus of the centres is the pair of right lines bisecting the ∠ˢ between the two given lines

7 (1) Let C be the centre of the given O, AB the given



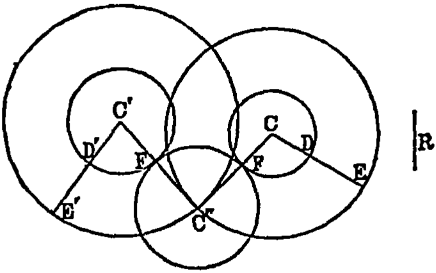
line, and R the radius It is required to describe a O that shall touch the ○ whose centre is C and the line AB, and have a radius equal to R

Sol —Take any point A in AB, and erect AD ⊥ to it and = R, draw DE ∥ to AB, from C draw any radius CF, and produce it to G, so that FG = R With C as centre, and CG as radius, describe a ○ cutting DE in E E is the centre of the required ○

Dem —Join CE, and draw EB ∥ to AD Now CG = CE, and CF = CH, ∴ FG = EH, but FG = R, ∴ EH = R, and EB = AD = R, ∴ EH = EB, and the ○, with E as centre and EB as radius, will pass through H Hence it will touch the given ○, the given line, and have a radius of given length

(2) Let C, C′ be the centres of the given ○s, and R the given radius

Sol —Draw any two radii CD, C′D′, and produce them to E, E′, so that DE, D′E′ are each equal to R, with C, C′ as centres, and CE, C′E′ as radii, describe two ○s Let them intersect in C″. C″ is the centre of the required ○



Dem —Join CC″, C′C″ Now CE = CC″, and CD = CF; hence DE = FC″, but DE = R (const), ∴ FC″ = R In like manner F′C″ = R, ∴ the ○ described with C″ as centre, and C″F as radius, will pass through F′, and touch the two ○s, and have the given radius

PROPOSITION XVII

2 Let O be the common centre From any points A, B, on the outer ○ tangents AC, BD are drawn to the inner one It is required to prove that AC = BD

Dem.—Join OA, OB, OC, OD. Now (XVI.) the ∠s at C, D are right; ∴ $OA^2 = OC^2 + CA^2$ and $OB^2 = OD^2 + DB^2$, but $OA^2 = OB^2$, and $OC^2 = OD^2$; ∴ $AC^2 = BD^2$; ∴ $AC = BD$.

3. Let ABCD be the quad. It is required to prove that $AB + CD = AD + BC$.

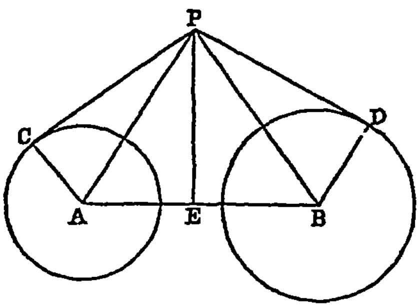
Dem.—Let E, F, G, H be the points of contact. Now (XVII., Ex. 1) $AE = AH$, and $BE = BF$; ∴ $AB = AH + BF$. In like manner $CD = DH + CF$; ∴ $AB + CD = AD + BC$.

4. Dem.—Let ABCD be the circumscribed ▭. Now $AB + CD = 2\,CD$, and $AD + BC = 2\,AD$; but $AB + CD = AD + BC$; ∴ $2\,CD = 2\,AD$; ∴ $CD = AD$. In like manner all the sides are equal. Hence ABCD is a lozenge.

Again, the line joining the centre to the intersection of tangents bisects the ∠ between the tangents; conversely, the line bisecting the ∠ between the tangents passes through the centre; therefore AC passes through the centre. Similarly, BD passes through the centre. Hence E is the centre.

5. Dem.—$OB = OD$, and OP common, and the base $BP = DP$; ∴ (I. VIII.) the ∠ $BOP = DOP$. Again, $OB = OD$, OF common, and the ∠ $BOF = DOF$; ∴ (I. IV.) the ∠ $OFB = OFD$. Hence each is a right ∠, and OP is ⊥ to BD.

6. Let A, B be the centres of ⊙s. Let P be a point from which the tangents PC, PD to the ⊙s are equal. It is required to prove that the locus of P is a right line.

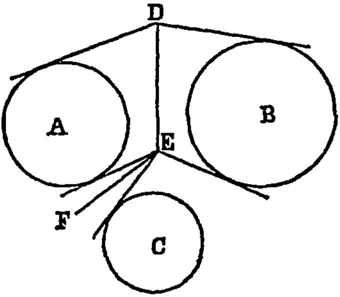


Dem.—Join AC, AP, BD, BP, and from P let fall a ⊥ PE on AB. Now $AP^2 = AC^2 + CP^2$; ∴ $CP^2 = AP^2 - AC^2$. In like manner $DP^2 = BP^2 - BD^2$, but $CP^2 = DP^2$; ∴ $AP^2 - AC^2 = BP^2 - BD^2$; ∴ $AP^2 - BP^2 = AC^2 - BD^2$, but $AC^2 - BD^2$ is given, since AC, BD are the radii of the ⊙s; ∴ $AP^2 - BP^2$ is given,

$AE^2 - EB^2$ is given, E is a given point, hence EP is given in position, and therefore the locus of P is the right line EP (*called the radical axis of the two* ○s)

Cor —*To construct the line* EP, *join the centres, divide the joining line in* E, *so that* $AE^2 - EB^2 = AC^2 - BD^2$, *and erect* EP ⊥ *to* AB

7 Let the three ○s be denoted by A, B, C It is required



to find a point such that the tangents from it to A, B, C shall be equal

Sol —Find a line DE, such that the tangents from any point of it to A and B will be equal (XVII, Ex 6), and find a line FE, such that the tangents from any point of it to A and C shall be equal E, where the lines DE, FE intersect, is evidently the required point.

8 Dem —OBP is a right-angled △, and BF is ⊥ to OP (XVII, Ex 5), (I XLVII, Ex 1) $OB^2 = OF \cdot OP$

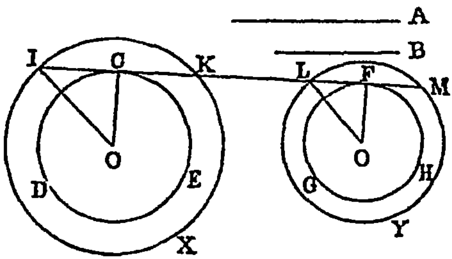
9 Let AB, AC be two fixed tangents, and EF a variable tangent cutting AB, AC in E, F, and touching the ○ in D Let O be the centre Join OE, OF It is required to prove that the ∠ EOF is constant

Dem —Join OB, OC, OD Now (I VIII) the ∠ EOD = EOB, EOD = ½ BOD In like manner FOD = ½ COD, EOF = ½ BOC, but the ∠ BOC is constant, since the tangents AB, AC are fixed, the ∠ EOF is constant.

10 (1) See "Sequel to Euclid," Book III, Prop VII.

(2) Draw a line cutting two ⊙ˢ, X, Y, so that the intercepted chords shall be of given lengths A, B

Sol —Let O, O′ be the centres of X, Y, R, R′ their radii Then with O, O′ as centres, describe ⊙ˢ CDE, FGH, the squares



of whose radii shall be equal to $R^2 - \frac{1}{4}A^2$, and $R'^2 - \frac{1}{4}B^2$ respectively, and draw the line IM a common tangent to both ⊙ˢ IM is the line required

Dem.—Let C, F be the points of contact Join OC, OI, OF, O′L Now $OC^2 = OI^2 - IC^2 = R^2 - IC^2$, but $OC^2 = R^2 - \frac{1}{4}A^2$ (const), ∴ $IC^2 = \frac{1}{4}A^2$ Hence $IC = \frac{1}{2}A$, but $IC = \frac{1}{2}IK$ (III iii), ∴ IK = A In like manner LM = B

PROPOSITION XXI

1 (1) Let ABC be a Δ, whose base BC, and vertical ∠ BAC, are given From B, C let fall ⊥ˢ BE, CF on AC, AB, and let them intersect in G It is required to find the locus of G

Dem —The four ∠ˢ A, F, G, E of the quad AFGE are together equal to four right ∠ˢ (I xxxii Cor 3), but the ∠ˢ E, F are right, ∴ the ∠ˢ A, G are together equal to two right ∠ˢ, but A is given (hyp), ∴ G is given, ∴ (I xv) the ∠ BGC is given And hence (xxi, Cor 2), the locus of G is a ⊙

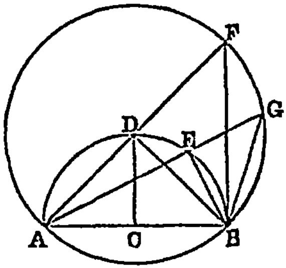
(2) Let the internal bisectors meet in D Now, the three ∠ˢ of the Δ ABC are equal to two right ∠ˢ, but the ∠ A is given, ∴ the sum of the ∠ˢ B, C is given, ∴ half their sum is given,

that is, DBC + DCB is given, ∴ the ∠ BDC is given, and hence (xxi, Cor 2) the locus of D is a ○

(3) Let the external bisectors meet in E Then, as before, the sum of the ∠s B, C is given, ∴ (I xxxii, Ex 14) the ∠ E is given Hence (xxi, Cor 2) the locus of E is a ○

(4) **Dem** —Let the external bisector of the ∠ C, and the internal bisector of B meet in F, then the ∠ BFC = ½ BAC (I xxxii, Ex 2), ∴ the ∠ BFC is given Hence (xxi, Cor 2) the locus of F is a ○

2 Let AB² be equal to the sum of the squares of the two lines



It is required to prove that their sum is a maximum when the lines are equal

Sol —Upon AB describe a semicircle ADB Bisect AB in C, and erect CD ⊥ to AB Join AD, BD In ADB take any other point E Join AE, BE Produce AD to F, so that DF = DB Join BF Produce AE to G, so that EG = EB, and join BG

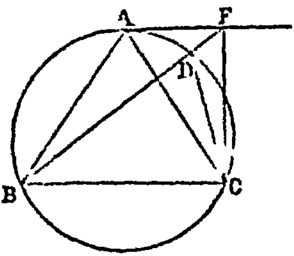
Dem —The ∠ DFB = DBF (I v), but BDF is a right ∠, ∴ DFB is half a right ∠ Similarly, EGB is half a right ∠, hence (xxi, Cor 1) the four points A, F, G, B are concyclic Now, since D is a point in a ○ from which the three equal lines DA, DB, DF are drawn to the circumference, D is the centre, ∴ AF is the diameter, but the diameter is the greatest chord, ∴ AF is greater than AG, that is, the sum of AD and DB is greater than the sum of AE and EB

3 Let there be two △s ADB, AEB on the same base AB, and having equal vertical ∠s, and let ADB be isosceles It is required to prove that the sum of the sides AD and DB is greater than the sum of the sides AE and EB (Diagram, Ex 2)

Dem —Produce AD to F, so that DF = DB Join BF Produce AE to G, so that EG = EB, and join BG Now the ∠ DFB = DBF (I v), but ADB = DFB + DBF (I xxxii), ∴ ADB = 2 DFB Similarly, AEB = 2 EGB, but ADB = AEB (hyp) ∴ DFB = EGB, and ∴ (xxi, Cor 1) the points A, F, G, B are concyclic, and it can be shown, as in Exercise 2, that AD + DB is greater than AE + EB

4 Dem —Let ABC be an inscribed Δ Then if any two sides AC, CB be unequal, by supposing the points A, B to remain fixed while C varies, the perimeter will be increased by making AC, CB equal Hence, when the three sides AB, BC, CA become all equal, the perimeter will be a maximum

Lemma —Let ABC, DBC be two Δs on the same base, inscribed



in a circle, of which ABC is isosceles It is required to prove that the area of ABC is greater than the area of BDC

Dem —Through A draw AF, touching the ○ Produce BD to meet it in F, and join CF Now the ∠ FAC = ABC (xxxii) = ACB, ∴ AF is ∥ to BC, hence (I xxxvii) the Δ BFC = BAC, but BFC is greater than BDC, ∴ BAC is greater than BDC Similarly it can be shown that BAC is greater than any other Δ inscribed in the ○, having BC for base, whose sides are unequal Hence the area of the isosceles Δ is a maximum

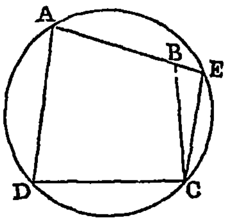
5 Let ABCDE be a polygon inscribed in a ○ It is required to prove that the area is a maximum when all the sides are equal

Dem —Join AC Now, if we suppose the point B to move about whilst the others remain fixed, when AB = BC, the Δ ABC will be a maximum (Ex 4), and therefore the area of the whole figure will be increased In like manner, if any other of the sides be unequal, we can increase the area by making them equal Hence the area will be a maximum when all the sides are equal

PROPOSITION XXII

1 Let ABCD be a quad whose opposite ∠s B, D are supplemental It is required to prove that it is cyclic

Dem —If not, let the ○ through A, D, C, intersect the line



AB produced in E Join CE Now the ∠s ADC, CBA are together equal to two right ∠s (hyp), and the ∠s ADC, CEA are equal to two right ∠s (xxii) Reject ADC, and we have the ∠ CBA = CEA, which is impossible (I xvi) Hence the ○ must pass through B

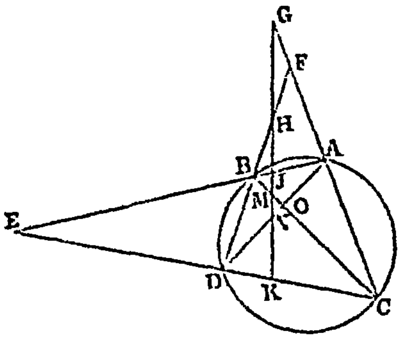
2 Let ABCDEF be a hexagon inscribed in a ○ It is required to prove that the sum of the alternate ∠s ABC, CDE, EFA is equal to four right ∠s

Dem —Join CF Now the ∠s ABC, CFA are together equal to two right ∠s (xxii), and the ∠s CDE, EFC, are equal to two right ∠s Hence, by addition, the ∠s ABC, CDE, EFA are equal to four right ∠s

3 (1) Let ABDC be a cyclic quad , and let the opposite sides meet in E, F Draw any line, GK, cutting the four sides, and making the ∠ EJK = EKJ It is required to prove that the ∠ GHF = HGF

Dem —The ∠s BDC and BAC are equal to two right ∠s (xxii), and the ∠s BAC, BAG equal to two right ∠s Reject the

∠ BAC, and we have the ∠ BDC = BAG, and the ∠ DKJ = AJG (hyp.), ∴ the remaining ∠ DHK = AGJ, that is, the ∠ GHF = HGF.



(2) Let GK cut the diagonals in M, N. It is required to prove the ∠ OMN = ONM.

Dem.—The ∠ LJK = EKJ (hyp.), and the ∠ ABC = ADC (xxi.), ∴ the remaining ∠ BMJ = DNK, that is, the ∠ OMN = ONM.

4. Bisect the ∠ AEC by ES, meeting the diagonals in Q, R. From O let fall a ⊥ OP on ES. It is required to prove that OP bisects the ∠ QOR.

Dem.—The ∠ ABC = BER + BRE (I. xxxii.), and ADC = DEQ + DQE, but (xxi.) ABC = ADC, ∴ BER + BRE = DEQ + DQE, but BER = DEQ (hyp.), ∴ BRE = DQE = OQR, and the ∠ OPR = OPQ. Hence the ∠ ROP = QOP.

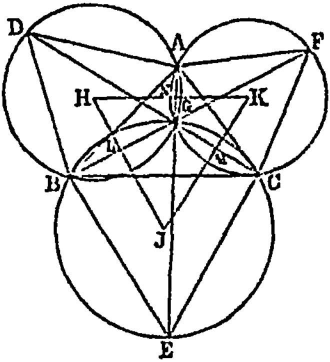
5. Let ABCDEF be a cyclic hexagon, having the side AB ∥ to DE, and BC to EF. It is required to prove that the side AF is ∥ to CD.

Dem.—Join CF. Now the ∠ ABC = DEF (I. xxix., Ex. 8), and since ABCF is a cyclic quad., the ∠s ABC, AFC are together equal to two right ∠s. For the same reason the ∠s DCF, DEF are equal to two right ∠s. ∴ the ∠s ABC and AFC = DCF and DEF, but ABC = DEF, ∴ AFC = DCF. And hence (I. xxvii.) AF is ∥ to CD.

6. Dem.—Join AB. Now the ∠ BAD = BFD (xxi.), and BAC = BEC, ∴ BFD = BEC. And hence (I. xxviii.) CE is ∥ to DF.

7. On the sides of any △ ABC, equilateral △s are described, BF and CD joined and intersecting in G. Join AG, EG. It is required to prove that AG and GE are in the same straight line.

Dem.—Since AB = AD, and AC = AF, and the ∠ BAD



= CAF, to each add BAC, therefore the ∠ DAC = BAF; hence (I. iv.) the ∠ ADC = ABF, and ACD = AFB. Now, because the ∠ ACG = AFG, AFCG is a cyclic quad.; hence the ∠s AFC, AGC are together equal to two right ∠s (xxii.); similarly ADBG is a cyclic quad., and the ∠s ADB, AGB are equal to two right ∠s; these four ∠s are together equal to four right ∠s, and the ∠s AGB, BGC, CGA are equal to four right ∠s. Reject the ∠s AGB, AGC, and we have the ∠ BGC equal to the sum of AFC and ADB. To each add BEC, and we have BGC + BEC = AFC + ADB + BEC; but these three ∠s are equal to two right ∠s, since each is an ∠ of an equilateral △; BGC, BEC are equal to two right ∠s, and hence BGCE is a cyclic quad.; the ∠ EGC = EBC; EGC is equal to an ∠ of an equilateral △, and therefore equal to AFC; but AFC and AGC are equal to two right ∠s; EGC and AGC are equal to two right ∠s, and hence (I. xiv.) AG and EG are in the same straight line. Therefore AE, BF, CD are concurrent.

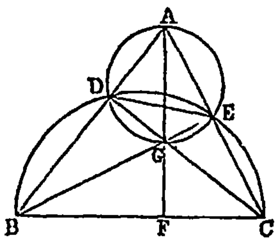
8. If we join the centres H, J, K, it is required to prove that HJK is an equilateral △.

Dem —Let HJ, JK, HK cut BG, CG, AG in the points L, M, N Now, because the ∠ˢ L, N are right (III, Cor 4), ∴ the ∠ˢ H, G are equal to two right ∠ˢ, and the ∠ˢ G, D are equal to two right ∠ˢ (Ex 8), hence the ∠ H = D, ∴ H is an ∠ of an equilateral Δ Similarly K is an ∠ of an equilateral Δ Hence the Δ HJK is equilateral

9 Let ABCD be the quad, O the centre of the inscribed circle, and E, F, G, H the points of contact Join O to A, B, C, D It is required to prove that the ∠ˢ AOB, DOC are supplemental

Dem —Join OE, OF, OG, OH Now the ∠ AOB = half sum of the ∠ˢ EOH, EOF (XVII, Ex 9), and the ∠ DOC = half sum of GOH, GOF, but the sum of EOH, EOF, GOH, GOF is four right ∠ˢ, ∴ AOB and DOC are together equal to two right ∠ˢ

10 Let ABC be a Δ, whose ⊥ˢ CD, BE intersect in G

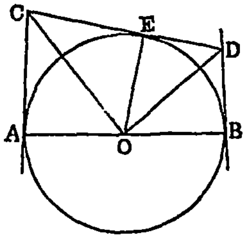


Join AG, and produce it to meet BC in F It is required to prove that AF is ⊥ to BC

Dem —Join DE Now, because each of the ∠ˢ ADG, AEG is right, ADGE is a cyclic quad, hence the ∠ DEG = DAG (XXI) Again, since the ∠ˢ BDC, BEC are right, the points B, D, E, C are concyclic, and therefore the ∠ DEB = DCB, ∴ DAG = DCB, and DGA = FGC (I XV), ∴ ADG = AFC, but ADG is a right ∠, ∴ AFC is a right ∠, and AF is ⊥ to BC

11 Let a variable tangent CD meet two ∥ tangents AC, BD Join the centre O to C, D. It is required to prove that the ∠ DOC is right

Dem —Draw the diameter AB, and join O to the point E where CD touches the ○



Now the ∠ DOC is equal to half the sum of the ∠ˢ EOB, EOA (xvii, Ex 9), but EOB and EOA are together equal to two right ∠ˢ, ∴ the ∠ DOC is right

12 See "Sequel to Euclid," Book III, Prop xii

13 Let ABCDEF be the hexagon, O the centre of the inscribed circle, and G, H, J, K, L, M the points of contact of the hexagon and circle Join O to the points A, B, C, D, E, F It is required to prove that the sum of the ∠ˢ AOB, COD, EOF is two right ∠ˢ

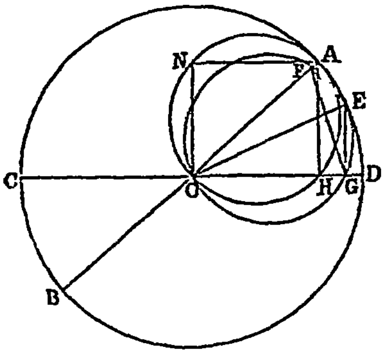
Dem —Join O to the points G, H, J, K, L, M Now, the ∠ AOB = ½ MOH (xvii, Ex 9), COD = ½ HOK, and EOF = ½ KOM, ∴ the sum of AOB, COD, EOF is two right ∠ˢ

PROPOSITION XXVIII

1 Let AB, CD be the two diameters given in position Take any point E in the circumference, and let fall ⊥ˢ EF, EG on AB, CD Join FG It is required to prove that FG is given in magnitude

Dem —(See diagram, Ex 2) Join OE, and from A let fall a ⊥ AH on CD Now, since the ∠ OHA is right, the ○ on OA as diameter will pass through H (xxxi), and because the ∠ˢ OFE, OGE are right, the ○ on OE as diameter will pass through F and G, but OA = OE, ∴ the ○ˢ on OA and OE are equal, and the ∠ AOH is in both these ○ˢ, ∴ the arc AH is equal to the arc FG (xxvi), and therefore the chord AH = FG, but AH is given in magnitude, since it is a ⊥ from the extremity of one of the diameters given in position on the other Hence FG is given in magnitude

2 Let OA, OD be two lines given in position, and FG a line



of given length sliding between them At the extremities of FG ⊥ˢ EF, EG are erected to OA, OD It is required to prove that the locus of E, where these ⊥ˢ meet, is a ○

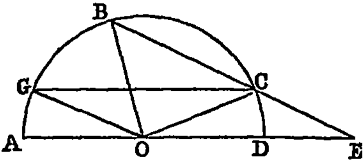
Dem —Join OE Erect ON ⊥ to OD, and equal to FG, draw NA ∥ to OD

Now, because ONA is a right ∠, the ○ described on OA as diameter will pass through N, for a similar reason, the ○ on OE as diameter will pass through F and G Now since ON and FG are equal, and subtend equal ∠ˢ OAN, FOG in the ○ˢ OAN, FOG, the ○ˢ are equal, therefore the diameters OA, OE are equal. Again, since ON = FG, ON is given, and AN is ∥ to OD, the point A is given, and hence the line OA is given in magnitude, but OE = OA, OE is given in magnitude, and the point O is given Hence the locus of E is a ○, having O as centre and OE as radius

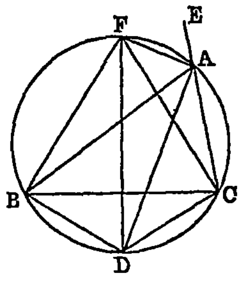
PROPOSITION XXX

1 Dem —Let O be the centre Through C draw CG ∥ to DA Join OB, OC, OG Now the ∠ GCO = COE (I xxix), but GCO = CGO, and CGO = AOG, DOC = AOG, the arc

DC = AG (xxvi) Again, the ∠ GOB is double GCB (xx), but GCB = AEB (I xxix), and AEB = COE, because CE = OC (hyp), ∴ GOB is double DOC, hence the arc GB is double CD, and therefore the arc AB is three times the arc CD



2 (1) Let AD be the internal bisector of the vertical ∠ of the △ ABC Join BD, CD It is required to prove that BD = CD



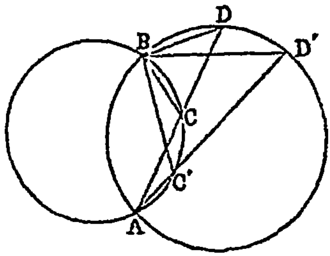
Dem —Because the ∠ BAD = CAD, the arc BD = CD (xxvi), and therefore the chord BD = CD (xxix)

(2) Produce CA to E Bisect the ∠ BAE by AF, meeting the circumference in F It is required to prove that the point F is equally distant from B and C

Dem —Join BF, CF Now the ∠s FBC and FAC are together equal to two right ∠s (xxii), and FAC and FAE are equal to two right ∠s (I xiii), ∴ the ∠ FBC = FAE Again, the ∠ BAF = BCF (xxi), but BAF = FAE, ∴ BCF = FAE, ∴ BCF = FBC, ∴ BF = CF

3 Dem —The ∠ ADB = AD′B (xxi), and the ∠ ACB = AC′B, but the ∠s ACB and DCB are together equal to two right ∠s,

and the ∠s AC'B, D'C'B are together equal to two right ∠s, ∴ the ∠ DCB = D'C'B, and hence (I xxxii, Cor 2) the remaining ∠s DBC, D'BC', are equal

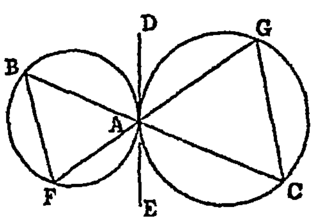


4 Dem —Join AD, DB Now because the ∠ ACD = BCD, the line AD = BD Again, the ∠s DBC, DAC are together equal to two right ∠s (xxii), and the ∠s DBC, DBF equal two right ∠s (I xiii), ∴ the ∠ DAE = DBF, and the right ∠s DEA, DFB are equal, ∴ (I xxvi) AE = BF Hence AC − CE = CF − CB, ∴ AC + CB = CF + CE = 2 CE, because CF = CE

PROPOSITION XXXII

1 Let the ⊙s touch in A Through A draw any line BAC It is required to prove that BAC divides the ⊙s into similar segments.

Dem —Through A draw a common tangent DE, take any

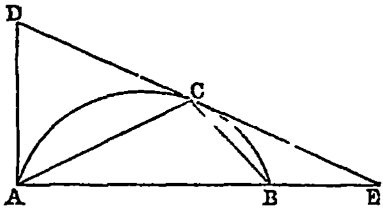


points F, G, in the ⊙s Join AF, BF, AG, CG Now the ∠ BAD = AFB (xxxii), and the ∠ CAE = AGC, but BAD = CAE (I xv), AFB = AGC, and hence the segments AFB, AGC are similar

2 Let the ○ˢ touch in A. Through A draw two lines BC, FG, meeting the ○ˢ in B, C, F, G Join BF, CG It is required to prove that BF, CG are ∥

Dem—Through A draw a common tangent DE Now it may be proved, as in Ex 1, that the ∠ AFB = AGC, hence (I xxvii), BF is ∥ to CG

3 **Dem**—Join AC, BC Now the lines CA, CD, CE are



equal (I xii, Ex 2), the ∠ AEC = EAC, but (xxxii) EAC = BCE, hence the ∠ BCE = BEC, BCE and BEC = 2 BEC, (I xxxii) the ∠ CBA = 2 BEC, but BEC = CAB, since CE = CA, CBA = 2 CAB Hence the arc AC = 2 CB

4 (1) See "Sequel to Euclid," Book III, Prop iii

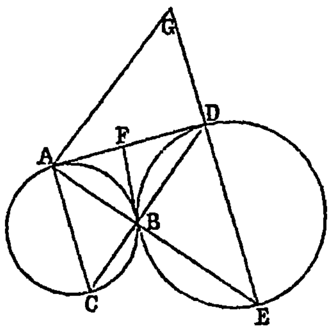
(2) **Dem**—Let GBF and ECH be the tangents to the ○ˢ at the points B, C Join CF, CG Now the ∠ CFG = FCH (I xxix.), but FCH = FGC (xxxii), GFC = FGC, and hence the chords GC, FC are equal

5 (1) Let the ○ˢ ABC, DBE touch at B Draw a common tangent AD Join AB, DB It is required to prove that the ∠ ABD is right

Dem—Draw a common tangent BF Now AF = BF (xvii, Ex. 1), the ∠ ABF = BAF, and because BF = DF, the ∠BDF = DBF, the ∠ ABD = BAD + BDA, and hence (I xxxii, Cor 7) the ∠ ABD is right

(2) **Dem**—Produce AB, DB to meet the circumferences in E, C Join AC, DE Produce ED to G, and draw AG ∥ to CD

Now, because the ∠ ABD is right, EBD is right, and therefore ED is a diameter, and hence (xix) the ∠ ADE is right, AD is ⊥ to EG Again, since AG, CD are ∥, the ∠ GAE = DBE,

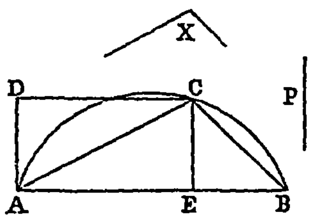


and is therefore a right ∠. Hence (I xlvii, Ex 2) AD^2 = DE DG = DE AC

PROPOSITION XXXIII

1 (1) Let AB be the base, X the vertical ∠, and P the ⊥

Sol —On AB describe a segment ACB containing an ∠ equal to X (xxxiii) Erect AD ⊥ to AB, and = P Through D draw



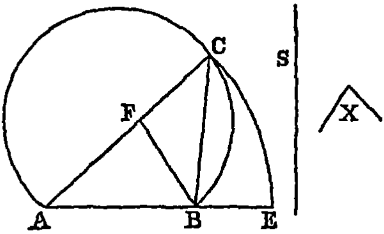
DC ∥ to AB, cutting the ○ in C. Join AC, CB ACB is the required △

Dem —Let fall a ⊥ CE on AB The vertical ∠ ACB = X, AB is the base, and the ⊥ CE = AD = P

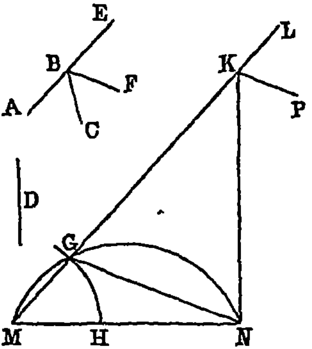
(2) Let the sum of the sides be equal to S

Sol.—On AB describe a segment ACB containing an ∠ equal ½ X Produce AB to E Cut off AE = S With A as centre, and AE as radius, describe a ○, cutting ACB in C Join AC, BC, and at the point B in the line BC make the ∠ FBC = FCB AFB is the required △

Dem —FC = FB (I vi), ∴ AC = AF + FB, but AC = AE = S, ∴ AF + FB = S, and the ∠ AFB = FBC + FCB (I xxxii) = 2 FCB = X



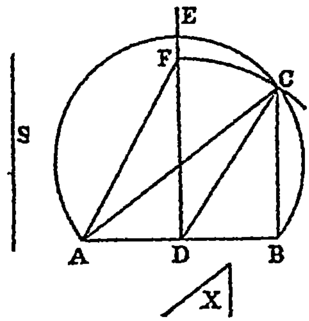
(2′) Let MN be the base, D the difference of sides, and ABC the vertical ∠.



Sol —Produce AB to E Bisect the ∠ CBE by BF On MN describe a segment MGN containing an ∠ = ABF, in MN take MH = D With M as centre, and MH as radius, describe a ○, cutting MGN in G Join MG, NG Produce MG, and at the point N in GN make the ∠ GNK = NGK MKN is the required △

Dem —Produce MK to L, and draw KP ∥ to GN Now KN = KG (I vi), ∴ MG is the difference between MK and NK; but MG = MH = D, ∴ the difference between MK and NK is equal to D Again, the ∠ PKN = GNK (I xxix), and LKP = KGN, but GNK and KGN are equal (const), ∴ PKN and LKP are equal, and since the ∠ MKP = MGN = ABF, the $\angle$ LKP = EBF, but LKP = NKP, and EBF = FBC, FBC = NKP Hence the $\angle$ MKN = ABC

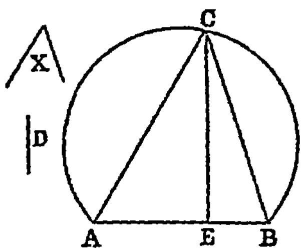
(3) Let AB be the given base, X the vertical $\angle$, and let the sum of the squares of the sides be equal to 2 S^2



Sol —On AB describe a segment containing an $\angle$ = X Bisect AB in D, and erect DE $\perp$ to AB, from A inflect AF on DE = S (I ii, Ex 2) With D as centre, and DF as radius, describe a ○, cutting ACB in C Join AC, BC ACB is the Δ required

Dem.—Join CD Now, DF = DC, $DF^2 = DC^2$, $AD^2 + DF^2 = AD^2 + DC^2$, AF^2, that is $S^2 = AD^2 + DC^2$, but $AC^2 + CB^2 = 2\ AD^2 + 2\ DC^2$ (II x, Ex 2) Hence $AC^2 + CB^2 = 2\ S^2$

(3) Let AB be the base, X the vertical $\angle$, and D^2 the difference of the squares of the sides

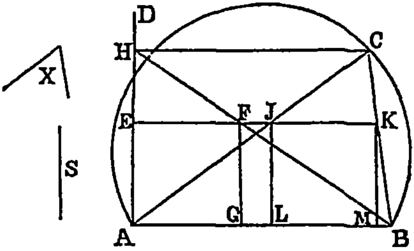


Sol —On AB describe a segment ACB containing an $\angle$ = X Divide AB in E, so that $AE^2 - EB^2 = D^2$ ("Sequel," Book I, Prop ix) Erect EC $\perp$ to AB, and join AC, BC ACB is the Δ required.

Dem —$AC^2 = AE^2 + EC^2$, and $BC^2 = BE^2 + EC^2$, $AC^2 - BC^2 = AE^2 - EB^2 = D^2$

(4) Let AB be the base, X the vertical ∠, and S the side of the inscribed square

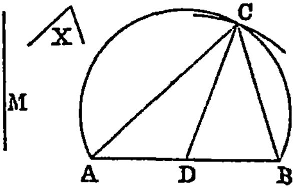
Sol —On AB describe a segment containing an ∠ = X Erect



AD ⊥ to AB In AD take AE = S On AE describe a square AEFG Join BF, and produce it to meet AD in H Through H draw HC ∥ to AB, meeting the ○ in C Join AC, BC, ACB is required △

Dem —Produce EF to meet AC, BC in J, K, and draw JL, KM ∥ to AE Now, JK = EF (I xxxviii, Ex 6), but EF = AE = JL JK = JL, hence the sides of JKLM are equal, and the ∠s are right (const.), it is a square, and is inscribed in the △ ABC

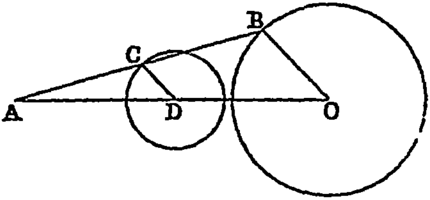
(5) Let AB be the base, M the median, and X the vertical ∠



Sol —On AB describe a segment ACB containing an ∠ = X, bisect AB in D With D as centre, and a radius equal to M, describe a ○, cutting ACB in C Join AC, BC, DC ACB is the △ required

Dem —Because D is the centre of the ○ cutting ACB, DC is the radius, but the radius is equal to M, DC = M, and it is the median bisecting the base AB

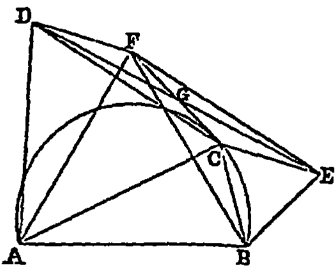
2 Let A be the fixed point, and O the centre of the given circle, Take any point B in the circumference of the ○ Join AB, and bisect it in C It is required to prove that the locus of C is a ○



Dem —Join AO, OB, and through C draw CD ∥ to OB

Now AO is bisected in D (I XL, Ex 3), but A and O are given points, the point D is given, and since CD is ∥ to OB, CD = ½ OB, but OB is a given line, CD is given, and the point D is given Hence the locus of C is a ○, having D as centre and DC as radius

3 Let AB be the base, and ACB the vertical ∠ About ACB describe a segment of a ○ containing an ∠ = ACB, then the

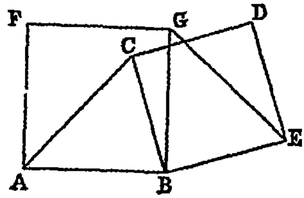


circle must pass through C On AC, BC describe equilateral △s ADC, BEC Join DE It is required to find the locus of the middle point of DE

Dem —On AB describe an equilateral △ AFB Join CF, DF, EF Now the ∠ BAF = DAC, the ∠ BAC = DAF, and since DA = AC, and BA = AF, we have DA and AF equal AC and AB, and the contained ∠s are equal, hence (I. IV) DF = CB = CE Similarly, DC = EF, DCEF is a ▭, hence (I XXXIV, Ex 1) DE, CF bisect each other in G Now F is a given point, and C a point on the circumference of the

O, and FC is bisected in G, (Ex 2) the locus o

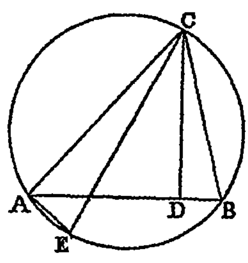
4. Let ACB be a △, whose base and vertical ∠ are gi
BC describe a square BEDC It is required to find the



E On AB describe a square ABGF Join EG Now AB and BC = GB and BE, and the contained ∠s are equal, (I iv) the ∠ ACB = BEG, BEG is a given ∠, and the base BG is given, since it is equal to AB, (xxi, Cor 2) the locus of E is a O

PROPOSITION XXXV

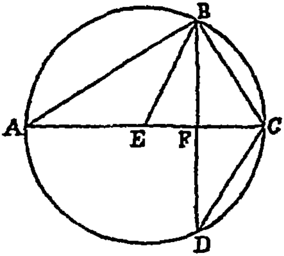
1 Let ACB be the △ About ACB describe a O Draw the diameter CE, and from C let fall a ⊥ CD on AB It is required to prove that AC CB = CD CE.



Dem —Join AE Now the ∠ CAE is right (xxxi), and is equal to CDB, and the ∠ AEC = ABC (xxi), (I. xxxii, Cor 2) the ∠ ACE = BCD, and hence (xxxv, Cor 3) AC CB = CD CE

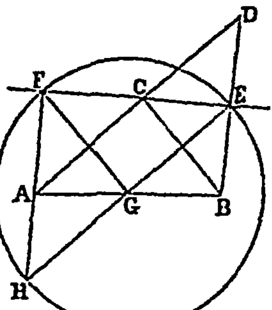
2 Let ABD be a O, of which AC is the diameter, let AB be the chord of an arc, then BC is the chord of its supplement join B to the centre E Let fall a ⊥ BF on AC, and produce it

2 Let A the circumference in D It is required to prove that
circle, T = BE . BD
AB, and
C is a



Dem —Join CD Now the ∠ BDC = BAC (xxi), but BAC = ABE, and BDC = DBC, ABE = DBC, hence the Δs ABE, DBC are equiangular, and (xxxv, Cor 3) AB BC = BE BD

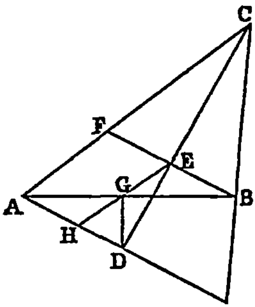
3 Let ABC be a Δ whose base and the sum of whose sides are given Produce AC to D, and bisect the ∠ BCD by EF From A, B let fall ⊥s AF, BE on EF It is required to prove that AF BE is given



Dem —Produce BE to meet AD Bisect AB in G Join EG, FG Now because the ∠ BCE = DCE, and CEB = CED, each being right, and CE common, (I xxvi) BE = DE, and BC = DC Now, since BC = DC, AD = AC + CB, hence AD is given, and because AB, DB are bisected in G, E, GE is ∥ to AD, and equal to half AD (I xl, Exs 2 and 5), that is, = ½ (AC + CB) Similarly, GF = ½ (AC + CB), the ○, with G as centre, and GE as radius, will pass through F, and will be a given ○ Produce EG to meet the circumference in H, and join AH Now because AG = GB, and GH = GE, and the

∠ AGH = BGE, (I IV) AH = BE, and the ∠ GAH = GBE To each add the ∠ GAF, and we have the ∠ˢ GAH, GAF = GBE, GAF, but GBE, GAF are equal to two right ∠ˢ, since BE and AF are ∥, GAH and GAF are equal to two right ∠ˢ, hence AH, AF are in the same straight line Now FGH is an isosceles Δ, (II VI Ex 6) AF AH = $FG^2 - AG^2$, but FG is given, since it is half the sum of AC and CB, and AG is given, because it is half AB Hence AF AH is given, that is, AF BE is given

4 Let ABC be a Δ whose base AB, and the difference of whose sides AC, CB is given Bisect the ∠ ACB by CD From



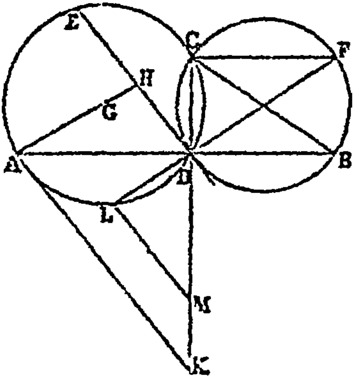
A, B let fall the ⊥ˢ AD, BE on CD It is required to prove that AD BE is given

Dem —Produce BE to meet AC in F Bisect AB in G Join EG, and produce it to meet AD in H Join GD Now because the ∠ BCE = FCE, and the ∠ BEC = FEC, and CE common, (I XXVI) CB = CF, and EB = EF, AF is the difference between AC and BC, and because EB = EF and GB = GA, GE is ∥ to AF, and equal to half AF (I XL, Exs 2 and 5) or half EH, GE = GH, and the three lines HG, EG, DG are equal (I XII, Ex 2), the Δ HGD is isosceles, hence (II VI Ex 6) AD AH = $AG^2 - GH^2$, but AG is given, since it is half AB, and GH is given, because it is equal EG = ½ AF, AD AH is given, that is, AD BE is given

5 Let ACD, BCD be two ○ˢ intersecting in C, D At D draw a tangent to the ○ BCD, meeting ACD in E From G, the centre of ACD, let fall a ⊥ GH on DE, and let it meet ACD in

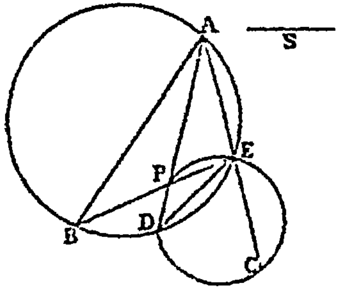
A Join AD, and produce it to meet ⊙ CD in B AB is the required line

Dem —Draw AK ∥ to DE. Join CD, and produce it to meet

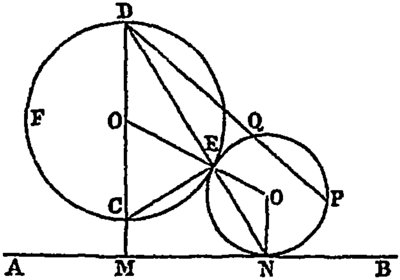


AK Take any other point L in ACD, and draw LM ∥ to DE Join LD, and produce it to meet BCD in F Join CF, CB Now the ∠ EDC = CBD (xxxii), but EDC = AKC (I xxix), ∴ AKC = CBD, ∴ AKBC is a cyclic quadrilateral, hence (xxxv, Cor 3) AD . DB = CD . DK In like manner LMFC is a cyclic quadrilateral, ∴ LD . DF = CD . DM, but CD . DK is greater than CD . DM, ∴ AD . DB is greater than LD . DF

8 Let AB, AC be two lines given in position, and P a given point. It is required through P to draw a transversal, so that PE . PB = S²



Sol —Join AP, and produce it to D, so that AP . PD = S² On PD describe a segment of a ⊙ PED, cutting AC in E, and containing an ∠ = BAD Join ED, EP, and produce EP to meet AB EPB is the required line given ○, through O draw DOCM ⊥ to AB, and in AP find Q so that DP DQ = DC DM Describe a ○ through P, Q touching AB in N (Ex 1(1)) This ○ shall be the one required



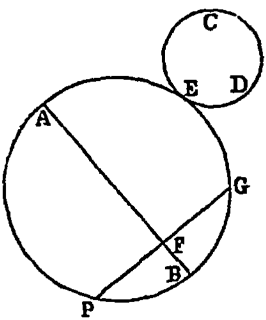
Dem —Join DN, cutting the ○ QPN in E Now (xxxvi) DP DQ = DE DN, but DP DQ = DM DC, DM DC = DE DN, CMNE is a cyclic quad, the ∠ DEC = CMN, and is right, hence E is a point on the ○ DFC

Again the ∠ O'EN = O'NE, and OED = ODE, but ODE = O'NE, O'EN = OED, and OE, O'E are in the same right line, the ○s touch at E Since we can describe two ○s through PQ touching AB, there are two solutions for this figure Also, if we had taken Q so that DP DQ = DM MC we would get two other solutions Hence there are four solutions to the problem.

3 Let AB be the line, CDE the ○, and P the point

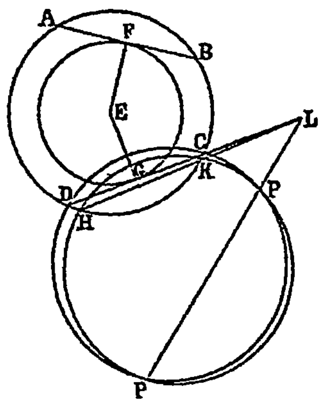
Sol —From P let fall a ⊥ PF on AB, and produce it until FG = PF, and through P and G describe a ○ PEG, touching CDE (Ex 1) PEG is the required ○

Dem —Because PG is bisected at right ∠s by AB, the centre PEG is in AB (iii)



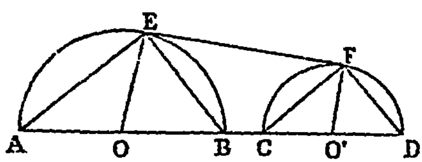
4 Let ABCD be the given ○, P, P the points

Sol.—Draw a line AB, cutting off an arc AB in ABCD equal to the given arc Let E be the centre From E draw EF ⊥ to AB With E as centre, and EF as radius, describe a ○ FG Through P, P′ describe a ○ PP KH, cutting ABCD, in KH Join HK, PP, and produce them to meet in L Through L draw LCGD, a tangent to FG, and cutting ABCD in C, D The ○ through P, P′, C will be the required one



Dem —Join EG Now because PP KH and DCKH are cyclic quads, PL LP′ = HL LK = DL LC, hence PP CD is a cyclic quad, the ○ through P, P′, C must pass through D, and since E is the centre of FG, EF = EG, AB = CD (xiv), and therefore the arc AB = CD Hence through P, P′ we have described a ○ PP′CD, intercepting an arc CD = AB, on a given ○ ABCD

5 Dem —Let O, O′ be the centres Join OE, O′F Now



since OE, O′F are each ⊥ to EF, they are ∥ to each other, hence the ∠ DOE = DO′F, but the ∠ BOE is (III xx) double of the

∠ BAE, and DO'F is double of DCF, hence the ∠ BAE = DCF In like manner, the ∠ABE = CDF Hence the △s ABE, CDF are equiangular

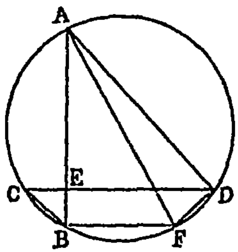
6 If r be the radius of the inscribed ○ of a right-angled triangle, by making the construction, we see at once that $2r$ is equal to the excess of the sum of the legs above the hypotenuse

Again, if ρ, ρ' be the radii of ○s touching the hypotenuse, the ⊥ from the right angle on the hypotenuse, and the ○ described about the right-angled △, it follows at once from the Demonstration, Book VI, Ex 59, that $\rho + \rho'$ is equal to the same excess Hence $2r = \rho + \rho'$

Miscellaneous Exercises on Book III

1 Let AB, CD, be two chords of a ○ intersecting at right △s It is required to prove that the sum of the squares of the four segments is equal to the square of the diameter

Dem —Draw BF ∥ to CD Join CB, FD, AF, AD Now $CB^2 = CE^2 + EB^2$, but CB = FD (xxvi, Cor 2), $FD^2 = CE^2 + EB^2$, and $AD^2 = AE^2 + ED^2$, $AD^2 + FD^2 = AE^2 + EB^2$

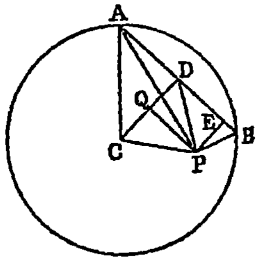


$+ CE^2 + ED^2$, but since the ∠ ABF = AED (I xxix), ABF is a right ∠, hence AF is the diameter, the ∠ ADF is right, $AF^2 = AD^2 + DF^2 = AE^2 + EB^2 + CE^2 + ED^2$

2 (1) Let AB, a chord of a given ○, subtend a right ∠ at a fixed point P From P, and C, the centre of the ○, let fall ⊥s PE, CD on AB It is required to prove that CD PE is constant

Dem —Join CP, CA, PD, and let fall a ⊥ PQ on CD Now AB is bisected in D (III), the lines AD, DP, DB are equal

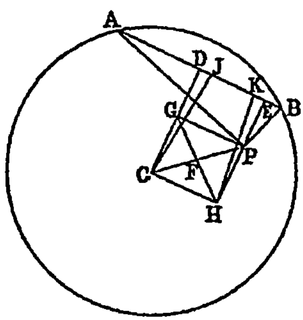
(I xii, Ex 2), and $AC^2 = AD^2 + DC^2 = DC^2 + DP^2$, but $DC^2 + DP^2$ is greater than CP by 2 CD DQ (II xiii), that is, by



2 CD PE, AC^2 is greater than CP^2 by 2 CD PE, but AC^2 and CP^2 are given, CD PE is given

(2) Join CP, and bisect it in F Erect FG ⊥ to CP, and equal to CF or PF Produce GF to H, so that FH = FG, and join CG, PG, CH, PH From C, G, H, P let fall ⊥s CJ, GD, HK, PE on AB It is required to prove that $GD^2 + HK^2$ is constant

Dem —Because CGPH is a square, $GD^2 + HK^2$ is greater than 2 CJ PE, by the area of CGPH ("Sequel," Book II Prop viii),



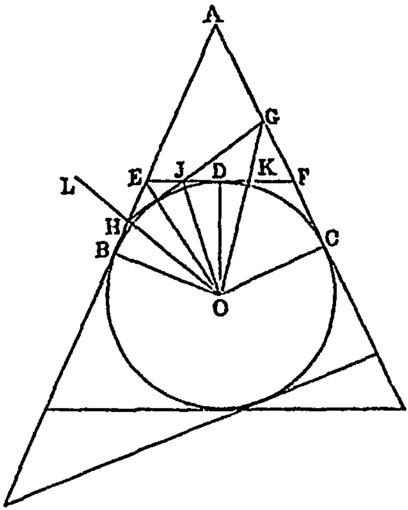
but 2 CJ PE is given (1), and the area of the square is given. Hence $GD^2 + HK^2$ is given

3 Let the ⊙s intersect in A, B Through B draw a line BCD, meeting the ⊙s in C, D Join AC, AD It is required to prove that AC = AD

Dem —Because the ⊙s are equal, the arcs AB are equal, the ∠s ACB, ADB are supplemental Hence the ∠s ACD, ADC are equal And hence AC = AD

4 (1) Let AB, AC be two fixed tangents, and EF a tangent cutting off with AB, AC, an isosceles Δ AEF AEF is greater than any other Δ AHG, made by a tangent HG, which does not cut off an isosceles Δ with AB, AC

Dem —Let EF, HG intersect in J Join OJ, OB, OC, OD, OG, OH, and produce OH to L Now, because AB = AC, and AE = AF, BE = CF, but BE = DE, and CF = DF, DE = DF, JF is greater than JE



Again, the ∠ HOG = BOD, because each = ½ BOC (xvii., Ex 9), and HOJ = ½ BOD, HOJ = JOG, and the ∠ HJO = KJO, and JO common, (I xxvi) JH = JK Now the ∠ LHG is greater than HGO, but LHG = GKJ, because they are the supplements of the equal ∠s OHJ, OKJ, GKJ is greater than JGK, JG is greater than JK, JG is greater than JH, and JF is greater than JE, the Δ FJG is greater than EJH To each add the figure AGJE, and we have the Δ AEF greater than AHG

(2) Let the tangent be drawn below the ○, making an isosceles Δ with the fixed tangents, then it can be shown, as in (1), that the isosceles Δ is less than the Δ formed by any other tangent which does not cut off an isosceles Δ with the fixed tangents

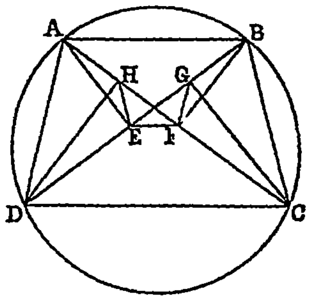
5 Dem —Join CF, DE, AB Now the ∠s ADE and ABE

are equal (xxi), and ACF, ABF equal, ∴ ADE, ACF are equal, ∴ CE is ∥ to DF, ∴ CDEF is a ▭, and ∴ (I xxxiv) CD = EF

6 See Book I , Miscellaneous Ex 45

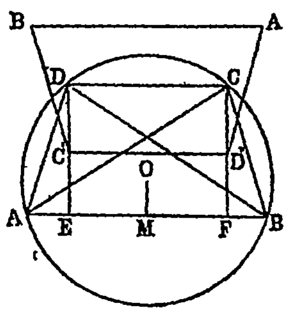
7 Let the sides of the cyclic quad ABCD be the diameters of four ○ˢ It is required to prove that those ○ˢ intersect in four concyclic points E, F, G, H

Dem —Draw the diagonals AC, BD, and let fall ⊥ˢ AE, BF, CG, DH on AC, BD Join HE, EF, FG Now, because the ∠ˢ AHD, CHD are right, the ○ˢ on AD, CD, as diameters, will pass through H In like manner the ○ˢ on the other sides will pass through E, F, G And since the ∠ˢ AHD, AED are right,



AHED is a cyclic quad , ∴ the ∠ˢ AHE, ADE are together equal to two right ∠ˢ (xxii), and the ∠ˢ AHE, FHE are equal to two right ∠ˢ, ∴ the ∠ ADE = FHE Similarly, BCF = EGF , but ADE = BCF (xxi), ∴ FHE = EGF And hence (xxi , Cor 1) the points E, F, G, H are concyclic

8 Let ABCD be a cyclic quad Draw the diagonals AC, BD



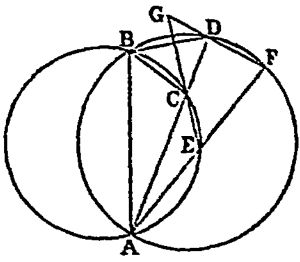
It is required to prove that the orthocentres of the Δs ADB, ACB, CAD, CBD are the angular points of a quad. which is equal to ABCD

Dem —From D and C let fall ⊥s DE, CF on AB Let C′, D′, be the orthocentres of the Δs ADB, ACB, and let A′, B′, be the orthocentres of the Δs BCD, ADC Join C D′, D A′, A′B, B′C′, and from O, the centre, let fall a ⊥ OM on AB

Now OM = ½ CD′ ("Sequel," Book I, Prop XII, Cor 3) Similarly OM = ½ C′D, ∴ CD′ = C′D, and they are parallel, hence DCD′C′ is a ▱, ∴ DC = D′C′ In a similar manner it can be shown that the other sides of A′B′C′D′ are respectively equal and ∥ to the remaining sides of ABCD Hence A′B′C′D′ = ABCD

9 Let the ⊙s intersect in A, B Through A draw ACD, AEF, cutting the ⊙s in C, E, D, F Join EC, FD, and produce them to meet in G It is required to prove that EGF is a given ∠

Dem —Join AB, BC, BD Now the ∠s BAE, BCE are equal



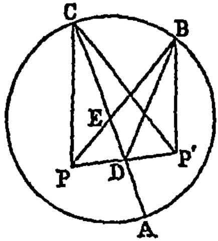
to two right ∠s (XXII), and BCE, BCG are equal to two right ∠s (I XIII), ∴ BAE = BCG Similarly BAE = BDG, ∴ BCG = BDG, and hence (XXI, Cor 1) the points B, C, D, G are concyclic, ∴ the ∠ CBD = CGD Again, the ∠s ACB, ADB are given, since they are in given segments, and the ∠ CBD is equal to ACB − CDB, ∴ CBD is a given ∠, that is, CGD is a given ∠

10 See "Sequel to Euclid," Book III., Prop X

11 Let P, P′ be the points in the ⊙

Sol —Join PP′ Bisect it in D Join D to the centre E, and produce it to meet the circumference in C, A C, A are the points required.

Take any other point B in the circumference Join BP, BP′ CP, CP′, BD Now because E is the centre DC is greater than DB, 2 DC^2 is greater than 2 DB^2 To each add 2 DP^2, and we have 2 DC^2 + 2 DP^2 greater than 2 DB^2 + 2 DP^2, but $CP^2 + CP'^2 = 2\ DC^2 + 2\ DP^2$ (II x , Ex 2), and $BP^2 + BP'^2 = 2\ DB^2 + 2\ DP^2$, $CP^2 + CP'^2$ is greater than $BP^2 + BP'^2$ Hence $CP^2 + CP'^2$ is a maximum In like manner it can be shown that $AP^2 + AP'^2$ is a minimum



12 Let ABCD (see fig , Ex 7) be the quad Draw AC one of the diagonals, and from B, D let fall ⊥s BF, DH on AC It is evident from the proof of Ex 7, that DH and BF are the common chords of the ⊙s on CD, AD, and on AB, CB as diameters, and that they are ∥

13 See "Sequel to Euclid," Book III , Prop xi

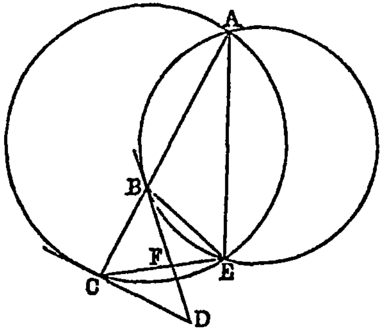
14 Let ACB be the Δ, and CD the internal bisector of the vertical ∠ It is required to prove that AC CB = CD^2 + AD DB

Dem —Describe a ⊙ about ACB Produce CD to meet the circumference in E, and join BE Now the ∠ ACE = BCE, and CAD = CEB (xxi), (I xxxii , Cor 2) the Δs ACD, BCE are equiangular, hence (xxxv Cor 3) AC CB = EC CD, but EC CD = ED DC + CD^2 (II iii), and ED DC = AD DB (xxxv), AC CB = CD^2 + AD DB

15 Draw BD, CD tangents to the ⊙s It is required to prove that BDC is a given ∠

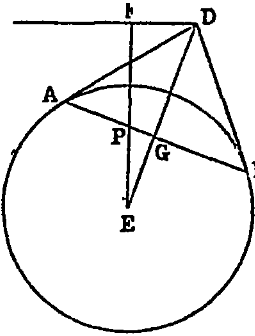
Dem.—Join AE, BE, CE Now the ∠ DCE = CAE (xxxii), and DBE = CAE, DCE = DBE, and CFD = BFE (I xv), CDF = BEF, but BEF = ABE − ACE (I xxxii), and

ABE and ACE are given ∠s; the ∠ BEF, that is, CDF, is given



16 Let AB, a chord of a given O, pass through a given point P, at A, B tangents AD, BD are drawn It is required to prove that the locus of D is a right line

Dem —Let E be the centre Join ED, EP Produce EP, and from D draw DF ⊥ to it Now, denoting the radius by

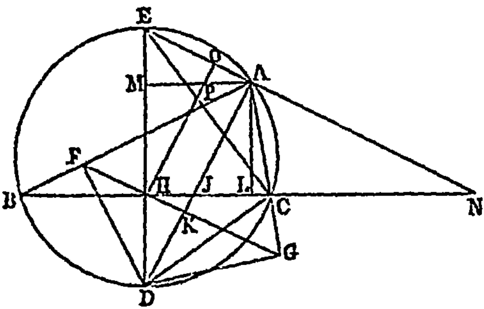


R, we have (xvii, Ex 8) DE EG = R^2, but because the ∠s DGP, DFP are right, DFPG is a cyclic quad, and DE EG = FE EP, FE EP = R^2, FE EP is given, and EP is given, EF is given, hence F is a given point, and FD is ⊥ to EF, FD is a line given in position Hence the locus of D is a right line

17 Let ABC be the Δ Describe a O about ABC Bisect the ∠ BAC by AJ, and produce it to meet the circum-

ference in D Through D draw the diameter DE From A let fall a ⊥ AL on BC Produce AC to G, and let fall ⊥s DF, DG on AB, AG, then CG = $\frac{1}{2}$ (AB − AC) (Dem of xxx, Ex 4) It is required to prove that HJ HL = CG^2

Dem —Join FH, GH, DC, CE, EA, and from A let fall a



⊥ AM on DE Now the ∠ EAD is right (xxxi), and EHJ is right, EAJH is a cyclic quad, ED DH = AD DJ, but because the ∠ ECD is right, and CH ⊥ to ED, ED DH = DC^2 (I xlvii, Ex 1), AD DJ = DC^2, and AD DK = DG^2, hence, by subtraction, AD JK = CG^2, and since the △s ADM, HJK are equiangular, we have (xxxv, Cor 3) AD JK = HJ AM = HJ HL Hence HJ HL = CG^2

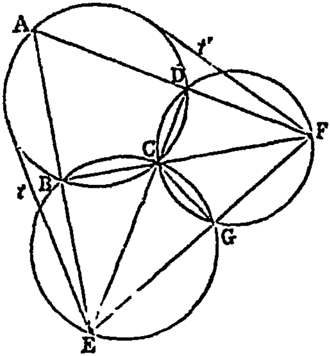
18 The rectangle contained by the distances of the point where the external bisector of the vertical ∠ meets the base, and the point where the ⊥ from the vertex meets it, from the middle point of the base, is equal to the square of half the sum of the sides

Let the same construction be made as in Ex 17 Join EA, and produce it to meet BC produced in N, then EA is the external bisector of the vertical ∠ (xxx, Ex 2) It is required to prove HN HL = AG^2

Dem —Through H draw HO ∥ to AD, meeting EN in O, and AM in P Now the ∠s NOH, AMD are equal, each being right, and the ∠ PAJ = PHJ (I xxxix), the ∠ MDA = ANH, the △s HNO, AMD are equiangular, (xxxv, Cor 3) HN AM = DA OH, but AM = HL, and OH = AK, HN HL = DA AK, but (I xlvii, Ex 1) DA AK = AG^2 Hence HN HL = AG^2

19 Let ABCD be a cyclic quad Produce AB, DC to meet in E, and AD, BC to meet in F Join EF, and from E, F draw tangents t, t' to the ○ described about ABCD It is required to prove that $EF^2 = t^2 + t'^2$

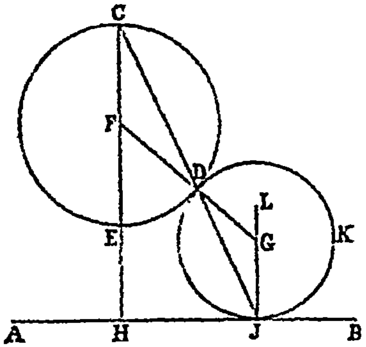
Dem —About the △ CDF describe a ○ CDFG, cutting EF



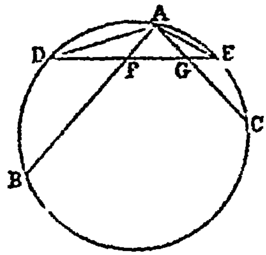
in G Join CG Now (xxii) the ∠s BAD, BCD are together equal to two right ∠s, and the ∠s DFG, DCG are equal to two right ∠s, the ∠s BAD, BCD, DFG, DCG are equal to four right ∠s, and the ∠s BCD, BCG, DCG are equal to four right ∠s Reject BCD, DCG, and we have the ∠ BCG = BAD + DFG To each add the ∠ BEG, and we get BCG + BEG = BAF + AFE + AEF, hence the ∠s BCG, BEG are equal to two right ∠s BCGE is a cyclic quad, FE EG = DE EC = t^2 (xxxvi), and EF FG = BF FC = t'^2, but EF^2 = FE EG + EF FG, $EF^2 = t^2 + t'^2$

20 Let AB be a given line, CDE a given ○, and DKJ a variable ○, touching CDE in D, and AB in J It is required to prove that JD produced passes through a given point

Dem —From the centre F let fall a ⊥ FH on AB, and produce it to meet the ○ in C Let G be the centre of DKJ Join FG, GJ, CD, DJ, and produce JG to L Now (xx) the ∠ LGD = 2 GJD = 2 GDJ, and the ∠ EFD = 2 FDC, but LGD = EFD (I xxix), GDJ = FDC, JD and DC are in one straight line, that is, the chord of contact JD produced passes through the fixed point C where the ⊥ from the centre of the given ○ on the given line meets the circumference

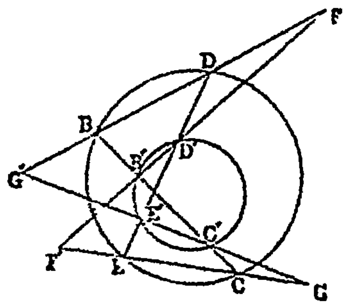


21 Dem.—Join DA, AE Now the ∠ DEA = DAB (xxvii.), and EAC = ADE, but AFG = FDA + FAD (I xxxii.) and AGF,



= GAE + GEA, AFG = AGF, and hence (I vi) the lines AF and AG are equal

22 Dem —Join BD, B D′, and produce them to meet in F

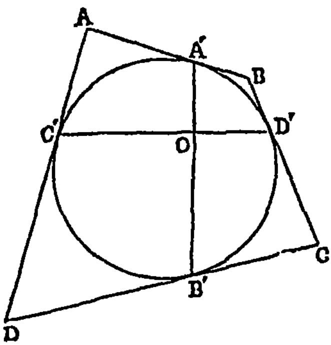


Join EC, E'C', and produce them to meet in G Produce FB' GE to meet in F', and FB, GE' to meet in G

Now the ∠ BDE = BCE (xxi), and B D'E' = B C'E', but B'D'E' = DD'F (I xv), and B C'E' = CC'G, hence the ∠ DFD' = CGC', and (xxi Cor 1) the four points F, G, F', G' are concyclic

23 Let ABCD be a cyclic quad, such that a circle can be inscribed in it It is required to prove that the lines A'B', C'D', joining the points of contact, are perpendicular to each other

Dem —Because AC' and BD' are tangents, if we produce

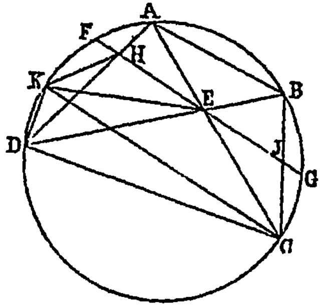


them until they meet, they will be equal, the ∠ AC'D' = BD'C' To each add the ∠ OD'C', and we have AC'D' + OD C' = BD C' + OD'C, but BD'C' + CD'C' equal two right ∠s, AC'D + OD C' equal two right ∠s Similarly, AA B + CB'A' equal two right ∠s, and (xxii) DAB + DCB equal two right ∠s, the sum of these six ∠s is six right ∠s, and these ∠s, together with the ∠s A OC + B OD' equal eight right ∠s, A OC' + B OD' equal two right ∠s, but A OC' = B'OD' Hence each is right, and therefore A B' and C D are ⊥ to each other

24 Let ABCD be a cyclic quad, AC, BD its diagonals intersecting in E Through E draw the minimum chord FG (xv, Ex 1) It is required to prove that EH = EJ

Dem —Through C draw CK ∥ to FG, and join KE, KH, KD Now, because FG is bisected in E, and CK is ∥ to FG, EC

= EK, and the ∠ JEC = HEK, but JEC = ECK, HEK = ECK, but ECK = ADK (xxi), HEK = ADK, and

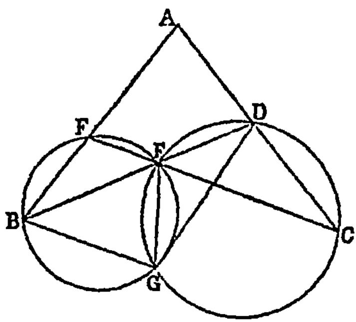


HEDK is a cyclic quad, the ∠ HDE = HKE, but HDE = ACB (xxi), HKE = ACB And the △s EHK, EJC have two ∠s and a side in one equal to two ∠s and a side in the other Hence (I xxvi) EH = EJ

25 See "Sequel to Euclid," Book VI, Sec i, Prop xv (3)

26 See "Sequel to Euclid," Book III, Prop xx, Cor 2

27 Let AB, AC, BD, CE be four lines forming four △s ABD, ACE, BEF, DCF About the △s BEF, DCF two ○s are



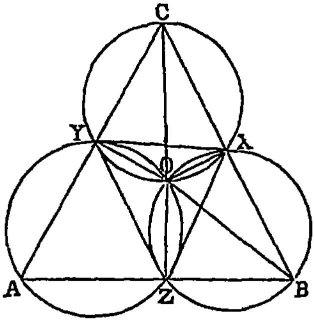
described intersecting in F, G It is required to prove that the ○s about the △s ABD, ACE will pass through G

Dem —Join GB, GF, GD Now the ∠ BEF = BAC + ACE, but ACE = FGD (xxi), BEF = BAC + FGD, BEF

+ BGF = BAD + BGD, but (xxii) BEF + BGF equal two right ∠s, BAD + BGD equal two right ∠s, hence the ○ about BAD will pass through G Similarly the ○ about ACE will pass through G

28 About AYZ, CXY describe ○s intersecting in O It is required to prove that the ○ about BXZ will pass through O

Dem —Join OX, OY, OZ Now the ∠s ZAY + ZOY equal two right ∠s (xxii), and YCX + YOX equal two right ∠s, those four ∠s equal four right ∠s, and the three ∠s ZOY,



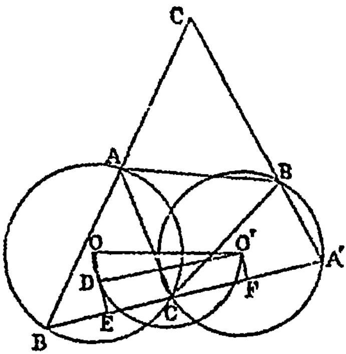
YOX, XOZ equal four right ∠s, hence the ∠ XOZ = ZAY + YCX, ZOX + ZBX = BAC + ACB + CBA, and equal two right ∠s Hence the ○ about BXZ will pass through O

29 **Dem** —Join OC, OB Now, because the points O and C are given, the line OC is given in position, and YC is given in position, the ∠ YCO is given, (xxi) the ∠ YXO is given. In like manner OXZ is given, hence the ∠ YXZ is given Similarly, it can be shown that the ∠s YZX and XYZ are each given

30 Let XYZ be a given △, and A, B, C three given points It is required to place a △ equal to XYZ whose sides shall pass through A, B, C

Sol —Join AB, AC, BC On BC, AC describe segments containing ∠s respectively equal to the ∠s X, Y Join O, O′, the centres On OO′ describe a semicircle, and in it place a chord O D = ½ XY Through C draw A B′ ∥ to O D Join B′ A, A′B, and produce them to meet in C′. A B′C′ is the required △.

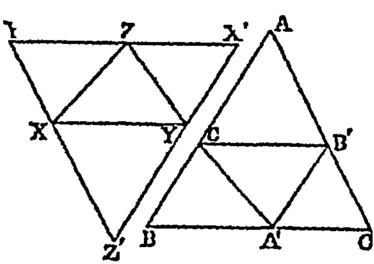
Dem.—From O′ let fall a ⊥ O′F on A′B′. Join OD, and produce it to meet A′B′ in E. Now the ∠ ODO′ is right (xxxi.),



OEF is right, hence (iii.) B′C is bisected in E, and CA′ is bisected in F. B′A′ = 2 EF = 2 O′D, = XY, and since the ∠s A′, B′ = X, Y respectively, the △ A B′C′ = XYZ.

31. Let XYZ be the given △, and AB, AC, BC the given lines. It is required to place a △ equal to XYZ, whose vertices shall be on AB, AC, BC.

Sol.—Through the points X, Y, Z, describe a △ X′Y′Z′ equal to ABC (Ex. 30), and in BC take BA′ = Y′Z, in BA take BC′

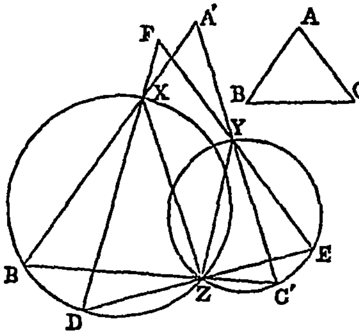


= Y′X, and in AC take B′C = X′Y. Join A′B′, B′C′, C′A′. A′B′C′ is the △ required.

Dem —Because A B = Y′Z, and BC′ = XY′, and the ∠ A′B C′ = XY′Z , (I IV) A′C′ = XZ Similarly A′B′ = YZ, and B′C′ = XY Hence the Δ A B′C′ = XYZ

32 Let ABC be the given Δ, and X, Y, Z the three points It is required to construct the greatest Δ equiangular to ABC, whose sides shall pass through X, Y, Z

Sol —Join XZ, YZ, and on them describe segments of ○ˢ containing ∠ˢ respectively equal to the ∠ˢ B, C Through Z draw



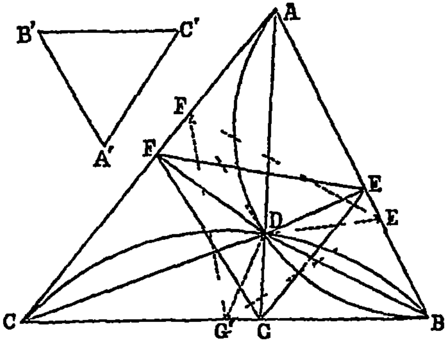
B C′ ∥ to the line joining the centres Join B′X, C′Y, and produce them to meet in A′ A B C is the Δ required

Dem —Through Z draw any other line DE Join DX, EY, and produce them to meet in F Now (XXI) the ∠ EDF = C′B′A′, and the ∠ DEF = B′C′A′, and the side B′C greater than DE Hence the Δ A B′C′ is greater than DEF ("Sequel," Book III , Props XV , XVI)

33 Let AB, AC, BC be the three given lines, and A′B′C′ the given Δ It is required to construct the minimum Δ equiangular to A′B C′, whose vertices shall be on AB, AC, BC

Sol —On BC describe a segment of a ○ containing an ∠ equal to the sum of the ∠ˢ A, A′ On AB describe a segment containing an ∠ equal to the sum of the ∠ˢ C, C′ From D let fall ⊥ˢ DE, DF, DG on AB, AC, BC Join FG, GE, EF EFG is the required triangle

Dem —The ∠ CDB = A + A′ (const), but CDB = A + DCF + DBE , A′ = DCF + DBE Again (const), FGCD and EBGD are cyclic quads , the ∠ FCD = FGD, and DBE = DGE, hence the ∠ FGE = FCD + DBE, hence the ∠ FGE = A′ Simi-



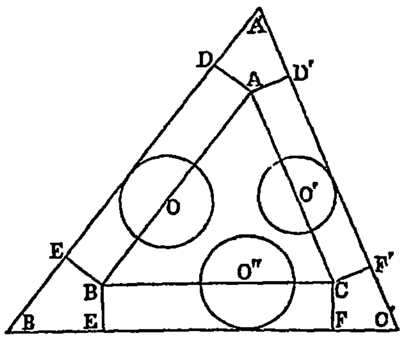
larly GFE = B′, and GEF = C′ Therefore the Δ FGE is equiangular to A′B′C

Draw any line DG′, and draw DF′, DE′, making each of the ∠ˢ FDF, EDE equal to GDG′ Join G F′, F′E′, E′G′ Now the ∠ FDF′ = GDG′ To each add FDG, and we have the ∠ F′DG′ = FDG To each add the ∠ F CG, and we get F′DG′ + F CG′ = FDG + FCG, but FDG + FCG = two right ∠ˢ, F′DG′ + F′CG′ = two right ∠ˢ, hence F′CG D is a cyclic quad , the ∠ F′G′D = FCD, but FCD has been shown to be equal to FGD, F′G D = FGD Similarly E′G′D = EGD, F′G′E′ = FGE In like manner G F E′ = GFE, and F′E′G′ = FEG Hence the Δˢ F′E G′, FEG are equiangular, and since DG′ is greater than DG, and DF′ greater than DF, and the ∠ G′DF′ = GDF, the side G′F′ is greater than GF, the Δ F′E′G is greater than FEG Hence FEG is a minimum.

34 Let O, O′, O″ be the centres of the given Oˢ It is required to construct the greatest Δ equiangular to a given one, whose sides shall touch the three circles

Sol —Through the points O, O′ O″, describe the maximum Δ ABC, equiangular to the given one (Ex 32) Draw tangents A B′, B′C′, C′A′ respectively ∥ to AB, BC, CA A′B C′ is the required Δ

Dem —From A, B, C let fall ⊥ˢ on the sides of the Δ A′B′C′ Because the ∠ˢ about B are together equal to four right ∠s, and that the ∠s EBA, E BC are each right, the ∠s EBE′, ABC are together equal to two right ∠s, but ABC is a given ∠, EBE′ is given, and the sides BE, BE′ are given, since they are equal to the radii of the ⊙s O, O″ Hence the figure EBE′B is given in magnitude Similarly the figures ADA′D′, CFC′F′ are given in magnitude Again, since the

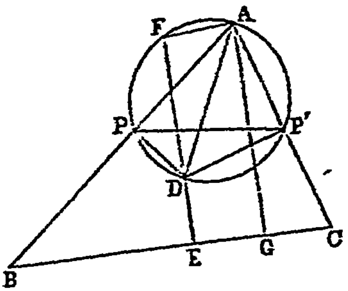


△ ABC is a maximum, the side BC is a maximum, therefore BCFE′ must be a maximum, because it is contained by BC and BE′, which is a given line, being equal to the radius of O″ In like manner each of the figures ABED, ACF′D′ is a maximum Hence the whole figure A B C′ is a maximum

35 Let AB, AC, two sides of a given △ ABC, pass through two fixed points P, P′ It is required to prove that the side BC touches a fixed circle

Dem —Join PP Describe a ⊙ about the △ APP Draw the diameter AD, and join DP, DP From D let fall a ⊥ DE on BC, and produce it to meet the ⊙ in F Join AF, and let fall a ⊥ AG on BC

Now since the points P, P′ are given, PP′ is a given line, and the ∠ PAP′ is given, hence (xxi, Cor 2) the circle PAP′ is given, and because the ∠s EBP, EDP are together equal to two right ∠s, and EDP, FDP are together two right ∠s, the ∠ FDP = EBP, and is therefore a given ∠, hence the arc PF is given, and F is a given point Again (xxxi) the ∠ AFD is right, and FEG is right, hence AFEG is a ▭, EF = AG, but AG is given, since it is the ⊥ from the vertex on the base of a given △, EF is given, and the point F is given, hence the locus of E is a ○, having F as a centre, and EF as

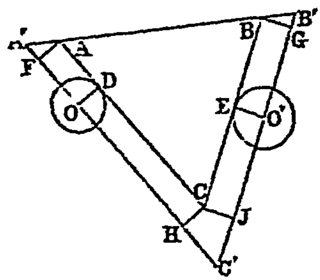


radius Hence the base BC touches a fixed ○

36 Let the sides CA, CB of the Δ ABC touch fixed ○s It is required to prove that AB touches a fixed ○

Dem.—Through the centres O, O′ draw ∥s A′C′, B′C′ to AC BC Join O, O to the points of contact D, E, and through A, B, C draw AF, BG, CH, CJ, ∥ to OD, O E

Now the ∠BAC = B′A′C ; ∴ BA′C is given, and the ∠ AFA′ is right, ∴ the Δ AA′F is given in species, and the side AF is given, being equal to OD, ∴ AA′, A F are each given Again, the ∠s ACB, HCJ are equal to two right ∠s, but ACB is given

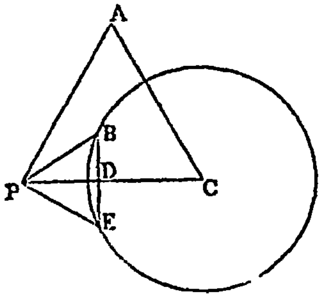


∴ HCJ is given, and the sides CH, CJ are given, ∴ HCJC′ is a given figure, ∴ C′H is given, and HF is given, being equal to AC, ∴ A′C is a given line Similarly B′C′ is given, and A′B′ is given, ∴ the Δ A′B′C′ is given And hence (Ex 35) A B′ touches a fixed ○

37 Let P be the given point, and C the centre of the ○.

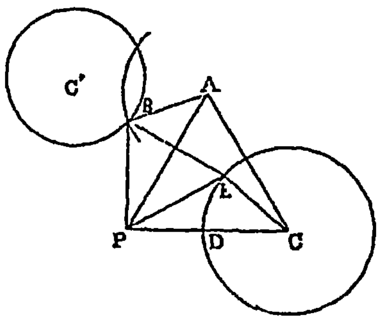
Sol —Join PC, and on it describe an equilateral △ PAC Draw PB, bisecting the ∠ APC From B let fall a ⊥ BD on PC, and produce it to meet the ○ in E Join EP EPB is the required △

Dem —BD = ED (III), and DP common, and the ∠ BDP = EDP, (I IV) PB = PE, and the ∠ BPD = EPD, but



BPD is ½ an ∠ of an equilateral △, EPD is ½ an ∠ of an equilateral △ Hence EPB is an ∠ of an equilateral △, and the ∠ PEB = PBE Hence the △ EPB is equilateral

38 Let P be the given point, and C, C′ the centres of the given ○s It is required to construct an equilateral △, having its vertex at P, and the extremities of its base on the circumferences of C and C

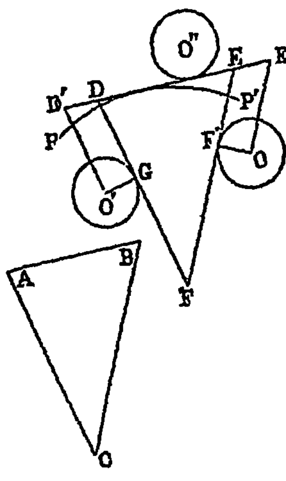


Sol —Join PC, and on it describe an equilateral △ PAC With A as centre, and a radius equal to CD, describe a ○, cutting the O whose centre is C' in B Join AB, and at the point C in CP make the ∠ PCE = BAP (I xxiii) Join BE, EP, PB BEP is the required Δ

Dem —Because AP = CP, and AB = CE, and the ∠ BAP = ECP, (I iv) the base BP = EP, and the ∠ BPA = CPE To each add the ∠ APE, the angle BPE = CPA, hence BPE is an ∠ of an equilateral Δ And since PB = PE, the Δ PBE is equilateral.

39 Let ABC be a given Δ It is required to place it so that its sides shall touch three given O• O, O', O''

Sol —If two sides of a Δ equal to ABC touch two O• O, O', the third must touch a fixed O (Ex 36) Let PP' be the fixed O

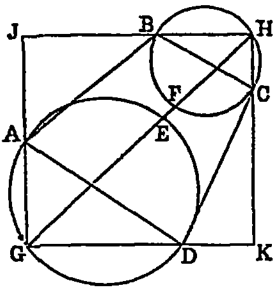


Draw DE a common tangent to O'' and PP' (xvii , Ex 10) Through O, O' draw OE', O'D', meeting DE produced, and making the ∠• OE'D', O'D'E' respectively equal to the ∠• CBA, CAB At O, O' draw OF', O G at right ∠• to OE', O'D , and through F', G draw EF, DF, touching the O• DEF is the Δ required.

Dem.—Because each of the ∠• E'OF', EF'O is right, E'O, EF are ∥, the ∠ DEF = DE'O, and equal CBA Similarly, EDF = CAB Hence DEF is the Δ required

40 Let ABCD be a given quad It is required to describe a square about it

Sol —On AD, BC, two opposite sides, as diameters, describe ○s AED, BFC Bisect the semicircles AED, BFC in E, F



Join EF, and produce to meet the ○s again in GH Join HB, GA, and produce them to meet in J Join GD, HC, and produce them to meet in K GJHK is the required square

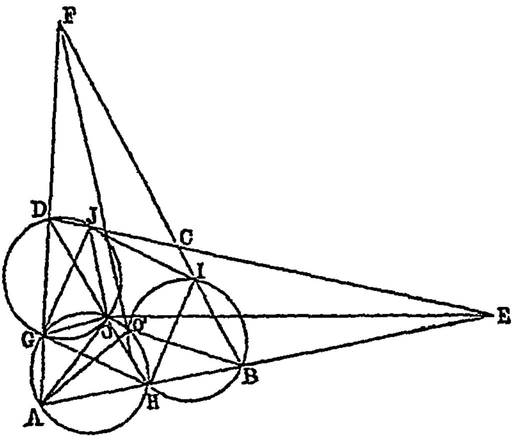
Dem —Because the arc AE = DE, the ∠ AGE = DGE, but the ∠ AGD is right (xxxi), AGE is half a right ∠ In like manner BHF is half a right ∠, AGE = BHF, JH = JG Similarly, KG = KH, hence the sides are equal, and the ∠s are evidently right Therefore GJHK is a square

Lemma —To find a point O in a △ ABC, such that the ∠ BOC may exceed the ∠ BAC by a given ∠ X, and that the ∠ AOC may exceed the ∠ ABC by a given ∠ Y

Sol —On BC describe a segment of a ○ containing an ∠ equal to BAC + X, and on AC describe a segment containing an ∠ equal to ABC + Y The point O, in which these segments intersect, is evidently the required one

41 Let ABCD be a given quad It is required to inscribe a square in it.

Sol —Produce AB, DC to meet in E, and AD, BC to meet in F In the △ AED find a point O, such that the ∠ AOD is equal to AED, together with a right ∠, and that the ∠ DOE is equal to DAE, together with half a right ∠ (Lemma), and in AFB find a point O′, so that the ∠ AO′B is equal to AFB, together with a right ∠, and the ∠ AO′F = ABF, together with half a right ∠ Describe a ○ through the points O, O′, A, cutting AF, AE in



G, H Through O, G, D describe a ○ cutting DE in J, and through O′, H, B describe a ○, cutting BF in I Join GJ, JI IH, HG GJIH is the required square

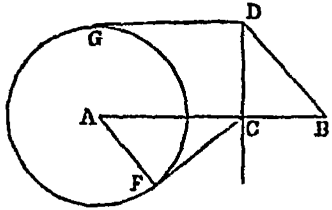
Dem —Join OG, OH, OJ Now the difference between the ∠s AOD and AED is equal to a right ∠ (const), and AOD − AED = EAO + ODE, hence EAO and ODE are together equal to a right ∠, but EAO = HGO (xxi), and ODE = OGJ, hence the ∠ HGJ is right. Similarly, by joining JH, it can be shown that GJH is half a right ∠, GH = GJ Similarly, it can be shown that the ∠ GJI is right, and that GJ = JI Hence GJIH is a square

42 (1) *Lemma*—To find the radical axis of a ○ and a point. Let A be the centre of the ○, and B the point.

Sol —Join AB, and divide it in C, so that $AC^2 - CB^2$ is equal to the square of the radius AE ("Sequel," Book III, Prop ix), Erect CD ⊥ to AB CD is the required radical axis

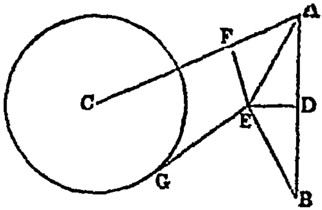
Dem.—Draw DG a tangent from any point D Join DB Draw CE a tangent, and join AE Now $AC^2 - CB^2 = AE^2$,

$AC^2 - AE^2 = CB^2$, that is, $CE^2 = CB^2$, $CE^2 + CD^2 = CB^2$



$+ CD^2$, but $CE^2 + CD^2 = GD^2$ ("Sequel," Book III, Prop xxi) and $CB^2 + CD^2 = DB^2$, $DG^2 = DB^2$ Hence CD is the radical axis (xvii, Ex. 6)

Sol —Let C be the centre of the ○, and A, B the points Join AB, and bisect it in D Erect DE ⊥ to AB Join AC,



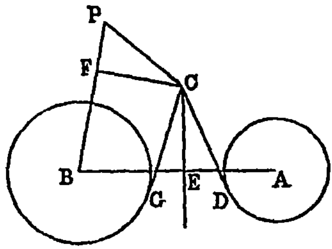
and find the radical axis FE (Lemma) of the ○ and the point A, and let it cut DE in E E is the centre of the required ○

Dem —From E draw the tangent EG to the ○ Join EA, EB EG = EA (Lemma), and EA = EB, EA, EB, EG are equal, and the ○, with E as centre and EA as radius, will pass through B, and cut the given ○ orthogonally in G ("Sequel," Book III, Prop xxii)

(2) *Lemma* —*To find the radical axis of two* ○s Let A, B be the centres Join AB, and divide in E, so that $AE^2 - EB^2$ is equal to the difference of the squares of the radii. Erect EC ⊥ to AB From C and E draw tangents CD, EH, CG, EJ to A and B Join AH, BJ Now $AE^2 - EB^2 = AH^2 - BJ^2$, $EH^2 = EJ^2$, $CE^2 + EH^2 = CE^2 + EJ^2$, hence

("Sequel," Book III, Prop xxi) $CD^2 = CG^2$ Hence EC is the radical axis of the ⊙s

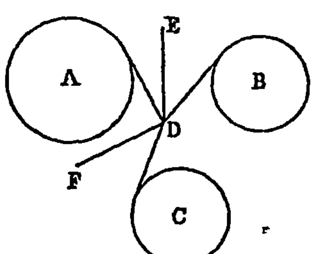
Sol —Let A, B be the centres, and P the point Join AB,



and find the radical axis CE (*Lemma*) Join BP, and find the radical axis CF of the point P, and the ⊙ B From C, where CE, CF intersect, draw tangents CD, CG to A and B Join CP C is the centre of the required ⊙

Dem —Since CE is the radical axis of the ⊙s A, B, CG = CD (*Lemma*), and because CF is the radical axis of the ⊙ B and the point P, CG = CP, CG, CD, CP are equal, and therefore the ⊙, whose centre is C, and radius CP, will cut the ⊙s A and B orthogonally ("Sequel," Book III, Prop xxi)

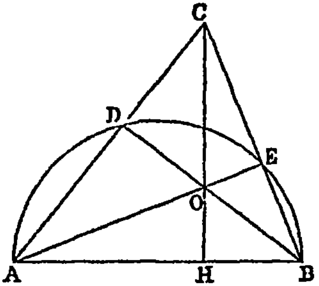
(3) Let A, B, C be the ⊙s Find DE the radical axis of A



and B, and DF the radical axis of A and C From D, where DE, DF intersect, draw tangents to A, B, C Now these tangents are equal, and the ⊙, with D as centre, and one of them as distance, will pass through the ends of the other two, and will cut the ⊙s A, B, C orthogonally ("Sequel," Book III, Prop xxi)

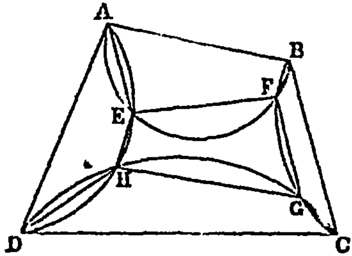
43 Dem —Join BD, AE, and let them intersect in O Join CO, and produce it to meet AB in H

Now (xxxi) each of the ∠ˢ ADB, AEB, is right, BD, AE are ⊥ˢ to AC, BC, hence (xxii, Ex 10) CH is ⊥ to AB Now (xxi, Cor 1) AHEC is a cyclic quad, (xxxvi)



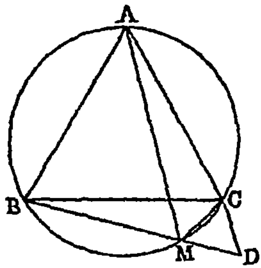
BC BE = BA BH And since BHDC is a cyclic quad, AC AD = AB AH Adding, we get AC AD + BC BE = AB (AH + BH) = AB²

44 Dem —Join AE, BF, CG, DH Now (xxii) the ∠ˢ AEF, ABF are together equal to two right ∠ˢ, and similarly the ∠ˢ AEH, ADH are together equal to two right ∠ˢ, hence the sum of those ∠ˢ is four right ∠ˢ, and the sum of the ∠ˢAEF, AEH,



FEH is four right ∠ˢ, the ∠ FEH = ABF + ADH In like manner the ∠ FGH = FBC + HDC, the ∠ˢ FEH and FGH = ABC and ADC, and are equal to two right ∠ˢ. Hence (xxii, Ex 1) EFGH is a cyclic quad

45 Dem —Describe a ○ about ABC Take any point M in the circumference Join MA, MB, MC It is required to prove that MA = MB + MC Produce BM to D, so that MD = MC Join CD Now (xxii) the ∠s BAC and BMC are together



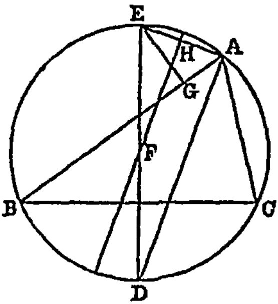
equal to two right ∠s, and BMC, DMC are together equal to two right ∠s, DMC = BAC, and is an ∠ of an equilateral △, and because MC = MD, MCD is an equilateral △

Again, because BMCA is a cyclic quad, the ∠ MBC = MAC, and ABC = AMC, but ABC = MDC, since each is an ∠ of an equilateral △, AMC = MDC, hence (I xxvi) the △s AMC, BDC are equal, AM = BD, that is, AM = MB + MC

46 (1) Let ABC be a △, the sum of whose sides AB, AC is given, and the ∠ BAC, both in magnitude and position About the △ ABC describe a ○ It is required to prove that the locus of its centre F is a right line

Dem —Bisect the arc BC in D Join AD Let fall a ⊥ DE on AB From F let fall a ⊥ FG on AD Now AE = ½ (AB + AC) (xxx, Ex 4), hence AE is a given line, E is a given point And since DE is ⊥ to AE, at a given point, DE is given in position, and because the ∠ BAD = ½ BAC, BAD is a given ∠, AD is given in position, and DE is given in position, D is a given point, and the point A is given, hence AD is a given line, and (iii) AD is bisected in G, G is a given point, and FG is a ⊥ from a fixed point to a line given in position, hence FG is given in position Hence the locus of F is the line FG

(2) Bisect the ∠ BAC by AD Erect DE ⊥ to BC DE is



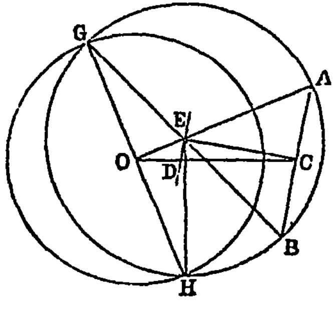
the diameter Join EA, and from E, F let fall ⊥s EG, FH on AB and AE

Now the line AG is given, for it is equal to ½ (AB – AC), EG, which is ⊥ to it, is given in position, and EA is given in position, since it is ⊥ to AD, E is a given point, and EA is bisected in H (III), FH is given in position Hence the locus of F is the line FH

47 (1) Let O be the centre of the given ○, and A, B the points It is required to describe a ○ which shall pass through A, B, and bisect the circumference of the given ○

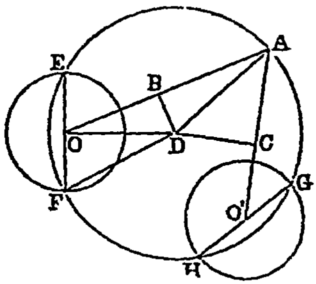
Sol —Bisect AB in C Join CO, and divide it in D, so that $CD^2 - OD^2 = R^2 - BC^2$ (R being the radius of the given ○) Erect DE, CE, ⊥s to OC, AB, and join AE, BE, OE E is the centre of the required ○

Dem —The Δs ACE, BCE are equal (I IV.), AE = BE, hence the ○, with E as centre, and AE as radius, will pass



through B Let it cut the given ⊙ in G, H Join OG, OH, EG, EH Now $CD^2 - OD^2 = R^2 - BC^2$, $CE^2 - OE^2 = R^2 - BC^2$, $BC^2 + CE^2 = R^2 + OE^2$, that is, $BE^2 = R^2 + OE^2$, $GE^2 = R^2 + OE^2$, but $OG = R$, $GE^2 = OG^2 + OE^2$, hence the ∠ EOG is right Similarly EOH is a right angle, OG and OH are in the same straight line, hence G H is the diameter of the given ⊙ Hence the circumference of the given ⊙ is bisected by the ⊙ ABH in the points G, H

(2) Let A be the given point, and O, O′ the centre of the given ⊙s It is required to describe a ⊙ passing through A which shall bisect the circumferences of the ⊙s whose centres are O, O′.



Sol —Join AO, and divide it in B, so that $AB^2 - BO^2 = R^2$ (R being the radius of the ⊙ whose centre is O) Join AO′, and divide it in C, so that $AC^2 - CO'^2 = R^2$ Erect BD, CD ⊥s to AO, AO Join AD With D as centre, and AD as radius, describe a ⊙ EAG, cutting the given ⊙s in the points E, F, G, H This is the ⊙ required

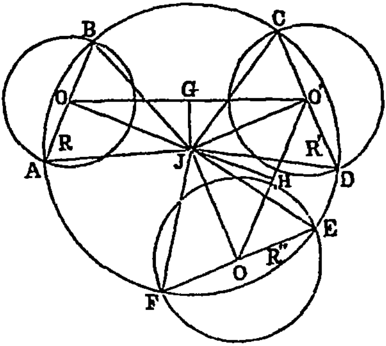
Dem —Join OE, OF, O′G, O′H, OD, FD Now $AB^2 - OB^2 = OF^2$ (const), $AD^2 - OD^2 = OF^2$, $AD^2 = OD^2 + OF^2$, that is, $FD^2 = OD^2 + OF^2$, the ∠ DOF is right Similarly, the ∠ DOE is right, OE and OF are in the same straight line Hence EF is the diameter of one of the given ⊙s In like manner GH is the diameter of the other given ⊙ Hence the circumferences of the given ⊙s are bisected by the ⊙ EAG

48 Let a ⊙, whose centre is D, bisect the circumferences of two given ⊙s in the points E, F, G, H It is required to find the locus of D (See last diagram)

Sol —Join EF, GH Now since the circumferences are bisected in E, F, G, H, the centres, must be in the lines EF, GH Bisect these lines in O, O′ Join OO, DO, DO′ From D let fall a ⊥ DJ on OO′ DJ is the locus of D

Dem —Join DF, DH Now (III) the ∠ DOF, DO′H are right, $DF^2 = DO^2 + OF^2$, and $DH^2 = DO'^2 + O'H^2$, but $DF^2 = DH^2$, $DO^2 + OF^2 = DO'^2 + O'H^2$, $DO^2 - DO'^2 = O'H^2 - OF^2$, but $O'H^2 - OF^2$ is given, since O′H and OF are the radii of two given ○s, $DO^2 - DO'^2$ is given, $OJ^2 - O'J^2$ is given, J is a given point, the line DJ is given in position Hence the locus of D is the line DJ

49 Let O, O′, O″ be the centres of the given ○s, and R, R′,

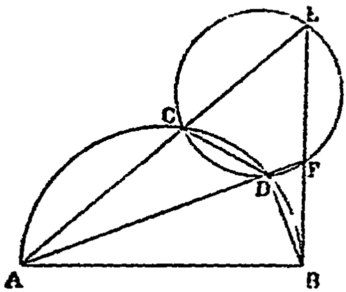


R″ their radii It is required to describe a ○ which shall bisect the circumferences of the given ○s

Sol —Join OO′, and divide it in G, so that $OG^2 - O'G^2 = R'^2 - R^2$ Join O′O″, and divide it in H, so that $O'H^2 - O''H^2 = R''^2 - R'^2$, and at G, H erect GJ, HJ, ⊥s to OO′, O′O″ The point J, where these ⊥s intersect, is the centre of the required ○

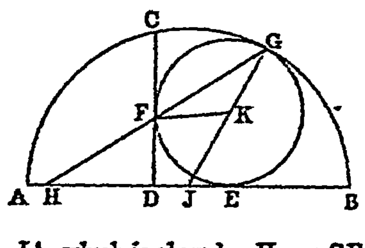
Dem —Join OJ, O′J, O″J Through O, O′, O″ draw AB, CD, EF at right angles to OJ, O′J, O″J, and join JA, JB, JC, JD, JE, JF Now $OA^2 = OB^2$, $OA^2 + OJ^2 = OB^2 + OJ^2$, $AJ^2 = BJ^2$, $AJ = BJ$ In like manner $CJ = DJ$, and $EJ = FJ$ Again, $OG^2 - O'G^2 = R'^2 - R^2$, $OG^2 + R^2 = O'G^2 + R'^2$, $OG^2 + JG^2 + R^2 = O'G^2 + JG^2 + R'^2$, that is, $OJ^2 + R^2 = O'J^2 + R'^2$, $AJ^2 = DJ^2$, $AJ = DJ$ Similarly, $BJ = EJ$, and $CJ = FJ$ Hence those six lines are equal, and the ○, with J as centre, and AJ as radius, will pass through the points B, C, D, E, F, and will bisect the circumferences of the given ○ˢ in those points.

50 Dem.—Join BC, CD, DB Now, since ABE is a right-angled △, and BC is ⊥ to AE, we have AE AC = AB^2



(I. XLVII, Ex. 1) In like manner AF . AD = AB^2, ∴ AE AC = AF AD Hence (XXXVI, Cor 1) the points C, E, F, D are concyclic.

51 (1) Dem —Let J, K be the centres of the ○ˢ Join JK, and produce it. JK produced must pass through G (XI) Join KF If GF does not pass through A, let it pass through H Now (XVIII) the ∠ CFK is right, and the ∠ CDB is right, ∴ FK and AB are ∥, ∴ the ∠ GFK = GHB, but GFK = FGK (I V), the ∠ JHG = JGH, hence JG = JH, but JG

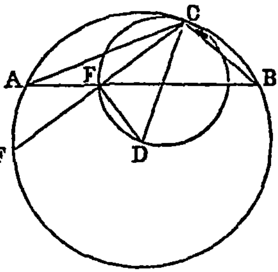


= JA, ∴ JH = JA, which is absurd Hence GF produced must pass through A

(2) Complete the ○ ACB, and produce CD to meet the circumference again in M Now (III.) DC = DM, the arc AC

$= AM$, hence (Ex 26) $AF \cdot AG = AC^2$, and (xxxvi) $AF \cdot AG = AE^2$, $\therefore AC^2 = AE^2$, $\therefore AC = AE$.

52 Let ACB be an obtuse angled Δ It is required to draw from C a line CE, so that $CE^2 = AE \cdot EB$

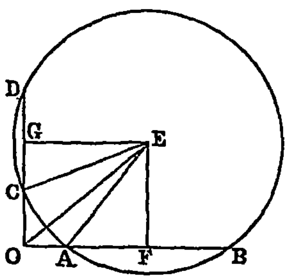


Sol —Describe a ○ about ACB Let D be its centre Join CD On CD as diameter describe a ○, cutting AB in E Join CE CE is the required line

Dem —Produce CE to meet the circumference again in F, and join DE

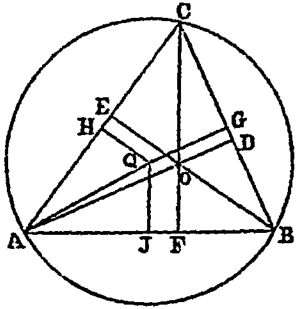
Now the $\angle$ CED is right (xxxi), $\therefore$ FED is right, hence (iii) CF is bisected in E, $\therefore FE \cdot EC = EC^2$, but (xxxv) $FE \cdot EC = AE \cdot EB$, $\therefore AE \cdot EB = CE^2$

53 Dem —From E let fall $\perp^s$ EF, EG on AB, CD, and join AE, CE Now $AF = BF$ (iii), $\therefore AB^2 = 4 AF^2$ Similarly, $CD^2 = 4 CG^2$, $\therefore AB^2 + CD^2 = 4 AF^2 + 4 CG^2$ Again (I xlvii), $OE^2 = OG^2 + EG^2 = EF^2 + EG^2$, $\therefore 4 OE^2 = 4 EF^2 + 4 EG^2$,



$\therefore AB^2 + CD^2 + 4 OE^2 = 4 AF^2 + 4 EF^2 + 4 CG^2 + 4 EG^2$, but $4\,AF^2 + 4\,EF^2 = 4\,AE^2 = 4\,R^2$, and $4\,CG^2 + 4\,EG^2 = 4\,CE^2 = 4\,R^2$ Hence $AB^2 + CD^2 + 4\,OE^2 = 8\,R^2$

54 (1) Let ABC be the △ From A, B, C let fall ⊥s AD, BE, CF on the sides, and intersecting in O It is required to

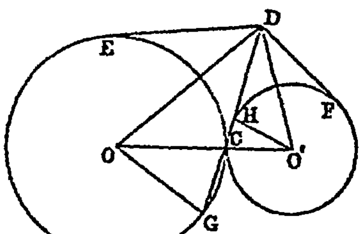


prove that $AB^2 + BC^2 + CA^2$ is equal to $2\,AO \cdot AD + 2\,BO \cdot BE + 2\,CO \cdot CF$

Dem —$AC^2 = AO^2 + OC^2 + 2\,AO \cdot OD$ (II xii), $BC^2 = CO^2 + OB^2 + 2\,CO \cdot OF$, and $AB^2 = AO^2 + OB^2 + 2\,BO \cdot OE$ Adding, we get $AB^2 + BC^2 + CA^2 = (2\,AO^2 + 2\,AO \cdot OD) + (2\,OB^2 + 2\,OB \cdot OE) + (2\,CO^2 + 2\,CO \cdot OF) = 2\,AO\,(AO + OD) + 2\,BO\,(BO + OE) + 2\,CO\,(CO + OF) = 2\,AO \cdot AD + 2BO \cdot BE + 2\,CO \cdot CF$

(2) Describe a ○ about ABC, and from its centre Q let fall ⊥s QG, QH, QJ on the sides, and join AQ Now (iii) $AJ = BJ$, ∴ $AB^2 = 4\,AJ^2 = 4\,AQ^2 - 4\,QJ^2$, but $AQ = R$, and $2\,QJ = OC$ ("Sequel," Book I, Prop xii, Cor 3), ∴ $AB^2 = 4\,R^2 - OC^2$ Similarly, $BC^2 = 4\,R^2 - OA^2$, and $CA^2 = 4\,R^2 - OB^2$ Hence $AB^2 + BC^2 + CA^2 = 12\,R^2 - (OA^2 + OB^2 + OC^2)$

55 Dem —Join the centres O, O′ Produce DC, and let it meet the ○s again in the points G, H Join OG, O′H



Now the ∠ DCO = OCG (I xv), but OCG = OGC, DCO = OGC, and O DC = ODG (hyp), the Δs ODG, O DC are equiangular, hence (xxxv, Cor 3) OG CD = O C DG Again, the ∠s O HD, O'HC are equal to two right ∠s, and the ∠s OCD, O'CD are equal to two right ∠s, and O'CD = O HC, the ∠ O'HD = OCD, and (hyp) O'DH = ODC, the Δs O'HD, OCD are equiangular hence (xxxv, Cor 3) O H CD = DH OC Multiplying these results, we get CD^2 = DH DG Now DG DC = DE^2 (xxxvi), and DH DC = DF^2, DG DH DC^2 = DE^2 DF^2, DC^4 = DE^2 DF^2, DC^2 = DE DF

BOOK IV.

PROPOSITION IV

1 Dem —CF = CD, OC common, and the base OF = OD, hence (I VIII) the ∠ OCF = OCD (Fig Prop IV),

2 Dem —BD = BE, CD = CF, AE = AF (III XVII), CB + AE = ½ (AB + BC + CA) = s, that is, a + AE = s, AE = ($s - a$) In like manner BD = ($s - b$), and CF = ($s - c$) (Fig, Prop IV)

3 Dem —From O′ let fall ⊥ˢ O′F, O′G, O′H on the sides AB, BC, CA of the Δ ABC Now, because the ∠ O′CG = O CH, and the ∠ O′GC = O′HC, and the side O′C commom, (I XXVI) O′G = O H Similarly, O′G = O′F, O F, O′G, O′H are equal, and the ○ with O′ as centre, and O′F as radius, will pass through G and H, and will touch the sides at F, G, H

4 Let D, E be the points in which CA, CB produced touch the ○ whose centre is O‴ It is required to prove that BE = ($s - a$)

Dem —Let J be the point of contact of AB and O‴ Now it may be proved, as in Ex 2, that CB + BJ = s, that is, CB + BE = s, but CB = a, hence BE = ($s - a$), and AD = $s - b$

5 (1) It is required to prove that the points O, O‴, A, B are concyclic

Dem —Let E be the point in which CB produced touches O‴ Now since the ∠ˢ ABC, ABE are bisected, the ∠ OBO‴ is equal to half the sum of the ∠ˢ ABC, ABE, and is therefore a right ∠ Similarly, OAO‴ is a right ∠, ∴ the ∠ˢ OAO‴, OBO‴ are together equal to two right ∠ˢ Hence (III XXII, Ex 1) the points O, O‴, A, B are concyclic

(2) It can be shown as in (1) that the ∠ˢ O AO″, O′BO″ are right ∠ˢ Hence (III XXI, Cor 1) the points O′, B, A, O″ are concyclic

6 It is required to prove that O is the orthocentre of the Δ O′O″O‴

Dem —Because the ∠ O″BO′′ is right, O″B is the ⊥ from O″ on O′O‴ Similarly, O′A, O‴C are the ⊥ˢ from O′, O‴ on O″O‴, O′O″ Hence the point O is the orthocentre of the Δ O′O ′O‴ Similarly for the others

7 See Book I, Miscellaneous Ex 36

8 Dem —It is shown, in Ex. 5, that the four points O, A, O‴, B are concyclic, hence (III xxi) the ∠ AO ″O = ABO, but ABO = CBO, CBO = AO″ C, and the ∠ ACO‴ = BCO, since ACB is bisected, hence (I xxxii, Cor 2), the Δˢ BOC, ACO‴ are equiangular, (III xxxv, Cor 3) CO CO ″ = BC AC = ab In like manner AO AO′ = bc, and BO BO″ = ca

10 Dem —From O′ let fall ⊥ˢ r' on AB, AC, BC Join O′A, O′B, O C Now br = 2 Δ ACO′, cr' = 2 ABO′, $r'(b+c)$ = twice the quad ACO′B, and ar' = 2 BO C, $r'(b+c-a)$ = 2 ABC, but $(a+b+c) = 2s$, $(b+c-a) = 2(s-a)$, $2r'(s-a)$ = 2 ABC Hence $r'(s-a)$ = area of the Δ ABC

11 From O, O′ let fall ⊥ˢ OK, O′H on AC It is required to prove that OK O H = $(s-b)(s-c)$

Dem —The line AH = s (Ex 4), and CH, CK are $(s-b)$ and $(s-c)$ (Exs 4 and 2) Now the ∠ OCO′ is right (Ex 5), the ∠ˢ OCK, O′CH are together equal to a right ∠, and since the ∠ O′HC is right, the ∠ˢ HO C, HCO′ make together a right ∠, the ∠ HO′C = OCK, and the ∠ O′HC = OKC, each being right, the Δˢ O HC, OKC are equiangular Hence (III, xxxv, Cor 3) OK O H = $(s-b)(s-c)$, that is, $rr' = (s-b)(s-c)$

12 Dem —Area of Δ ABC = rs (Ex. 9), and $r'(s-a)$ = area of ABC (Ex 10), r r s $s-a$ = square of area of ABC, but $rr' = s-b$ $s-c$ (Ex 11) Hence square of area of ABC = s $s-a$ $s-b$ $s-c$

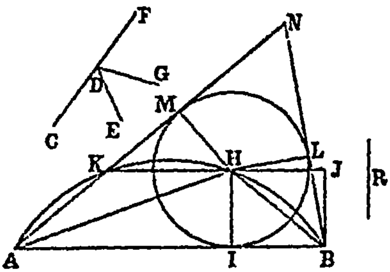
13 Dem —Let the area of ABC be denoted by Δ Now rs = Δ (Ex 9), and r $s-a$ = Δ (Ex 10) Similarly r' $s-b$ = Δ, and r''' $s-c$ = Δ, hence (r r' r'' r''')(s $s-a$ $s-b$ $s-c$) = Δ^4, but (s $s-a$ $s-b$ $s-c$) = Δ^2 (Ex 12) Therefore r r' r' r'' = Δ^2

14 Dem —From O‴ let fall ⊥ˢ O‴D, O‴D′ on CB, CA. Now the ∠ O‴D′C is right, and the ∠ D′CO‴ is half a right ∠, the ∠ CO‴D′ is half right, (I vi) D′O ″ = D′C, but D′O‴ = r''' and D′C = s (Ex 4), r'' = s Similarly it can be shown, if we let fall ⊥ˢ OE, OE′ from O on CB, CA, that E′C

= E O, but E′O = s, and E′C = $(s - c)$ (Ex 2), $r = (s - c)$ In like manner $r = (s - b)$, and $r'' = (s - a)$

15 (1) Let AB be the base, CDE the vertical ∠, and R the radius of the in ○ It is required to construct the △

Sol —Produce CD to F, and bisect the ∠ EDF by DG On AB describe a segment of a ○ containing an ∠ = CDG Erect BJ ⊥ to AB and = R Through J draw JH ∥ to AB, and cutting the ○ in H Join AH, BH, and let fall a ⊥ HI on AB At the points A, B, in the lines AH, BH, make the ∠s HAK, HBL respectively equal to the ∠s HAB, HBA, and produce AK, BL to meet in N ANB is the required △



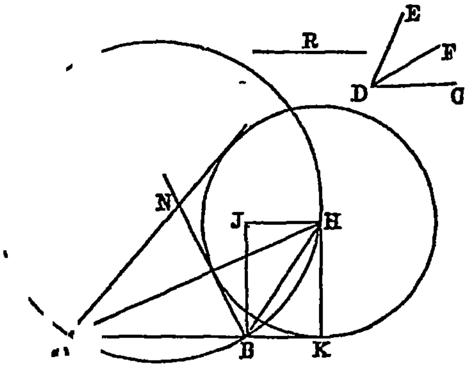
Dem —From H let fall ⊥s HM, HL on AN, BN Now in the △s HIB, HLB we have the ∠s HIB, HBI = HLB, HBL, and the side HB common, (I xxvi), HI = HL Similarly HI = HM, hence the ○ with H as centre, and HI as radius, will pass through L and M, and its radius = R, for HI = BJ = R

Again, the ∠s of the △ HAB are equal to two right ∠s, and the ∠s CDG, FDG are equal to two right ∠s, but the ∠ AHB = CDG, the ∠ FDG = HAB + HBA, and because the ∠s of the △ ANB are two right ∠s, the ∠s of ANB are equal to the ∠s CDG + FDG, but the ∠s NAB + NBA = 2 (HAB + HBA) = 2 FDG = FDE Hence the remaining ∠ ANB = CDE

(2) Let AB be the base, CDE the vertical ∠, and R the radius of the ex ○ which touches the base and one of the sides produced

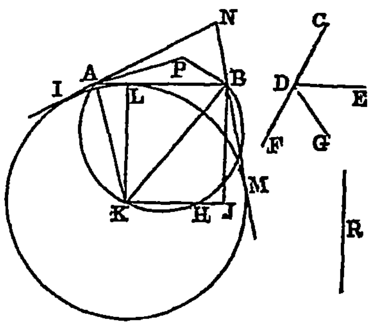
Sol —Bisect the ∠ CDE by DF, and on AB describe a segment containing an ∠ = CDF Erect BJ ⊥ to AB and = R

I" ∥ to AB, and from H, where it meets the ○,
AB produced. With H as centre, and HK



a ○ From A, B draw tangents to this ○,
B is the required Δ
BH Now HK = JB = R, and because H
ex ○ of the Δ ANB, AH, BH are the
nal ∠ NAB and the external ∠ NBK (I
ie ∠ AHB = ½ ANB, but AHB = ½ CDE

e base, CDE the vertical ∠, and R the radius
touches the base externally and the sides
produced.



Sol —Produce CD to F, and bisect the ∠ EDF by DG On AB describe a segment of a ○ containing an ∠ = EDG Erect

BJ ⊥ to AB, and make it equal to R. Through J draw JK ∥ to AB, and cutting the ○ in K From K let fall a ⊥ KL on AB With K as centre, and KL as radius, describe a ○ Through A, B draw tangents IN, MN to this ○, meeting in N ANB is the Δ required.

Dem —Join KA, KB Since K is the centre of the ex-○ of the Δ ABN, the ∠ AKB = ½ (NAB + ABN) (I xxxii, Ex 14), but AKB = ½ FDE (const), NAB + ABN = FDE, hence the ∠ ANB = CDE, and LK = BJ = R.

PROPOSITION V.

2 Dem —Because each of the ∠ˢ APB, AQB is right, AQPB is a cyclic quad, and AP, BQ are chords in the ○, hence (III xxxv) OA . OP = OB OQ Similarly OB OQ = OC OR (Diagram 2, Ex. 1)

3 Dem —The ∠ AOF = DOC (I xv), and AFO = CDO, each being right, FAO = OCD, but OCD = GAF (III xxi), ∴ FAO = GAF, and AFO = AFG, each being right, and AF common. Hence (I xxvi) OF = GF (Diagram, Ex 1)

5 Dem.—In the Δ O'O''O''' the lines O'A, O''B, O'''C are ⊥ˢ from O, O'', O''' on O'O'', O'''O, O'O' (iv, Ex 6), and the points A, B, C are the feet of these ⊥ˢ, hence (Ex 4), the ○ about ABC is the nine points ○ of the Δ O'O''O''' In like manner it is the nine-points ○ of the Δˢ OO'O'', OO''O''', OO'O''' (Diagram, Ex 3, Prop iv)

6 Dem —Because the lines IF, IH, IK are equal (Ex 4), and the ∠ KFH is right, HK is the diameter of the ○ about the Δ KFH, ∴ IK, IH are in one straight line, and since KH is ∥ to OP, and CK to PH, POKH is a ▭, ∴ CK = PH, but CO = 2 CK, ∴ CO = 2 PH (Diagram, Ex 4)

7 Dem —IF = ½ PG This is proved in Ex 4

PROPOSITION X

1. Dem —The ∠ ACD = CBD + CDB (I. xxxii), but CBD = 2 CAD (x), and CDB = CAD. Hence the ∠ ACD = 3 CAD

2 Dem —The ∠s of the Δ ABD are equal to two right ∠s, but each of the ∠s ABD, ADB is equal to 2 BAD, hence the ∠ BAD is $\frac{1}{5}$ of two right ∠s, that is, $\frac{1}{10}$ of four right ∠s, the arc BD is $\frac{1}{10}$ of the whole circumference Hence the line BD is a side of a regular decagon

3 Dem —Let A be the centre Join AB, AD, AE, AF, and join BF, cutting AD in G Now since BD is a side of a regular inscribed decagon, ABD is an isosceles Δ, having each of its base ∠s double of the verticle ∠ (Ex 2), the ∠ BAD is $\frac{1}{5}$ of two right ∠s, the ∠ BAF is $\frac{3}{5}$ of two right ∠s, hence the ∠ AFB is $\frac{1}{5}$ of two right ∠s, the ∠ AGF is $\frac{3}{5}$ of two right ∠s, AF = GF, that is, BF – BG = R Now the ∠ DBG is $\frac{1}{5}$ of two right ∠s, and BDG is $\frac{2}{5}$, BGD is $\frac{2}{5}$, BG = BD Hence BF – BD = R

4 Dem —Because ACDE is a cyclic quad, the ∠s ACD, AED are together equal to two right ∠s (III xxii), and the ∠s ACD, BCD are together = to two right ∠s, the ∠ AED = BCD, that is, AED = CBD, but AED = ADE, and CBD = ADB, ADE = ADB, and AD common Hence (I xxvi) DE = DB

Again, the ∠ ACE = ADE (III xxi), and the ∠ CDA = CEA, but (x) CDA = CAD = DAE, CEA = DAE, and the side AE = AD Hence (I xxvi) the Δs ACE, ADE are congruent.

5 Dem —Let O be the centre of the ○ ACD Join OA, OC Now (Ex 4) AEC is an isosceles Δ, having each base ∠ double of the vertical ∠, and since the ∠s of the Δ AEC are together equal to two right ∠s, the ∠ AEC is $\frac{1}{5}$ of two right ∠s, hence (III xx) the ∠ AOC is $\frac{2}{5}$ of two right ∠s, that is, $\frac{1}{5}$ of four right ∠s Hence AC is the side of a regular pentagon

PROPOSITION XI

1 Let ABCDE be a regular pentagon inscribed in a ○, and let its diagonals CE, AD intersect in A′, BD, CE in B′, CA, BD in C′, AC, BE in D′, and BE, AD in E′ It is required to prove that A′B′C′D′E′ is a regular pentagon

Dem.—Because the arc AE = BC (xi), the ∠ ECA = BAC, CE is ∥ to AB, hence (I xxix) the ∠s EB'B, B'BA, are together equal to two right ∠s, for the same reason the ∠s CA A, A'AB are equal to two right ∠s, but the ∠ DBA = DAB, hence the ∠ A'B'B = B'A'A In like manner the ∠s at C, D', E' are equal Hence the figure A'B C'D'E' is equiangular

Again, because the arc BC = DE, the ∠ BDC = DCE, the side B'C = B'D, and (I xv) the ∠ CB'C' = A'B D, and the ∠ B'C C = B A'D, because they are the supplements of the equal ∠s B'C'D', B A'E', hence the side C B = A'B' Similarly, the other sides of A'B C'D'E' are equal Hence it is a regular pentagon

2 Produce AE, CD to meet in A', ED, BC in B', DC, AB in C', CB, EA in D', BA, DE in E' Join A B', B C', &c It is required to prove that A'B'C'D'E' is a regular pentagon

Dem —In the Δs ABD, CBC, the ∠ ABD' = CBC', and the ∠ D'AB = BCC', being the supplements of equal ∠s, and the side AB = CB, hence (I xxvi) BD = BC', and the ∠ AD'B = BC'C Similarly, AD' = AE', EE' = EA', DA' = DB, and CB' = CC' Again, because the ∠ ABC = EAB, the ∠ D BA = D'AB, D A = D'B Now in the Δs D'AE', D'BC', we have the sides AD', AE' = BD', BC', and the contained ∠s equal, hence the base D E' = D'C' In like manner the other sides are equal Hence the figure is equilateral Again, we proved the ∠ BD'C' = BC'D, and the ∠ AD'B = BC'C, and the ∠ AD'E' is = CC B', since the Δs AD E', CC'B' are equal in every respect. Hence the ∠ E D C' = D C B' In like manner the other ∠s are equal. Hence the pentagon A B'C'D'E is regular

3 Let AD, BE, two consecutive diagonals of a regular pentagon, intersect in E' It is required to prove that BE EE' = E B²

Dem —Join CE, and describe a O about the Δ AE'B

Now because DE = BC, the ∠ DCE = BEC, DC is ∥ to BE Similarly, BC is ∥ to AD, hence (I xxxiv) DC = BE', but DC = AB (hyp), AB = BE' Again, because AE = DE, the ∠ ABE = EAD, and hence (III xxxii) AE is a tangent to the O ABE', (III xxxvi) BE EE' = AE² = AB² = E'B² Hence BE is cut in extreme and mean ratio in E'

4 Let AB be a side of a regular pentagon It is required to construct it.

Sol —Erect BC ⊥ to AB, and make it equal to ½ AB Join AC, and produce it to D, so that CD = CB On AB describe an isosceles △ ABE, having each of its equal sides equal to AD About the △ ABE describe a ○ Bisect the ∠ˢ BAE, ABE by the lines AF, BG, meeting the circumference in F and G Join AG, GE, EF, FB ABFEG is the required pentagon

Dem —From AC cut off CH = CB or CD Now DA . AH + $CH^2 = AC^2$ (II vi), but $CH^2 = BC^2$ and $AC^2 = AB^2 + BC^2$, ∴ DA . AH = AB^2 = DH^2, ∴ AD is divided in extreme and mean ratio in H Therefore, since AE = AD, if we divide AE in extreme and mean ratio, the greater segment would be equal to AB, and hence (x.) AEB is an isosceles △, having each base ∠ double the vertical ∠, but the base ∠ˢ are bisected by the lines AF, BG, ∴ the ∠ˢ EAF, FAB, ABG, GBE, AEB are equal, ∴ the chords EF, BF, AG, EG, AB are equal Hence ABFEG is a regular pentagon

5 Let ABC be a right ∠ It is required to divide it into five equal parts

Sol —Draw BD, making the ∠ ABD equal to the vertical ∠ of an isosceles △ having each of its base ∠ˢ double the vertical ∠ Bisect the ∠ ABD by BE, each of the ∠ˢ ABE, DBE is ⅕ of a right ∠ Draw BF, BG, making the ∠ˢ DBF, FBG each equal to EBD Then the ∠ ABC is divided into five equal parts by the lines BE, BD, BF, BG

PROPOSITION XV

1 (1) Let ABCDEF be the hexagon Join AC, AE, CE It is required to prove that the area of the hexagon is double the area of the △ ACE

Dem —Let the diagonals of the hexagon intersect in O Now the △ˢ OCD, OED are equilateral, and hence OCDE is a lozenge, and CE is its diagonal, ∴ OCDE = 2 OCE Similarly OABC = 2 OAC, and OAFE = 2 OAC Hence ABCDEF = 2 ACE.

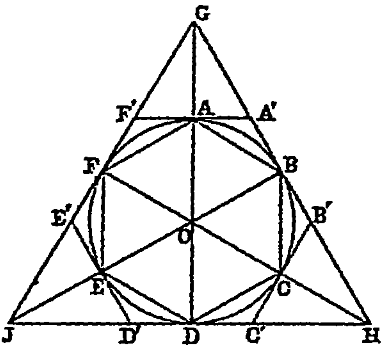
(2) See Book I, Prop I, Ex 4

2 Let AB be the diameter, and O the centre Produce AB to C, so that BC = BO From C draw tangents CD, CE to the ⊙, and join DE It is required to prove that the △ CDE is equilateral

Dem.—Join OD, OE, BD, BE Now (III xviii) the ∠ CDO is right, (Book I, Prop xli, Ex 2) the lines BD, BO, BC are equal, but OB = DO, the △ ODB is equilateral, and because each of the ∠s CDO, CEO is right, CDOE is a cyclic quad., the ∠s DOE, DCE are together equal to two right ∠s, but each of the ∠s DOB, BOE is an ∠ of an equilateral △, DCE is an ∠ of an equilateral △, and because CD = CE, the △ CDE is equilateral

3 (1) Let ABCDEF be the hexagon, and GHJ the equilateral △ It is required to prove that the area of the △ is double the area of the hexagon

Dem.—Let the diagonals of the hexagon intersect in O Join



AG, CH, EJ Now, because AB = AF, AG common, and the base GB = GF, (I viii) the ∠ BAG = FAG, and the ∠ OAB = OAF, the ∠s FAG, OAF are together equal to two right ∠s, hence (I xiv) OA and AG are in the same straight line

Again (III xviii), the ∠ OFG is right, the ∠s FOG, FGO make one right ∠, but the ∠ AFO = FOA, the ∠ AFG = AGF, AF = AG, but AO = AF, AO = AG, hence

(I xxxvii) the △ AFO = AFG, the △ OFG = 2 OFA Similarly, OBG = 2 OBA, OFGB = 2 OFAB In like manner OBHD = 2 OBCD, and OFJD = 2 OFED Hence the △ GHJ = 2 ABCDEF

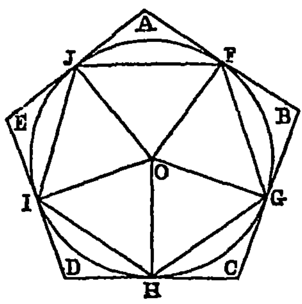
(2) Let A′B′C′D′E′F′ be the circumscribed hexagon It is required to prove that the area of ABCDEF is three-fourths the area of A′B′C′D′E′F′

Dem —Because each of the ∠ˢ F′AO, F′FO is right (III xviii), the ∠ˢ AF′F, AOF are together equal to two right ∠ˢ, and the ∠ˢ AF′F, AF′G are together equal to two right ∠ˢ, hence the ∠ AF′G = AOF, AF′G is an ∠ of an equilateral △ In like manner AA′G is an ∠ of an equilateral △, GF′A′ is an equilateral △, and because GA is ⊥, it bisects the base, AF′ = AA′, A′F′ or GF′ = 2 AF′ = 2 FF′, hence the △ FGA = 2 FF′A, FGA = 3 FF′A, hence (1) AOF = 3 FF′A, AOF = $\frac{3}{4}$ OFF′A In like manner AOB = $\frac{3}{4}$ OAA′B, &c Hence ABCDEF = $\frac{3}{4}$ A′B′C′D′E′F′

Exercises on Book IV.

1 (1) Let ABCDE be a regular polygon circumscribing a ○ It is required to prove that the corresponding inscribed polygon is regular

Dem —Let O be the centre Join OF, OG, OH, OI, OJ



Now (III xviii) the ∠ˢ OHD, OID are right, ∴ the ∠ˢ IDH, IOH are together equal to two right ∠ˢ In like manner the ∠s GCH, GOH are together equal to two right ∠s, but IDH = GCH (hyp), ∴ the ∠ IOH = GOH In the same way it can be shown that all the ∠s at O are equal Hence the arcs are all equal, and therefore the five chords FG, GH, HI, IJ, JF are all equal

(2) *Proved as in Book IV, Prop xii*

2 Let the circumscribing △ ABC be isosceles Let AB, BC, CA touch the ○ in E, D, F It is required to prove that the △ DEF is isosceles

Dem —Let O be the centre Join OD, OE, OF Now the ∠s ODB, OEB are right (III xviii), ∴ the ∠s EBD, EOD are together equal to two right ∠s Similarly the ∠s FCD, FOD are together equal to two right ∠s, but the ∠ EBD = FCD (hyp), ∴ the ∠ EOD = FOD, ∴ the arc ED = FD, ∴ the chord ED = FD And hence the △ DEF is isosceles

3 Let the ∠ BAC = EDF It is required to prove that both △s are equilateral

Dem —Because the △s are isosceles, and the ∠ BAC = EDF, their remaining ∠s are equal, ∴ the ∠ ABC = EFD, but EFD = EDB (III xxxii), ∴ EBD = EDB, and EDB = BED, ∴ EBD is an ∠ of an equilateral △ Similarly FCD is an ∠ of an equilateral △ Hence ABC and DEF are equilateral △s

4 Let ACB be an ∠ of an equilateral △ It is required to divide it into five equal parts

Sol —Describe a ○ about the △ ABC, and in it inscribe a regular polygon of fifteen sides (xvi), then five of those sides will be in the arc AB Let D, E, F, G be the points of division Join CD, CE, CF, CG Now since the arcs AD, DE, EF, FG, GB are equal, the ∠s ACD, DCE, ECF, FCG, GCB are equal

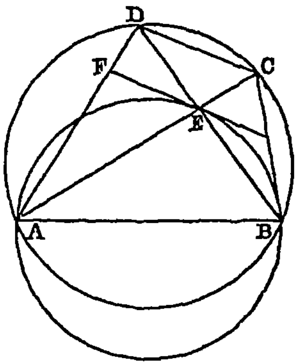
5 Let ABC be a sector of a given ○ It is required to inscribe a ○ in it

Sol —Bisect the ∠ BAC by AD, meeting the arc BC in D Through D draw EF a tangent to the sector Produce AB, AC to meet this tangent in E, F Bisect the ∠ AEF by EG, meeting AD in G G is the centre of the required ○

Dem —From G let fall ⊥s GH, GJ on AE, AF Now (III xviii) the ∠ EDG is right, and the ∠ EHG is right (const), and the ∠ DEG = HEG, and EG common, ∴ (I xxvi) GD = GH. Similarly GH = GJ. Hence the ○, with G as centre and GD as radius, will pass through H and J.

6 Dem.—Describe a ○ about ABC, and through A draw AF touching this ○. Now (III. xxxii.) the ∠ FAC = ABC, but ABC = ADE (I. xxix.), ∴ FAC = ADE, ∴ the ○ about ADE will touch AF in A. Hence the ○s touch each other in A.

7 Dem.—Let EF be the tangent at E to the ○ about ABE. Now the ∠ FEA = EBA (III. xxxii.), but EBA = DCA (III. xxi.). Hence the ∠ FEA = DCA, and ∴ the lines EF, CD are ‖.

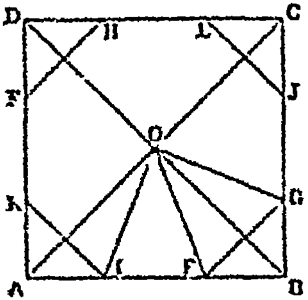


8 Let ABCD be a given square. It is required to describe a regular octagon in it.

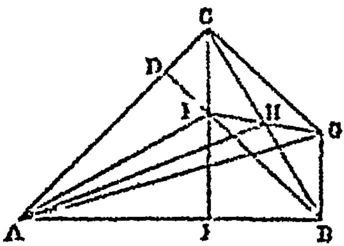
Sol.—Draw the diagonals AC, BD, intersecting in O. Cut off AE, AF = AO, BI, BJ = BO, CG, CH = CO, DK, DL = DO. Join EG, JL, HF, KI. EGJLHFKI is the octagon required.

Dem.—Join OG, OE, OI. Now, because AE = AO, and the ∠ EAO is half a right ∠, ∴ each of the ∠s AEO, AOE is three-fourths of a right ∠, and the ∠ AOB is right, ∴ EOB is one-fourth of a right ∠. Similarly, each of the ∠s GOB, AOI is one-fourth of a right ∠, hence EOI is half a right ∠, and we have seen that AEO is three-fourths of a right ∠, ∴ EIO is three-fourths of a right ∠, ∴ OI = OE. And because the ∠ EOB

= GOB, and EBO = GBO, and the side BO common, OG = OE



= OI Now OG = OI, and OE common, and the ∠ GOE = IOF, the bases EG, FI are equal In like manner all the sides are equal Again, because BE = BG, the ∠ BEG = BGE, ∴ their supplements GEI, LGJ are equal In like manner all the ∠s are equal Hence the octagon is regular

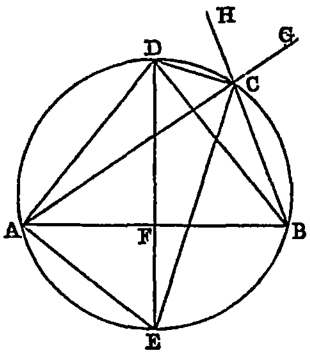


9 Let AB, AC be two given lines, and BC a line of given length sliding between them From B, C ⊥s BD, CE are let fall on AC, AB, intersecting in F It is required to find the locus of F

Sol —At B, C erect ⊥s BG, CG to AB, AC Join FG, cutting BC in H Join AF, AG, AH Now, because CE and GB are ⊥ to AB, and CG, BD to AC, CGBF is a ▭, hence (I xxxiv, Ex 1) BH = CH, and FH = GH. Again, since BC is a line of given length sliding between two fixed lines, AB, AC, and BG, CG are ⊥s at its extremities, (III xxviii, Ex. 2) the locus of G is a ○, having A as centre, and AG as radius, hence AG is a given line, and (I xlvii) $AC^2 + CG^2 = AG^2$, and $AB^2 + BG^2 = AG^2$, $AC^2 + CG^2 + AB^2 + BG^2$ is given, but (II x., Ex 2) $BG^2 + CG^2 = 2\ CH^2 + 2\ HG^2$, and $AB^2 + AC^2 = 2\ CH^2 + 2\ AH^2$, $4\ CH^2 + 2\ AH^2 + 2\ GH^2$ is given, but $4\ CH^2 = CB^2$, $4\ CH^2$ is given, and $2\ AH^2 + 2\ GH^2$ is given, $AF^2 + AG^2$ is given, but AG^2 is given, AF is given, hence AF is a line of given length, and since A is a fixed point, the locus of F is a ○ having A as centre, and AF as radius

10 Let ABC be the △ About ABC describe a ○ Let DF be a ⊥ to AB at its middle point Produce DF to meet the circumference in E Join AD, BD, CD, CE It is required to prove that CE is the internal, and CD the external bisector of the ∠ ACB

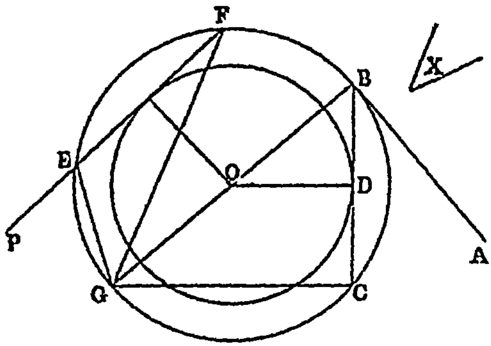
Dem —Produce BC to H, and join AE, BE Now (I xv) AE = EB, the arc AE = BE, hence the ∠ ACE = BCE Therefore CE is the internal bisector of the ∠ ACB Again (I iv), AD = BD, and the ∠ FAD = FBD, and because



ABCD is a cyclic quad, the ∠s BAD and BCD are together equal to two right ∠s, and the ∠s BCD, DCH are together equal to two right ∠s, the ∠ BAD = DCH, and (III xxi) the ∠ ACD = ABD, and ABD = BAD, hence ACD = DCH Therefore CD is the external bisector of the ∠ ACB

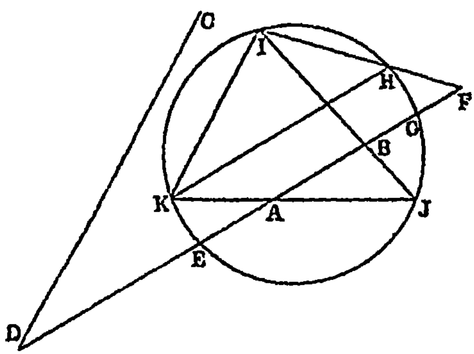
11 Sol —Draw any tangent AB to the ○ At the point B make the ∠ ABC = X From the centre O draw OD ⊥ to BC, and with O as centre, and OD as radius, describe a ○ Through P draw a tangent to this ○, cutting BCE in E and F PEF is the line required

Dem —Take any point G in BCE, and join BG, CG, EG, FG



Now (III xiv) EF = BC, ∴ the ∠ EGF = BGC, but (III xxxii) BGC = ABC = X Hence the ∠ EGF = X

12 Let IJK be the given ○, A, B the given points, and CD

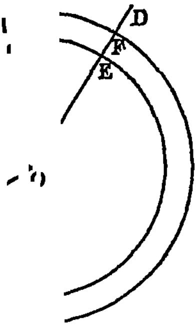


the given line It is required to inscribe a Δ in IJK, having two sides passing through A, B, and the third ∥ to CD

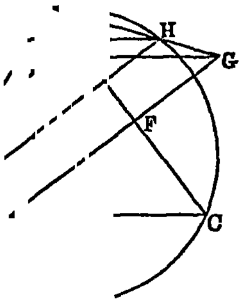
Sol —Join BA, and produce it to meet CD Produce EB to F, and make AB : BF = EB : BG Through F draw FHI, cutting

∠ = CDF (Ex. 11) Join IB
Join JA, and produce it to
h is the required Δ
use AB BF = EB BG,
oints A, J, F, I are concyclic,
= KHI (III xxi), AFI
re ∥, and since the ∠ HKI

ts, no three of which are col-
○ which shall be equidistant



ough A, B, C Let O be its
E Bisect ED in F With



ribe a ○ GHI This is the

produce them to meet the

O GHI Because OF = OI, and OE = OC, . EF = CI, but EF = DF, CI = DF In like manner BH and AG are equal to DF Hence the O through G, H, I, F is equally distant from the points A, B, C, D

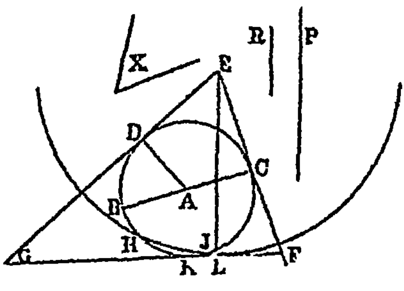
14 Let ABC be a given O It is required to inscribe a Δ in it, whose sides shall pass through three given points D, E, F

Analysis —Let ABC be the Δ Join EF Through B draw BH ‖ to EF, and meeting the O in H. Join AH, and produce it to meet EF produced Join GD Through H draw HJ ‖ to GD, and meeting the O in J Join JB, and produce GD to meet JB in I Now because BH is ‖ to EG, the ∠ CEG = CBH, but CBH = CAH (III xxi), CEG = CAH, hence ECGA is a cyclic quad, EF FG = AF . FC, but AF FC is given, since F is a given point, EF . FG is given, but EF is given, FG is given, G is a given point

Again, the ∠ ABJ = AHJ (III xxi), but AHJ = AGI (I xxix), ABJ = AGI, hence BIAG is a cyclic quad, GD DI = AD DB, but AD DB is given, GD DI is given, DI is given, hence I is a given point. Now the ∠ JHB = IGE (I xxix, Ex 8), but the ∠ IGE is given, since the lines EG, IG are given in position, the ∠ JHB is given Hence the question reduces to Ex 11

15 (1) Let R be the radius of the in-O, X the vertical ∠, and P the ⊥ from the vertical ∠ on the base It is required to construct the Δ

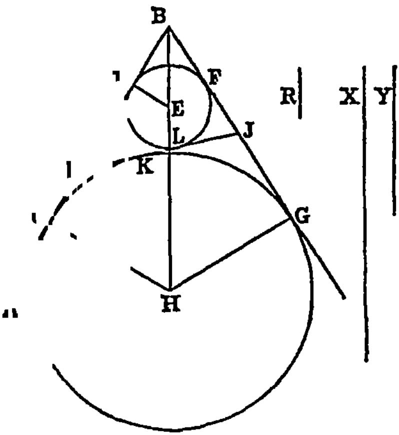
Sol —With any point A as centre, and a radius equal to R, describe a O Draw BC a diameter, and at the point A in AB



make the ∠ BAD = X. Through C, D draw EF, EG tangents to the O With E as centre, and a radius equal to P, describe ion tangent, touching the ⊙s in K, L

v the ∠ELF is right, EL is the ⊥ from
, and it is equal to P (const.), and AD,
is equal to R Again, each of the
II xviii), the ∠s CAD, CED are
ght ∠s, and the ∠s CAD, BAD are
it ∠s, CED = BAD = X
of the sides, Y the base, and R the

and from it cut off BC = ½ (X + Y) and
AB, and make it equal to R With E



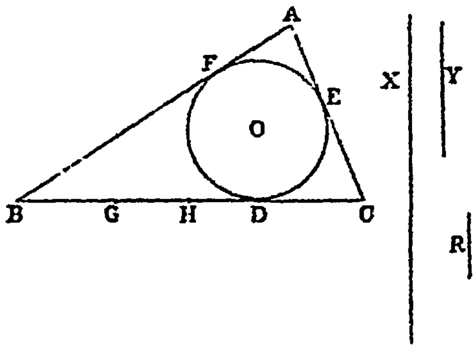
as centre, and ED as radius, describe a ⊙ Draw BG, touching this ⊙ at F Join BE, and produce it Erect CH ⊥ to AB, and meeting BE produced in H From H draw HG ⊥ to BG With H as centre, and HC as radius, describe a ⊙ Draw IJ a common tangent, touching the ⊙s in K and L BIJ is the required △

Dem —ED, the radius of the in-⊙, is equal to R, and (iv, Ex 2) IJ + BD = ½ (IB + BJ + IJ), and (iv, Ex 4) BC = ½ (IB + BJ + IJ), hence IJ = CD = Y Again, BC = ½ (IB + BJ + IJ), and BC = ½ (X + Y) (const), hence (IB + BJ + IJ) = (X + Y), (IB + BJ) = X

(2′) Let X be the base, Y the difference of the sides, and R the radius of the in-○

Sol —With any point O as centre, and a radius equal to R, describe a ○ In the circumference take a point D Through D draw a tangent BC From DB cut off DG = Y Bisect DG in H, and make BH, CH each equal to ½ X Through B, C draw AB, AC tangents to the ○ ABC is the Δ required

Dem —BC = BH + CH = X, and AB = AF + FB, and AC = AE + EC Hence AB − AC = FB − EC = BD − CD = BD − BG = DG = Y If we take the radius of an ex-○ the proofs are similar to those in (2), (2′)



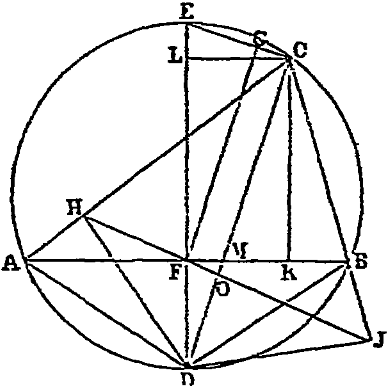
(3) (Diagram, Prop IV, Ex. 3) —Let O′, O″, O‴ be the centres of the ex-○s Join them, and let fall ⊥s O′A, O″B, O‴C Join AB, BC, CA ABC is the Δ required.

Dem —Produce AB, AC to F and H Let O be the point where the ⊥s intersect Now because each of the ∠s O CO‴, O AO‴ is right, AOCO‴ is a cyclic quad, the ∠ ACO″ = AOO‴ Similarly the ∠ BCO′ = BOO′, but AOO‴ = BOO′, and ACO″ = O CH, BCO′ = O′CH, hence CO is the external bisector of the ∠ ACB Similarly, BO is the external bisector of the ∠ ABC Hence O′ is the centre of the ex-○ touching BC externally and the other sides produced In like manner O″, O‴ are the centres of the other ex-○s

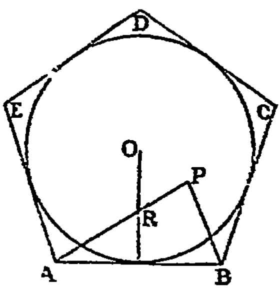
16 (1) Dem —From D let fall ⊥s DH, DJ on AC and CB produced Join DA, DB, HF, FJ, the points H, F, J are collinear (III xxxii, Ex 12) Join DC, CE, and through F draw

FG ∥ to DC Now because the ∠ ACB is bisected by CD HC = ½ (AC + CB) (III xxx, Ex. 4), and since the ∠ DHC is right, DC . CO = HC^2 (I xlvii, Ex 1), that is, DC . FG = HC^2 Again (III xxxi), the ∠ DCE is right, ∴ EGF is right, and CLD is right, ∴ EGF = CLD, and (I xxix.) the ∠ EFG = LDC, ∴ the Δs DCL, EFG are equiangular, hence (III xxxv, Cor 3) DC . FG = DL . FE, ∴ DL . FE = HC^2

(2) From C let fall a ⊥ CK on AB Now (III, Ex 17) FM . FK is equal to the square of half the difference of AC and CB, that is, equal to AH^2.



Again, the ∠ ELC = DFM, each being right, and because DCE is right, the ∠s CED, CDE are together equal to a right ∠; and the ∠s LEC LCE are equal to a right ∠, ∴ LCE = CDE, hence the Δs DFM, CLE are equiangular, ∴ (III. xxxv, Cor 3) DF . LE = LC . FM = FK . FM = AH^2

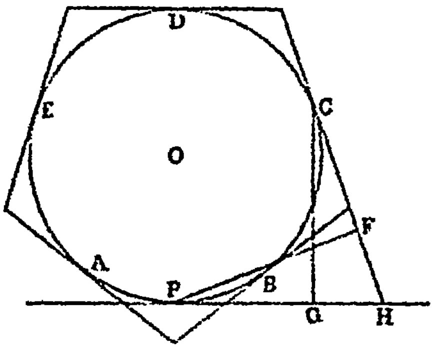


17 Let the regular polygon of *n* sides be a pentagon ABCDE,

P a point within it, and p_1, p_2, &c., the ⊥s from P on the sides. Let O be the centre of the in-○, and R its radius. It is required to prove that $(p_1 + p_2 + p_3 + p_4 + p_5) = 5R$.

Dem.—Join AP, BP, &c., and let the sides be denoted by s. Now sp_1 = twice the △ APB, sp_2 = twice the △ BPC, &c.; hence $s(p_1 + p_2 + p_3 + p_4 + p_5)$ = twice the area of the pentagon. Again, Rs = twice the △ AOB, &c.; ∴ $5Rs$ = twice the area of the pentagon; ∴ $s(p_1 + p_2 + p_3 + p_4 + p_5) = 5Rs$. Hence $(p_1 + p_2 + p_3 + p_4 + p_5) = 5R$. Similarly for a regular polygon of any number of sides.

18. Let A, B, C, D, E be the angular points of a regular polygon of five sides. About ABCDE describe a ○, and through

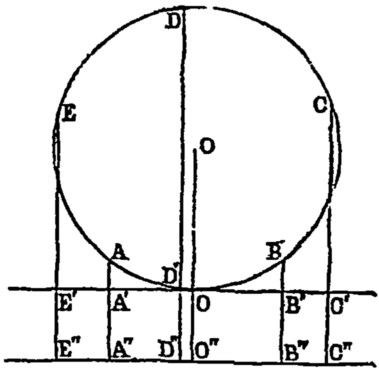


A, B, C, D, F draw tangents to this ○. It is required to prove that the sum of the ⊥s from A, B, C, D, E on any line is equal to five times the ⊥ from O, the centre of the ○, on the same line.

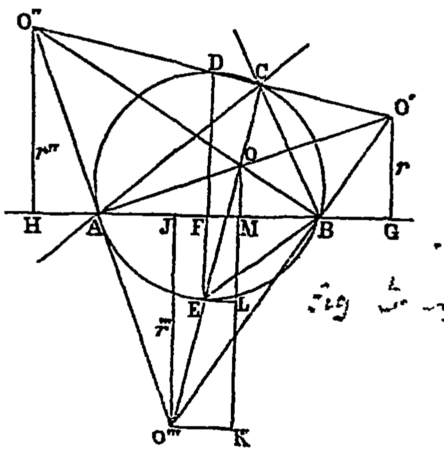
(1) Dem.—Let the line be a tangent at any point P in the circumference. From P, C let fall ⊥s PF, CG on the tangents through C and P. Produce CF to meet PG in H. Now in the △s CGH, PFH, the ∠ CGH = PFH, and PHC is common, and the side CH = PH; hence (I. xxvi.) CG = PF. Similarly, the ⊥s from A, B, D, E on the tangent at P are equal to the ⊥s from P on the tangents through those points; but (Ex. 17) the sum of the ⊥s from P on the sides of ABCDE is equal to 5 R; hence the sum of the ⊥s from A, B, C, D, E on PH = 5 R; that is, equal to five times the ⊥ from O on PH; and similarly for a regular polygon of any number of sides.

(2) Let the line not touch the ○

Dem.—From A, B, C, D, E let fall ⊥ˢ AA″, BB″, CC″, DD″, EE″ on the line, and from O let fall OO″ At O′, where OO′ cuts the ○, draw a line ∥ to C′E′, and let the ⊥ˢ from A, B, C, D, E cut this line in A′, B′, C′, D′, E′ Now (1) AA′ + BB′ + CC′



+ DD′ + EE′ = 5 OO′, and A′A″ + B′B″ + C′C″ + D′D″ + E′E″ = 5 O′O″ Hence, by addition, we get AA″ + BB″ + CC″ + DD″ + EE″ = 5 OO″



19 (1) Let ABC be a Δ inscribed in a ○ Bisect the base AB in F, and erect a ⊥ DFE, meeting the ○ in D, E, then

(III. III.) DE is the diameter. Join CD, CE. CD and CE are the external and internal bisectors of the ∠ ACB (Ex. 10). Produce AB to G, H. Bisect the ∠s CBG, CAH by BO′, AO″, meeting CD produced in O′, O″. Produce O′B, O″A to meet in O‴. O′, O″, O‴ are the centres of the ex-○s (IV., Ex. 3). Produce CE. CE produced will pass through O‴ (I. XXVI., Ex. 8). From O‴ let fall a ⊥ O‴J on AB. Join AO′, meeting CE in O. From O draw OK ∥ to O‴J, and from O″ and E draw O‴K and EL ∥ to AB. From O′, O″ let fall ⊥s O′G (r'), O″H (r'') on GH. Join BE.

Now, since AO′, BO′, CO′ meet in O′, and that BO′, CO′ are two external bisectors, hence (I. XXVI., Ex. 8) AO′ is the internal bisector of the ∠ BAC. Similarly, BO″ is the internal bisector of the ∠ ABC.

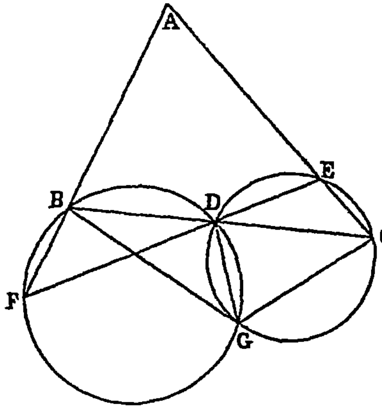
Again, AG, BH are each equal to s (IV., Ex. 4), ∴ AH = BG, ∴ HF = GF, hence HG is bisected in F, ∴ (I. XL., Ex. 8) O′G + O″H = 2 DF, that is, $r' + r'' = 2$ DF. And because the ∠ ECB = ACE, (III. XXI.) ECB = ABE, and CBO = ABO, hence (I. XXXII.) EOB = EBO, ∴ EB = EO, but the ∠ OBO‴ is right, ∴ (I. XII., Ex. 2), EB = EO‴, hence OO‴ is bisected in E, and EL is parallel to O‴K, ∴ (I. XL., Ex. 3) OK is bisected in L, and divided unequally in M, hence KM − OM = 2 LM, that is, $r''' - r = 2$ EF, and we have proved $r' + r'' = 2$ DF, ∴ $r' + r'' + r''' - r = 2$ DE $= 4$ R. Hence $r' + r'' + r''' = 4\text{R} + r$.

(2) It has been shown that $r''' - r = 2$ EF, but EF is $= \mu$, hence $r''' - r = 2\mu$. Similarly $r' - r = 2\mu'$, and $r'' - r = 2\mu''$, hence $r' + r'' + r''' - 3r = 2(\mu + \mu' + \mu'')$, that is, $4\text{R} + r - 3r = 2(\mu + \mu' + \mu'')$. And therefore $(\mu + \mu' + \mu'') = 2\text{R} - r$.

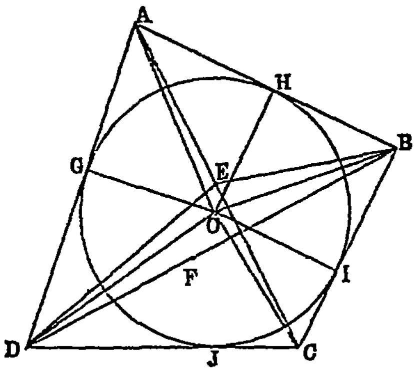
(3) Dem.—$\mu + \delta = \mu' + \delta' = \mu'' + \delta'' = \text{R}$, hence we have $\mu + \mu' + \mu'' + \delta + \delta' + \delta'' = 3\text{R}$, that is, $2\text{R} - r + \delta + \delta' + \delta'' = 3\text{R}$. And hence $\delta + \delta' + \delta'' = \text{R} + r$.

20. Dem.—Let G be the second point of intersection. Join GB, GC, GD. Now (III. XXII.) the sum of the ∠s DGC, DEC is two right ∠s, but DEC = EAF + AFE, and AFE = BGD (III. XXI.), ∴ BGC + BAC is equal to two right ∠s, hence BACG is a cyclic quad., ∴ the ○ through B, A, C will pass through G. And the locus of G is a ○.

21. Let ABCD be the quad., E, F the middle points of the diagonals, and O the centre of the in-○. It is required to prove that the points E, O, F are collinear.



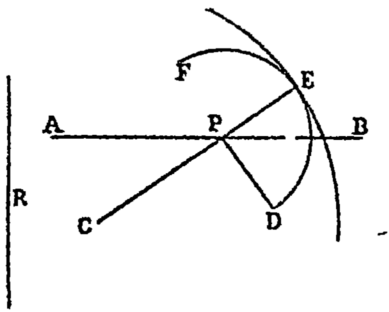
Dem —Join EB, ED, and join O to the points of contact G, H, I, J



Now (I xxxviii) the Δ ABE = CBE, and the Δ ADE = CDE; AEB + CDE = ½ ABCD, hence the sum of the areas of AEB and CDE is given, and their bases AB, CD are given, (I, Ex. 29) the locus of E is a straight line, and F is a point on the locus, since it can be shown in the same manner that AFB + CFD = ½ ABCD Again, the Δ OAG = OAH, and OIB = OBH, the area of OAB is half the area of the figure GABIO Similarly, OCD = ½ GOICD, hence OAB + OCD = ½ ABCD, and (I, Ex. 29) O is a point on the locus, that is, the points E, O, F are on the same straight line

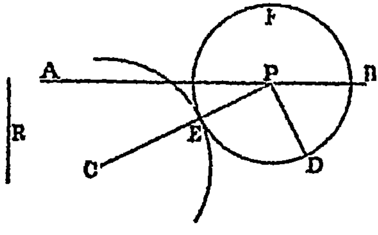
22 (1) Let AB be a given line, C, D two given points It is required to find a point P on AB, so that CP + DP = R (a given line)

Sol —With C as centre, and a radius equal to R, describe a ○, and describe a second ○ DEF, having its centre on AB, passing through D, and touching the first ○ internally in E (III xxxvii, Ex 3) Let P be its centre P is the required point



Dem —Join CP, and produce it, then (III xi) CP produced passes through E Join PD Now PE = PD, ∴ CP + PD = CE = R

(2) It is required to find a point P, so that CP − DP = R



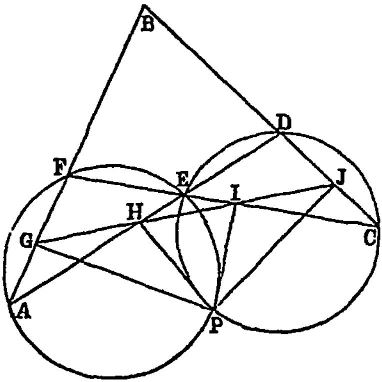
Sol —With C as centre, and a radius equal to R, describe a ○, and describe a second ○ DEF, having its centre on AB, passing through D, and touching the first ○ externally in E Let P be its centre P is the required point

Dem —Join CP, DP Now CP = CE + EP, CP − EP = CE = R, that is, CP − DP = R

23 Let AB, AD, CB, CF be the four lines About the Δ^s AFE, CDE describe ʘ^s, let them intersect in P P is the point required

Dem —From P let fall ⊥^s PG, PH, PI, PJ on the sides of the Δ^s AFE, CDE

Now (III xxii, Ex 12) the feet of the ⊥^s on the sides of the Δ AFE are collinear Similarly the feet of the ⊥^s on the sides



of the Δ CDE are collinear Hence the feet of the ⊥^s PG, PH, PI, PJ are collinear, and these are the ⊥^s on the four given lines AB, AD, CB, CF

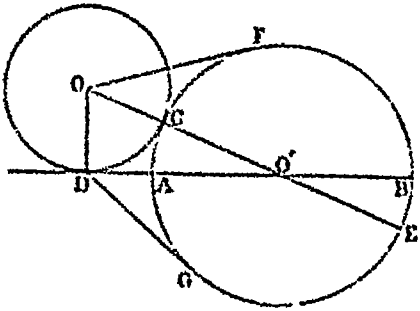
24 See "Sequel to Euclid," Book III, Prop xiv

25 See "Sequel to Euclid," Book III, Prop xiv, Cor

26 (1) See "Sequel to Euclid," Book III, Prop v

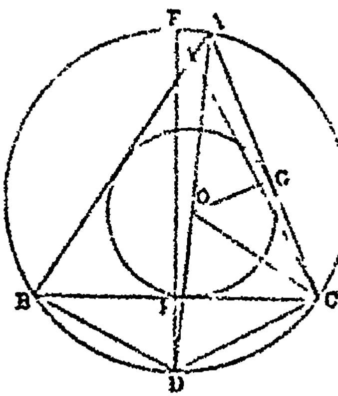
(2) Dem —Let AB be the diameter of the semicircle ACB Produce BA to D, and let a ʘ whose centre is O touch ACB in C, and BD in D Join OD, OO′ OO′ passes through C (III xii) Produce OO′ to meet ACB in E, and from O, D draw OF, DG tangents to ACB Now EO OC = OF^2 (III xxxvi) = OD^2 + DG^2 ("Sequel," Book III, Prop xxi), and OC^2 = OD^2

Subtracting, we get (EO − OC) OC, that is, EO . OC = DG^2, that is, 2 Rr = DG^2 = DA . DB.



27. *Lemma*.—If a Δ ABC have a ○ inscribed in it, and another circumscribed to it, the rectangle contained by the diameter of the circum-○ and the radius of the in-○ is equal to the rectangle contained by the segments of any chord of the circum-○ passing through the centre of the in-○.

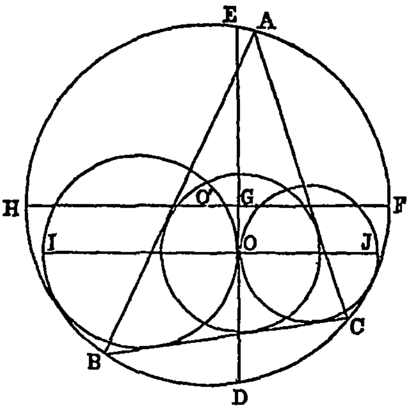
Dem.—Let O be the centre of the in circle. Join AO, and produce it to meet the circum-○ in D. From D let fall a ⊥ DF on BC, and produce it to meet the circumference in E. Join EC, OG, OC, BD, CD. Now the arc BD = CD, ∴ the



chord BD = CD; hence BF = CF, ∴ DE is the diameter of the circum-○, ∴ the ∠ DCE is right, and (III. xviii.) the ∠ OGA is right, and (III. xxi.) the ∠ DEC = OAG, hence the $\triangle^s$ DEC, OAG are equiangular, (III xxxv, Cor 3) ED OG = OA DC, but DC = DO (Dem, Ex 19 (1)) Hence ED OG = OA OD

Let ABC be the $\triangle$, O, O′ the centres of the in- and circum-$\bigcirc^s$, and ρ, ρ' the radii of two $\bigcirc^s$ touching each other at O, and touching the circum-$\bigcirc$ It is required to prove that $\frac{1}{\rho}+\frac{1}{\rho'}=\frac{2}{r}$, r being the radius of the in-$\bigcirc$

Dem —Through O draw a common tangent to those $\bigcirc^s$, meeting the circum-$\bigcirc$ in D, E Join the centres of the $\bigcirc^s$ whose radii are ρ, ρ', and produce to meet the circumferences in I, J



Through O′ draw HF ∥ to IJ, and cutting DE in G Now FG 2ρ = EO OD ("Sequel," Book III, Prop vi),

$$FG=\frac{EO\ \ OD}{2\rho}\quad \text{Similarly, } HG=\frac{EO\ \ OD}{2\rho'},\qquad FH=\frac{EO\ \ OD}{2\rho}$$

$$+\frac{EO\ \ OD}{2\rho}\quad \text{Again, } 2Rr=EO\ \ OD\ (\textit{Lemma}),\qquad 2R=\frac{EO\ \ OD}{r},$$

$$\frac{EO\ \ OD}{2\rho}+\frac{EO\ \ OD}{2\rho}=\frac{EO\ \ OD}{r},\ \text{therefore } \frac{1}{2\rho}+\frac{1}{2\rho'}=\frac{1}{r},$$

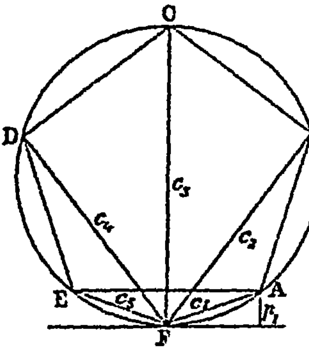
$$\frac{1}{\rho}+\frac{1}{\rho'}=\frac{2}{r}$$

28 *Lemma* —Let AB be the diameter of a $\bigcirc$, AD a tangent.

From C, any point in the circumference, a ⊥ CD is let fall on AD, and AC joined It is required to prove that AB CD = AC²

Dem —Through C draw CE ∥ to AD, meeting AB in E Join BC Now (I XLVII, Ex 1) AB AE = AC², but AE = CD, . AB CD = AC².

Dem —Let the polygon be a regular pentagon ABCDE Take any point F in the circumference At F draw a tangent to the O Join F to the angular points of the pentagon, and let the joining



lines be denoted by c_1, c_2, &c From the angular points let fall ⊥ˢ p_1, p_2, &c, on the tangent, and let the radius be denoted by R

Now (*Lemma*) 2 Rp_1 = c_1^2, and 2 Rp_2 = c_2^2, &c, 2 R (p_1 + p_2 . p_5) = ($c_1^2 + c_2^2$ c_5^2), but ($p_1 + p_2$ p_5) = 5 R (Ex 18), 10 R² = ($c_1^2 + c_2^2$ c_5^2) And similarly for a figure of any number of sides

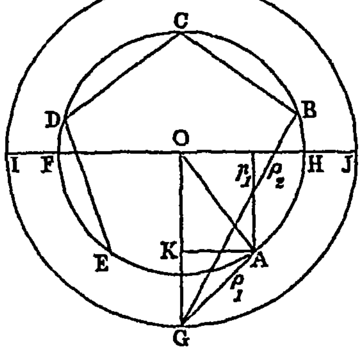
29 This is a special case of the next exercise

30 If any point G in the circumference of any concentric O be joined to the angular points of an inscribed regular polygon, the sum of the squares of the joining lines is equal to n times the square of the radius of the concentric O, together with n times the square of the radius of the circum-O, that is, $\rho_1^2 + \rho_2^2 +$ $+ \rho_5^2 = 5\ R^2 + 5\ r^2$

Dem —Let O be the common centre Through O draw the diameter From A let fall a ⊥ p_1 on IJ, and draw AK parallel to IJ

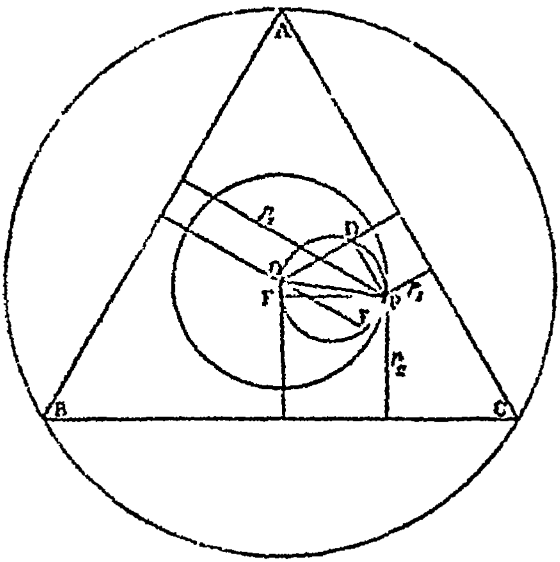
Now $AG^2 = OG^2 + OA^2 - 2\ OG \cdot OK$ (II xiii), that is, $\rho_1^2 = R^2 + r^2 - 2\ Rp_1$. Similarly, $\rho_2^2 = R^2 + r^2 + 2\ Rp_2$, &c, the sign of $2\ Rp_2$ being positive, since the ⊥ is let fall from above the line. Adding, we get, since the terms by which 2 R is multiplied cancel each other, $\rho_1^2 + \rho_2^2 + \ldots + \rho_5^2 = 5\ (R^2 + r^2)$

31 Let ABC be an equilateral Δ inscribed in a ○. From P, any point in the circumference of a concentric ○, ⊥s p_1, p_2, p_3,



are let fall on the sides of ABC. It is required to prove that $p_1^2 + p_2^2 + p_3^2$ = three times the square of the radius of the in-○, together with three half times the square of the radius of the concentric ○.

Dem —From O, the centre, let fall ⊥s on the sides of ABC. Through P draw PD ∥ to AC, meeting the ⊥ from O on AC in D, draw PE ∥ to BC, meeting the ⊥ from O on BC in E. Produce the ⊥ from O on AB to F, and draw PF ∥ to AB. Join OP. Now, since the ∠s ODP, OEP, OFP are right, the ○ on OP as diameter will pass through D, E, F, and because PD is ∥ to AC, and PE ∥ to BC, ∴ (I xxix, Ex 8) the ∠DPE = ACB = an ∠ of an equilateral Δ, ∴ DE is ⅓ of the circumference of DEF. In like manner, EF, DF, are each ⅓ of the circumference of DEF, ∴ D, E, F are the angular points of an equilateral Δ inscribed in DEF, and ∴ (Ex 28) $OD^2 + OE^2 + OF^2 = 6\left(\frac{OP}{2}\right)^2 = \frac{3\ OP^2}{2}$. Again, $p_1 = (r - OD)$, r being the radius of the in-O, $p_1^2 = r^2 - 2r \cdot OD + OD^2 = (r^2 + OD^2) - 2r(r - p_1)$, and $p_2^2 = (r^2 + OE^2) - 2r(r - p_2)$, and $p_3^2 = (r^2 + OF^2) - 2r(r - p_3)$, $\therefore p_1^2 + p_2^2 + p_3^2 = 3r^2 + \frac{3\,OP^2}{2} - 2r\{3r - (p_1 + p_2 + p_3)\}$, but $(p_1 + p_2 + p_3) = 3r$ (Ex 17) Hence $p_1^2 + p_2^2 + p_3^2 = 3r^2 + \frac{3\,OP^2}{2}$ And in general, in the case of a figure of n sides, the sum of the squares of the ⊥s will equal $nr^2 + \frac{n\,OP^2}{2}$.



32 & 33 These are special cases of Ex 31.

35 Let A, B, C, D be the four concyclic points From O, the centre of the O, let fall ⊥s Oα, Oβ, Oγ, Oδ on the sides of ABCD, then (III iii) the sides of the quad are bisected in α, β, γ, δ From α, γ let fall ⊥s αF, γF on AB, CD, and let them intersect in O' Join βO', and produce it to meet AD in G It is required to prove that βG is ⊥ to AD

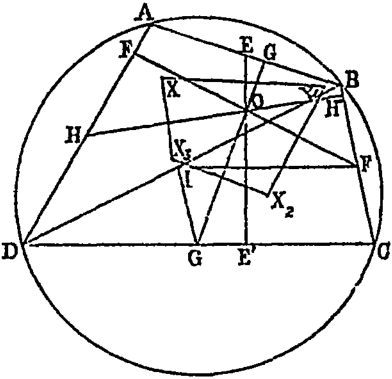
Now (XII) the $\triangle$ AOP = BOP, $\therefore$ the $\angle$ AOP = $\frac{1}{2}$ AOB, but AOB $= \frac{4 \text{ rt } \angle^s}{5}$, $\therefore$ AOP $= \frac{2 \text{ rt } \angle^s}{5}$ In like manner the $\angle$ A′OP $= \frac{2 \text{ rt } \angle^s}{6}$, $\therefore$ the $\angle$ AOA′ $= \frac{2 \text{ rt } \angle^s}{30}$

Let C be the point where OA′ cuts the ○ Then if we divide the arc CP into five equal parts in the points D, E, F, G, and join OD, &c , and produce to meet AB in the points D′, E′, F′, G′, the $\angle^s$ A′OD′, D′OE′, &c , will be each $\frac{1}{30}$ of two right $\angle^s$ Again, the line OA is greater than OD′ (I xix , Ex 4) Cut off OH = OD′ Join A′H Then (I iv) A′D′ = A′H, and the $\angle$ OD′A′ = OHA′, $\therefore$ the $\angle$ OD′E′ = AHA′, but OD′E′ is greater than OAD′ (I xvi), $\therefore$ AHA′ is greater than A′AH, and hence AA′ is greater than A′H, that is, than A′D′ Similarly, A′D′ is greater than D′E′, D′E′ greater than E′F′, &c , hence 5 AA′ is greater than A′P To each add 5 A′P, and we have 5 AP greater than 6 A′P, $\therefore$ 5 AB is greater than 6 A′B′, but 5 AB is the perimeter of the pentagon, and 6 A′B that of the hexagon Hence the perimeter of the pentagon is greater than that of the hexagon, and in general the greater the number of the sides, the less the perimeter

37 By the last exercise the area of a pentagon is less than the area of a square, but the area of a square is equal to the square of the diameter Hence the area of a pentagon is less than the square of the diameter Similarly for other polygons

38 Dem —Let the four concyclic points be A, B, C, D Bisect the joining lines in E, F, G, H Join BD, and bisect it in I Then (v, Ex 4) the nine-points ○ of the $\triangle$ ABD will pass through the points H, E, I Similarly, the nine-points ○ of the $\triangle$ ABC will pass through E, F, and the middle point of AC Hence two of the nine-points ○^{s} will pass through E In like manner two of them will pass through each of the points F, G, H From E, F, G, H let fall $\perp^s$ EE′, FF′, GG′, HH′ on the opposite sides, these $\perp^s$ will co-intersect in a point O (Ex 35) Join IF, IG Now, because each of the $\angle^s$ AG′O, AF′O is right, $\therefore$ the $\angle^s$ F′AG′, F′OG′ are together equal to two right $\angle^s$, and the $\angle^s$ BAD, BCD are equal to two right $\angle^s$, $\therefore$ the $\angle$ F′OG′ = BCD, that is, the $\angle$ FOG = BCG, but (I xxxiv) BCG = FIG, $\therefore$ FOG = FIG, and hence the ○ through the points F, G, I, must pass through O In like manner each of the four nine-

points ○ˢ must pass through O Now, since two of these ○ˢ pass through E and O, if we bisect EO, and erect XX_1 ⊥ to it, their centres must be in XX_1 Similarly, the centres of each other pair must be in the lines X_1X_2, X_2X_3, X_3X. Hence the points X, X_1, X_2, X_3 must be the centres And because each of the lines



XX_1, CD is ⊥ to EE′, they are ∥ to each other Similarly, the remaining sides of $XX_1X_2X_3$ are ∥ to the remaining sides of ABCD, hence the ∠ˢ X and X_2 are equal to the ∠ˢ A and C, but A and C are together equal to two right ∠ˢ, X and X_2 are equal to two right ∠ˢ Hence the points X, X_1, X_2, X_3 are concyclic

39 Let AB, AC be two fixed lines, having their included ∠ BAC equal to an ∠ of an equilateral △, and let BC be a third line forming a △ with AB, AC Bisect BC, AC, AB in D, E, F Join DE, EF, FD The ○ through D, E, F is the nine-points ○ of the △ ABC (v, Ex 4) It is required to prove that the locus of its centre O is a right line

Dem —Join OA, OE, OF Now DE, DF are respectively ∥ to AB, AC (I xl, Ex 2), AEDF is a ▭, the ∠ FDE = FAE, but FOE = 2 FDE (III xx), FOE = 2 FAE, hence FOE is twice an ∠ of an equilateral △, FOE + FAE are equal to two right ∠ˢ, hence AEOF is a cyclic quad Again, because OE = OF, the arc OE = OF, and (III xxvi) the ∠ OAE = OAF, the ∠ FAE is bisected Hence the line OA is given in position, and since O is a point on it, the locus of O is a right line

41 Let AB, AC be two lines given in position, P a given

point, and let the line FG be equal to the given perimeter It is required to draw a transversal through P, so that the Δ formed with AB and AC shall have its perimeter equal to FG

Sol —Bisect FG in H In AB take AD = GH, and erect DO ⊥ to AB Bisect the ∠ BAC by AO, and let fall a ⊥ OE on AC Then (I xxvi) the Δˢ ADO, AEO are equal in every respect, OD = OE, hence the ○, with O as centre and OD as radius, will pass through E, and touch the lines AB, AC in D, E Through P draw MN, touching this ○, and cutting AB, AC in M, N AMN is the Δ required. For (IV, Ex 4) AD is equal to half the perimeter of AMN Hence the perimeter is equal to 2 AD, or FG

42 (1) Let BAC be the vertical ∠, X its bisector, and FG the perimeter

Sol —Bisect the ∠ BAC by AP, and make AP = X Through P draw MN, cutting off a Δ AMN whose perimeter is equal to FG (Ex 41)

(2) Let BAC be the vertical ∠, FG the perimeter, and X the ⊥

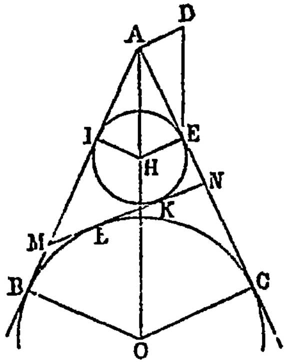
Sol.—Bisect FG in H, in AB take AD = GH, erect DO ⊥ to AB, bisect the ∠ BAC by AO, and from O let fall OE ⊥ to AC, then the ○, with O as centre, and *OD* as radius, will pass through E, and will touch AB, AC, in D, E With A as centre, and a radius equal to X, describe a ○, cutting AB, AC in P, Q Draw a common tangent to the two ○ˢ, meeting AB, AC in M, N AMN is the required Δ

Dem —Join AR, R being the point where MN touches the ○ PQ Now (III xviii) the ∠ ARN is right, AR is a ⊥, and it is equal to X, and as in Ex 41, the perimeter of the Δ AMN = FG

(3) Let BAC be the vertical ∠, FG the perimeter, and R the radius of the in-○

Sol —Bisect BAC by AO Draw AD ⊥ to AC, and make it equal to R Through D draw DE ∥ to AO, and where it meets AC draw EH ∥ to AD From H let fall HI ⊥ on AB Take AB = ½ FG, erect BO ⊥ to AB, and from O let fall a ⊥ OC on AC Now, as in Ex. 41, HE = HI, and OB = OC, hence the ○ˢ with H, O as centres, and HE, OC as radii, will pass through the points I, B Draw a common tangent, touching the ○ˢ in K and L, and cutting AB, AC in M, N AMN is the required Δ

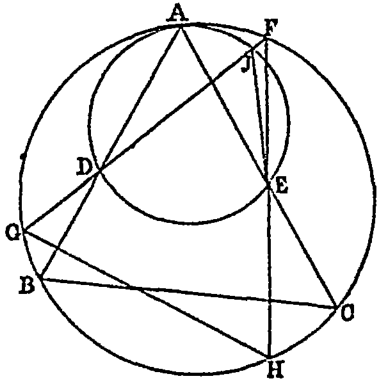
For, as before, the perimeter of AMN = FG And since ADEH is a ▭, EH = AD = R



43 (1) Let ABC be the given ○, D, E the points It is required to inscribe a △ in ABC, so that two sides may pass through D, E, and the third be a maximum

Sol —Describe a ○ passing through D, E, and touching ABC in A (III xxxvii, Ex 1) Join AD, AE, and produce to meet ABC in B, C Join BC ABC is the required △

Dem —Take any other point F in ABC Join FD, FE, and



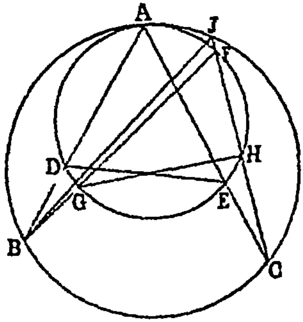
produce to meet ABC in GH Join GH, JE, J being the point where FG cuts the ○ ADE Now the ∠ DJE is greater than

DFE, the ∠ DAE is greater than DFE, the arc BC is greater than GH Hence the chord BC is greater than GH

(2) Let ADE be the given ○, B, C the points

Sol —Through B, C describe a ○ ABC, touching ADE in A Join AB, AC, cutting the ○ ADE in D, E Join DE ADE is the required Δ

Dem —Take any point F in ADE Join BF, CF, cutting the ○ ADE in GH Join GH Produce CF to meet ABC in J Join BJ Now the ∠ BFC is greater than BJC, that is, greater than BAC, the arc GH is greater than DE. Hence the chord GH is greater than DE

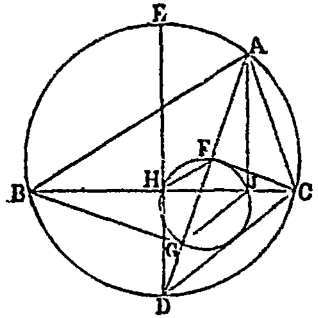


44 Let Δ represent the area of the triangle

Now $r' = \dfrac{\Delta}{s-a}$ (IV, Ex 10), $r'' = \dfrac{\Delta}{s-b}$, $r'r'' = \dfrac{\Delta^2}{(s-a)(s-b)}$, but $\Delta^2 = s \;\; s-a \;\; s-b \;\; s-c$ (IV, Ex 12), therefore $r'r'' = \dfrac{s \;\; s-a \;\; s-b \;\; s-c}{(s-a)(s-b)} = s \;\; s-c$ Similarly, $r''r''' = s \;\; s-a$, and $r'''r' = s \;\; s-b$ Hence $r'r'' + r''r''' + r'''r' = s\,\{3s - (a+b+c)\}$, $= s\,\{3s - 2s\} = s \;\; s = s^2$

45 Let ABC be a Δ inscribed in a ○ Draw the diameter DE ⊥ to BC Join AD AD is the internal bisector of the vertical ∠ From A let fall a ⊥ AJ on BC From B and C let fall ⊥s BG, CF on AD, and let H be the point where DE bisects BC It is required to prove that the points F, H, G, J are concyclic

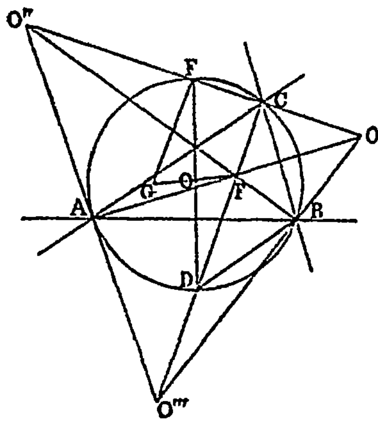
Dem —Join FH, GJ, CD Now, since each of the ∠s BGA, BJA is right, BGJA is a cyclic quad , ∴ the ∠ BAG = BJG



And because DHFC is a cyclic quad , the ∠ DCH = DFH , but (III xxi) DCH = BAD, ∴ DFH = BJG Hence the points F, H, G, J are concyclic

46 Let ABC be the Δ whose base AB and vertical ∠ ACB are given

Describe a ○ about ACB Let O be its centre Draw DE, the diameter, ⊥ to AB Join CD, CE. CD, CE are the inter-



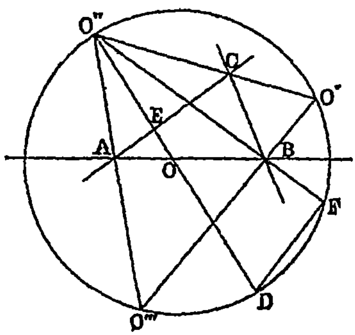
nal and external bisectors of the ∠ ACB (III xxx , Ex 2) Bisect the external ∠ CAB by AO ', meeting CE produced Produce CD, O″A to meet in O‴ Join O‴B Produce O‴B, O″C, to meet in O′ O′B is the external bisector of the ∠ CBA (I xxvi, Ex 8), O′, O″, O‴ are the centres of the ex-○s Join O′A, O″B, intersecting CD in F Join FO Draw EG ∥ to CD, meeting FO produced in G G is the centre of the ○ passing through O′, O″, O‴ It is required to find its locus

Dem —Join BD Now, because F is the orthocentre of the △ O O″ O‴ (IV iv, Ex 6), O the centre of its nine-points ○ (IV v, Ex 5), and EG the ⊥ from the middle point of O′O″,

OF = OG (IV v Ex 4), and since the ∠ GEO = FDO (I xxix.), and GOE = FOD, EG = DF, but DF = DB (Dem of Ex 27), and DB is given, EG is given, and the point E is given Hence the locus of G is a ○, having E as centre and EG as radius

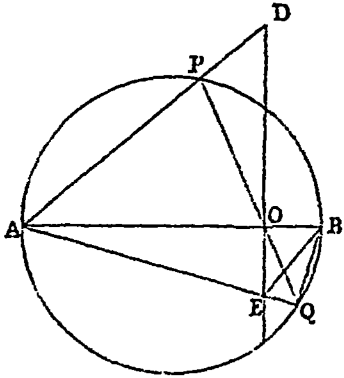
47 Let ABC be the △, O′, O″, O‴ the centres of the ex-○s

Dem —Describe a ○ about the △ O′O″O‴ Let O be its centre Join O O, and produce it to meet the circumference in



D, and cutting AC in E We shall prove that O″O is ⊥ to AC Join O″B, and produce it to meet the circumference in F Join DF Now the ∠ O″FD is right (III xxxi), and O″BO‴ is right, since O″B is ⊥ to O′O‴, O′O‴ is ∥ to FD, (III xxvi, Cor 2) the arc O‴D = O′F, hence the ∠ O‴O″D = O′O″F, and the ∠ O″AE = O″O′B (I, Ex 36), the ∠ O″EA = O″BO′, but O″BO′ is right, O″EA is right, hence O″O is ⊥ to AC Similarly, if we join O′O, O‴O, they will be ⊥ to BC, AB Hence the three ⊥s are concurrent

48 Dem —Join BE, BQ, BD Now (III xxxi) the ∠ AQB is right, and EOB is right (hyp), OEQB is a cyclic quad, the ∠ OQB = OEB, but OQB = PAB (III xxi),



OEB is equal to PAB And hence the ○ through the points A, E, B will pass through D

49 Let ABC be a △ whose sides a, b, c are in arithmetical progression, a being the greatest, and c the least It is required to prove that $6Rr = ac$

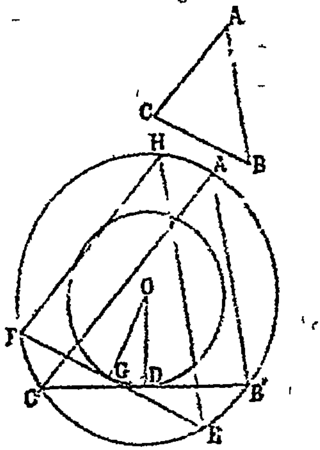
Dem —Let p denote the length of the ⊥ from B on CA, and R the radius of the circum-○ Now $2\,Rp = ac$ (III xxxv., Ex 1), $2\,R\;bp = abc$, but bp is equal to twice the area, that is, equal to $2\,\Delta$ (suppose), $2\,R\;2\,\Delta = abc$, $R = \frac{abc}{4\,\Delta}$, and since the sides are in A P, $a + c = 2\,b$, $a + b + c = 3b$, but $(a + b + c) = 2s$, therefore $2s = 3b$ Again (IV Ex 9), $r = \frac{\Delta}{s}$, and multiplying this and the equation $R = \frac{abc}{4\Delta}$ we get $Rr = \frac{abc}{4s}$,

$Rr = \frac{abc}{6\,b}$, and hence $6\,Rr = ac$

50 Let A′B′C′ be the ○, and AB, BC, CA three lines in the form of a △ It is required to inscribe in A′B′C′ a △ whose sides shall be ∥ to the sides of ABC

Sol —Take a point A′ in the circumference, and draw A′B′ ∥ to

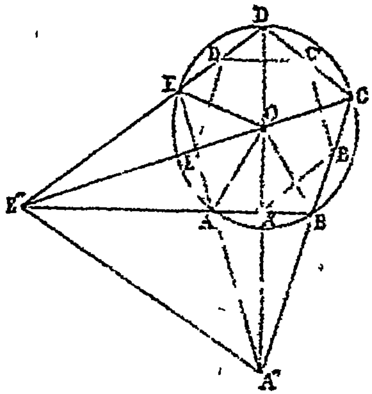
AB, and A'C' ∥ to AC Join B C. If B C' is ∥ to BC, the thing required is done If not, from the centre O let fall a ⊥ OD on



B'C' With O as centre, and OD as radius, describe a ○. Draw EF, touching this ○, and ∥ to BC (III xvi., Ex 2) Join O to G, the point of contact Draw FH ∥ to C A', and join HE HFE is the Δ required.

Dem —Because OG = OD, EF = B C' (III xiv), ∴ the arc EF = B'C', hence the arc FC' = B'E, but FC' = HA' (III xxvi, Cor 2), B'E = HA, HE is ∥ to A'B'; that is ∥ to AB, and FH is ∥ to A'C, that is to AC, and EF is ∥ to BC Hence the sides of the Δ HEF are ∥ to the sides of ABC

51 Dem —Let O be the centre of the circum ○ Join DA



OE″, OA, OB, OC, &c Now the △ A″OE″ + DOC − (A″OC + E″OD) = 4 A′OE′ (Book I, Ex 52), that is, AA″E′ + AOA″ + AOE′ + DOC − (BOC + A″OB + EOD + EOE′) = 4 A′OE′, but evidently AOA″ = A″OB, AOE″ = EOE″, and DOC = EOD, AA″E″ − BOC = 4 A′OE, and BOC = A′OE′ + A′AE′ Adding, we get A″AE″ − A′AE′ = 5 A′OE′ = pentagon A′B′C′D′E′

52 (1) **Dem** —Let ABCDE be the equilateral inscribed polygon

Now, since the sides are equal, the arcs are equal, therefore the whole arc EABC = DEAB, hence the ∠ CDE = BCD Similarly, the ∠ BCD = ABC, &c Hence the polygon is regular

(2) **Dem** —Let ABCDE be the equilateral circumscribed polygon, F, G, H, I, J the points of contact, and O the centre Join OA, OB, OF, OG, OH

Now ID = HD, IE = HC, JE = GC, AJ = BG, AF = BF Now since AF = BF, OF common, and the ∠ AFO = BFO, the ∠ OAF = OBF, the ∠ BAE = ABC Similarly all the ∠s are equal Hence the polygon is regular

53 (1) Let ABCDE be the equiangular circumscribed polygon, F, G, H, I, J the points of contact, and O the centre Join OA, OB, OG, OH

Now since the ∠ CBA = EAB, their halves are equal, that is, the ∠ OBF = OAF, and the ∠ OFB = OFA, each being right, and the side OF common, (I xxvi) BF = AF, that is, AB = 2 AF Similarly, AE = 2 AJ, but AF = AJ, AB = AE In like manner all the sides are equal Hence the polygon is regular

(2) **Dem** —Let ABCDE be the inscribed polygon, and O the centre Join OA, OB, OC, OD, OE Now the ∠ ABC = EAB (hyp), but the ∠ OBA = OAB, since OA = OB, therefore the ∠ OBC = OAE, that is, OCB = OEA, but the ∠ BCD = AED, OCD = OED, that is, ODC = ODE Now, in the △s ODC, ODE, the ∠s OCD, ODC are equal to the ∠s OED, ODE, and the side OD common, hence (I xxvi) DC = DE Similarly all the sides are equal Hence the polygon is regular

54. The sum of the ⊥s drawn to the sides of an equiangular polygon X from any point P inside the figure is constant.

Dem.—Suppose a regular polygon Y of the same number of sides as X constructed so as to include X, and have its sides parallel to those of X. Then, if the ⊥s from P on the sides of X be produced to meet the sides of Y, their sum is constant (Book IV., Ex. 17), but the excess of the latter sum over the former is constant. Hence the former is constant.

55. Dem.—If the radii be r', r'', r''', we have, denoting the area of the triangle by Δ (Book IV., Prop. IV., Ex. 10),

$$r' = \frac{\Delta}{s-a},\quad r'' = \frac{\Delta}{s-b},\quad r''' = \frac{\Delta}{s-c};$$

$$r'(r'' + r''') = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-a)(s-c)};$$

but (Book IV., Prop. IV., Ex. 12) $\Delta^2 = s\,.\,s-a\,.\,s-b\,.\,s-c$,

$$r'(r'' + r''') = s\,.\,s-c + s\,.\,s-b = sa,$$

and (Book IV., Ex. 44) $\sqrt{r'r'' + r''r''' + r'''r'} = s$,

$$\therefore\ a = \frac{r'(r'' + r''')}{\sqrt{r'r'' + r''r''' + r'''r'}}.$$

Similarly,

$$b = \frac{r''(r''' + r')}{\sqrt{r'r'' + r''r''' + r'''r'}},$$

$$c = \frac{r'''(r' + r'')}{\sqrt{r'r'' + r''r''' + r'''r'}}.$$

BOOK V.

Miscellaneous Exercises.

1 (1) Let a be greater than b It is required to prove that $\frac{a-x}{b-x}$ is greater than $\frac{a}{b}$.

Dem —Subtract, and we get $\frac{ab-bx-ab+ax}{b(b-x)}$, that is $\frac{(a-b)x}{b(b-x)}$, but since a is greater than b, $\frac{(a-b)x}{b(b-x)}$ is positive Hence $\frac{a-x}{b-x}$ is greater than $\frac{a}{b}$.

(2) To prove that $\frac{a}{b}$ is greater than $\frac{a+x}{b+x}$

Dem —Subtract, and we get $\frac{a}{b}-\frac{a+x}{b+x}=\frac{ab+ax-ab-bx}{b(b+x)}$ $=\frac{(a-b)x}{b(b+x)}$, but because a is greater than b, $\frac{(a-b)x}{b(b+x)}$ is positive Hence $\frac{a}{b}$ is greater than $\frac{a+x}{b+x}$

2 The proof of this is similar to that of Ex 1

3 Let a, b, c, d be the four magnitudes, then if $a \quad b \quad c \quad d$, it is required to prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

Dem —Because $a \quad b \quad c \quad d$, we have $a+b \quad b \quad c+d \quad d$ (XVIII), that is $\frac{a+b}{b}=\frac{c+d}{d}$ Again, $a-b \quad b \quad c-d \quad d$ (XVII), that is, $\frac{a-b}{b}=\frac{c-d}{d}$ Dividing, we get, $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

4 Let a, b, c, d, and e, f, g, h, be the two sets of four magnitudes that are proportionals, that is, $a \quad b \quad c \quad d$, and $e \quad f \quad g \quad h$ It is required to prove that $ae \quad bf \quad cg \quad dh$

Dem.—Because $a \quad b : \quad c \quad d$, we have $\frac{a}{b} = \frac{c}{d}$ Similarly, $\frac{e}{f} = \frac{g}{h}$ Multiplying together, we get $\frac{ae}{bf} = \frac{cg}{dh}$, that is, $ae \quad bf \quad cg \quad dh$

5 It is required to prove that $\frac{a}{e} \quad \frac{b}{f} \quad \frac{c}{g} \quad \frac{d}{h}$.

Dem.—As in (4), we have $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$, $\therefore \frac{a}{b} - \frac{e}{f} = \frac{c}{d} - \frac{g}{h}$, but $\frac{a}{b} - \frac{e}{f} = \frac{af}{be} = \frac{a}{e} - \frac{b}{f}$ and $\frac{c}{d} - \frac{g}{h} = \frac{ch}{dg} = \frac{c}{g} - \frac{d}{h}$, $\frac{a}{e} - \frac{b}{f} = \frac{c}{g} - \frac{d}{h}$ Hence $\frac{a}{e} \quad \frac{b}{f} \quad \frac{c}{g} \quad \frac{d}{h}$

6 Let a, b, c, d be the four magnitudes

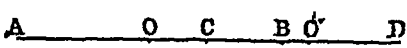
Dem —$a \quad b \quad c \quad d$, $\frac{a}{b} = \frac{c}{d}$, $\frac{a^2}{b^2} = \frac{c^2}{d^2}$, that is, $a^2 \quad b^2 \quad c^2 \quad d^2$ Similarly $a^3 \quad b^3 \quad c^3 \quad d^3$

7 Let a, b, c, d , a, b, c, d', be the two sets of magnitudes It is required to prove that $d = d'$

Dem —$a \quad b \quad c \quad d$, and $a \quad b \quad c \quad d'$, $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{b} = \frac{c}{d'}$, $\frac{c}{d} = \frac{c}{d'}$ Hence $d = d'$

8 Dem —Since the three magnitudes are continual proportions we have $\frac{a}{b} = \frac{b}{c}$ and $\frac{b}{c} = \frac{b}{c}$ Multiplying these equalities, we get $\frac{a}{c} = \frac{b^2}{c^2}$, that is, $a \quad c \quad b^2 \quad c^2$ Again, $\frac{a}{b} = \frac{b}{c}$, $\left(\frac{a}{b} - 1\right) = \left(\frac{b}{c} - 1\right)$, $\frac{a-b}{b} = \frac{b-c}{c}$, and therefore $\frac{(a-b)^2}{b^2} = \frac{(b-c)^2}{c^2}$, that is, $(a-b)^2 \quad (b-c)^2 \quad b^2 \quad c^2$ Hence we have $a \quad c \cdot (a-b)^2 \quad (b-c)^2$

9 Dem —AC CB AD DB (hyp), AC − CB AC



+ CB AD − DB AD + DB , that is, 2 OC 2 OB 2 OB 2OD Hence OC OB OB OD

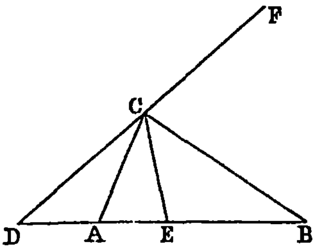
Dem —Because CD ıs bisected ın O , and produced to O, ıve (II vı) OD OC + $O'C^2 = OO'^2$, but OD OC = OB^2 9) , $OB^2 + O'C^2 = OO'^2$, that ıs, $OO'^2 = OB^2 + O'D^2$

' Dem —AC CB AD DB (hyp), AC AB − AC ' D AD − AB, $\frac{AC}{AB - AC} = \frac{AD}{AD - AB}$, . AC (AD − AB) '' (AB − AC) , AC AD − AC AB = AD AB − AD AC

sposıng, we get 2 AC AD = AB (AC + AD) Divide by AC AD, and we have $\frac{2}{AB} = \frac{1}{AD} + \frac{1}{AC}$.

Dem —BD BC AD AC (hyp.) Working, as ın I, we get $\frac{2}{CD} = \frac{1}{BD} + \frac{1}{AD}$

Dem —AC CB AD BD (hyp), AC BD = CB . AD, C BD + CB AD = 2 CB AD Again, AB CD = (AC 3) (CB + BD) = AC BD + AC CB + CB^2 + CB BD C BD + CB (AC + CB + BD) = AC BD + CB AD JB AD

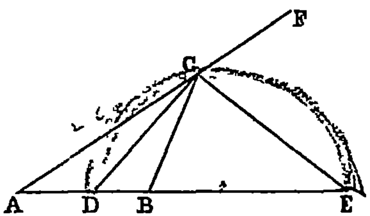


hence (Ex 1) DC CE DB BE, DA AE DB BE
Hence AB is cut harmonically in E and D

6 Let AB be the base, AC and CB the sides

Sol —Bisect the ∠ ACB by CD Produce AC to F, and bisect the ∠ BCF by CE, meeting AB produced in E

Now AD DB AC CB (III), but the ratio AC CB is given (hyp), the ratio AD DB is given, D is a given point. Again, AC CB AE EB (Ex 1), the ratio AE EB is given, and AB is given, hence the point E is given And



because the ∠ ACD = BCD, and FCE = BCE, the ∠ DCE is right, hence the ○ on DE as diameter will pass through C, and because the points D, E are given, it will be a given ○ It divides the base in the points D, E harmonically, in the ratio of AC CB, and is the locus of the vertex It is called the "Apollonian locus"

7 Dem —b c CD DB (III), $b + c$ c CD + DB DB, but CD + DB = CB = a, $b + c$ c a DB

$(b + c)$ DB $= ac$, hence DB $= \dfrac{ac}{b+c}$ Similarly, D B $= \dfrac{ac}{b-c}$

Adding, we get DD′ $= \dfrac{ac}{b+c} + \dfrac{ac}{b-c} = \dfrac{2\,abc}{b^2 - c^2}$

8 (1) Dem —In the last Exercise we got $DD' = \frac{2abc}{b^2 - c^2}$

$$\frac{1}{DD'} = \frac{b^2 - c^2}{2abc} \quad \text{Similarly, } \frac{1}{EE'} = \frac{c^2 - a^2}{2abc}, \text{ and } \frac{1}{FF'} = \frac{a^2 - b^2}{2abc}$$

Adding, we get $\frac{1}{DD'} + \frac{1}{EE'} + \frac{1}{FF'} = 0$

(2) Dem —From (1) we have

$$\frac{1}{DD'} = \frac{b^2 - c^2}{2abc}, \quad \frac{a^2}{DD} = \frac{a^2b^2 - c^2a^2}{2abc}$$

Similarly,

$$\frac{b^2}{EE} = \frac{b^2c^2 - a^2b^2}{2abc}, \text{ and } \frac{c^2}{FF'} = \frac{c^2a^2 - b^2c^2}{2abc}$$

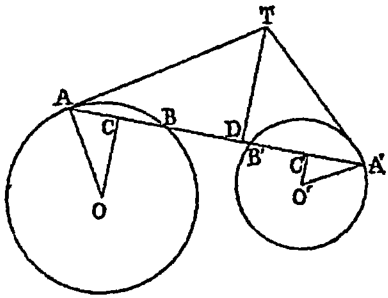
Adding, we have

$$\frac{a^2}{DD} + \frac{b^2}{EE'} + \frac{c^2}{FF'} = 0$$

PROPOSITION IV

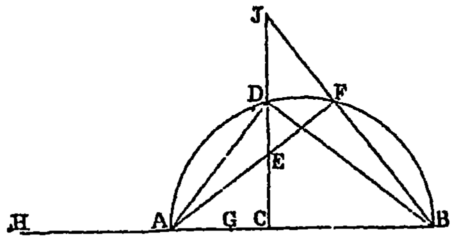
1 Dem —Let O, O be the centres Join OA, O'A', and let fall ⊥• OC, O C' on AA' From T let fall a ⊥ TD on AA

Now in the Δ• ACO, ADT we have the ∠• ACO, ADT equal, and the ∠ OAT is right (III xviii), and is equal to the sum of the ∠• OAC, AOC Reject OAC, and we have AOC = TAD ,



the Δ• OAC, ADT are equiangular , hence (iv) OA AC AT TD , alternation, OA AT AC TD Similarly, O A A'T A'C' TD , but AC = A'C , since AB, A'B are

= AB BC, but AB BC = BD^2, that is = to GH AH, JB BF = GH AH, but BF = AH (const), . JB = GH, and JF = AG



Again, AC CE JF EF (IV), alternation, AC JF CE EF, but JF = AG Hence AC AG CE EF

PROPOSITION X

3 See "Sequel," Book VI, Prop V, Sect III

4 Let the sum of the squares of the lines be equal to the squares on AB, and their ratio that of m n

Sol —On AB as diameter describe a ○ ABC Divide AB in D, so that AD DB m n (Ex 1) Bisect the arc ACB in C Join CD, and produce it to meet the circumference in E Join AE, BE AE, BE are the required lines

Dem —$AB^2 = AE^2 + BE^2$, and (III XXVII) the ∠ AEB is bisected, hence AE EB AD DB, but AD DB m n Hence AE EB m n

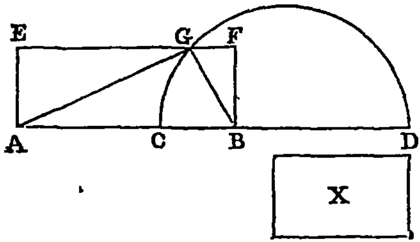
5 Let the difference of the squares of the lines be equal to AB^2, and their ratio that of m n

Sol —Divide AB internally and externally in C and D, in the ratio of m n (Ex 1) On CD as diameter describe a semicircle Let O be its centre Erect BE ⊥ to AD, meet the ○ in E, and join AE AE and BE are the required lines

Dem —Join OE, CE Now (I XLVII) $AE^2 - BE^2 = AB^2$ And because AB is divided harmonically in C and D, and CD is bisected in O, OB, OC, OA are in geometrical progression (Book V, Ex 9) Hence OA OB = OC^2 = OE^2, the ∠ AEO is right, the ∠ OAE = BEO, but ECO = CEO (I V) Hence (I XXXII) the ∠ AEC = CEB, (III) AE EB AC CB, that is, *m* *n*.

6 (1) *Let AB be the base, m n the ratio of the sides, and* the rectangle X the area

Sol —Divide AB internally and externally in C and D, in the

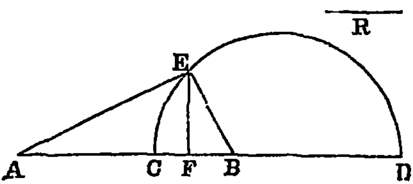


ratio *m* *n* (Ex 1) On CD as diameter describe a ○, to AB apply a ▭ AF, whose area is 2 X Let its side EF cut the ○ in G Join AG, BG AGB is the △ required

Dem.—AG GB AC CB (Dem of Ex 5), that is as *m* *n*, and the ▭ AF = 2 AGB, but AF = 2 X, AGB = X.

(2) Let AB be the base, *m* *n* the ratio of the sides, and R^2 the difference of the squares of the sides

Sol —Divide AB as in (1) On CD as diameter describe a ○ Divide AB in F, so that $AF^2 - BF^2 = R^2$ ("Sequel," Book I, Prop IX) Erect FE ⊥ to AD, cutting the ○ in E Join AE, BE AEB is the △ required



Dem —AE EB AC CB, that is as *m* *n*, and $AE^2 - EB^2 = AF^2 - FB^2 = R^2$

(3) Let AB be the base, *m* *n* the ratio of the sides, and $2 R^2$ the sum of the squares of the sides

Sol —Divide AB as in (1), and on CD as diameter describe a

○ CDE Bisect AB in F Erect FG ⊥ to AD From A inflect AG on FG, and equal to R With F as centre, and FG as radius, describe a ○, cutting CDE in E Join AE, BE AEB is the △ required

Dem —Join FE Now as in (1) AE BE *m* *n*, and FG = FE (const), $FG^2 = FE^2$, $AF^2 + FG^2$, that is, $AG^2 = AF^2 + FE^2$, $2\,AG^2$, that is, $2\,R^2 = 2\,AF^2 + 2\,FE^2$ Hence (II x, Ex 2) $AE^2 + BE^2 = 2\,R^2$

(4) Let AB be the base, *m* *n* the ratio of the sides, and X the vertical ∠

Sol —Divide AB as in (1) On CD as diameter describe a ○ CDE, and on AB describe a ○ AEB, containing an ∠ = X Join AE, BE AEB is the △ required

Dem —AE BE *m* *n*, and the vertical ∠ AEB = X

(5) Let X be the difference of the base angles

Sol —Divide AB as in (1), and on CD describe a ○ CDF Erect CE ⊥ to AD, and at C, in the line CE, make the ∠ ECF = ½ X Join AF, BF AFB is the △ required

Dem —AF BF *m* *n*, and the difference between the ∠ˢ ACF, BCF is equal to 2 ECF = X, but ACF = CBF + CFB and BCF = CAF + CFA, and CFA = CFB Hence CBF − CAF = ACF − BCF = X

PROPOSITION XI

1 **Dem** —Join OB, B′C, &c Now in the △ˢ OAB, BB′C, we have OA AB B′B BC, and the right ∠ OAB = B′BC, hence (VI) the △ˢ are equiangular, the ∠ ABO = BCB′, hence OB, B′C are ∥ Similarly B C, C′D are ∥ Now, since the lines AO, BB′, CC′ are ∥, we have (II, Ex 1) OB′ B′C′ AB BC, and because OB, B′C, C′D are ∥, OB′ B′C′ BC CD, hence AB BC BC CD In like manner BC CD CD DE Hence AB, BC, CD, &c, are in continued proportion

2 **Dem** —Because B′M is ∥ to AΩ, the △ˢ OMB′, OAΩ are equiangular, OM MB′ OA AΩ, but OM = OA − AM = AB − BB′ = AB − BC, and MB′ = AB, and OA = AB Hence AB − BC · AB AB AΩ

PROPOSITION XIII

1 (Diagram to Prop VIII)

Sol.—Let AB, BD be the two lines On AB describe a semicircle At D erect DC ⊥ to AB, and meeting the semicircle in C Join BC BC is a mean proportional between AB, BD

Dem —Join AC Now the Δ^s ABC, BCD are equiangular (VIII), AB BC BC BD Hence BC is a mean proportional between AB and BD

2 Sol —Let O be any point taken within a ○ ABC, O′ the centre Join OO′, and produce both ways to meet the circumference in A, B Through O draw CD ⊥ to AB CD is bisected in O (III III) Through O draw any other chord FE OC is a mean proportional between OF and OE

Dem.—Join CF, DE Now, because the Δ^s OCF, OED are equiangular (III XXI), we have (IV) OF OC OD OE, but OD = OC, OF OC OC OE Hence OC is a mean proportional between OF and OE

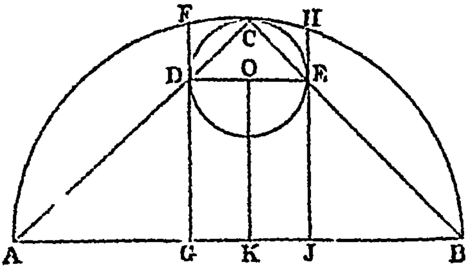
3 Let ABC be a ○, O any external point From O draw a secant OAB, and a tangent OC to the ○ It is required to prove that OC is a mean proportional between OB and OA

Dem —Join AC, BC Now in the Δ^s OAC, OBC, we have the ∠ OCA = OBC (III XXXII), and the ∠ BOC common, hence the Δ^s are equiangular, BO OC OC OA Hence OC is a mean proportional between OB and OA

4 Dem —Let AB be the chord of the arc Join AE, AC, CB Now because the arc AC = BC, the ∠ CAB = CBA, but CBA = AEC (III XXI), AEC = CAD, and the ∠ ACD is common, the Δ^s ACE, ACD are equiangular, EC AC AC . CD Hence AC is a mean proportional between CE and CD

5 Let ACB be a ○ whose diameter is AB, FG, HJ two parallel chords, CDE a ○ touching ACB internally in C, and FG, HJ in D, E From O, the centre of CDE, let fall a ⊥ OK on AB It is required to prove that OK is a mean proportional between AG and JB

Dem.—Join OD, OE, CD, CE CD, CE produced must pass



through A, B (III, Ex 61) Now (III xviii) the ∠ ODG is right, and DGB is right, OD is ∥ to AB Similarly OE is ∥ to AB, OD, OE are in the same straight line Again, since the ∠ AGD is right, the ∠s GAD, GDA are equal to a right ∠, and because ACB is right (III xxxi), the ∠s CAB, CBA are equal to a right ∠, hence the ∠ GDA = JBE, and the ∠ DGA = EJB, the Δs ADG, JEB are equiangular, hence AG GD EJ JB, but GD and EJ are each equal to OK, AG OK OK JB Hence OK is a mean proportional between AG and JB

6 Let ADB be a semicircle whose diameter is AB, CEF a ○ touching ADB in F and AB in C Through O, its centre, draw the diameter CF, and produce it to meet ADB in D It is required to prove that CF is a harmonic mean between AC and CB

Dem —AB CO = CD² ("Sequel," III, Prop v), but AC CB = CD², AB CO = AC CB, · $CO = \frac{AC\ \ CB}{AC + CB}$,

$2\,CO = \frac{2AC\ \ CB}{AC + CB}$ Hence (V, Miscellaneous, Ex 11) 2 CO, that is CF, is a harmonic mean between AC and CB

7 Let ACB be a ○ whose diameter is AB, FG, HJ, two parallel chords meeting the ○ in F, H, and the diameter in G, J Describe a ○ CDE touching ACB externally in C, and GF, JH produced in D, E From O, its centre, let fall a ⊥ OK on AB It is required to prove that OK is a mean proportional between AJ and GB

The proof is the same as in Ex 5

PROPOSITION XVII

2 Dem —Describe a ○ about the △. Produce AC to G, and bisect the external ∠ BCG by CD, meeting AB produced in D′ Produce D′C to meet the ○ in F, and join AF Now the ∠ BCD′ = GCD′, and GCD′ = FCA, BCD′ = FCA, and since the ∠ˢ CBD, CBA are together equal to two right ∠ˢ, and the ∠ˢ CFA, CBA are equal to two right ∠ˢ, the ∠ CBD′ = CFA, the △ˢ AFC, BCD′ are equiangular, AC CF D′C CB (IV), hence AC CB = D′C CF Again AD′ D′B = FD′ D C, but FD′ D′C = FC CD′ + CD′² (II III) = AC CB + CD² Hence AD′ D B − CD² = AC CB

4 Dem —Let O be the centre of the ex-○, touching AB externally, and the other sides produced Join O′C, cutting the circum-○ in E Through E draw EF, the diameter of the circum-○ Join O′B, EB, FB, O′G, G being the point where CB produced touches the ex-○

Now the ∠ˢ O′GC, EBF are equal, each being right, and the ∠ O CG = EFB (III xxi), the △ˢ O CG, BFE are equiangular, hence (IV) FE EB O′C O G, and EB = EO (Dem of IV, Ex 19), hence FE EO′ O C O G, FE O′G = EO′ O′C, that is, the rectangle contained by the diameter of the circum-○, and the radius of the ex-○, is equal to the rectangle contained by the segments of any chord of the circum-○ passing through the centre of the ex-○

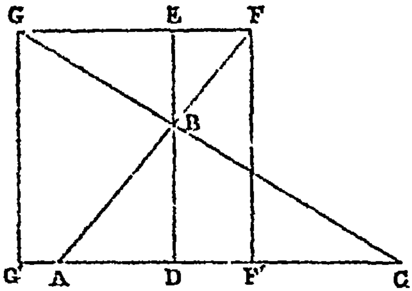
7 Dem —Produce AD to meet the circumference in G, then (Ex 6) we have AB AE + AC AF = AG AD, but AG AD = GD DA + DA² (II III), and GD DA = BD DC (III xxxv) Hence AB AE + AC AF = BD DC + DA²

9 Dem —Let ABC be the △, and FGG′F′ the inscribed square, F and G being on AB and BC From B let fall a ⊥ BD on AC, cutting the side FG of the square in E

Now AC FG AB FB (IV), but AB FB BD BE (IV), AC FG BD BE Hence, putting *b* for base,

p for ⊥, and s for side of square, we have $b \quad s \qquad p \quad p - s$, $bp - bs = sp$ Hence $bp = (b + p)\, s$

10 Dem —Let ABC be the Δ, and FGG'F' the escribed



square From B let fall a ⊥ BD on AC, and produce it to meet FG in E

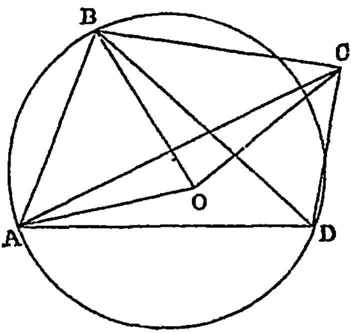
Now AC FG ∵ AB BF (IV), but AB BF BD BE (IV), AC FG BD BE, that is, putting s' for the side of the square, $b \quad s' \qquad p \quad s - p$ Hence $bs' - bp = sp$, $bp = s\ (b - p)$

11 From P let fall a ⊥ PC on the chord AB, and from A, B let fall ⊥s AD, BE on DE, the tangent at P It is required to prove that CP^2 = AD BE

Dem —Join AP, BP Now in the Δs APD, BPC, the ∠ APD = PBC (III xxxii), and the ∠ ADP = BCP, the Δs are equiangular, hence (IV) AP AD BP PC, alternation, AP BP AD PC In like manner for the Δs APC, BPE, we have AP BP PC BE, AD PC PC BE Hence CP^2 = AD BE

12 Dem —In the Δs AOD, BOC, the ∠ AOD = BOC, and the ∠ OAD = OBC (III xxi), hence (IV) AD AO BC BO, alternation, AD BC ∵ AO BO Multiplying each by AB, we get AD AB AB BC AO BO Similarly AB BC BC CD ∴ BO CO, &c Hence the four rectangles are proportional to the four lines

14 Dem —Draw the diagonals AC, BD Make the ∠ ABO = DBC, and BAO = BDC Join OC



Now the Δ• ABO, DBC are equiangular, AB AO BD DC, AB CD = AO BD Again, since AB BO BD BC, alternation, AB BD BO BC, and since the ∠ ABO = DBC, the ∠ ABD = OBC, hence (IV) the Δ• ABD, OBC are equiangular, (IV) AD BD OC BC, hence AD BC = BD OC Now we have proved AB CD = AO BD, AD BC = OC BD, and AC BD = AC BD, hence the three rectangles are proportional to the sides AO, OC, AC of the Δ AOC, and since the Δ• AOB, CDB have been shown to be equiangular, the ∠ AOB = BCD, and because the Δ• BOC, ABD are equiangular, the ∠ COB = BAD Hence the ∠ AOC is equal to the sum of the ∠• BAD, BCD

15 Let ABCD be a cyclic quad, AC, BD its diagonals At P, any point in the circumference of the circum-O, draw a tangent to the O, and let fall ⊥• PE, PF, PG, PL on AB, BD, AC, CD It is required to prove that PF PG = PE PL

Dem —From A, B, C, D let fall ⊥• AH, BI, CJ, DK on the tangent at P Now PF^2 = BI DK (Ex 11), and PG^2 = AH CJ, PF^2 PG^2 = BI DK AH CJ In like manner PE^2 PL^2 = BI AH DK CJ, PF^2 PG^2 = PE^2 PL^2 Hence PF PG = PE PL

16 Dem —The ∠ APB is right (III xxxi), DPE is right, and equal to ECB, and PED = CEB, PDE = CBE Now since PDE = CBE, and ACD = ECB, the Δ• ADC, EBC

are equiangular, hence AC CD CE CB (IV), AC CB = CD CE, but AC CB = CF^2 (XVII), CD CE = CF^2 Hence CF is a mean proportional between CD and CE

PROPOSITION XIX

1 Let ABC, DEF be the two △s Now AB = $\frac{3}{2}$ DE (hyp), AB DE 3 2, AB^2 DE^2 9 4, but ABC DEF AB^2 DE^2 (XIX) Hence the △ ABC DEF 9 4

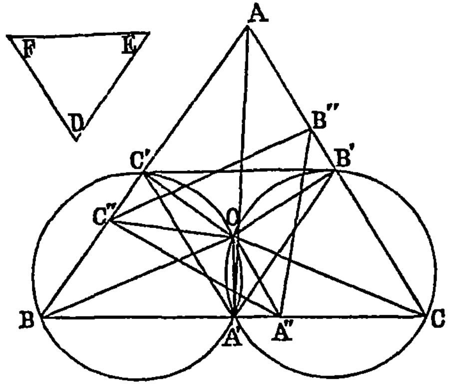
2 Let AB be a side of the inscribed polygon, O the centre of the ○ Join OA, OB, and bisect the ∠ AOB by OPP', meeting the chord AB in P, and the arc AB in P' Through P' draw a tangent to the ○, and produce OA, OB to meet it in A'B', then evidently A B is a side of the circumscribed polygon

Now, if each of the polygons have n sides, and we denote their areas by S and S', we have the △ AOB = $\frac{S}{n}$, and A'OB' = $\frac{S'}{n}$, hence AOB A'OB S S', but (XIX) AOB A'OB' AO^2 $A'O^2$, that is, OP^2 OP'^2 (IV), or OP^2 OA^2, hence S S' ' OP^2 OA^2, S' − S S AP^2 OA^2, that is, as 4 AP^2 4 OA^2, that is, as AB^2 is to the square of the diameter, but S is less than the square of the diameter (IV, Ex 37) Hence S' − S is less than AB^2

PROPOSITION XX

4 Dem —Let AB, BC, CA be three given lines in the form of a △ Inscribe in ABC a △ A'B'C' similar to the △ FDE. About the △s A'BC', A'B'C describe ○s intersecting in O, then the ○ about AB C will pass through O (III, Ex 28) Join OA', OB, OC, OB', OC', AA' Now (III XXI) the ∠ BOA' = BC'A', and COA' = CB'A', the ∠ BOC is equal to the sum of the ∠s BC'A', CB'A', but BC A' = BAA' + AA C, and CB A' = CAA' + AA B, the ∠ BOC = C AB' + C'A'B', but C A'B' = FDE, hence BOC = C AB' + FDE, but the ∠ FDE is given, and C'AB' is given, the ∠ BOC is given, and the base BC is given, hence the ○ described about the △ BOC is given in position Similarly, the ○s about the △s AOB, AOC are given in position, hence O is a given point Hence, if we inscribe another △ A"B"C" similar to FDE in ABC, the ○s described about the

△• A″BC″, B″CA″, C′AB″ will co-intersect in O, and if we join the angular points to O, the ∠• OC″A″, OA′C″ will be equal to



the ∠• OBA′, OBC′, that is, equal to the ∠• OC′A′, OA′C′, hence the △• OC′A′, OC A′ are equiangular, and therefore (Ex 2) O is the centre of similitude of the △• A′B′C′, A″B″C″

5 Let ABCDE, A′B′C′D′E′ be two similar figures, having the sides AB, BC ‖ to the sides A′B′, B′C It is required to prove that the other homologous sides are ‖

Dem —Join AA′, BB′, and produce them to meet in F Now the ∠ BAF = B′A′F (I xxix), but since the figures are similar, the ∠ BAE = B A E′, hence the ∠ FAE = FA′E, and therefore the line AE is ‖ to A′E′ Similarly, it can be shown that the other homologous sides are ‖

6 Let ABC, DEF be the homothetic figures Join BE, AD, and produce them to meet in G Join CF It is required to prove that CF produced will pass through G

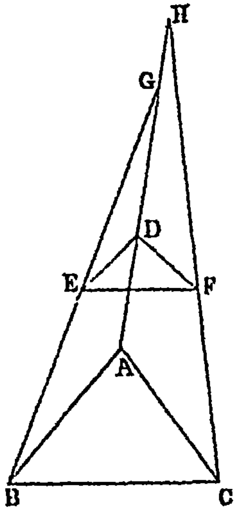
Dem —If not, let it pass through H Produce AG to H

Now the ∠ GED = GBA (I xxix), and the ∠ GDE = GAB, hence (iv) AG AB DG DE, but AB AC DE DF,

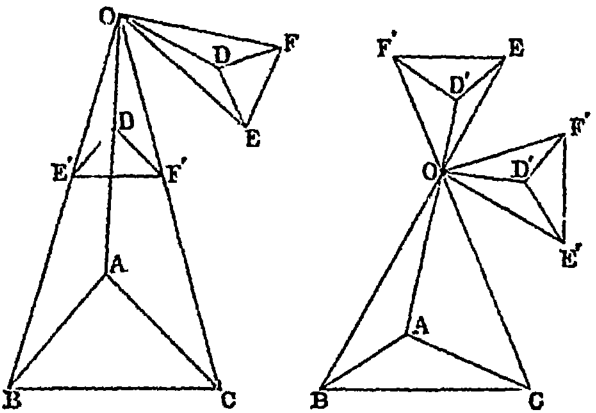
AG AC DG DF, alternation, AG DG AC DF

Again, since the △• HAC, HDF are equiangular, we have AH AC DH DF, alternation, AH DH AC DF,

AH DH AG DG, hence (V xvii) AD DH AD DG, and therefore DH = DG, which is absurd Hence CF produced must pass through G



7 Dem —Let ABC, DEF be the two similar figures, O their centre of similitude Join OA, OB, OC, OD, OE, OF From OA, OB, OC cut off OD′, OE′, OF′ equal, respectively, to OD, OE, OF, and join E′F′, F′D′, D′E′ Now since OD′ = OD, OE′ = OE, and the ∠ D′OE′ = DOE (hyp), ∴ DE = D′E′,

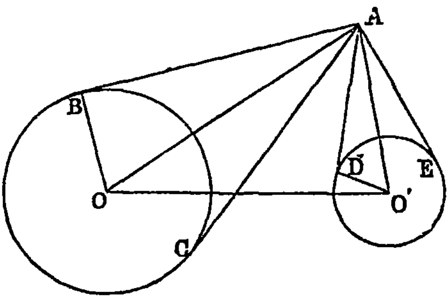


and the ∠ OED = OE′D′, but OED = OBA (hyp), ∴ OE′D′

= OBA, D'E and AB are parallel Similarly, D'F' is ∥ to AC and equal to DF, and E'F' is equal to EF and ∥ to BC, hence the figure DEF may be turned round O so as to take up the position D'E'F' In like manner the figure may be turned round in the opposite direction, as in the second diagram

10 Dem —Let O, O' be the centres of the ○ˢ, and A one of their centres of similitude Join OO', and from A draw AB, AC, AD, AE tangents to the ○ˢ Join OA, OB, O'A, O'D

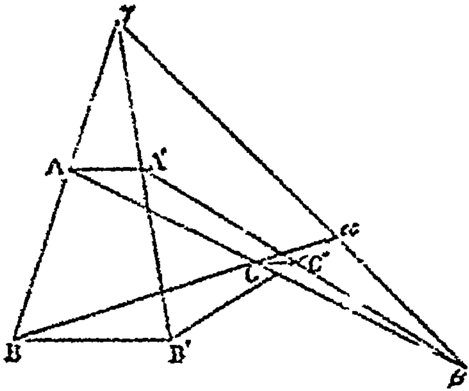
Now since A is the centre of similitude, the ∠ BAC = DAE, therefore their halves are equal, that is, the ∠ BAO = DAO', and the right ∠ˢ ABO, ADO' are equal, the Δˢ ABO, ADO' are equiangular, hence AO OB AO' O D, alternation, AO AO' OB O D, but the ratio OB O'D is given, since



OB and O'D are given lines, hence the ratio AO A'O is given Now in the Δ OAO' we have the base OO' given, and the ratio of the sides Therefore (III, Ex 6) the locus of A is a circle

PROPOSITION XXI

1 Dem —Let AA', BB', CC' be corresponding sides of the similar rectilineal figures, then since the figures are homothetic, these sides are parallel Join BA, B A', and produce to meet in γ, then because AA', BB are corresponding sides of the homothetic figures, γ will be their centre of similitude In like manner, if we join BC, B'C', and produce to meet in α, AC, A C to meet in β, α and β will be centres of similitude



Now (IV) $\frac{B\gamma}{\gamma A'} = \frac{BB'}{AA'}$. Similarly, $\frac{C\alpha}{\alpha B'} = \frac{CC'}{BB'}$, and $\frac{A\beta}{\beta C'} = \frac{AA'}{CC'}$;

but the product of $\frac{BB'}{AA'}$, $\frac{CC'}{BB'}$, $\frac{AA'}{CC'}$ is unity; ∴ the product of $\frac{B\gamma}{\gamma A'}$, $\frac{C\alpha}{\alpha B'}$, $\frac{A\beta}{\beta C'}$ is unity. And hence ("Sequel," Book VI., Prop. IV., Cor. 1, p. 69) the points α, β, γ are collinear.

PROPOSITION XXIII

1. *Dem.*—Let ABC, DEF be the △s having the ∠ ABC = DEF. Complete the ▱s ABCG, DEFH. Now the △ ABC : DEF :: ABCG : DEFH; but ABCG : DEFH :: AB . BC : DE . EF (XXIII.). Hence ABC : DEF :: AB . BC : DE . EF.

2. Let ABCD, EFGH be two quads whose diagonals AC, BD, EG, FH intersect in I, J, making the ∠ CIB = GJF. It is required to prove that ABCD : EFGH :: AC . BD : EG . FH.

Dem.—The area of ABCD is equal to the area of a △ having two sides equal to AC, BD, and the contained ∠ equal to CIB (I. XXXIV., Ex. 7); and EFGH is equal to a △ having two sides equal to EG, FH, and the contained ∠ equal to GJF; but (Ex. 1) these △s are to one another as AC . BD : EG . FH. Hence ABCD : EFGH :: AC . BD : EG . FH.

'OPOSITION XXX

'ht-angled Δ whose sides are in continued
ving AB BC BC CA From C let
It is required to prove that AB is divided
tio in D

BC BC CA, AB AC = BC² Again
BD = BC², AC = BD, and AB AD
BD² Hence AB is divided in extreme

e, we can prove AC = BD and AD = BC

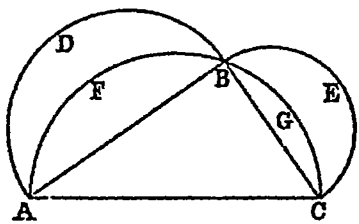
scribe a O about the Δ FHD Let O be
and produce it to meet the circumference
prove that DI² = 6 FD²

roduce FH, and let fall a ⊥ DJ on it
quare, AF = AH, the ∠ AHF = AFH,
, AHF is half a right ∠, BHL is
L is a right ∠, HLB is half a right ∠,
∠ DLJ = BLH, DLJ is half a right ∠,
and DJL is a right ∠, JDL is half a right ∠, and JL = JD, JL² = JD², and DL² = 2 DJ²

Again, since AB = DB, and BH = BL, DL = AH, but AB is divided in extreme and mean ratio in H, BD is divided in extreme and mean ratio in L, and hence (II xi, Ex 4) BD² + BL² = 3 DL² = 6 DJ², hence BD² + BH², that is, DH² = 6 DJ² Again (III xxii), the ∠ˢ FHD, FID are together equal to two right ∠ˢ, and the ∠ˢ FHD, DHJ are equal to two right ∠ˢ, the ∠ FID = DHJ, and the right ∠ IFD = HJD the Δˢ IFD, DHJ are equiangular, ID DF DH DJ, ID² DF² DH² DJ², but DH² = 6 DJ² Hence ID² = 6 DF²

PROPOSITION XXXI

Dem —Let ABC be the semicircle, of which AB, CB are supplemental chords On AB, CB describe semicircles ADB, BEC Now (xxxi) the semicircle ABC is equal to the sum of the semicircles ADB, BEC Take away the common segments



AFB, BGC, and we have the Δ ABC equal to the sum of the crescents ADBF, BECG

Exercises on Book VI.

1 Let ACB be a fixed Δ, DE a ∥ to AB Draw the diagonals AE, BD, intersecting in O Join CO, and produce it to meet AB in H It is required to prove that CH bisects AB.

Dem.—Through O draw FG ∥ to AB Now (II) AE EO BD DO, but, by similar Δ^s, AE EO AB OG, and BD DO AB OF, hence AB OG AB OF, and therefore OG = OF Now ACB is a Δ, and FG, a ∥ to the base, is bisected by CO Hence AB is bisected by CO

2 Let O be the centre of the ○, and P the given point From P draw PA to any point A in the ○ Divide AP at B in a given ratio It is required to find the locus of B

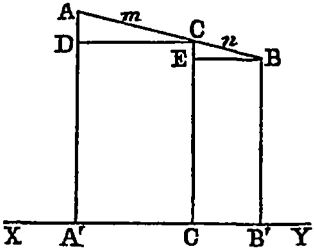
[Sol —Join OP, OA, and draw BC ∥ to AO

Now PB BA PC CO (II), but the ratio PB BA is given, PC CO is given, and therefore C is a given point Again, by similar Δ^s, we have PA AO PB BC, alternation, PA PB AO BC, but the ratio PA PB is given, AO · BC is given, but AO is given, BC is given, and the point C is given Hence the locus of B is a ○, having C as centre and CB as radius

3 Dem —Through B, C draw BE, CD ∥ to XY

Now, by similar Δ^s, AC AD CB CE, alternation, AC CB AD CE, AD CE *m* *n*, but AD = AA′ − A′D = AA′ − CC′, and CE = CC′ − C′E = CC′ − BB′, hence AA′

– CC′ CC′ – BB′ ‘ m n, n AA′ – n CC′ = m CC′ – m BB′, and hence m BB′ + n AA′ = (m + n) CC′



4 See "Sequel," Book VI, Prop II, Section 1

5 See "Sequel," Book VI, Prop IV, Section 1

6. Dem —Let the rectangle AB AC = k^2 Produce AB to meet the circumference in D Now, if t denote the tangent drawn from A to the ○ (III XXXVI), AB AD = t^2, AB AD AB AC t^2 k^2, that is, AD AC t^2 k^2, but the ratio t^2 k^2 is given, AD AC in a given ratio, and hence (Ex 2) the locus of C is a ○

7 Dem —Join O, the centre of the in-○, to the points F, G, H, where the sides AB, AC, BC touch the ○ Join OC

Now since AF = AG, BF = BH, and CG = CH, AB – AC = BF – CG = BH – CH = 2 DH Again, $AB^2 - AC^2 = BE^2 - EC^2$ (I XLVII), that is (AB + AC) (AB – AC) = (BE + EC) (BE – EC), (AB + AC) 2 DH = BC 2 DE, hence (AB + AC) BC DE DH Again (III) AB AC BL LC, (AB + AC) AC BC LC, (AB + AC) BC AC LC Again, AC LC AO OL (III), but AO OL HE HL (II), AC LC HE HL, hence (AB + AC) BC HE HL, that is, DE DH HE HL, and hence DE HL = HE HD

8 Dem —Let O′ be the centre of the ex-○, touching BC produced in K Now (AB + AC) BC AC LC (Ex 7), that is, as AO OL, (AB + AC + BC) BC AL OL LE LH, 2 BK 2 BD LE LH, hence LH BK = BD LE

9 See Book VI, Prop XVII, Exs 3, 4

10 Dem —From Ex 9 we have $d^2 = R^2 - 2Rr$, $d'^2 = R^2 + 2Rr'$, $d''^2 = R^2 + 2Rr''$, and $d'''^2 = R^2 + 2Rr'''$, $d^2 + d'^2 + d''^2 + d'''^2 = 4R^2 + 2R(r' + r'' + r''' - r)$, but (Book IV,

Ex 19) $(r' + r'' + r''' - r) = 4R$. Hence $d^2 + d'^2 + d''^2 + d'''^2 = 4R^2 + 2R \cdot 4R = 12R^2$.

11. (1) **Dem.**—Let the sides of the Δ be denoted by a, b, c.

Now (IV. iv., Ex 9) $rs = \Delta$, $\therefore s = \frac{\Delta}{r}$. Again, $ap' = 2\Delta$ (II. i., Cor 1), $\therefore a = \frac{2\Delta}{p'}$. Similarly, $b = \frac{2\Delta}{p''}$, and $c = \frac{2\Delta}{p'''}$; $\therefore (a + b + c)$, or $2s = \frac{2\Delta}{p'} + \frac{2\Delta}{p''} + \frac{2\Delta}{p'''}$, $\therefore s = \frac{\Delta}{p'} + \frac{\Delta}{p''} + \frac{\Delta}{p'''}$; but $s = \frac{\Delta}{r}$, hence $\frac{1}{r} = \frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}$.

(2) $(s - a)\, r' = \Delta$ (IV. iv., Ex 10), $\therefore (s - a) = \frac{\Delta}{r'}$. Again, from (1) we have $(b + c - a) = \frac{2\Delta}{p''} + \frac{2\Delta}{p'''} - \frac{2\Delta}{p'}$, but $(b + c - a) = 2(s - a)$, $\therefore (s - a) = \frac{\Delta}{p''} + \frac{\Delta}{p'''} - \frac{\Delta}{p'}$; that is, $\frac{\Delta}{r'} = \frac{\Delta}{p''} + \frac{\Delta}{p'''} - \frac{\Delta}{p'}$. Hence $\frac{1}{r'} = \frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}$.

(3) Subtract (2) from (1), and we get $\frac{2}{p'} = \frac{1}{r} - \frac{1}{r'}$.

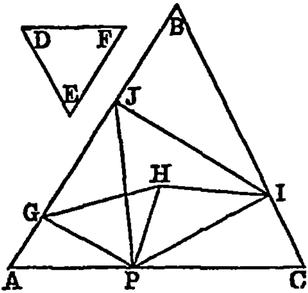
(4) Interchange in (2), and we have $\frac{1}{p'} + \frac{1}{p'''} - \frac{1}{p''} = \frac{1}{r''}$; interchange again, and $\frac{1}{p'} + \frac{1}{p''} - \frac{1}{p'''} = \frac{1}{r'''}$. Add, and we get

$$\frac{2}{p'} = \frac{1}{r''} + \frac{1}{r'''}.$$

12. Let ABC be a given Δ, and P a given point in one of the sides. It is required to inscribe in ABC a Δ equiangular to DEF, and having one of its angular points at P.

Sol.—From P let fall a $\perp$ PG on AB. Make the $\angle$ PGH = EDF, and GPH = DEF. Erect HI $\perp$ to PH, meeting BC in I; join PI, and make the $\angle$ IPJ = GPH, and join IJ. JPI is the Δ required.

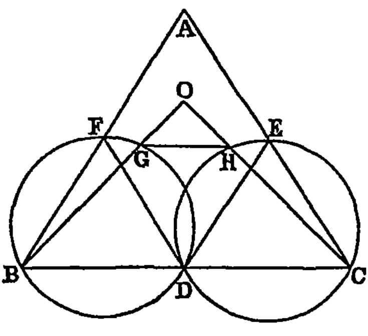
Dem —Because the ∠ GPH = JPI, GPJ = HPI, and the



right ∠ PGJ = PHI, hence the Δ' PGJ, PHI are equiangular, GP PJ HP PI, alternation, GP HP PJ PI, and the ∠ GPH = JPI, hence (VI) the Δ' GPH, JPI are equiangular, but GPH, DEF are equiangular Hence JPI is equiangular to DEF, and it has one of its angles at the given point P

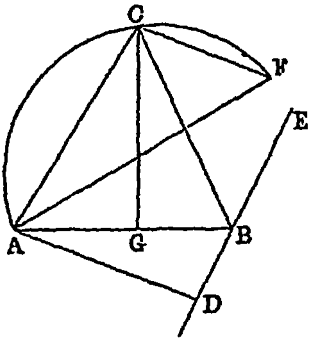
13 Let ABC be a given Δ, D, E, F three fixed points in its sides, and BOC a Δ of given species described on BC It is required to prove that the locus of O is a ○

Dem —Join DF, DE Describe ○' about the Δ' DBF, DCE, cutting OB, OC in G, H Join GH Now since the



points D, F are given, the line DF is given, and the ∠ DBF is given (hyp), hence the ○ about DBF is given, and the ∠ DBO is given by the given conditions, hence the arc DG is given, and therefore G is a given point. In like manner H is a given point, . the line GH is given, and the ∠ GOH is given Hence the locus of O is a ○

14 (1) Dem —Let the point B move along DE From A let fall a ⊥ AD on DE Draw AF, making the ∠ DAF = CAB

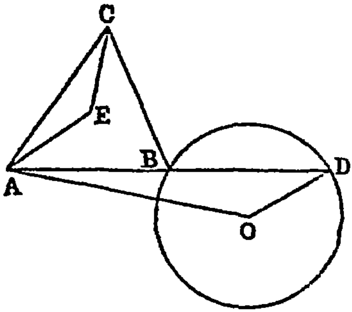


From C draw CF ⊥ to AC, and let fall a ⊥ CG on AB Now because the ∠ CAG is given, and the ∠ AGC is a right ∠, the Δ ACG is given in species, therefore the ratio AC CG is given, hence the ratio AC AB CG AB is given, but CG AB is given, therefore AC AB is given

Again, since the ∠ DAF = BAC, DAB = CAF, and the right ∠ ADB = ACF, therefore the Δ* DAB, CAF are equiangular, hence AD AB AC AF, AB AC = AD AF, but AB AC is given, AD AF is given, and AD is given, AF is given, and since the ∠ DAF is given, AF is given in position, and the ∠ ACF is right Hence the locus of C is a ○

(2) Let the point B move along a ○ Produce AB to meet the circumference in D Let O be the centre Join OA, OD Make the ∠ EAO = CAB, and ACE = ADO

Now (1) the rectangle AB AC is given, and AB AD is given (III xxxvi), therefore the ratio AB AC AB AD is given, the ratio AC AD is given Again, since the Δ* ACE, ADO are equiangular, AC AE AD AO, alternation, AC AD AE AO, but the ratio AC AD is given, the ratio AE AO is given, and AO is given, since it is drawn from a fixed point to the centre of a fixed ○, AE is given in magnitude, and it is given in position, because it is drawn making a given ∠ with a given line; hence the point E is given. And because the Δ^s AOD, AEC are equiangular, AO OD AE EC, but the ratio AO OD is given, ∴ the



ratio AE EC is given, and AC is given EC is given, and the point E has been shown to be fixed. Hence the locus of C is a ⊙, having E as centre and EC as radius

15 (1) Let the vertex A remain fixed Let the locus of B be a right line DB It is required to find the locus of C

Sol —From A let fall a ⊥ AD on DB. Make the ∠ DAG = CAB Let fall CG ⊥ on AG, and join DG

Now because the ∠ CAB=DAG, the ∠ CAG=DAB, and the right ∠ CGA=BDA, hence the Δ^s CAG, DAB are equiangular, ∴ AC AG AB AD, alternation, AC AB AG AD, but the ratio AC AB is given, since the Δ ABC is given in species, the ratio AG AD is given, and AD is given in magnitude, because it is a ⊥ from a given point on a given line, AG is given in magnitude, and it is also given in position, since the ∠ DAG is equal to a given ∠ CAB, G is a fixed point, and CG is at right $\angle^s$ to a given line at a given point. Hence the locus of C is the line CG

(2) Let the point B move along a ⊙, let O be its centre Join AO, BO, and draw AD, making the ∠ DAO = CAB Draw CD, making the ∠ ACD = ABO Now the Δ^s ACD, ABO are equiangular, AC AD AB AO, alternation, AC AB AD AO, but the ratio AC AB is given, the ratio AD AO is given, and AO is given, AD is given And since it makes the ∠ DAO=CAB with a given line AO, AD is given in position, hence the point D is given Again, in the Δ^s AOB, ADC we have AO OB AD DC, but the ratio AO OB is given,

the ratio AD DC is given, and AD is given, DC is given, and the point D is given Hence the locus of C is a ○, having D as centre and DC as radius

16 (1) Dem —Bisect the sides BC, CA, AB in D, E, F Join AD, BE, CF, let them intersect in O Produce AD to G, so that DG = OD Join BG Draw EH ∥ to AG, and produce BG to meet in H

Now since BD = CD, the △ BDO = CDO, and the △ BDA = CDA, the △ BOA = COA In like manner, COA = COB, the △s BOC, COA, AOB are equal, AOB = $\frac{1}{3}$ ABC And because OG = OA, the △ BOG = AOB, hence BOG = $\frac{1}{3}$ ABC And since the △s BOG, BEH are similar, BOG BEH OB^2 BE^2 (XIX), BOG BEH 4 9, that is, $\frac{1}{3}$ ABC BEH 4 9, hence 4 BEH = 3 ABC, ABC = $\frac{4}{3}$ BEH Again, it is evident that the sides of the △ BEH are equal to the medians of ABC, hence, denoting the medians by α, β, γ, and their half sum by σ, we have (IV IV, Ex 12) the △ BEH

$$= \sqrt{\sigma \quad \overline{\sigma-\alpha} \quad \overline{\sigma-\beta} \quad \overline{\sigma-\gamma}}$$

Hence the △ ABC is equal to

$$\tfrac{4}{3}\sqrt{\sigma \quad \overline{\sigma-\alpha} \quad \overline{\sigma-\beta} \quad \overline{\sigma-\gamma}}$$

(2) Dem —Let Δ denote the area of the triangle, then (IV IV, Ex 12) $\Delta^2 = s \quad \overline{s-a} \quad \overline{s-b} \quad \overline{s-c}$, $16\,\Delta^2 = (a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)$

Again, denoting the ⊥s by p', p'', p''', we have $ap' = 2\,\Delta$, $bp'' = 2\,\Delta$, and $cp''' = 2\,\Delta$, $(a+b+c) = \frac{2\,\Delta}{p'} + \frac{2\,\Delta}{p''} + \frac{2\,\Delta}{p'''}$

$= 2\,\Delta\left(\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}\right)$, and, substituting, we get

$$16\,\Delta^2 = 2\,\Delta\left\{\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}\right\}2\,\Delta\left\{\frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}\right\}$$

$$2\,\Delta\left\{\frac{1}{p'''} + \frac{1}{p'} - \frac{1}{p''}\right\}2\,\Delta\left\{\frac{1}{p'} + \frac{1}{p''} - \frac{1}{p'''}\right\},$$

hence

$$\frac{1}{\Delta^2} = \left(\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}\right)\left(\frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}\right)$$

$$\left(\frac{1}{p'''} + \frac{1}{p'} - \frac{1}{p''}\right)\left(\frac{1}{p'} + \frac{1}{p''} - \frac{1}{p'''}\right);$$

and hence

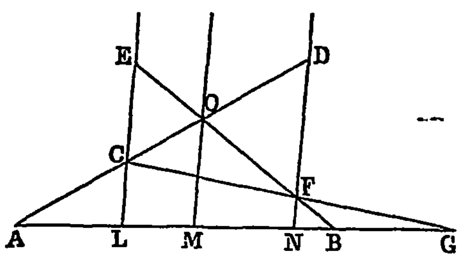
$$\Delta = \frac{1}{\sqrt{\left(\frac{1}{p'}+\frac{1}{p''}+\frac{1}{p'''}\right)\left(\frac{1}{p''}+\frac{1}{p'''}-\frac{1}{p}\right)\left(\frac{1}{p'''}+\frac{1}{p'}-\frac{1}{p''}\right)\left(\frac{1}{p'}+\frac{1}{p'}-\frac{1}{p'''}\right)}}$$

17 Let the ○ˢ ABC, DBE touch at B Draw a common tangent AD Join, AB, DB, and produce them to meet the ○ˢ in E, C Join DE, AC DE, AC are the diameters of the ○ˢ (III xiii, Ex 4)

Now the ∠ ADC = AED (III xxxii), and the right ∠ CAD = ADE, therefore the Δˢ CAD, ADE are equiangular Hence CA AD AD DE, that is, AD is a mean proportional between AC and DE

18 Let CL, OM, FN be the three ∥ lines Take any point O in OM Join AO, BO, and produce them to meet FN, CL in D, E Join AB, cutting the ∥ˢ in L, M, N Join CF, and produce it to meet AB produced in G It is required to show that G is a given point

Now in the Δ AOB the line OFG cuts the three sides in C, F, G, hence ("Sequel," Book VI, Prop iv, Sect. 1), $\frac{AC}{CO}\ \frac{OF}{FB}\ \frac{BG}{GA} = 1$, but $\frac{AC}{CO} = \frac{AL}{LM}$ (ii), and the ratio $\frac{AL}{LM}$ is given, $\frac{AC}{CO}$ is given In like manner, $\frac{OF}{FB}$ is given, $\frac{BG}{GA}$ is given Hence the line AB is divided externally in G in a

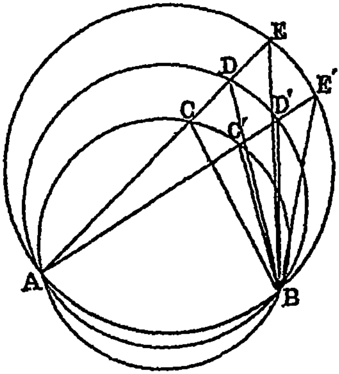


given ratio, G is a given point Hence CF passes through a fixed point Similarly, DE passes through a fixed point

19 Let a system of ○ˢ pass through two fixed points A, B From A draw any two secants, cutting the ○ˢ in C, D, E, C′, D, E′ It is required to prove that CD DE C D D′E

Dem —Join BC, BD, BE, BC′, BD′, BE′.

Now the ∠ ACB = AC′B (III xxi), DCB = D′C′B, and CDB = C′D′B , the Δˢ CDB, C′D′B are equiangular, hence



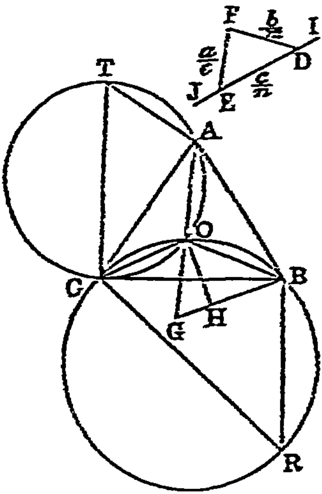
CD DB C′D′ D′B In like manner, the Δˢ DEB, D′E′B are equiangular, and BD DE BD′ D′E Hence *ex aequali* CD DE C′D′ D′E′

20 Let ABC be a Δ, the sides being denoted by *a*, *b*, *c* It is required to find a point O in ABC, such that the diameters of the ⊙ˢ about the Δˢ OAB, OBC, OCA may be in the ratios of three given lines *l*, *m*, *n*

Sol —Construct a Δ EDF whose sides EF, FD, DE shall be in the ratios $\frac{a}{l}$, $\frac{b}{m}$, $\frac{c}{n}$ Produce ED to I, J On CB describe a segment of a ⊙ COB containing an ∠ = IDF, and on AC a segment AOC containing an ∠ = JEF O, where these segments intersect, is the required point

Dem —Join OA, OB, OC Produce AO, and draw BG ∥ to OC From O let fall a ⊥ OH on BG Draw CR, CT, the diameters of the ⊙ˢ Join BR, AT

Now the sum of the ∠ˢ AOC, GOC is two right ∠ˢ, and the sum of FEJ, FED is two right ∠ˢ, hence GOC = FED , but GOC = OGB (I xxix), OGB = FED Again, the ∠ˢ COB, GBO equal two right ∠s, and IDF, EDF equal two right ∠s; ∴ GBO = EDF. Hence the ∠s OBG, DEF are equiangular.



Because the ∠s CTA, COA = two right ∠s (III. xxii.), and COA, COG equal two right ∠s, the ∠ COG = CTA. ∴ OGH = CTA, and OHG = CAT, each being right, ∴ the △s CAT, OGH are equiangular; ∴ $\frac{CT}{CA}=\frac{OG}{OH}$. Again, the ∠s COB, OBG equal two right ∠s, and COB, CRB equal two right ∠s, ∴ OBH = CRB, and the right ∠ CBR = OHB, ∴ the ∠s CBR, OHB are equiangular, ∴ $\frac{CR}{CB}=\frac{OB}{OH}$. Hence $\frac{CT}{b}:\frac{CR}{a}$ :: OG : OB, but OG : OB :: $\frac{a}{l}:\frac{b}{m}$; ∴ $\frac{CT}{b}:\frac{CR}{a}$ :: $\frac{a}{l}:\frac{b}{m}$, ∴ $\frac{CT}{m}=\frac{CR}{l}$. Hence CR : CT :: l : m. In like manner it can be shown that CT is to the diameter of the ○ about OAB as m to n.

21. Sol.—Describe a ○ about ABCD. Join CB, CD, BD. Divide BD at E in a given ratio, and join CE, AC.

Now the points A, C are given, AC is given in position, and AD is given in position, hence the ∠ DAC is given; but (III xxi) DAC = DBC, DBC is a given ∠ In like manner, the ∠ BDC is given, the ∠ DCB is given, hence the Δ DBC is given in species, . DB BC is given, and DB BE is given (hyp), BC BE is given, and the ∠ CBE is given Hence the Δ EBC is given in species Now EBC is a Δ of given form One of its vertices, C, is fixed, another, B, moves along a line AB Hence (Ex 15) the locus of E is a straight line

22 Dem —Produce CB, AD to meet in H Draw DF ∥ to BE, meeting BH in F Let CD and BE intersect in G

Now, because DF is ∥ to BG, we have DF BG CF CB, but DF = BF, · BF BG CF CB

Again, since the lines CA, BE, FD are parallel, we have (ii, Ex. 1) BF DE CF AD, and, by similar Δ*, ED EG · AD AC, hence, *ex aequali*, BF EG CF AC, but AC = CB, BF EG CF CB But it has been proved that BF BG CF CB, therefore BG = EG

Lemma —Take any point O within a Δ ABC Join OA, OB, OC, and produce AO to meet BC in A′ It is required to prove that the Δ OBC ABC OA′ AA

Dem —From A, O let fall ⊥* AD, OE on BC

Now the Δ ABC = ½ BC AD, and the Δ OBC = ½ BC OE hence ABC OBC AD OE, but AD OE AA′ OA′, ABC OBC AA′ OA′.

23 Dem —The Δ* OBC + OCA + OAB = ABC Divide by ABC, and we have

$$\frac{OBC}{ABC} + \frac{OCA}{ABC} + \frac{OAB}{ABC} = 1;\ \text{but}\ \frac{OBC}{ABC} = \frac{OA'}{AA'}\ (\textit{Lemma}),$$

and similarly for the others Hence

$$\frac{OA'}{AA} + \frac{OB'}{BB} + \frac{OC}{CC'} = 1$$

24 Dem —AB BC Δ AOB BOC (i), and A′B′ B′C′ Δ A OB′ Δ B′OC, (Book V, Ex 5)

$$\frac{AB}{AB} \quad \frac{BC}{BC} \qquad \frac{AOB}{AOB'} \quad \frac{BOC}{B'OC},$$

but (XXIII, Ex 1),

$$\frac{AOB}{A'OB'} \quad \frac{BOC}{B'OC'} \qquad \frac{AO}{A'O}\ \frac{OB}{OB'} \cdot \frac{OB}{B'O}\ \frac{OC}{OC'}, \qquad \frac{AB}{AB'} \cdot \frac{BC}{BC}$$

$$\frac{AO}{A'O}\ \frac{OB}{OB} \quad \frac{BO}{BO}\ \frac{OC}{OC'}, \qquad \frac{AB}{A'B} \quad \frac{BC}{BC} \qquad \frac{AO}{AO} \quad \frac{OC}{OC'}.$$

Hence $$\frac{AB}{A'B}\ \frac{OC}{OC} = \frac{BC}{BC}\ \frac{OA}{OA}$$

And similarly, $$\frac{BC}{BC'}\ \frac{OA}{OA'} = \frac{CA}{C'A'}\ \frac{OB}{OB'}$$

25 (1) **Dem** —Draw the diagonals AC, BD Bisect them in F, E Join FE, and produce both ways to meet AD, BC, and DC produced in H, G, I Now, in the Δ BDC, the line EI cuts the three sides in E, G, I Hence ("Sequel," Book VI Prop IV, Sect I)

$$\frac{BE}{ED}\ \frac{DI}{IC}\ \frac{CG}{GB} = 1,$$

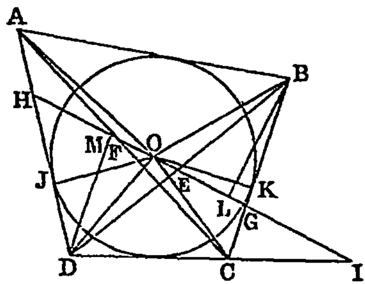
but

$$\frac{BE}{ED} = 1, \qquad \frac{DI}{IC}\ \frac{CG}{GB} = 1, \qquad \frac{DI}{IC} = \frac{GB}{CG}.$$

In like manner, from the Δ ADC, we get

$$\frac{DI}{IC} = \frac{HD}{AH} \qquad \text{Hence} \quad \frac{GB}{CG} = \frac{HD}{AH}.$$

(2) **Dem** —Join O, the centre, to A, B, C, D And also to



J, K, where AD, BC touch the ○ Now, since OK = OJ, we have (I) the △ OBC OAD BC AD Let fall ⊥ˢ BL, DM on OG, OH, then (I xxvi) the △ˢ BEL, DEM are equal, BL = DM and the △ OBG OHD OG OH In like manner OCG OHA OG OH Adding, we have OBC OAD ∴ OG OH, but it was shown that OBC OAD BC AD Hence BC AD OG OH

(3) Dem —Consider the △ ECI It is intersected by AB, hence ("Sequel," Book VI, Prop IV, Sect 1)

$$\frac{EG}{GI}\ \frac{IB}{BC}\ \frac{CA}{AE} = 1, \text{ but } \frac{CA}{AE} = 2, \quad \frac{EG}{GI}\ \frac{IB}{BC} = \frac{1}{2}$$

Again, consider the △ AEK, it is intersected by CD,

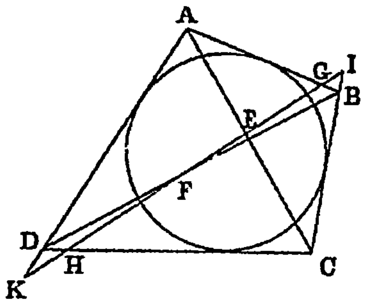
$$\frac{EH}{HK}\ \frac{KD}{DA}\ \frac{AC}{CE} = 1,$$

and, as before,

$$\frac{EH}{HK}\ \frac{KD}{DA} = \frac{1}{2}, \quad \frac{EG}{GI}\ \frac{IB}{BC} = \frac{EH}{HK}\ \frac{KD}{DA}$$

Now AD, BC are opposite sides, and they are cut by EF in K, I, hence (1) they are cut proportionally,

$$\frac{CB}{IB} = \frac{AD}{DK}, \text{ and } \quad \frac{EG}{GI} = \frac{EH}{HK},$$



that is, EG GI EH HK, and the first is to the sum of the first and second as the third is to the sum of the third and fourth Hence EG EI EH EK

26 It is required to prove that AD DB AC CB AD² : AC²

Dem —AD DB, AC CB are rectangular figures, and since AD DB AC CB (III), these figures are similar, hence (XIX) AD DB AC CB AD^2 AC^2 In like manner AC CB AD′ D′B AC^2 AD'^2

(1) Dem —If AD DB, AC CB, and AD′ D′B, are in A P, the difference between AD DB and AC CB is equal to the difference between AC CB and AD′ D B, but AC CB – AD DB = CD^2 (XVII, Ex 1), and AD′ D′B – AC CB = CD'^2, CD^2 = CD'^2, CD = CD′, the ∠ CDD′ = CD D, but the ∠ DCD is right; each of the ∠s CDD′, CD′D is half a right ∠, hence the ∠ CDA is a right ∠ and a-half Now the ∠ CDA = CBD + BCD, and CDB = CAD + ACD, hence CDA – CDB = CBD – CAD, but the difference between CDA and CDB is a right ∠ Hence the difference between CBD and CAD is a right ∠

(2) Dem —If the three rectangles be in G P, the squares of the lines DB, BC, BD are in G P, DB, BC, BD are in G P, BC is a mean proportional between DB and BD, but the ⊥ is a mean proportional between the segments of the hypotenuse (VIII, Cor 1) Hence BC is a ⊥, and hence the ∠ ABC is right

(3) Dem —If the rectangles AD DB, AC CB, AD′ D′B are in H P, the 1st 3rd difference between 1st and 2nd difference between 2rd and 3rd, but difference between 1st and 2nd = CD^2 (XVII, Ex 1) and difference between 2nd and 3rd = CD'^2, AD DB AD′ D′B CD^2 CD'^2, but, by similar figures, AD DB AD′ D′B DB $D'B^2$, hence CD^2 CD'^2 DB^2 $D'B^2$, CD CD′ DB D′B, and (III) the ∠ DCD is bisected, the ∠ DCB is half a right ∠, but the ∠ ACD = DCB, the ∠ ACB is right. Hence the sum of the ∠s CAB, CBA is a right ∠

28 Dem —Denote the radii of the ⊙s by ρ, ρ', then (VI IV) DC D C ρ ρ', and A C BC ρ ρ, DC D C A′C BC, DD D C A′B BC (V XVII) In like manner DD′ D′C AB′ B C, DD^2 $D'C^2$ A′B AB BC B C, but D C^2 = BC B C (III XXXVI) Hence DD^2 = AB′ A′B

29 Dem.—Because A′O is ‖ to BO″, AO′ OO″ AB A′B (II), that is, R (R – ρ) AB A′B Similarly, R (R – ρ') AB AB′, R^2 (R – ρ) (R – ρ) AB^2 A′B AB′, but

$A'B \cdot AB' = DD'^2$ (Ex 28) Hence $R^2 \cdot (R - \rho)(R - \rho') \cdot AB^2 \cdot DD'^2$

30 Dem —Let A, B, C, D be the points in which the four ⊙s, whose radii are ρ_1, ρ_2, ρ_3, ρ_4 respectively, touch the fifth, whose radius is R Join AB, BC, CD, DA, AC, BD, then putting $\overline{12}^2$ for DD'^2, we have, from Ex 29, $AB^2 \cdot \overline{12}^2 \cdot R^2 \cdot (R - \rho_1)(R - \rho_2)$, hence

$$AB^2 = \frac{\overline{12}^2 \cdot R^2}{(R - \rho_1)(R - \rho_2)}, \qquad AB = \frac{\overline{12} \cdot R}{\sqrt{(R - \rho_1)(R - \rho_2)}}$$

Similarly,

$$CD = \frac{\overline{34} \cdot R}{\sqrt{(R - \rho_3)(R - \rho_4)}}, \quad AD = \frac{\overline{14} \cdot R}{\sqrt{(R - \rho_1)(R - \rho_4)}},$$

and

$$BC = \frac{\overline{23} \cdot R}{\sqrt{(R - \rho_2)(R - \rho_3)}}$$

Now, by Ptolemy's theorem (xvii, Ex 13) AB · CD + BC · AD = AC · BD Therefore

$$\frac{\overline{12} \cdot \overline{34} \cdot R^2}{\sqrt{(R-\rho_1)(R-\rho_2)(R-\rho_3)(R-\rho_4)}} + \frac{\overline{23} \cdot \overline{14} \cdot R^2}{\sqrt{(R-\rho_2)(R-\rho_3)(R-\rho_1)(R-\rho_4)}}$$

$$= \frac{\overline{13} \cdot \overline{24} \cdot R^2}{\sqrt{(R-\rho_1)(R-\rho_3)(R-\rho_2)(R-\rho_4)}},$$

and hence

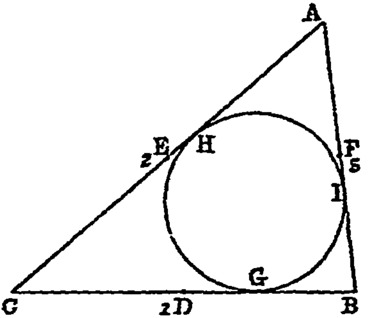
$$\overline{12} \cdot \overline{34} + \overline{23} \cdot \overline{14} = \overline{13} \cdot \overline{24}$$

31 Dem —Bisect the sides of the △ ABC in the points D, E, F Inscribe a ⊙ in ABC, touching the sides in G, H, I Let the sides opposite the angular points be denoted by a, b, c

Now if we consider the points D, E, F as infinitely small ⊙s, DE, EF, FG are common tangents to the ⊙s 1, 2, 2, 3, 3, 1, hence we have $\overline{12} = DE = \frac{1}{2} AB = \frac{1}{2} c$ Similarly, $\overline{23} = \frac{1}{2} a$, $\overline{31} = \frac{1}{2} b$

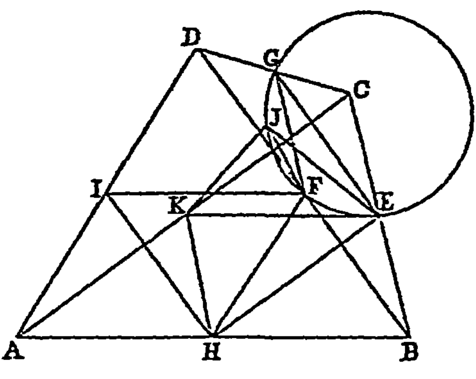
Let the inscribed ⊙ be denoted by 4 Now $BD = \frac{1}{2} BC = \frac{1}{2} a$, and $BG = (s - b)$ (IV iv, Ex. 2), $DG = \frac{1}{2} a - (s - b) = \frac{1}{2}(b - c)$, that is, $\overline{14} = \frac{1}{2}(b - c)$ In like manner, $\overline{24} = \frac{1}{2}(c - a)$, and $\overline{34} = \frac{1}{2}(a - b)$ Now if we substitute these values in the condition of the last question, we find that it is fulfilled Hence the

O through the middle points of the sides of the Δ touches the in-O Similarly, it touches the ex-O*



32 Let A, B, C, D be the four points, join them, and join AC, BD Bisect BC, BD, CD in E, F, G Bisect AB, AD in HI Describe a O through the points E, F, G, and another O through H, I, F, let them intersect in J It is required to prove that the O* through the middle points of the Δ* ABC, ADC will also pass through J

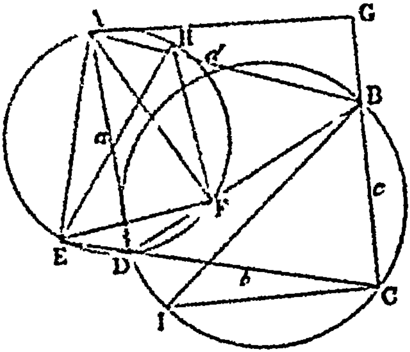
Dem —Bisect AC in K. Join KE, KH, EH, GE, EJ, JF, FH, HI, IF, JK.



Now because CB, CD are bisected in E, G, EG is ∥ to BD. Similarly, GF is ∥ to BC, hence BEGF is a ▭; the ∠ FGE = FBE, but FGE = FJE (III xxi), FJE = FBE Again, as before, HIFB is a ▭, . the ∠ HIF = HBF, but

HIF = HJF (III. xxi.), ∴ HJF = HBF, ∴ the whole ∠ HJE = HBE, but HBE = HKF, since HKEB is evidently a ▭, ∴ HJE = HKE, hence the four points H, K, J, E are concyclic, and the ○ through H, K, E will pass through J. Similarly, the ○ through K, I, G will pass through J. Hence the four nine-points ○s have a common point.

33. Dem.—From A let fall ⊥s AE, AF, AG on CD, DB, CB. Now because the ∠s AED, AFD are right, AEDF is a cyclic quad., and AD is the diameter of its circum ○. Draw another diameter EH. Join EF, FH. About the △ BDC describe a ○. Draw its diameter BI, and join IC. Now (III. xxii.) the sum of the ∠s EHF, EDF is two right ∠s, and the sum of EDB,



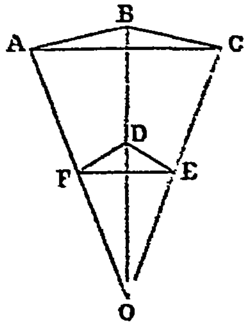
CDB is two right ∠s, hence the ∠ EHF = CDB, but (III. xxi.) CDB = CIB, hence EHF = CIB, and the right ∠ EFH = ICB, ∴ the △s EFH, ICB are equiangular, hence EH : EF :: IB : BC, ∴ EH . BC = EF . IB, that is, ac = EF . IB. Similarly, bd = FG . IB, and BD = EG . IB. Hence EF, FG, EG are proportional to ac, bd, BD'.

34. OEDF is a four-sided figure, OD, EF its diagonals. If OF . DE + OE . DF = OD . EF, it is required to prove that OEDF is a cyclic quad.

Dem.—Produce OD, OE, OF, to B, C, A until each of the rectangles OD . OB, OE . OC, OF . OA is equal to the square of a given line, say R². Join AB, BC, AC.

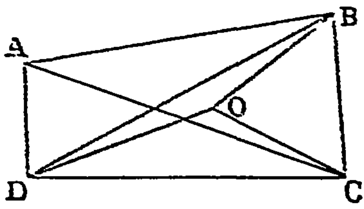
Now OD . OB = OE . OC, ∴ OB : OC :: OE : OD, and the ∠ BOC is common to the two △s OBC, OED, hence (vi.) they are equiangular, and BC : OB :: ED : OE, alternation, BC : ED

$OB \cdot OE$, ∴ $BC \cdot ED$ OB $OD \cdot OE \cdot OD$, that is, BC ED · R^2 OE OD, hence $\frac{ED}{OE \quad OD} = \frac{BC}{R^2}$. In like manner $\frac{DF}{OD \cdot OF} = \frac{AB}{R^2}$, and $\frac{EF}{OE \cdot OF} = \frac{AC}{R^2}$. Now $ED \cdot OF + DF \quad OE = OD \quad EF$ (hyp), ∴ $\frac{ED}{OE \quad OD} + \frac{DF}{OD \cdot OF} = \frac{EF}{OE \quad OF}$; that is, $\frac{BC}{R^2} + \frac{AB}{R^2} = \frac{AC}{R^2}$; ∴ $AB + BC = AC$, but this could not be true unless AB



and BC are in one straight line, ABC is a straight line; ∴ the sum of the ∠s ABO, CBO is two right ∠s, but ABO = DFO, and CBO = DEO, ∴ DFO + DEO = two right ∠s. Hence OEDF is a cyclic quad.

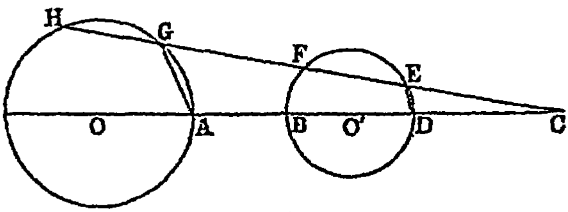
Alternative Proof—Given AB . CD + BC AD = AC BD it is required to prove that ABCD is a cyclic quad



Dem —If the ∠ CBD = CAD, then (III xxi, Cor 1) the four points A, B, C, D are concyclic But if the ∠ CBD be not equal to CAD, make CBO = CAD, and take BO so that BC . AD = AC BO, join CO, DO Now, since BC AD = AC BO, (VI vi) the Δ^s BCO, ACD are similar, the ∠BCO = ACD, and the ∠ BCA = DCO Also DC CA OC CB, DC OC AC CB, and the Δ^s DCO, ACB are similar, OD CD AB AC, AC OD = AB CD, but AC OB = BC AD, adding we get AC (OB + OD) = AB CD + BC AD = (hyp) AC BD, OB + OD = BD, which (I xx) is absurd, the ∠ CBD must be = CAD, and · (III xxi, Cor 1) ABCD is a cyclic quad

Lemma —If C be the external centre of similitude of two ⊙^s, CH any line passing through C, and cutting both ⊙^s in the points E, F, G, H, it is required to prove that CG FC = AC BC

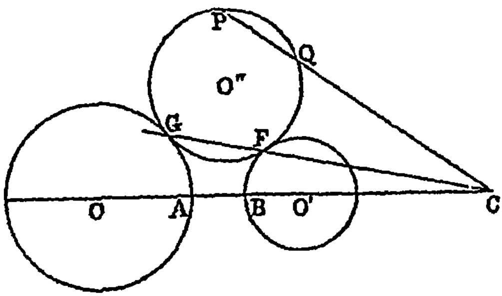
Dem —Join AG, DE



Now AC DC GC EC, · AC BC BC DC GC FC EC FC, but BC DC = EC FC Hence AC BC = GC FC

35 (1) Let O, O′ be the centres of the given ⊙^s, and P the point

Sol —Join OO′, and produce Let C be the external centre of

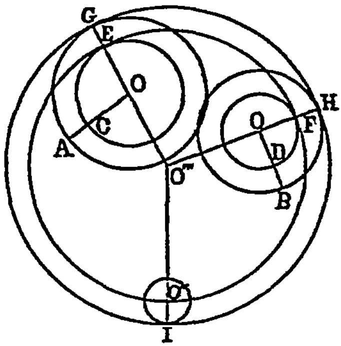


similitude Join PC, and find the point Q, so that PC QC

= AC BC Describe a ⊙ passing through P, Q, and touching the ⊙ whose centre is O in G (III xxxvii, Ex. 1) This is the required ⊙

Dem —Join GC, cutting the ⊙ whose centre is O′ in F Now (const.) PC QC = AC BC, and (*Lemma*) AC BC = GC FC, ∴ PC QC = GC FC Hence the ⊙ through the points P, Q, G passes through F, and touches the ⊙ whose centre is O′

(2) Sol —Let O, O′, O″ be the centres of the given ⊙s Draw any two radii OA, O′B Cut off AC, BD, each equal to the radius of O″. With O as centre and OC as radius, describe a ⊙ With O′ as centre and O′D as radius, describe a ⊙ Now (1) describe



a ⊙ touching those two in E, F, and passing through the point O″ Let O‴ be its centre Join O‴O, O‴O′, O‴O″, and produce them to meet the circumference of the given ⊙s in the points G, H, I The ⊙ through G, H, I will be the required ⊙

Dem —Because OG = OA and OE = OC, EG = AC, but AC = O″I, ∴ EG = O″I, and O‴E = O‴O″, hence O‴G = O‴I In like manner, O‴H = O‴I Hence the ⊙ described with O‴ as centre, and O‴G as radius, will pass through H, I, and touch the given ⊙s in the points G, H, I

36 Let O, O′ be the centres of the fixed ⊙s, and C their centre of similitude, and let any variable ⊙ O″ touch O, O′ in G, F

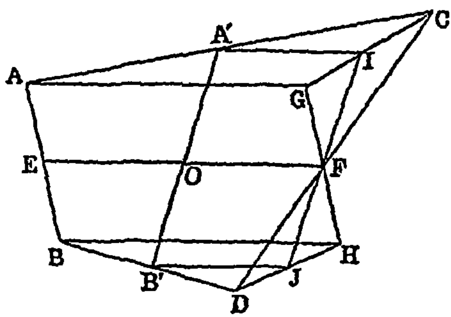
From C draw CD a tangent to O″ It is required to prove that CD is of constant length (See Diagram to Ex 35 (1))

Dem —Join GF, and produce it to pass through C

Now CD^2 = GC CF (III xxxvi), and GC CF = AC CB (*Lemma* to 35), hence CD^2 = AC CB, but AC CB is constant, since A, C, B are fixed points Hence CD is constant

37 Dem —Draw DD′ a common tangent to the two fixed O^s Join AD, BD′, and produce them, they must meet on the circumference of O″ For, if not, let AD meet the circumference of O″ in P, and BD′ meet it in Q Join O ′O, O″O′, and produce them, O′ O, O O′ must pass through A, B (III xi) Join OD, O′D′, O P, O″Q Now the ∠ O″AP = O″PA, and OAD = ODA, ODA = O″PA, hence OD is ∥ to O ′P Now the ∠ ODD′ is right (III xviii), hence O″P is ⊥ to DD′ Similarly, O″Q is ⊥ to DD′, which is impossible, unless Q coincide with P Hence BD′ must pass through P

38 Join A′B′ Take a fixed point C in AC, and in BD find a



point D, so that as AA′ AC BB′ BD Join AB, and divide it in E in a given ratio Join CD, and divide it in F in the same ratio Join EF, cutting A′B′ in O It is required to prove that A′O OB′ AE EB

Dem —Through F draw GH ∥ to AB, and draw AG, BH, each ∥ to EF Join CG, DH Draw A′I ∥ to AG, and B′J ∥ to BH Join IF, JF

Now, by construction, AA′ : AC ∙ : BB′ ∙ BD, ∴ AC : A C . . BD . B′D. And hence, by similar Δ^s, GC . IC ∙ ∙ DH : DJ; but GC CF DH . DF Hence IC CF .: DJ DF, and the contained ∠^{s} ICF, JDF are equal, ∴ the ∠^{s} ICF, JDF are equiangular, . . the ∠IFC = JFD ; ∴ IF, FJ are in the same straight line

Again, from similar Δ^s, AG AI AC . A C, and BH . B′J . BD B′D, hence AG . A′I BH B J, but AG = BH; ∴ A I = B J hence IJ is ∥ to A′B′, . A′O ∙ OB : IF : FJ; that is, . CF FD, or . AE . EB Hence the locus of the point in which A B′ is divided in the ratio of AE EB is the right line EF.

39 Dem.—It was proved in the last Exercise that A′O . OB′ . AE EB In like manner, EO . OF ∙ AA′ . A C Now putting G, H for A′, B′, we have GO . OH . AE EB, and EO : OF ∙ AG GC.

Lemma —If a given line AC be divided in B, so that AB BC^4 is a maximum ; it is required to prove that BC = 4AB

Dem.—Divide BC into four equal parts in E, F, G; then each of the parts BE, EF, FG, GC is equal to $\frac{BC}{4}$, hence

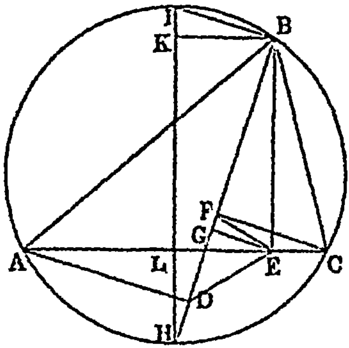
BE . EF FG GC = $\frac{BC^4}{256}$ Multiply each by AB, and we get

AB BE EF FG GC = $\frac{AB\ BC^4}{256}$, but (hyp) AB . BC^4 is a maximum , AB BE EF FG GC is a maximum , ∴ AB, BE, EF, FG, GC are all equal (" Sequel," Book II., Prop XII., Cor) Hence BC = 4AB

Similarly, if it be required to divide AC in B, so that AB BC^n may be a maximum, BC = nAB

40 *Analysis* —Let ABC be the required ∠ Bisect the vertical ∠ ABC by BH. From A, C let fall ⊥^{s} AD, CF on BH, and from B let fall a ⊥ BE on AC. Join DE, EF. Draw HI, the diameter Join BI Draw BK ∥ to AC, and let fall a ⊥ EG on HB

Now the ∠ ADB = AEB, each being right, hence the four points A, D, E, B are concyclic, the ∠ EDF = BAC Again, because each of the ∠s BEC, BFC is right, BFEC is a cyclic quad, the sum of the ∠s BFE, BCE is two right ∠s, and the sum of BFE, DFE is two right ∠s, the ∠ DFE = BCA, . the △s ABC, DEF are equiangular And since their ⊥s



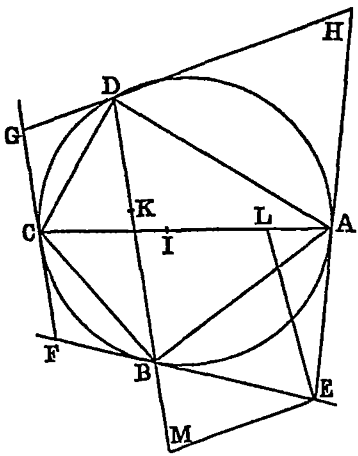
are BE, EG, ABC DEF BE^2 EG^2, but BE^2 EG^2 - HI^2 IB^2, or HI IK, BE^2 EG^2 HI IK, ABC DEF HI IK, ABC IK = DEF HI Now DEF is a maximum (hyp), and HI is a given line, because it is the diameter of the ○, ABC IK is a maximum Now ABC = ½ base perpendicular = AL BE, or AL KL, . AL KL IK is a maximum Now whatever AL is, the rectangle KL IK is a maximum when IL is bisected in K, and then

KL KI = ¼ IL^2, AL $\frac{IL^2}{4}$ is a maximum, AL IL^2 is a

maximum, AL^2 IL^4 is a maximum, but AL^2 = HL LI, HL IL^5 is a maximum And (*Lemma*) IL = 5 HL Hence the method of construction is evident

41 Let AC, BD, the diagonals of the inscribed quad, intersect in O At the points A, B, C, D draw tangents to the ○ Let them meet in E, F, G, H, then EFGH is a circumscribed quad It is required to prove that its diagonals EG, FH must pass through O

Dem —If possible let EG not pass through O , but cut AC, BD in I, K Produce AE, CF to meet in J (not represented in the diagram) Through E draw EL ∥ to GF, and EM ∥ to GH Produce DB to meet EM Now because JA = JC, being tangents, the ∠ JCA = JAC , but ELA = JCA (I xxix), EAL = ELA, and EA = EL In like manner EB = EM , but EA = EB ,



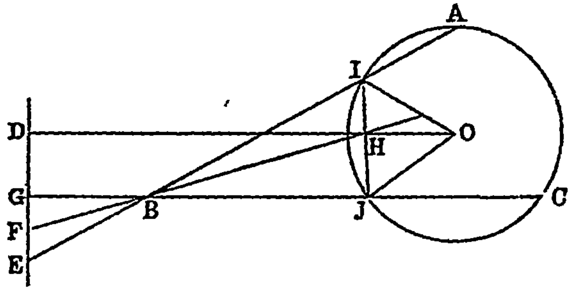
. EL = EM Now since the Δˢ GCI, ELI are equiangular, GC GI EL EI , alternation, GC EL GI EI , but GC = GD, and EL = EM , GD EM GI EI , and because the Δˢ GKD, MKE are equiangular, GD EM GK EK , GI EI GK EK, which is impossible unless the points I, K coincide Hence GE must pass through O In like manner FH must pass through O

42 (1) Sol —Let A, B be the given points, W the given ○, and X the given line Through A, B describe any ○ cutting W in C, D Join AB, CD, and produce them to meet in E Through E draw EFG ∥ to X, and cutting W in F, G The ○ through A, B, F, G is the one required

Dem —AE EB = CE ED, and CE ED = GE EF , AE EB = GE EF Hence the four points A, B, F, G are concyclic, and the common chord FG is ∥ to X

(2) Sol —Let O be the given point Make the same construction as before, and instead of drawing EFG ∥ to X, join EO, and produce it to cut W in F, G Then, as in (1), EFG is a common chord, and it passes through O, the given point

43 Sol —Let O be the centre of the ○, ABC the ∠, and DE the given line Produce AB, CB to meet DE in E, G Bisect GE in F Join FB From O let fall a ⊥ OD on DE, and meeting FB produced in H Through H draw IJ ∥ to DE,



meeting AB, CB in I, J Join OI, OJ Now because the lines GJ, FH, EI pass through B, and are cut by the ∥s GE, IJ, GF FE IH HJ, but GF = FE, IH = HJ, and since IJ is ∥ to DE, and OD meets them, the ∠ OHJ = ODE, OHJ is a right ∠, OHI is right, and (I iv) OJ = OI, and the ○, with O as centre, and OJ as radius, will pass through I, and its chord IJ is ∥ to the given line DE

44 Let ABCDE be a polygon of an odd number of sides Take any point O within it. Join AO, BO, CO, DO, EO, and produce them to meet the opposite sides in A, B, C′, D′, E It is required to prove that the product of AD′, BE′, CA′, DB′, EC′ is equal to the product of A D, B′E, C′A, D′B, E′C

Dem —Join AC, AD Now the Δ AOD A′OD AO A′O (I), and AOC A OC AO A O, AOD A′OD AOC A′OC, alternation, AOD AOC A OD A′OC, but A′OD A′OC DA′ A′C Hence

$$\frac{DA'}{A'C} = \frac{AOD}{AOC}$$

, BD, CE, CA, DB, DA, EC,

$$\frac{BD'}{D'A} = \frac{BOD}{DOA}, \quad \frac{CE'}{EB} = \frac{COE}{BOE}$$

er, we find that the numerators of
ie denominators Hence the pro-
rst terms is equal to the product of
B'E C'A D'B E'C = A'C

let the sides touch the ○ in the

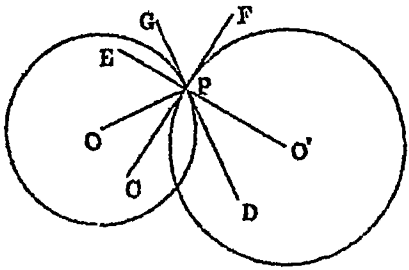
Now AB' = AC', BA', = BC',
BA' = A'C C'B B'A, and hence
' are concurrent

ents AA', BB', CC', and produce
them in A', B', C It is required
C' are collinear

Dem —The ∠ B BC = BAB' (III xxxii), and the ∠ BB'C is common, the Δ^s AB B, BB C are equiangular, ∴ AB' AB BB BC, alternation, AB BB' AB BC, AB'^2 BB'^2 ∴ AB^2 BC^2, but BB'^2 = AB' B C (III xxxvi), AB'^2 AB' B'C AB^2 BC^2, AB' B'C AB^2 BC^2 Hence, denoting the sides of the Δ ABC by a, b, c, we have AB B C c^2 a^2 Interchange, and we get BC' C'A a^2 b^2, and CA' A'B b^2 c^2 Multiply these together, and we have AB' BC' CA', B C C A A B $c^2a^2b^2$ $a^2b^2c^2$, AB' BC' CA' = B'C C A A'B, and hence (Ex 5) the points A', B', C' are collinear

47 Dem —Produce the sides, and draw AA', BB', CC', bisecting the external ∠^s Now (III, Ex 1) AB B C AB BC Interchange, and we have BC' C'A BC CA Interchange again, and CA' A'B CA AB Now, multiply together, and AB' BC' CA B'C C'A A'B AB BC CA BC CA AB, but the third term is equal to the fourth, the first is equal to the second, that is, AB' BC' CA' = B'C C'A A'B, and hence (Ex 5) the points A', B', C' are collinear

Lemma.—Let two ⊙s, whose centres are O, O′, cut in P. Join

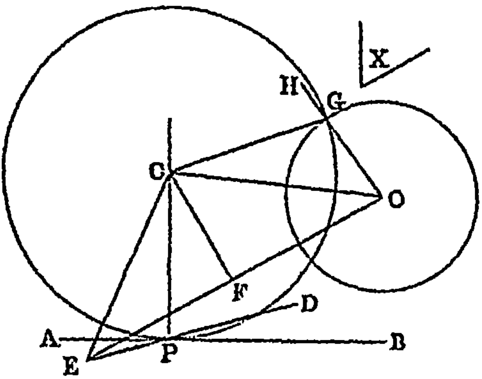


OP, O′P. Produce O′P to E. Draw CP, DP tangents to the ⊙s. It is required to show that the ∠ EPO = CPD.

Dem.—Produce CP, DP to F and G. Now the ∠ O′PF is right (III. xviii.), hence (I. xv.) CPE is right, and OPD is right; ∴ CPE = OPD. Reject OPC, and EPO = CPD.

48. Let AB be a given line, P a given point, O the centre of the given ⊙, and X a given ∠. It is required to describe a ⊙, touching AB in P, and cutting O at an ∠ equal to X.

Sol.—Erect PC ⊥ to AB. Draw DP, making the ∠ CPD = X. Produce DP to E, cut off EP equal to the radius of O. Join EO.



Bisect it in F. Erect FC ⊥ to EO, meeting PC in C. With C as centre, and CP as radius, describe a ⊙, cutting O in G. This is the required ⊙.

Dem —Join EC, CO, CG, OG Now because EF = OF, and FC common, and the ∠ EFC = OFC, (I IV) EC = OC, and CP = CG, being radii, and EP = OG (const), the ∠ EPC = OGC, but DPC and EPC are supplements, and HGC, OGC are supplements, HGC = DPC, but DPC = X, and HGC is equal to the ∠ between the ○ˢ (*Lemma*) Hence the ∠ between the ○ˢ is equal to the given ∠, and the ○ PG touches AB in P

49 See "Sequel," Book IV, Prop III, Cor 2

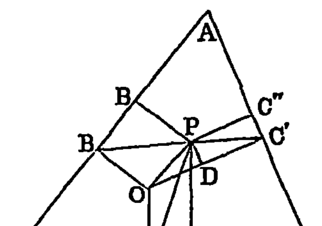
50 See "Sequel," Book I, Prop XVII

51 See "Sequel," Book II, Prop X

52 Let O be the centre of mean position of the feet of ⊥ˢ from it on the sides From O let fall ⊥ˢ OA′, OB′, OC′ on the sides Take any other point P within the △, and let fall ⊥ˢ PA″, PB″, PC″ It is required to show that $OA'^2 + OB'^2 + OC'^2$ is less than $PA''^2 + PB''^2 + PC''^2$

Dem —Join OP, PA′, PB′, PC′ Now, because O is the centre of mean position of A′, B′, C′, we have (Ex. 51) $A'P^2 + B'P^2 + C'P^2 = OA'^2 + OB'^2 + OC'^2 + 3OP^2$, but $A'P^2 = A'A''^2 + A''P^2$, $B'P^2 = B'B''^2 + B''P^2$, and $C'P^2 = C'C''^2 + C''P^2$, $A'A''^2 + B'B''^2 + C'C''^2 + A''P^2 + B''P^2 + C''P^2 = OA'^2 + OB'^2 + OC'^2 + 3OP^2$

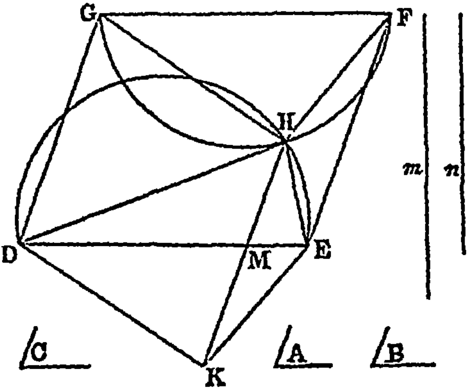
From P let fall a ⊥ PD on OC′, then OP^2 is greater than PD^2,



that is, greater than $C'C''^2$ In like manner it is greater than $A'A''^2$, and greater than $B'B''^2$, $3OP^2$ is greater than $A'A''^2 + B'B''^2 + C'C''^2$, and hence $A''P^2 + B''P^2 + C''P^2$ is greater than $OA'^2 + OB'^2 + OC'^2$

53 (1) Let A, B be the opposite ∠ˢ, *m*, *n* the diagonals, and C the angle between the diagonals

Sol —Construct a ▭ DEFG, having two adjacent sides



DE, DG respectively equal to *m* and *n*, and their included ∠ = to C On DE describe a segment of a ○ containing an ∠ equal to A, and on FG describe a segment containing an ∠ equal to B, let them intersect in H Join HD, HE, HF, HG Through H draw HK ∥ and = to EF Join DK, EK DHEK is the required quad

Dem —The ∠ DHE = A, and EF = HK (I xxxiv), but EF = GD, HK = GD, and it is ∥ to it, HKDG is a ▭, . HG is ∥ to DK, and HF is ∥ to EK, hence the ∠ GHF = DKE, but GHF = B, DKE = B, and (I xxix) the ∠ HME = GDE, but GDE = C, HME = C

64 Let a ○, whose centre is O′, roll inside another ○, whose centre is O, and whose diameter is twice that of O′ Take a fixed point P in the circumference of O′ It is required to find its locus

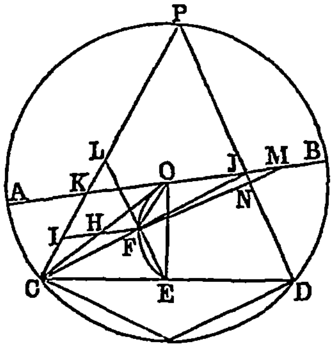
Sol —Let R be the *point of contact* *Join OP, OR, O P*, and produce OP to meet the circumference in Q, and bisect the ∠ RO′P by O S meeting the ○ O′ in S

Now the ∠ RO P = 2 ROP (III xx), the ∠ RO′S = ROQ, and the arc RS RQ O′R OR, but OR = 2O′R, RQ = 2RS, RP = RQ Now since the arc RP = RQ, the point P must have coincided with Q Hence the line OQ is the locus of P

65 Sol —Take any point G in the arc CD Join CG, DG From the centre O let fall a ⊥ OE on CD, and on OE describe a segment OFE containing an ∠ equal to CGD Join OC Bisect it in H. Through H draw HF ∥ to AB, cutting the segment

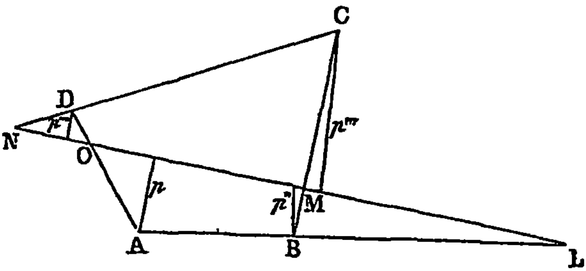
OFE in F Join OF, and through C draw CP ‖ to OF P is the required point

Dem —Let CP intersect AB in K Join PD, cutting AB in J Produce FH to meet CP in I Join EF, and produce it to



meet CP in L Join CF, FJ The points C, F, J are collinear, if not, let CF, FM be in a straight line Now (III xxii) the ∠s CGD, CPD equal two right ∠s, OFE, CPD equal two right ∠s, and OFE, OFL equal two right ∠s, OFL = CPD, that is, CLE = CPD, hence EL is ‖ to PD Again, in the Δ COM, since CO is bisected in H, CM is bisected in F (I xl, Ex 3), and similarly, in the Δ CDN, CN is bisected in F, FN = FM, which is absurd, hence CF produced must pass through J, and CF = FJ Now, in the Δ CJK, CJ is bisected in F, and OF is ‖ to CP, KJ is bisected in O, that is, OK = OJ

56 Let ABCD be a polygon of four sides Produce AB, CD



to L, N, and draw a transversal LMON, cutting the four sides. From A, B, C, D let fall ⊥s p', p'', p''', p'''' on LMON

Now, since the △s $Ap'L$, $Bp''L$ are equiangular,

$$\frac{AL}{BL}=\frac{p'}{p''} \text{ (IV.)}$$

For the same reason,

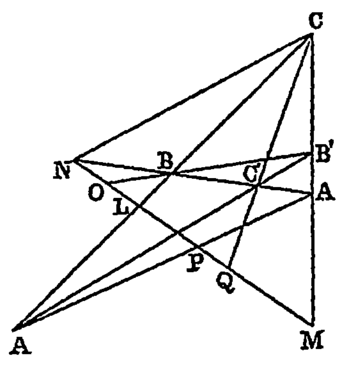
$$\frac{BM}{CM}=\frac{p''}{p'''},\ \frac{CN}{DN}=\frac{p'''}{p''''},\ \text{and}\ \frac{DO}{AO}=\frac{p''''}{p'}.$$

Multiplying together, we get

$$\frac{AL.BM.CN.DO}{BL.CM.DN.AO}=\frac{p'p''p'''p''''}{p'p''p'''p''''}.$$

Hence AL.BM.CN.DO = BL.CM.DN.AO. And similarly for a figure of any number of sides.

57. Let the transversal LMN cut the sides of the △ ABC in the points L, M, N. Bisect LN, NM, ML in O, P, Q. Join AP, OB, CQ, and produce them to meet the sides of the △ ABC in A′, B′, C′, respectively. It is required to prove that the points A′, B′, C′ are collinear.



Dem.—The sides of the △ AMN are cut by OBB′,

$$\frac{AB'}{BM}\cdot\frac{MO}{ON}\cdot\frac{NB}{BA}=-1 \text{ (Ex. 5)}.$$

And the △ CLM is cut by OB′, $\quad\frac{MB}{BC}\cdot\frac{CB}{BL}\cdot\frac{LO}{OM}=-1.$

Multiplying together, we have $\frac{AB'}{BC}\cdot\frac{CB}{BA}\cdot\frac{NB}{BL}=1$, interchange, and $\frac{BC'}{C'A}\ \frac{AC}{CB}\ \frac{LC}{CM}=1$, interchange again, and $\frac{CA'}{A'B}\ \frac{BA}{AC}\ \frac{MA}{AN}=1$

Multiply these results together, and we get

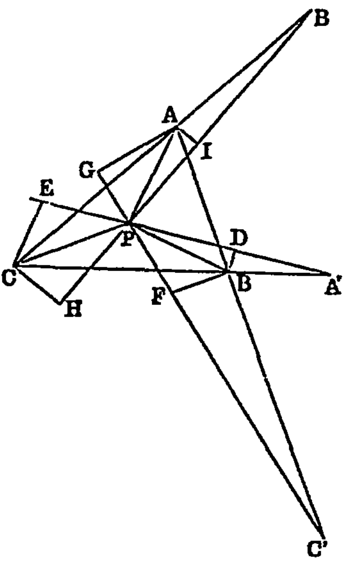
$$\frac{AB'}{BC}\ \frac{BC'}{CA}\ \frac{CA'}{A'B}\ \frac{NB}{BL}\ \frac{LC}{CM}\ \frac{MA}{AN}=1,$$

but $\frac{NB}{BL}\ \frac{LC}{CM}\ \frac{MA}{AN}=-1$ (Ex 5), $\qquad \frac{AB'}{BC}\ \frac{BC'}{CA}\ \frac{CA'}{A'B}=-1$

And hence the points A', B', C' are collinear

58 Let ABC be the △ Join PA, PB, PC, and erect at P ⊥ˢ A'E, B'H, C G to PA, PB, PC, intersecting the sides BC, CA, AB, respectively, in A', B', C' It is required to show that the points A, B , C' are collinear

Dem —From A, B, C let fall ⊥ˢ AG, AI on C G, B'H , BD, BF on A'E, C G , CE, CH on A'E, B H



Now, because each of the ∠ˢ APA', BPB' is right, the ∠ API = BPD, and AIP = BDP , hence the △ˢ AIP, BDP are equiangular In like manner, the △ˢ AGP, CEP are equiangular, and CPH, BPF are equiangular

Again, since the Δ^s CA'E, BA'D are equiangular,

$$\frac{CA'}{A'B} = \frac{CE}{BD},$$

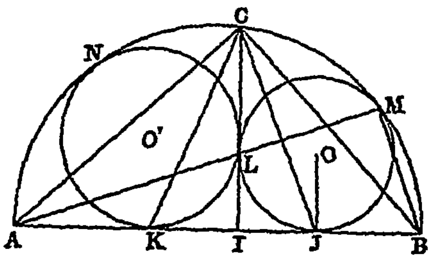
Similarly, $$\frac{AB'}{B'C} = \frac{AI}{CH}, \text{ and } \frac{BC'}{C'A} = \frac{BF}{AG},$$

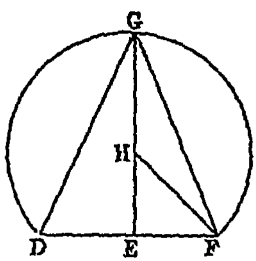
therefore $$\frac{CA'}{A'B}\,\frac{AB'}{B'C}\,\frac{BC'}{C'A} = \frac{CE}{BD}\,\frac{AI}{CH}\,.\,\frac{BF}{AG},$$

hence $$\frac{CA'}{A'B}\,\frac{AB'}{B'C}\,\frac{BC'}{C'A} = \frac{CE}{BD}\,\frac{AI}{CH}\,\frac{BF}{AG}\,\frac{PB}{PB}\,\frac{PC}{PC}\,\frac{PA}{PA},$$

but $AI \; BP = BD \; AP$, since the Δ^s AIP, BDP are equiangular, and $PA \; CE = AG \; PC$, and $PC \; BF = PB \; CH$, therefore $CA' \; AB' \; BC' = A'B \; B\,C \; C\,A$ And hence (Ex 4) the points A', B', C are collinear.

59 Let ACB be a given semicircle It is required to divide it into two parts by a ⊥ on the diameter AB, so that the radii of the ⊙s inscribed in them may have a given ratio DE . EF



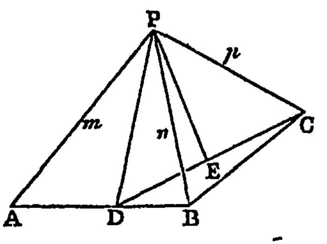


Sol —On DF describe a segment containing an ∠ equal to half a right ∠. Erect EG ⊥ to DF Join DG, FG At the point F in FG draw FH, making the ∠ HFG = HGF In the semicircle ıng the ∠ABC = EFH Let fall the ⊥ CI on AB ıred lıne

the figures CJBM, CIAN descrıbe ⊙ˢ touchıng nd the arcs BC, AC ın the poınts J, K, L, M, N theır centres Joın OJ, CJ, OL, O'L, AL, LM , L, M, are collınear (III Ex 51) Joın BM, Now the ∠ LIB ıs rıght, and LMB ıs rıght , ILMB ıs a cyclıc quad , BA AI ut BA AI = AC^2 (I xlvii , Ex 1), and MA AL xxvi), $AC^2 = AJ^2$, AC = AJ , the ∠ACJ CJ = JBC + JCB , but ACI = IBC (viii), ICJ ke manner, the ∠ ICK = ACK , hence the ∠ KCJ ∠ Now ın the Δˢ EHF, ICB the ∠BIC = FEH, FH (const), ICB = EHF , but ICB = 2 ICJ, EGF , ICJ = EGF, and CIJ = GEF , CJI the Δˢ CIJ, GEF are equıangular And because KCJ, and GFD = CJK, GDF = CKJ , hence GFD are equıangular, KI IJ DE EF , O L OL Hence O L OL DE EF

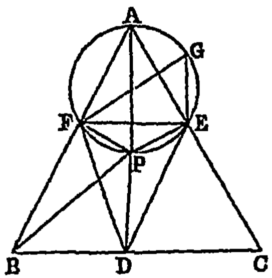
A, B, C be fixed poınts, and P a varıable poınt, of P, ıf $mAP^2 + nBP^2 + pCP^2$ ıs gıven

AP, BP, CP, AB, BC Dıvıde AB ın D, so that $mAD = nDB$ Joın DP Now $mAP^2 + nBP^2 = mAD^2 + nDB^2 + (m + n) DP^2$ (Book II , Ex 12) Joın DC, and dıvıde ıt ın E, so that $(m + n) DE = pEC$ Joın EP , then $(m + n) DP^2 + pPC = (m + n) DE^2 + pEC^2 + (m+n + p) EP^2$, and $mAP^2 + nBP^2$



$+ pPC^2 = mAD^2 + nDB^2 + (m + n) DE^2 + pEC^2 + (m + n + p) EP^2$, but $mAP^2 + nBP^2 + pPC^2$ ıs gıven (hyp), $mAD^2 + nDB^2 +$ &c ıs gıven , but $mAD^2 + nDB^2$ ıs gıven, and $(m + n) DE^2$, and pEC^2 ıs gıven , $(m + n + p) EP^2$ ıs gıven, and $(m + n + p)$ ıs gıven , EP² is given, ∴ EP is given, and E is a given point Hence the locus of P is a ○, having E for centre and EP for radius

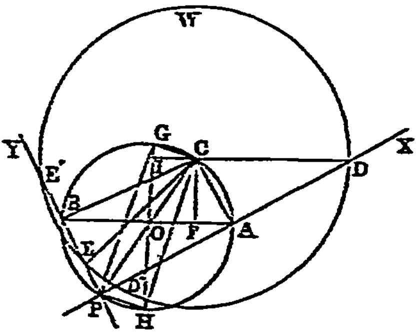
60 Dem —Let P be the point From P let fall ⊥ˢ PD, PE, PF on the sides of the △ Join DE, EF, FD, AP, BP, CP Now because the ∠ˢ AEP, AFP are right, AEPF is a cyclic quad., then AP is the diameter of the circum-○ Draw FG, another diameter Join GE Now the ∠ FGE = FAE (III xxi), but FAE is a given ∠, ∴ FGE is a given ∠, and the



∠ FEG is given, being right, ∴ the △ FGE is given in species, hence $\frac{EF}{FG}$ is given, but FG = AP, ∴ $\frac{EF}{AP}$ is given, ∴ $\frac{EF^2}{AP^2}$ is given, let it be equal to m, then $EF^2 = mAP^2$ In like manner, $FD^2 = nBP^2$, and $DE^2 = pCP^2$, but $EF^2 + FD^2 + DE^2$ is given (hyp), ∴ $mAP^2 + nBP^2 + pCP^2$ is given And hence (*Lemma*) the locus of P is a ○

61 Let the ○ W make given intercepts DD′, EE′ on two fixed lines PX, PY It is required to prove that the rectangle CG . CH contained by the ⊥ˢ from the centre C on the bisectors of the ∠ˢ formed by the lines PX, PY is given

Dem —From C let fall ⊥ˢ CA, CB on DD′, EE′ Join CD, CE Now $AC^2 + AD^2 = CD^2$, and $BC^2 + BE^2 = CE^2$, ∴ $AC^2 + AD^2 = BC^2 + BE^2$, ∴ $AD^2 - BE^2 = BC^2 - AC^2$, but AD, BE are the halves of DD′, EE′ (III iii), and are given (hyp), ∴ $BC^2 - AC^2$ is given Now since the ∠ˢ CAP, CBP are right, CAPB is a cyclic quad Describe a ○ about it Join AB, the line bisecting AB perpendicularly will be the diameter Let it be GH Join GP, HP, these are the internal and external bisectors of the ∠ EPD (III xxx, Ex 2) Join CP, CH and let fall ⊥s CF, CI on AB, GH. Now $BC^2 = BF^2 + FC^2$ and $AC^2 = AF^2 - FC^2$, ∴ $BC^2 - AC^2 = BF^2 - FA^2$; but $BC^2 - AC^2$ is given, ∴ $BF^2 - FA^2$ is given, that is (BF − FA) (BF − FA) is given, but BF − FA = AB, and BF − FA = 2 OF ∴ AB . OF is given, that is, AB . CI is given. And because the ∠ APB is given, the ratio of AB to the diameter is given

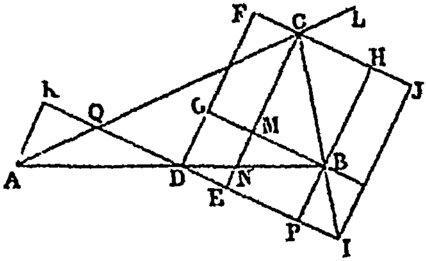


(Dem. of Ex. 60); that is, AB : GH is given; ∴ the ratio AB . CI : GH . CI is given; but AB . CI is given, ∴ GH . CI is given. And since the Δs GCH, ICH are equiangular, GH . CI = GC . CH. Hence GC . CH is given.

62. Let ABC be a Δ, whose base and the difference of whose base ∠s is given. Draw CE, CF, the internal and external bisectors of the vertical ∠. Bisect AB in D, and let fall ⊥s DE, DF on CE, CF. It is required to prove that the rectangle DE . DF is given.

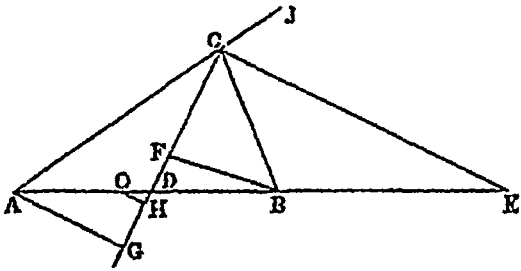
Dem.—Draw BG, BH ∥ to CF, DF. Produce CB to meet DE produced in I. Draw IJ ∥ to CE, and let fall a ⊥ AK on ID produced.

Now the ∠ NCB = ACN, and LCJ = ICJ, ∴ the ∠ NCJ is right, and DFC is right, ∴ DF is ∥ to CN, ∴ FCMG is a ▭. Now the ∠ ANC = NCB − CBN, and BNC = NCA − CAN; ∴ (ANC − BNC) = (NCB + CBN) − (NCA − CAN); but NCB = NCA; ∴ (ANC − BNC) = (CBN − CAN); but (CBN − CAN) is given (hyp.), ∴ (ANC − BNC) is given, and their sum is given, hence each is given, but DNE = BNC, ∴ DNE is given, and DEN is right, ∴ EDN is given; hence the line IK is given in position, PB ⊥ to KJ is given in position And because the ∠ QCE = ICE, and CEQ = CEI, and CE common, EQ = EI, and the ∠ EQC = EIC, but LQC



= AQK, EIC or BIP = AQK, and AKQ = BPI, each being right, and the side AK = BP, KQ = IP To each add QP, and we have KP = QI, hence (Ax 7) KD = QE, KQ = DE, DE = IP, hence the figure GC = BJ, but BJ = BE (I xliii), GC = BE, hence the rectangle DC = BD that is, the rectangle DE DF = BD, but BD is a given rectangle Hence DE DF is given

63 Let ABC be the Δ Bisect the ∠ ACB by CD From A, B let fall ⊥s AG, BF on CD Produce AC, and bisect the



∠ BCJ by CE, meeting AB produced in E Bisect AB in O, and let fall a ⊥ OH on CD It is required to prove that AG FB = OH CE

Dem —Now AD DB AE EB (III, Ex 3), hence (Book V, Ex 9) OD OB OB OE, that is, OD OE = OB^2, but (II III) OD . OE = OD^2 + OD DE, and (II v.)

$OB^2 = AD \quad DB + OD^2$, hence $OD \quad DE = AD \quad DB$, ∴ AD OD DE DB, but AD OD AG OH, and DE DB CE FB, AG · OH CE FB And hence AG FB = OH CE

64 The rectangle contained by the ⊥ˢ from the extremities of the base on the external bisector of the vertical angle is equal to the rectangle contained by the internal bisector and the ⊥ from the middle of the base on the external bisector

Let ACB be the Δ Produce AC to J, and bisect the ∠ BCJ by ECG, meeting AB produced in E From A, B let fall ⊥ˢ AG, BF on EG Bisect the ∠ ACB by CD Bisect AB in O, and let fall a ⊥ OH on EG It is required to prove that AG BF = OH CD

Dem —AE EB + $OB^2 = OE^2$ (II vi), but OE^2 = OD OE + DE OE (II ii), hence AE EB + OB^2 = OD OE + DE OE; hence AE EB = DE . OE (see Ex 63), AE DE OE EB Hence, by similar Δˢ, AG CD OH BF, AG BF = OH CD

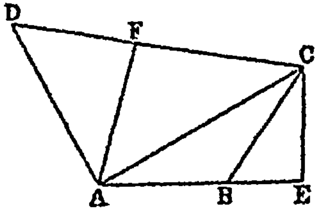
65 **Dem** —From C let fall a ⊥ CD on AB Now the Δˢ ACD, BCD, ABC are similar (viii), then, if R, R′, ρ, are the radii of the ⊙ˢ inscribed in these Δˢ, AC, BC, AB are proportional to R, R′, ρ, but $AC^2 + BC^2 = AB^2$, $R^2 + R'^2 = \rho^2$, and $\rho^2 = (s - c)^2$ (IV iv, Ex 14), that is, $R^2 + R'^2 = (s - c)^2$

66 **Sol** —Through A, C draw two ∥ lines AF, CE, and through B, D draw two ∥ lines BF, DE, meeting the ∥ through A, C in F, E Join EF, and produce it to meet AD in O

Dem —Because BF is ∥ to DE, the Δˢ ODE, OBF are equiangular, hence OD OB OE OF, and since the Δˢ OCE OAF are equiangular, OE OF OC OA, OD OB OC OA Hence OA OD = OB OC

67 **Sol** —Let a, b, c, d be the four sides Find a fourth proportional to $(2ab + 2cd)$, $\{(c^2 + d^2) - (a^2 + b^2)\}$, and b Let it be BE Produce EB to A, so that AB = a Erect EC ⊥ to AE With B as centre, and a radius equal to b, describe a ○ cutting EC in C Join BC, AC, and on AC describe a Δ ACD having its sides CD, AD equal to c and d ABCD is the required quad

Dem —From A let fall a ⊥ AF on CD Now because BE is a fourth proportional to $(2ab + 2cd)$, $\{(c^2 + d^2) - (a^2 + b^2)\}$, and b, we have $(2ab + 2cd)\,BE = \{(c^2 + d^2) - (a^2 + b^2)\}\,b$. Now $AC^2 = AB^2 + BC^2 + 2AB \cdot BE$ (II xii), that is, $AC^2 = a^2 + b^2$



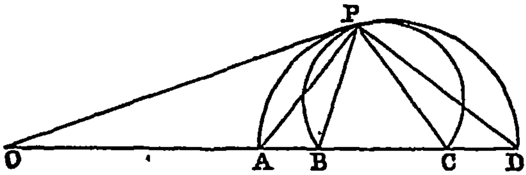
$+ 2a \cdot BE$, and $AC^2 = c^2 + d^2 - 2c \cdot DF$ (II xiii), ∴ $c^2 + d^2 - 2c \cdot DF = a^2 + b^2 + 2a \cdot BE$, ∴ $c^2 + d^2 - (a^2 + b^2) = 2a \cdot BE + 2c \cdot DF$, hence $(2ab + 2cd)\,BE = (2a \cdot BE + 2c \cdot DF)\,b$, ∴ $2cd \cdot BE = 2bc \cdot DF$, ∴ $d \cdot BE = b \cdot DF$, ∴ $d : DF :: b : BE$, that is, AD : DF :: BC : BE, and the ∠ AFD = BEC ∴ the Δs ADF, CBE are equiangular ∴ the ∠ ADF = CBE. To each add ABC, and we have the ∠s ADC, ABC equal to ABC, EBC, ∴ ADC + ABC equal two right ∠s. Hence ABCD is a cyclic quad.

68. Let A, B be the centres of the ⊙s. From a point C tangents CF, CE are drawn to the ⊙s A, B, so that CF : CE :: a : b. It is required to find the locus of C.

Sol.—Join AF, BE, AC, BC, and let the radii be denoted by R, R′. Now since CF : CE :: a : b, $CF^2 : CE^2 :: a^2 : b^2$, that is, $AC^2 - R^2 : BC^2 - R'^2 :: a^2 : b^2$, ∴ $b^2AC^2 - b^2R^2 = a^2BC^2 - a^2R'^2$, ∴ $b^2AC^2 - a^2BC^2 = b^2R^2 - a^2R'^2$. Join AB, and produce it to D, and make AD : BD :: a^2 : b^2, then $b^2AD = a^2BD$. Now, joining CD, and putting b^2 for m, and a^2 for n, we have (Book II, Ex. 13) $b^2AC^2 - a^2BC^2 = b^2AD^2 - a^2DB^2 + (b^2 - a^2)\,CD^2$, and ∴ (Ax. 1) $b^2AD^2 - a^2DB^2 + (b^2 - a^2)\,CD^2 = b^2R^2 - a^2R'^2$, and transposing, we get $(a^2 - b^2)\,CD^2 = b^2(AD^2 - R^2) - a^2(DB^2 - R'^2)$, ∴ $(a^2 - b^2)\,CD^2$ is given, ∴ CD is given, and the point D is given. Hence the locus of C is a ⊙.

69. Sol.—Describe ⊙s about the Δs APD, BPC. Draw OP a tangent to the ⊙ APD, meeting DA produced in O. Now the ∠ OPA = PDA (III xxxii), and the ∠ APB = CPD (hyp.), ∴ the ∠ OPB = ADP + CPD = ACP; hence OP touches the ⊙ BPC. Now (III xxxvi) OA . OD = OP², and OB . OC = OP², ∴ OA . OD = OP . OC, ∴ O is a given point (Ex. 66), and A, D

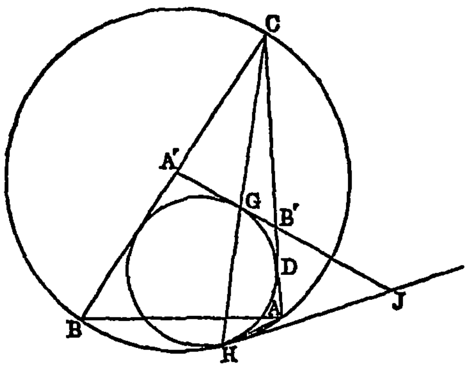
are given points, . OA OD is given, OP² is given, OP



is given Hence the locus of P is a ○, having O as centre and OP as radius

70 If a ○ ACB be circumscribed to a Δ, and a ○ GBH be inscribed, touching the sides AC, BC in D, F, and the circumscribed ○ in H It is required to prove that CD is a fourth proportional to the semi-perimeter of the Δ ABC, and the sides CA, CB

Dem —Join CH, and draw HJ a tangent to the ○ ABC, at G draw a tangent A′J to the ○ DFH Join AH



Because JG = JH, the ∠ JHG = JGH, but JGH = GB′C + B′CG, JHG = GB′C + B′CG, and AHJ = GCB′ (III xxxii), GHA = GB′C To each add GB′A, and we have GB C + GB′A = GB′A + GHA, GB A + GHA equal two right ∠s, hence GB′AH is a cyclic quad, and therefore HC CG = AC CB′, but HC CG = CD² (III xxxvi), AC CB′ = CD² Again, the ∠ CHA = A′B′C, but CHA = CBA (III xxi), · CBA

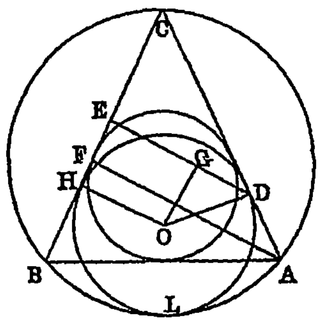
= CB'A', and the ∠ A'CB' is common, the Δ^s ABC, A'B'C are equiangular, and, denoting their semi-perimeters by s, s', we have (xx., Cor 1) s s' BC B'C, s s' CA BC B'C CA, that is, s s' CA BC CD^2, but $CD^2 = s'^2$ (IV iv, Ex 4), s s' CA BO s^2. Hence s · CA BC · s', or, s CA · CB CD

71 It is an obvious modification of 70

•73 Let the sides AC, BC of the Δ ABC, circumscribed to a given ○, be given in position, but the third side AB variable About ABC describe a ○ It is required to prove that the ○ about ABC touches a fixed ○

Dem —Describe a ○ touching the sides AC, BC in D, H, and the ○ about ABC in L Let O be its centre Join OD, OH Let fall a ⊥ AF on BC Draw DE ∥ to AF, and let fall a ⊥ OG on DE

Now s CB CA CD (Ex 70), but CA CD AF DE, therefore s CB AF DE, s . DE = CB AF = twice the area



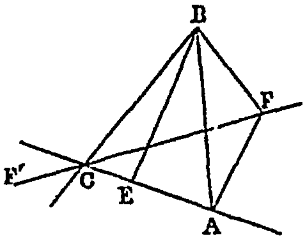
of the Δ ABC = 2 rs (IV iv, Ex 9), · DE = $2r$, but $2r$ is given, DE is given, and because the ∠ ECD is given (hyp), and the ∠ E is right, the Δ ECD is given in species, . the ratio ED DC is given, but ED is given, DC is given, D is a given point

Again, because the ∠ ODC is right, and = ECD + CDE, ODG = ECD Hence ODG is given, and OGD is right, the Δ OGD is given in species, the ratio OD DG is given, but OD = OH = GE, the ratio EG GD is given, but ED is given, · EG, that is OD, is given, and the point D has been shown to be given Hence the ○, with O as centre, and OD as radius, is a fixed ○, and the ○ about ABC touches it in L

74 Let AC, BC be the two sides given in position

Sol —Bisect the ∠ ACB by FF′ In CF find a point F, such that $CF^2 = CA$ CB F is one of the required points

Dem —Join AF, BF, and let fall a ⊥ BE on AC Now



because the area of the △ ACB is given, CA EB is given, and since the ∠ BCE is given, and the ∠ BEC is right, the △ BCE is given in species, the ratio CB BE is given, the ratio CB CA BE CA is given, but CB $CA = CF^2$ (const), and BE CA is given, CF^2 is given, CF is given, and F is a given point Again, because CA $CB = CF^2$, CA CF CF CB, and the ∠ ACF = BCF, (vi) the ∠ CFA = CBF To each add the sum of the ∠s CFB, BCF, and we have the sum of the ∠s of the △ CBF equal to the ∠s AFB and BCF, AFB and BCF are equal to two right ∠s, but the ∠ BCF is given, AFB is given Hence the base AB subtends a constant ∠ at a given point F In like manner it can be shown that it subtends a constant ∠ at F′, constructed by making CF = FC

75 Let ABCD be the cyclic quad (See Diagram, Ex 67)

Dem —Draw the diagonal AC Produce AB, and let fall the ⊥s AF, CE on CD, AB

Now, since the sides AB, BC, CD, DA are denoted by a, b, c, d, we have (II xii) $AC^2 = a^2 + b^2 + 2a$ BE and (II xiii) $AC^2 = c^2 + d^2 - 2c$ DF, $c^2 + d^2 - 2c$ DF $= a^2 + b^2 + 2a$ BE,

. $c^2 + d^2 - (a^2 + b^2) = 2a$ BE $+ 2c$ DF, and because the Δs BCE, ADF are equiangular, BC BE AD DF, that is,

b BE d DF, b DF $= d$ BE, DF $= \frac{d}{b}$ BE, and

hence we have $c^2 + d^2 - (a^2 + b^2) = 2a$ BE $+ \frac{2cd}{b}$ BE

$$= \frac{2(ab+cd)}{b} \text{ BE}, \quad \text{BE} = \frac{b\{(c^2+d^2) - (a^2+b^2)\}}{2(ab+cd)}.$$

Again, $CE^2 = BC^2 - BE^2 = b^2 - \frac{b^2\{(c^2+d^2)-(a^2+b^2)\}^2}{4(ab+cd)^2}$

$$= b^2\left\{1 - \frac{\{(c^2+d^2)-(a^2+b^2)\}^2}{4(ab+cd)^2}\right\}$$

$$= b^2 \quad \frac{4(ab+cd)^2 - \{(c^2+d^2)-(a^2+b^2)\}^2}{4(ab+cd)^2}$$

$$= b^2 \quad \frac{\{(c+d)^2-(a-b)^2\}\{(a+b)^2-(c-d)^2\}}{4(ab+cd)^2}$$

$$= b^2 \quad \frac{\{(c+d+a-b)(c+d-a+b)(a+b+c-d)(a+b-c+d)\}}{4(ab+cd)^2}.$$

Hence, putting $(a+b+c+d) = 2s$, and substituting, we get

$$CE^2 = \frac{16b^2 \ (s-a)(s-b)(s-c)(s-d)}{4(ab+cd)^2},$$

$$CE = \frac{2b\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}$$

Now AB $= a$, and AB CE $= 2\Delta$ ABC

$$2ABC = \frac{2ab\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ab+cd)},$$

$$ABC = \frac{ab\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}$$

Similarly, ACD $= \frac{cd\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ab+cd)}$

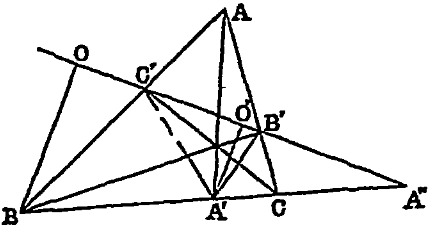
Hence the quadrilateral ABCD

$$= \frac{(ab+cd)\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ab+cd)}$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

76 Dem —Produce BC, C B′ to meet in A″ Let fall ⊥s A O′, BO on A″C′

Now AB′ BC′ CA′ = A′B B′C C A (Ex 4), and AB′ BC′ CA″ = A″B B′C C′A (Ex 5) Divide, and we get $\frac{CA'}{CA''} = \frac{A'B}{A''B}$; A″B A′C = A″C A′B, A′ B A′C + A″C . A′B = 2A″B A′C, that is ("Sequel," Book II, Prop VII), A″A′ CB = 2A″B A′C.



Now the Δ ABC ABB′ AC AB′ (I), and ABB′ BC′B′ AB BC, and BC B′ A′B C′ BO A′O′ BA″ A′A″, that is, BC 2A C, since A A′ . CB = 2A″B A′C, Δ ABC A′B C AB BC CA 2AB′ BC′ CA

77 Dem —Draw the diameter AE Join BE, and let fall a ⊥ AD on BC Now (XVII, Ex. 5) AE AD = AB AC, AE AD BC = AB BC CA, but AD BC = twice the Δ ABC, 2AE ABC = AB BC CA, hence (Ex 76) ABC A′B′C′ 2AE ABC 2AB′ BC CA′, 1 A′B C . AE AB BC CA′, AE A′B C′ = AB′ BC′ CA, and hence

$$AE = \frac{AB' \ BC' \ CA'}{A B C}$$

78 Dem —Let the sides of the quad be denoted by a, b, c, d Now (III XVII, Ex 3) $(a+c) = (b+d)$, $2(a+c) = (a+b+c+d)$ Hence, putting $(a+b+c+d) = 2s$, we have

$2(a+c)=2s$, $(a+c)=s$, $a=(s-c)$ Similarly, $b=(s-d)$, $c=(s-a)$, $d=(s-b)$, and (Ex 75), we have area of quad $=\sqrt{(s-a)(s-b)(s-c)(s-d)}$, area $=\sqrt{abcd}$ Hence the square of the area $= abcd$

79 Dem —Join BF, CF, BE Let the ratio BD AD be denoted by m n Now the ΔABC ABE AC AE (I) AB BD (hyp), that is, as $(m+n)$ m, and ABE BDE $(m+n)$ m, and BDE BDF $(m+n)$ m Multiplying together, we have ABC BDF $(m+n)^3$ m^3, hence BDF $=\frac{\text{ABC}\ m^3}{(m+n)^3}$ In like manner ECF $=\frac{\text{ABC}\ n^3}{(m+n)^3}$ Again (XXIII, Ex 1), ABC . ADE $(m+n)^2$ mn, ADE $=\frac{\text{ABC}\ mn}{(m+n)^2}$

Now the Δ BFC = ABC − BDF − CEF − ADE =

$$\text{ABC}\left\{1-\frac{m^3}{(m+n)^3}-\frac{n^3}{(m+n)^3}-\frac{mn}{(m+n)^2}\right\}=\text{ABC}\ \frac{2mn}{(m+n)^2}$$

Hence the Δ BFC = twice the Δ ADE

80 Let ABCD be a quad Join AC, BD, and bisect them in E, F Through E, F draw EG, FG ∥ respectively to BD, AC Bisect AD, CD in H, I Join GH, GI It is required to prove GIDH = ½ ABCD

Dem —Join HF, IF, IH Now, because AD, BD are bisected in H, F, HF is ∥ to AB, and the Δ DHF = ¼ ADB (I XL, Ex 2) In like manner, DFI = ¼ DBC, DHFI = ¼ ABCD Again, HI is ∥ to AC, and FG is ∥ to AC, HI is ∥ to FG, (I XXXVII) the Δ HFI = HGI To each add HDI, and HDIF = HGID, HGID = ¼ ABCD In like manner, if we bisect BC in J, and join GJ, GICJ = ¼ ABCD, &c

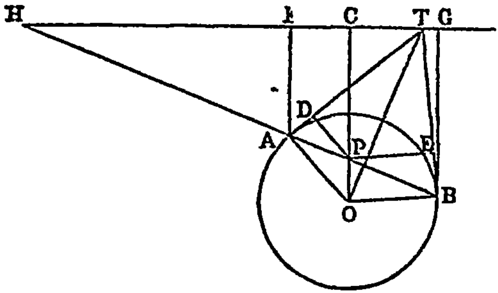
81 Dem —Let O, O be the centres of the ○s touching the semicircle internally and externally respectively, and also touching CE, DF Join OO′, and produce it to meet AB in G, O′G is evidently ⊥ to AB Complete the ○ on AB, and produce EC, FD to meet it again in H, I

Now AC DB = OG² (XIII, Ex 5), and AD . CB = O′G² (XIII, Ex 7), hence AC CB AD DB = OG² O′G², but AC CB = CE², and AD DB = DF², therefore CE² DF² = OG² O′G² And hence CE DF = OG O G

82 Let ABCDE be the inscribed regular polygon Take any point P in the circumference Join PA, PB, PC, PD, PE, and let those lines be denoted by ρ_1, ρ_2, ρ_3, ρ_4, ρ_5 It is required to prove that $\rho_1 + \rho_3 + \rho_5 = \rho_2 + \rho_4$

Dem —Join BD Let the sides of the polygon be denoted by s, and the diagonals by d Now, considering the polygon ABDP formed by ρ_1, ρ_2, ρ_4, we have (xvii, Ex 13) $\rho_1 d + \rho_4 s = \rho_2 d$. Similarly, we have $\rho_4 d = \rho_3 s + \rho_5 s$, and $\rho_5 d + \rho_3 s = \rho_4 d$ Adding, we get $(\rho_1 + \rho_3 + \rho_5)d = (\rho_2 + \rho_4)d$ Hence $\rho_1 + \rho_3 + \rho_5 = \rho_2 + \rho_4$

83 Let O be the centre of the given ○, P the given point, AB any chord passing through P, PD, PE ⊥s on the tangents AT, BT It is required to prove that the sum of the reciprocals of PD, PE is constant



Dem —Join OP, produce it, and from T let fall the ⊥ TC on OP produced Produce BA to meet TC in H, and let fall the ⊥s AF, BG

Now ("Sequel," Book III, Prop xxviii) CT is the polar of P, and AT is the polar of A Hence ("Sequel," Book III, Prop xxvii.) since PD and AF are ⊥s on the polars, OA : OP :: AF : PD, therefore $\frac{1}{PD} = \frac{OA}{OP} \cdot \frac{1}{AF}$

In like manner, $$\frac{1}{PE} = \frac{OB}{OP} \cdot \frac{1}{BG}$$

Hence, denoting the radius of the ○ by r, and the distance OP by d, we have

$$\frac{1}{PD} + \frac{1}{PE} = \frac{r}{d} \left(\frac{1}{AF} + \frac{1}{BG}\right)$$

Again, since P is the pole of the line GH, the line HB is cut harmonically, HP is a harmonic mean between HA and HB, but AF, PC, BG are proportional to HA, HP, HB, hence PC is a harmonic mean between AF and BG,

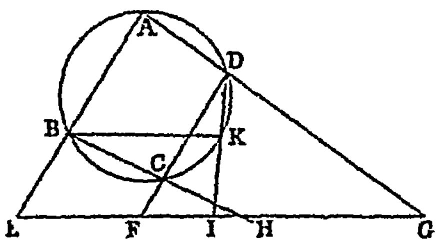
$$\frac{2}{PC} = \frac{1}{AF} + \frac{1}{BG}, \quad \frac{1}{PD} + \frac{1}{PE} = \frac{r}{d} \cdot \frac{2}{PC}.$$

Hence the proposition is proved

84 Let ABCD be a cyclic quad, whose sides AB, CD, AD pass through three collinear points E, F, G Join BC, and produce it to meet EG in H It is required to prove that H is a fixed point

Dem —Through B draw BK ∥ to EG Join DK, and produce it to meet EG

Now the ∠s ADK, ABK equal two right ∠s (III xxii), but ABK = AEG (I xxix), AEI and ADI are equal to two right ∠s, hence AEID is a cyclic quad, EG GI = AG GD, but AG GD is given, ∴ EG GI is given, and EG is given, GI

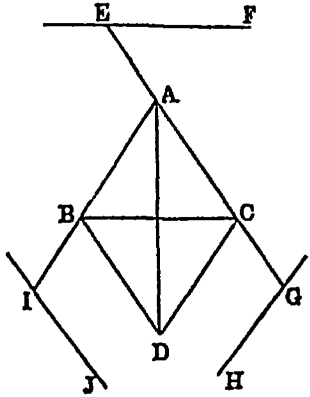


is given, I is a given point Again, the ∠ IDF = KBC (III. xxi), but KBC = CHF, IDF = CHF, and the points D, C, I, H are concyclic, hence DF FC = HF FI, but DF FC is given, HF FI is given, and FI is given, FH is given And hence H is a given point

85 (1) Suppose the polygon to be a △ Let BCD be a △ whose sides are ∥ to three given lines EF, GH, IJ, and let the loci of its angular points B, C, be right lines AB, AC It is required to prove that the locus of D is a right line

Dem —Join AD Produce CA to meet EF in F

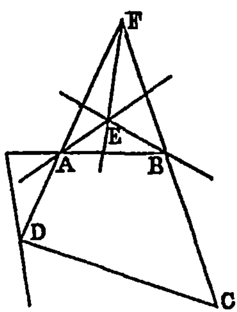
Now the ∠ BCA = FEA, BCA is a given ∠, and the ∠ BAC is given, since the lines AB, AC are given in position, hence the $\triangle$ ACB is given in species, the ratio AC CB is given



Similarly, the ratio BC CD is given, . the ratio AC CD is given, and the $\angle$ ACD is given, hence the $\triangle$ ACD is given in species, the $\angle$ CAD is given, and the line AC is given in position, therefore the line AD is given in position Hence the line AD is the locus of D

(2) Let the polygon be the quad ABCD, having its sides ∥ to four given lines, and the loci of the $\angle$s A, B, D right lines

Dem —Let the loci of A, B meet in E Produce DA, CB to meet in F. Join EF



Now AFB is a $\triangle$, whose three sides are ∥ to three given lines, and the loci of A, B are right lines Hence (1) the locus of F is the line EF, which is therefore given in position

Again, DFC is a Δ, having its sides ∥ to three given lines, and having straight lines for the loci of D and F Hence—(1) the locus of C is a right line In like manner it can be proved for a figure of any number of sides

86 Let BAC be a Δ whose vertical ∠ BAC and its bisector AD are given It is required to prove that $\frac{1}{AC} + \frac{1}{AB}$ is given

Dem —Describe a ○ about ABC Produce AD to meet the circumference in E Join EC, and let fall a ⊥ EF on AB

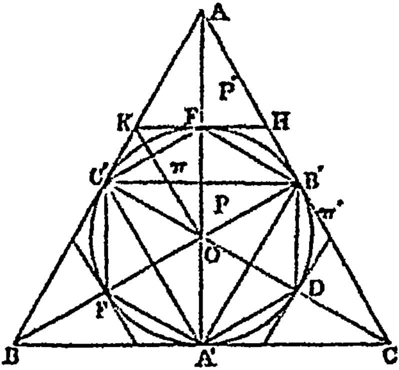
Now AF = ½ (AB + AC) (III xxx, Ex 4) And since the ∠ BAC is bisected by AE, FAE is a given ∠, and the ∠ AFE is right, the Δ AFE is given in species, $\frac{AF}{AE}$ is given, $\frac{2AF}{AE}$, that is, $\frac{AB + AC}{AE}$ is given, and AD is given (hyp), $\frac{AB + AC}{AD \quad AE}$ is given Again, the ∠ ABC = AEC (III xxi), and BAD = CAE, the Δ• BAD, CAE are equiangular, AB AD AE AC, hence AB AC = AD AE, $\frac{AB + AC}{AB \quad AC}$ is given, that is, $\frac{AB}{AB \quad AC} + \frac{AC}{AB \quad AC}$ is given Hence $\frac{1}{AC} + \frac{1}{AB}$ is given

87 (1) Let the polygons be the Δ• A′B′C′, ABC Bisect the arcs A′B′, B′C, C′A in the points D, E, F Join A D, DB′, B E, EC′, C′F, FA′ This hexagon is the corresponding polygon of double the number of sides It is required to prove that the hexagon is a geometric mean between the Δ• ABC, A′B′C′

Dem —Join AO, A′O, BO, B O, CO, C O Let OC intersect A′B′ in N

Now we have the Δ OB′C OB D OC OD (I), and OB′D OB′N OD ON, but OC OD OD ON, hence OB′C OB′D OB′D : OB′N, that is, the Δ OB D is a geometric mean between the Δ• OB′C, OB′N, but the hexagon is six times OB′D, ABC six times OB′C, and A′B′C′ six times OB′N Hence, denoting the areas by P, P′, Π, we see that Π is a geometric mean between P and P′

(2) At the points D, E, F draw tangents to the ○, the figure, whose sides are those tangents, and the parts cut off by them



from the sides AC, CB, BA, is a circumscribed polygon of double the number of sides

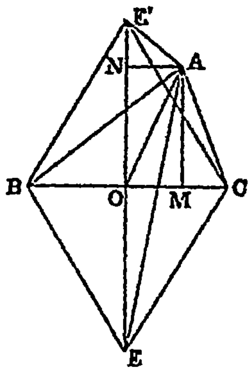
Dem —Join OK Now, since A'C is ∥ to OK, AO OA' AK KC, but OA' = OE, AO EO AK KC Again (1), the △ AOC EOC AO EO, and AKE EKC AK KC, AOC' EOC AKE EKC' Now consider the figures AOC', OEKC', and OEC AOC is the first, OEC' the third, and OEKC the second, and we have shown AOC EOC AKE EKC, that is, the 1st 3rd (1st − 2nd) (2nd − 3rd), OEKC' is a harmonic mean between OEC' and AOC', but OEKC is $\frac{1}{2}$ of π', OEC' is $\frac{1}{6}$ of π, and AOC' $\frac{1}{6}$ of P Hence π' is a harmonic mean between π and P In the same manner the proposition may be proved for a polygon of any number of sides

88 *Lemma* —If upon the base BC of a △ ABC two equilateral △s BCE, BCE' be described on opposite sides, and their vertices E, E' joined to A, then (1) if S denote the area of ABC, $AE'^2 - AE^2 = 4 S \sqrt{3}$, (2) $AE^2 + AE'^2 = AB^2 + BC^2 + CA^2$

(1) Dem —Join EE', intersecting BC in O Join AO, and draw AM, AN ⊥ to BC, EE' Now $AE'^2 - AE^2 = E'N^2 - NE^2 = 4EO \cdot ON = 4\sqrt{3}\ OC \cdot ON$, but OC ON = area of the △ ABC = S, $AE'^2 - AE^2 = 4\sqrt{3}\ S$

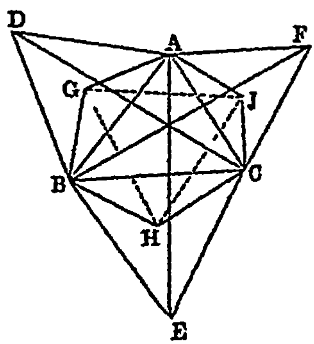
(2) $AE^2 + AE'^2 = 2 AO^2 + 2OE^2 = 2 AO^2 + 6 OC^2$ Again, $AB^2 + AC^2 = 2 AO^2 + 2 OC^2$, and $BC^2 = 4 OC^2$, $AB^2 + BC^2$

$+ OA^2 = 2\,AO^2 + 6\,OC^2$ Hence $AE^2 + AE'^2 = AB^2 + BC^2 + CA^2$



Let ABC be the Δ G, H, J the circumcentres of the equilateral Δˢ constructed outwards on its sides Join AG, AJ, BG, BH, CJ, CH, and GH, HJ, JG

Now the ∠ EBH = ABG, because each is half an ∠ of an



equilateral Δ, to each add HBA, and we have the ∠ EBA = HBG

Again, $EB^2 = 3\,BH^2$, and $AB^2 = 3\,BG^2$,   EB BA   BH BG   Hence the Δˢ EBA, HBG are equiangular, EB^2 EA^2 BH^2 HG^2, but $EB^2 = 3\,BH^2$, $EA^2 = 3\,GH^2$

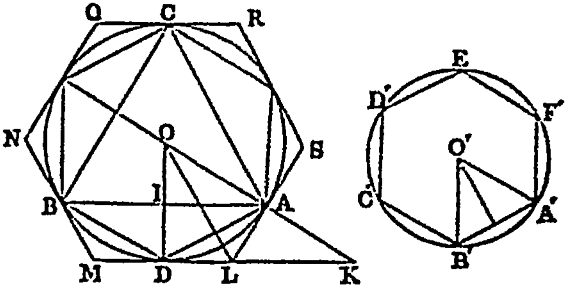
In like manner it may be proved, if G′, H′, J′ be the circumcentres of the equilateral △s constructed inwards on the sides of ABC, that $AE'^2 = 3\,G'H'^2$. Hence $AE^2 - AE'^2 = 3\,(GH^2 - G'H'^2)$.

Again, denoting the areas of the equilateral △s GHJ, G′H′J′ by Σ, Σ′, we have $\Sigma = \frac{GH^2\sqrt{3}}{4}$, $\Sigma' = \frac{G'H'^2\sqrt{3}}{4}$, ∴ $4\sqrt{3}\,(\Sigma - \Sigma') = 3\,(GH^2 - G'H'^2)$, but $4\sqrt{3}\,S = AE^2 - AE'^2$ (*Lemma*), ∴ $\Sigma - \Sigma' = S$.

89 From last demonstration we have $AE^2 + AE'^2 = 3(GH^2 + G'H'^2)$, but $AE^2 + AE'^2 = AB^2 + BC^2 + CA^2$ (*Lemma*), ∴ $3(GH^2 + G'H'^2) = AB^2 + BC^2 + CA^2$, or the sum of the squares of the sides of the two equilateral △s GHJ, G′H′J′ is equal to the sum of the squares of the sides of the △ ABC.

90 (1) Let ABC be a regular polygon of three sides, the radii of whose circumscribed and inscribed ⊙s are denoted by R, r, A′B′C′D′E′F′ a regular polygon of the same area, and double the number of sides, the radii of whose circumscribed and inscribed ⊙s are R′, r′. It is required to prove that $R' = \sqrt{Rr}$.

Dem.—Join OA (R), O′A′ (R′), and let fall a ⊥ OI (r) on AB. Produce OI to meet the ⊙ in D. Join AD, BD, O′B′. Now (i.) the △ OAD : OAI :: OD : OI, that is, as R : r, but OAI



= O′A′B′, ∴ OAD : O′A′B′ :: R : r, but (xix.) OAD : O′A′B′ :: OA² : O′A′², that is, as R² : R′², hence R : r :: R² : R′², ∴ $RR'^2 = R^2 r$, ∴ $R'^2 = Rr$. And hence $R' = \sqrt{Rr}$.

(2) It is required to prove that $r' = \sqrt{\frac{r(R+r)}{2}}$.

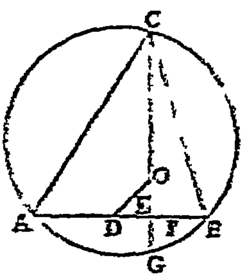
gents SR, NQ, EQ. Join OL.

Now OA : OI :: OK : OD; but OK : OD :: EL : LD; ∴ OA : OI :: EL : LD; that is, $R : r :: EL : LD$, ∴ $(R - r) : r :: ED : LD$; and ED : LD :: △OED : OLD; ∴ $(R - r) : r :: OED : OLD$; ∴ $(R - r) r : 2r'^2 :: OED : 2OLD$, or OALD. Again (xx.), $r'^2 : R^2 :: OAI : OED$. Hence, multiplying these proportions, we get $(R - r) r : 2R^2 :: OAI : OALD$, but OAI : OALD :: ABC : LMNQRS, that is, OAI : OALD :: A'B'C'D'E'F' : LMNQRS; that is, as $r'^2 : R^2$, ∴ $(R - r) r : 2R^2 :: r'^2 : R^2$,

∴ $(R - r) r = 2r'^2$. Hence $r' = \sqrt{\dfrac{(R-r)r}{2}}$.

In the same way the proposition may be proved for a polygon of any number of sides.

91. *Dem.*—Let fall a ⊥ CE on AB; then CE = AB (hyp.). Describe a ⊙ about ABC, and produce CE to meet it in G. Let O be the orthocentre. Cut off BF = OE. Bisect AB in D. Join OD. Now since BF = OE, and AB = CE, ∴ AF = CO. Now $AF . FB + DF^2 = DB^2$ (II. v.); that is, $CO . OE + DF^2 = DB^2$. Again, $AE . EB + DE^2 = DB^2$; ∴ $CE . EG + DE^2 = DB^2$; ∴ $CE . EO + DE^2 = DB^2$; ∴ $(CO + OE) EO + DE^2 = DB^2$;



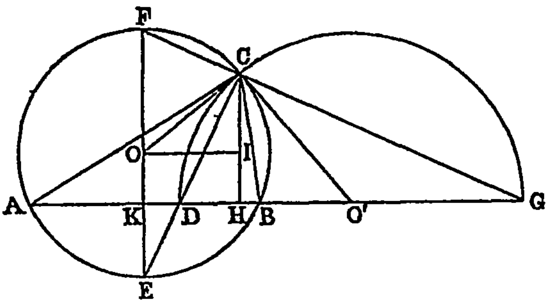
∴ $CO . EO + EO^2 + DE^2 = DB^2$, ∴ $CO . EO + OD^2 = DB^2$; ∴ $OD^2 = DF^2$, ∴ OD = DF, and OE = FB (const.). Hence OD + OE = DF + FB = DB.

92. Let ABC be any △. Describe a ⊙ about ABC. Draw the diameter EF ⊥ to AB. Join CE, CF; these are the internal

and external bisectors of the ∠ ACB Produce FC, AB to meet in G Let fall a ⊥ CH on AB, it is evident that the ○ on DG as diameter will be the locus of C when the base and ratio of the sides are given Let O, O′ be the centres Join OC, O′C It is required to prove that $AC^2 - CB^2$ 4 times area OC O′C

Dem —Through O draw OI ∥ to AB

Now the ∠ FOC = 2FEC (III xx), but FOC = OCI;



OCI = 2FEC, and CO′D = 2CGD Now the ∠ KDE = CDG, and DKE = DCG, KED = CGD, OCI = CO′H, and the right ∠ OIC = CHO′, the Δs OCI, O′CH are equiangular, OC O C OI CH, that is, OC O′C KH CH Again, $AC^2 - CB^2 = AH^2 - BH^2 = (AH + HB)(AH - HB) = 2AK$ 2KH = 4AK KH, but area of ABC = AK CH, four times area = 4AK CH, hence $AC^2 - CB^2$ 4 times area KH CH but KH CH OC O′C Hence $AC^2 - CB^2$ 4 times area OC O′C

Lemma —To construct a ▭, being given the diagonals and one of the ∠s

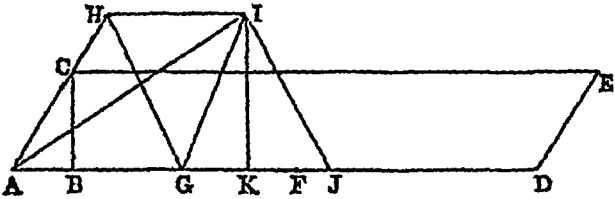
Sol —Let AB, CD be the diagonals, and E one of the ∠s On CD describe a segment CFD containing an ∠ equal to E Bisect CD in G With G as centre, and a radius equal to ½ AB, describe a ○, cutting CFD in F Join FG, and produce it to H Cut off GH = GF Join CF, DF, CH, DH CFDH is the required ▭, for it has the ∠ CFD = E, and its diagonal FH = AB

93 Let BAC be one of the ∠s, and AB the difference between its diagonals

Sol —Erect BC ⊥ to AB, to AC apply a ▭ ACED equal to four times the given area, and having BAC one of its ∠ˢ Bisect BD in F Construct a ▭ AHIG, having one of its diagonals, AI = AF, and the other, HG = FD, and the ∠ BAC for one of its ∠ˢ (*Lemma*) AHIG is the required ▭

Dem —Through I draw IJ ∥ to HG, and let fall a ⊥ IK on AD

Now $AI^2 = AG^2 + GI^2 + 2AG \cdot GK$ (II xii), and (II xiii) $IJ^2 = JG^2 + GI^2 - 2JG \cdot GK = AG^2 + GI^2 - 2AG \cdot GK$, ∴ $AI^2 - IJ^2 = 4AG \cdot GK$ Again, AB = AF − FD, and AD = AF + FD, ∴ $AB \cdot AD = AF^2 - FD^2$, but AF = AI, and FD = IJ, ∴ $AF^2 - FD^2 = AI^2 - IJ^2$, ∴ $AB \cdot AD = 4AG \cdot GK$ Again, since the △ˢ ABC, GKI are equiangular, we have AB : BC

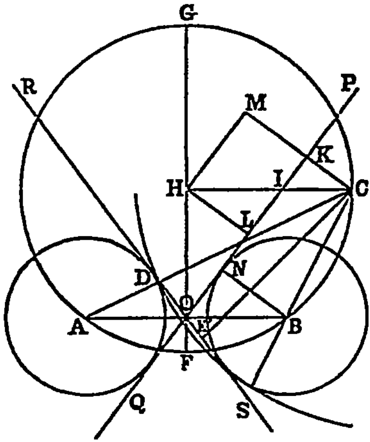


∷ GK : KI, ∴ AB · AD : BC · AD ∷ 4AG · GK : 4AG · KI, hence BC · AD = 4AG · KI Now BC · AD = ▭ AE, and 4AG · KI = 4 times ▭ AI, ∴ ▭ AE = 4 times ▭ AI, but AE = 4 times the given area (const) Hence AI is equal to the given area

94 Let A, B be the centres of two equal ○ˢ, C the centre of a variable ○, which is touched externally by A in D, and internally by B in E Let O be the point of intersection of two transverse tangents PQ, RS From C let fall ⊥ˢ CK, CK′ on PQ, RS It is required to prove that OK · OK′ is constant.

Dem —Join CB, and produce it to E Join CA. Describe a ○ passing through the points C, A, B Draw the diameter FG, passing through O, ⊥ to AB Let fall a ⊥ CH on FG. Produce CK, and draw HM ∥ to PQ Let fall a ⊥ HL on PQ Join BN, and let the sides BN, ON, OB of the △ ONB be denoted by *a*, *b*, *c*

Now AC = AD + DC, and BC = CE − BE, ∴ AC − BC = 2AD, ∴ AD = ½ (AC − BC), that is, *a* = ½ (AC − BC), hence (iv, Ex. 16) $a^2 = OF\ \ GH$, and since AB is bisected in O, AO = OB, and AO OB = OF OG, OB^2, that is, $c^2 = OF\ \ OG$, and



ON^2, that is, $b^2 = OF\ \ OH$ Now since the Δ' ONB, HMC are equiangular, and that HM = LK, we have $c\ \ b\ \ HC\ \ LK$,

$$LK = \frac{b\ \ HC}{c} \quad \text{In like manner } OL = \frac{a\ \ OH}{c},$$

$$\cdot\ OK = \frac{b\ \ HC}{c} + \frac{a\ \ OH}{c}$$

Similarly,
$$OK' = \frac{b\ \ HC}{c} - \frac{a\ \ OH}{c},$$

$$OK\ \ OK' = \frac{b^2\ \ HC^2}{c^2} - \frac{a^2\ \ OH^2}{c^2} = \frac{OF\ \ OH\ \ FH\ \ HG - a^2\ \ OH^2}{c^2}$$

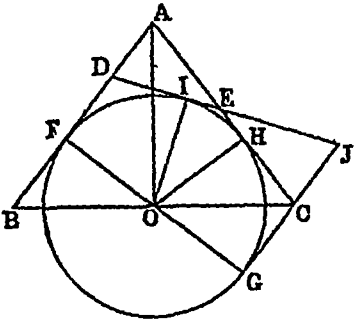
$$= \frac{a^2\ \ OH\ \ FH - a^2\ \ OH^2}{c^2} = \frac{a^2\ \ OH\,(FH - OH)}{c^2} = \frac{a^2 b^2}{c^2},$$

but a^2 is constant, since a is the radius of the ○, and c^2 is constant, because c is half the base of the Δ ACB, $\frac{a^2b^2}{c^2}$ is constant Hence OK OK′ is constant

95 *Analysis*—Let ABC be a Δ whose base AB is given in magnitude and position, and vertical ∠ C is given in magnitude, and P the given point in AB, whose distance CP from the vertex is equal to ½ (AC + CB) Describe a ○ about the Δ ACB Bisect the ∠ ACB by CD Let fall a ⊥ DE on AB, then, be-

cause AB and the ∠ ACB are given, the ○ is given, and since the ∠ ACB is bisected by CD, the arc AB is bisected in D, hence D is a given point. Again, because the ∠ ACB is given, its half, the ∠ DCE, is given, and the ∠ DEC is right, hence the △ DCE is given in species, the ratio of DC CE is given, but CE = CP, because each is equal to $\frac{1}{2}$ (AC + CB), hence the ratio of DC CP is given, and the points D, P are given Hence the locus of the point C is a circle, and therefore the point C, where this locus cuts the ○ ACB, is given

96 Let O be the middle point of the base, F, H the points of contact of AB, AC with the ○ Join OF, OH. Produce FO to

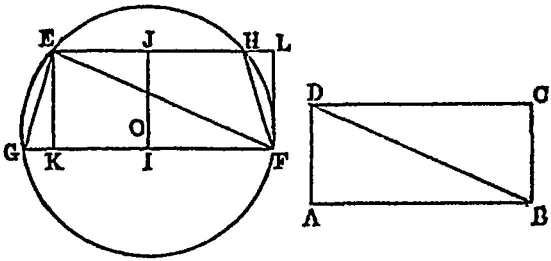


meet the ○ in G Join CG, then, since OC = OB, and OG = OF, and the ∠ COG = BOF, CG is equal to BF, and the ∠ OGC = OFB, and is therefore a right ∠, hence CG is a tangent Again, because the ∠ AOC is right, and OH is ⊥ to AC, AH HC = OH^2, but AH = AF, and HC = CG, hence AF CG = OH^2 In like manner, if I be the point of contact of DE with the ○, DF JG = OI^2, but $OH^2 = OI^2$, AF CG = DF JG, hence AF DF JG CG, AD DF JC CG, AD JC DF CG or FB, but, by similar △s, AD JC AE EC, AE EC DF FB, hence, *componendo*, AC CE DB FB, hence AC BF = BD CE, but AC and BF are each given, the rectangle BD CE is given

97 Let AB equal half the sum of the opposite sides, and the area equal the rectangle ABCD

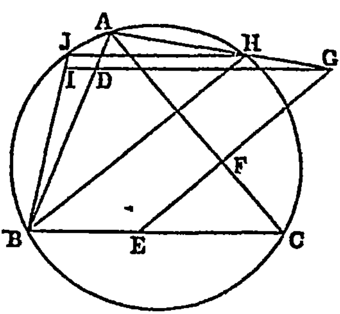
Sol —Join BD, and in the ○ place EF = BD At the point F in EF make the ∠ EFG = ABD Join EG, and draw EH ∥ to FG EHFG is the required trapezium

Dem.—From the centre O let fall a ⊥ OI on FG, and produce o meet EH in J Let fall a ⊥ EK on FG Produce EH, draw FL ∥ to EK Because EF = BD, and the ∠ EFK



BA, and the right ∠ EKF = DAB, FK = AB, 2AB = ' + 2IK, that is = FG + EH Again, the ∠ˢ EGF and F equal two right ∠ˢ, and EHF, LHF equal two right ∠ˢ, EGK = LHF, and the right ∠ EKG = HLF, and the side = FL, the Δˢ EGK, FLH are equal To each add figure EHFK, and EHFG = ELFK Hence EHFG = CD

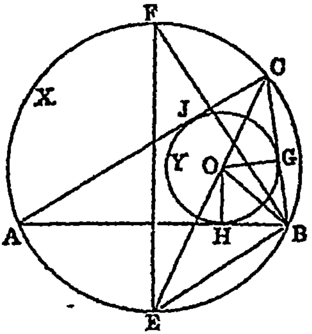
8 *Analysis* —Let the polygon be the Δ ABC, whose sides through the points D, E, F Join EF, and produce it



ough B draw BH ∥ to EF Join AH, and produce it to t EF in G Now the ∠ GAC = HBC (III xxi), and HBC

= GEO (I xxix), GEO = GAO , GAEC is a cyclic quad , EF FG = AF FC, but AF FC is given , EF FG is given, and EF is given , hence G is a given point Join GD, and produce it. Through H draw HJ ∥ to GD Join JB Now the ∠ AHJ = ABJ, and AHJ = AGI, ABJ = AGI, AGBI is a cyclic quad , hence GD DI = AD DB, and is given , but GD is given, DI is given, and I is a given point , and since JH, BH are respectively ∥ to IG, EG, the ∠ JHB = IGE , but IGE is given, since the lines IG, EG are given in position, the ∠ JHB is given, the arc JB is given, the chord JB is given, and we have shown that I is a given point Hence the question reduces to III xv , Ex 2 Similarly for a polygon of any number of sides

99 Let the O• X, Y be so related that the rectangle contained by the diameter of X, and the radius of Y, is equal to the rectangle contained by the segments of any chord of X passing



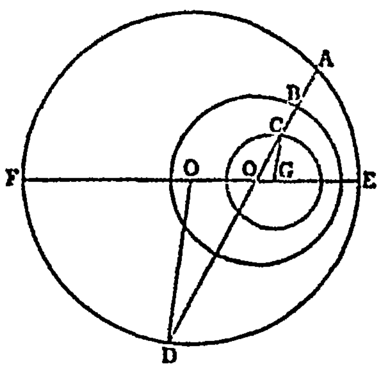
through the centre of Y ,-then, if from any point in the circumference of X we draw tangents CA, CB to Y, and join AB, it is required to prove that AB touches Y

Dem —Let O be the centre of Y. Join CO, and produce it to meet X in E Through E draw EF the diameter of X Join BE, BF, BO Join O to G, the point of contact, and let fall a ⊥ OH on AB Now the ∠ EFB = ECB (III xxi), and the right ∠ EBF = OGC, the Δ• EFB, OCG are equiangular, EF EB OC OG, EF OG = EB OC, but (hyp) EF OG = OC OE, EB = OE, the ∠EOB = EBO, EBO = OCB + OBC , that is, EBA + ABO = OCB + OBC , that is, ACE + ABO = OCB + OBC, but ACE = OCB, ∴ ABO = OBC, and the right ∠ OHB = OGB, and the side OB common, ∴ OH = OG, but OG is the radius, ∴ OH is the radius, and hence AB touches Y. Similarly, wherever we take the point in the circumference of X, and draw tangents to Y, the base will touch Y.

Lemma—If any point A is taken in the circumference of a ○, and A joined to O, the centre of another ○, and if we divide AO in C, so that OA . OC = r^4, r being the radius of O. It is required to prove that the locus of C is a ○.

Dem.—Suppose one ○ inside the other. Let O be the centre of the larger ○. Produce AO to meet O in D. Join DO′, OO′, and produce OO′ to meet O in E, F. Through C draw CG ∥ to DO′.

Now OA . OC = r^2, and OA . OD = OE . OF, ∴ OD : OC :: OE . OF : r^2, but the ratio OE . OF : r^2 is given, since r is the



radius of a given ○, and OF . OF is a given rectangle, ∴ the ratio OD : OC is given. And because the Δs ODO′, OCG are equiangular, OD : OC :: OO : OG, ∴ the ratio OO : OG is given, but OO′ is given, ∴ OG is given, hence G is a given point. Again, OD : OC :: O′D : GC, ∴ the ratio O′D : GC is given, but O′D is given, since it is the radius of a given ○, ∴ GC is given, and we have shown that G is a given point. Hence the locus of C is a ○.

Def —The point C is called the *inverse* of the point A, and the ○ through C the *inverse* of the ○ through A with respect to the ○ through B

100 Let G, H, J be the points where Y touches the sides of the △ ABC Join HG, GJ, JH It is required to prove that the ○ inscribed in the △ GHJ touches a given ○

Dem —Join OA, OB, OC, cutting JH, HG, GJ in L, M, N Then since L, M, N are the middle points of the sides of the △ GHJ, the ○ through these points will be the nine-points ○ of GHJ, and will (Ex 31) touch its in-○ Again, the ○ through LMN will evidently be the inverse of X with respect to Y (*Lemma*), and will be a given ○ Hence the in-○ of the △ GHJ touches a given ○

101 See "Sequel," Book VI, Prop XII, Sect IV, Cor 2

102 Sol —Let A, B, C be the given points, join them, and on AB, AC describe segments of ○s containing ∠s equal to one-third of four right ∠s Let them intersect in D D is the point required

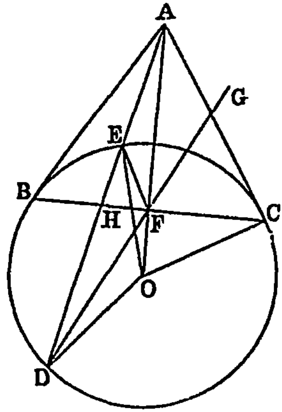
Dem —Join AD, BD, CD, and through D draw EF ⊥ to CD, meeting AC, BC in E, F Now the ∠ ADC = BDC, and EDC = FDC, ADE = BDF, hence ("Sequel," Book I, Prop XXI, Cor 1), the sum of AD and DB is a minimum, and CD, being a ⊥, is less than any other line from C to EF Hence the sum of the lines AD, BD, CD is a minimum

103 Let AB, AC be the tangents, and O the centre Join BC Join AO, cutting BC in F Through A draw AD, cutting the ○ in E, D, and BC in H It is required to prove that AD is divided harmonically

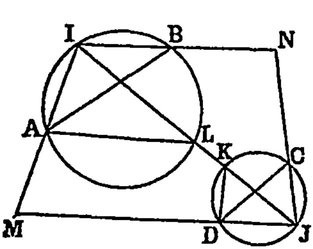
Dem —Join OC, OD, OE, OF Join DF, and produce it

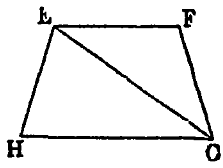
Now (I XLVII, Ex 1) AO . OF = $OC^2 = OD^2$, ∴ AO : OD :: OD : OF, and the ∠ AOD common, hence (VI) the ∠ ADO = OFD, but because OD = OE, the ∠ ODE = OED, OFD = OED, OFED is a cyclic quad, the ∠s EDO and EFO equal two right ∠s, but the ∠s EFO, EFA equal two right ∠s, EFA = EDO, and EDO = OFD, EFA = OFD, and AFB = OFB, DFH = EFH, hence the ∠ EFD is bisected internally by FH, and the ∠ OFD = AFG, and OFD = EFA, EFA = AFG Hence EFD is bisected externally by FA, and therefore ED (III, Ex 3) is divided harmonically in the points H, A

104 Let A, B, C, D be the four points, and EFGH the given quad It is required to construct a quad similar to EFGH whose sides shall pass through the points A, B, C, D



Sol —Join AB, and on it describe a segment AIB, containing an angle equal to FEH Join CD, and on it describe a segment



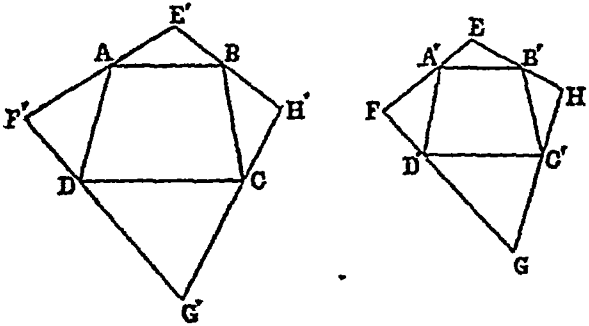


CJD, containing an ∠ equal to FGH Join EG At the point A in AB make the ∠ BAL = FEG, and at the point D in DC make the ∠ CDK = EGF Join KL, and produce it to meet the ⊙s in I, J Join IA, IB, and produce Join JC, JD, and produce INJM is the required quad

Dem —For the ∠ BIL = BAL = FEG, and the ∠ CJK = CDK = EGF, ∴ the △s INJ, EFG are similar And because the ∠ BIA = FEH, ∴ MIJ = HEG Similarly, MJI = HGE, ∴ the △s MIJ, HEJ are similar Hence the quads are similar

105 Let ABCD be the given quad, and EF, FG, GH, HE the given lines

Sol —Construct the quad E'F'G'H' similar to EFGH, whose sides pass through the points A, B, C, D (103) Divide EF in A', so that EA' A'F EA AF', and divide EH in B', so



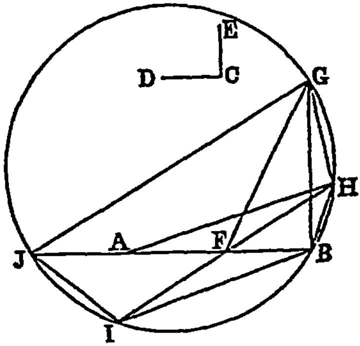
that EB' B'H E'B BH', and similarly for the other sides Join A'B', B'C', C'D', D'A' It is evident that A'B'C'D' is similar to ABCD

106 Let AB be the base, and DCE the difference of the base angles

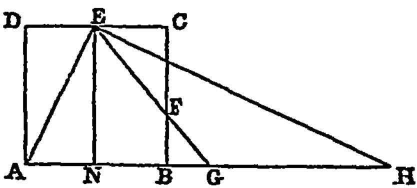
Sol —Bisect AB in F Draw BG, making the ∠ FBG = DCE, and the rectangle FB BG equal to the rectangle under the sides Join FG Bisect the ∠ BFG by FH, and make FH a mean proportional between FG and FB Join AH, BH ABH is the required Δ.

Dem —Produce HF to I, so that IF = FH Through G draw GJ ∥ to HI, and produce BA to meet it Join IJ, IB, GH Now (I xxix) the ∠ HFB = GJF, and GFH = FGJ, but HFB = GFH (const), GJF = FGJ, and FG = FJ Now the ∠ GHI = JIH To each add HGJ, and we have GHI + HGJ = JIH + HGJ; JIH + HGJ are equal to two right ∠s, hence HIJG is a cyclic quad And since FG FB = FH² (const), and FG = FJ, and FH² = FH FI, ∴ FJ FB = FH FI, JIBH is a cyclic quad Hence the five points F, I, B, H, G are in a ○. Now the ∠ HBG = IBJ, but IBJ = BAH, HBG = BAH; ∴ FBG, that is DCE, is the difference between HAB and HBA, Again, the Δs IBF, GBH are equiangular, IB BF GB

BH, IB BH = BF BG, that is, AH BH = BF BG This construction is due to HAMILTON



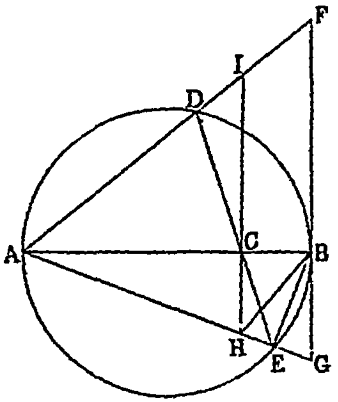
107 Let the line EF produced meet AB produced in G, cut off GH = EG Join EH, and let fall the ⊥ EN Now since (hyp) the ∠ AEF = EAB, the Δ AEG is isosceles, AG = EG,



and EG = GH, hence the ∠ AEH is right, AN NH = EN², but EN = 2 AN, since CD is bisected, .. NH = 2 EN = 2 AB, . AH $= \frac{5\,AB}{2}$, hence AG $= \frac{5\,AB}{4}$, BG $= \frac{AB}{4}$ Hence EC = 2 BG, CF = 2 FB

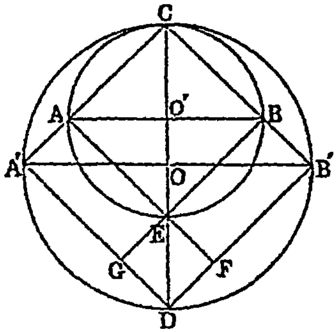
108 Let C be a fixed point in the diameter AB, DE a chord passing through C Join AD, AE At B draw FG a tangent to the ○, and produce AD, AE to meet it in F and G It is required to prove that BF BG is constant.

Dem —*Through C draw HI ∥ to BG*, meeting AF, AG in I, H Join BE, BH Now the ∠ BCH is right, and BEH is right, CBEH is a cyclic quad, the ∠ BEC = BHC, but BEC = BAD (III xxi), BHC = BAD, the four points B, H, A, I are concyclic, hence IC CH = AC CB, and because the Δs ACI, ABF are equiangular, AC AB IC BF, and since the Δs ACH, ABG are equiangular, we have AC AB CH BG, AC^2 AB^2 IC CH BF BG, that is AC^2



AB^2 AC CB BF BG, but the first three terms of this proportion are constant Hence the fourth, BF BG is constant

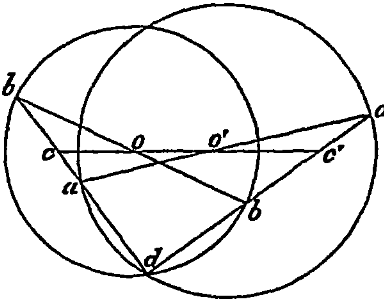
109 Let O, O′ be the centres of the ⊙s, and C the point of contact



Dem —Join OO′, and produce it, OO′ must pass through C Let E, D be the other points in which it meets the ⊙s Join AD, BD, AE, BE, and let AE, BE meet B D, A D in F, G Now each of the ∠s CBE, CB D is right (III xxxi), BG is ∥ to B′D, and BB′ = GD In like manner AA′ = GE, $AA'^2 + BB'^2 = GE^2 + GD^2 = DE^2$, but DE^2 is constant, since it is equal to the square of the difference of the diameters Hence $AA'^2 + BB'^2$ is constant.

110 Let *d* be the point of intersection

Dem —Join *aa*, *bb'*, those lines must pass respectively through



the centres *o'*, *o* (hyp) Now the sides of the Δ *bdb'* are cut by *cc* in the points *c*, *o*, *c'*, hence (VI, Ex 5),

$$\frac{dc}{cb} \quad \frac{bo}{ob'} \quad \frac{b'c'}{c'd} = 1, \text{ but } bo = ob', \quad \frac{dc}{cb} \quad \frac{b'c'}{c'd} = 1, \quad \frac{dc}{cb} = \frac{c'd}{b'c'}$$

In like manner, from the Δ *ada'*, we get

$$\frac{dc}{ca} = \frac{dc'}{a'c'}, \quad \frac{ca}{cb} = \frac{a'c'}{b'c'},$$

that is, *ac* *cb* *a'c'* *b'c'* Hence *ab* *cb* *a'b'* *b'c'*

111 "Sequel," Diagram, p 32 By "Sequel," Prop VIII, Cor 3, p 32, we have AB QR = EP^2 Similarly AB, multiplied by the diameter of the ○ touching EP, the semicircle ACB, and the semicircle on AP as diameter, is equal to EP^2 Hence the ○s are equal

112 Let P be the given point, AB the chord, and CA, CB the tangents

Dem —Let O be the centre Join OA, OB, OP, OC, PE Bisect OP in D Join DE

Now because OA = OB, OC common, and the base CA = CB, the ∠ AOC = BOC, and since AO = BO, OE common, and the ∠ AOE = BOE, the base AE = BE Now $AO^2 = AE^2 + EO^2$, but (I XII, Ex 2) the lines AE, EB, EP are equal, $AO^2 = OE^2 + EP^2 = 2\ OD^2 + 2\ DE^2$ (II X, Ex 2), but AO^2 is given, $2\ OD^2 + 2\ DE^2$ is given, and $2\ OD^2$ is given, since OP is given, DE is given, and D is a fixed point Hence the locus of E is a ○, having D as centre, and DE as radius Now (I XLVII, Ex 1) CO OE = $OA^2 = R^2$, C, E

$(a - b)$ or $2GD = BH = 2GE$, $GD = GE$ In like man GD = GF, and GD = GI, hence the lines GE, GD, GF, GI equal, and the ○, with G as centre, and GD as radius, will p through E, F, I Let it cut BD in K Now (III xxxv BD BK = BF BI But since AG = GB, and DG = GK, = KB Also BI = HC = AE Hence BD AD = BF AB

114 See "Sequel," Book VI, Prop x, Sect i, Cor 1

115 See "Sequel," Book VI, Prop x, Sect ii, Cor 2

116 *Analysis* —Let P be the required point Join AP, B CP, DP Now (hyp) the ∠ APC is bisected, AB AP PC (III), but the ratio AB BC is given, AP is given, and the base AC is given, hence (III, Ex 6) the loc of P is a ○ Similarly for the △ BPD, the locus of P another ○ Hence the point in which these ○s intersect is t point required

117 Let ABC be a △ whose sides are denoted by a, b, Bisect the ∠ ACB by CD, and let CD be denoted by γ Now (II we have a b BD DA, $(a + b)$ b BA AD, that $(a + b)$ b c AD, $AD = \dfrac{bc}{a+b}$ Similarly, $BD = \dfrac{ac}{a+}$

$$BD\ DA = \frac{abc^2}{(a+b)^2}, \text{ but } ab = BD\ DA + CD^2 \text{ (xvii, Ex 1}$$

$$ab = \frac{abc^2}{(a+b)^2} + CD^2, \text{ that is, } ab - \frac{abc^2}{(a+b)^2} = CD^2, \text{ that}$$

$$ab\left\{1 - \frac{c^2}{(a+b)^2}\right\} = CD^2, \text{ hence } \gamma^2 = ab\left\{\frac{(a+b)^2 - c^2}{(a+b)^2}\right\}$$

$$= \frac{ab(a+b+c)(a+b-c)}{(a+b)^2} = \frac{4ab\ s\ s-c}{(a+b)^2}$$

In like manner, denoting the bisectors of the ∠s A, B by α, respectively, we have

$$\alpha^2 = \frac{4bc\ s\ s-a}{(b+c)^2}, \text{ and } \beta^2 = \frac{4ca\ s\ s-b}{(c+a)^2},$$

hence

$$\alpha^2\beta^2\gamma^2 = \frac{64a^2b^2c^2\ s^2(s\ s-a\ s-b\ s-c)}{(a+b)^2(b+c)^2(c+a)^2} = \frac{64a^2b^2c^2\ s^2\ (\text{area})^2}{(a+b)^2(b+c)^2(c+a}$$

Hence, $$\alpha\beta\gamma = \frac{8abc\ s\ \text{area}}{(a+b)(b+c)(c+a)}$$

118 Let Aa', Bb, Cc be the bisectors of the ∠s, then (II we have

$$c\ a\ \ Ab\ \ bC, \quad c\ \ c+a\ \ Ab'\ \ b, \quad Ab = \frac{bc}{c+a}$$

In like manner,

$$Bc' = \frac{ca}{a+b}, \text{ and } Ca = \frac{cb}{b+c},$$

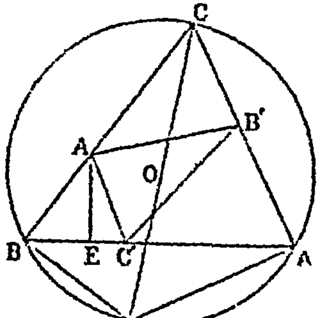
$$\therefore\ Ab' \cdot Bc' \cdot Ca' = \frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

119 Let ABC be a Δ Draw any three lines Aa, Bb, Cc, intersecting in D Describe a ○, passing through the points a, b, c, and cutting the sides of the Δ ABC in A′, B′, C′ It is required to prove that the lines AA′, BB′, CC′ are concurrent

Dem —Now we have Ab . AB′ = Ac . AC′, Bc . BC′ = Ba . BA′, and Ca . CA′ = Cb . CB′, ∴ (Ab . Bc . Ca) (AB′ . BC′ . CA′) = (aB . bC . cA) (A′B . B′C . C′A), but Ab . Bc . Ca = aB . bC . cA (Ex 4) ∴ AB′ . BC′ . CA′ = A′B . B′C . C′A And hence the lines AA′, BB′, CC′ are concurrent

120 Dem —Describe a ○ about ABC Let O be the centre Join CO, and produce it to meet the circumference in D Join DA, DB, and from A′ let fall a ⊥ A′E on AB

Now if we denote the sides by a, b, c, and the parts A′B, B′C, C′A, by x, y, z, we have $(a-x)(b-y)(c-z)$ = AB′ . BC′ . CA′, and xyz = A′B . B′C . C′A ∴ $abc - (abz + bcx + cay) + ayz + bzx + cxy$ = AB′ . BC′ . CA′ + A′B . B′C . C′A Again, since the Δs BA′E, ACD are equiangular, we have BA′ : A′E :: CD : CA,

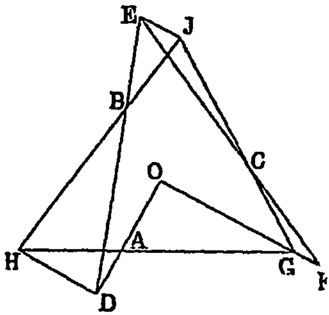


that is (denoting CD by δ), x : A′E :: δ : b, ∴ bx = δ . A′E,
∴ bx . BC′ = δ . A′E . BC′ = δ . 2 A′BC′, that is, $bx(c-z)$ = δ . 2 A′BC′, ∴ ($bcx - bxz$) = δ . 2A′BC′ in like manner ($cay - cxy$) = δ . 2 B′CA′ and ($abz - ayz$) = δ . 2 C′AB′, and "Sequel,"

(Book VI, Prop v, Sect 1) $abc = \delta$ 2 ABC, , $abc - (bcx + cay + abz) + (ayz + bzx + cxy) = \delta$ 2 A'B C' Hence AB' BC' CA = A'B B C C A = δ 2 A'B'C'

121 Let A, B, C be the fixed points, and the given ratio that of 2 1

Sol —Take any point O Join OA, and produce it to D, so that OA = 2 AD Join DB, and produce to E until DB = 2 BE Join EC, and produce it to F, so that EC = 2 CF Join OF, and divide it in G, so that OG = 8 FG Join GA, and produce it, and through D draw DH ∥ to OG Join HB, and produce it, and



through E draw EJ ∥ to HD Join JC, GC GHJ is the required Δ

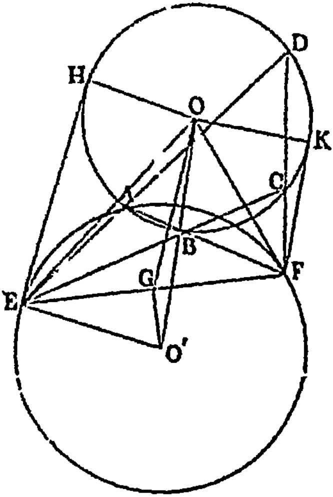
Dem —The Δs OAG, DAH are equiangular, OA AD OG DH, OG = 2 DH, but OG = 8 GF, DH = 4 GF Similarly, from the Δs BDH, BEJ we get DH = 2 JE, JE = 2 GF, and EC = 2 CF (const), and the ∠ JEC = GFC, hence (vi) the ∠ JCE = GCF, and therefore JC and GC are in the same straight line, and evidently the sides are divided in the points A, B, C in the given ratio Similarly for any polygon of an odd number of sides, and for any given ratio

122 Let ABCD be a cylic quad whose third diagonal EF is a chord of another given ○ Bisect EF in G It is required to prove that the locus of G is a ○

Dem —Let O, O' be the centres Join OG, O G, O'E From E, F draw tangents EH, FK to O Join OH, OK, EO, FO, OO'

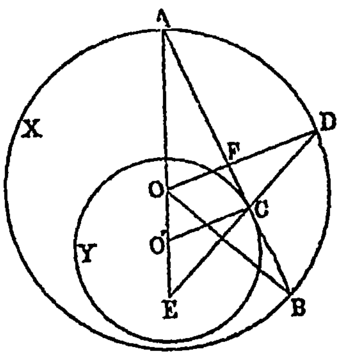
Now $4\,EG^2 + 4\,GO'^2 = 4\,EO'^2$, that is, $EF^2 + 4\,GO'^2 = 4\,EO'^2$, but $EF^2 = EH^2 + FK^2$ (III, Ex 19), and $OH^2 + OK^2 = 2\,OH^2$

Adding, we get $EO^2 + OF^2 + 4\,GO'^2 = 4\,EO'^2 + 2\,OH^2$, that is (II x, Ex 2), $2\,EG^2 + 2\,GO^2 + 4\,GO'^2 = 4\,EO'^2 + 2\,OH^2$, and $2\,EG^2 + 2\,GO'^2 = 2\,EO'^2$ Subtracting, we have $2\,GO^2 + 2\,GO'^2$



$= 2\,EO'^2 + 2\,OH^2$, $GO^2 + GO'^2 = EO'^2 + OH^2$, but EO'^2 and OH^2 are given, $GO^2 + GO'^2$ is given Therefore OGO' is a Δ whose base is given, and the sum of the squares of its sides Hence (II x, Ex 3) the locus of G is a ⊙

123 Let X, Y be the ⊙s, and let AB, a chord of X, touch Y



in C Bisect the arc AB in D Join DC It is required to prove that DC passes through a given point

Dem —Join OO, and produce OO′, DC to meet in E E is the given point Join OA, OB Now OA = OB, OF common, and the ∠ AOF = BOF, hence the ∠ AFO = BFO, the ∠ AFO is right, and FCO′ is right, OD is ∥ to O C, hence (II) the Δ• DOE, CO′E are equiangular, DO CO′ OE O′E, hence the ratio OE O E is given, the ratio OO O E is given, but OO is given, O′E is given, and O is a given point, E is a given point

124 Let ABC be a given Δ From a point P, within it, let fall ⊥• PD, PE, PF on the sides BC, CA, AB Join DE, EF, FD, and let the area of DEF be given It is required to prove that the locus of P is a ○

Dem —Join AP, BP, CP Because each of the ∠• AEP, AFP is right, AFPE is a cyclic quad Bisect AP in G G is the centre of the ○ Similarly, BDPF, CDPE are cyclic quads, and H, J, the middle points of BP, CP, are the centres of their circum-○• Join DH, HF, FG, GE, EJ, JD Produce FG, and let fall a ⊥ EK on it Because AG = GP, the Δ AGF = PGF, AFP = 2 PGF In like manner, AEP = 2 EGP, hence the quad AEPF = 2 EGFP Similarly, BFPD = 2 FHDP, and CDPE = 2 PDJE, hence the area of the figure EGFHDJ is given, but the area of FDE is given (hyp), hence the sums of the areas EGF, FHD, DJE is given Again, the ∠ FGE = 2 FAE (III xx), the ∠ FGE is given, the ∠ KGE is given, and the ∠ GKE is right, hence the Δ EGK is given in species, the ratio EG EK is given, the ratio EG FG EK FG is given, but EK FG = 2 Δ EGF, and EG FG = FG^2, the Δ EGF has a given ratio to FG^2, and FG^2 has a given ratio to AP^2, since AP = 2 FG, $\frac{EGF}{AP^2}$ is given Suppose it equal to l, hence EGF = l AP^2 In like manner, FHD = m BP^2, and DJE = n CP^2, but we have shown that the sum of EGF, FHD, DJE is given, hence l $AP^2 + m$ $BP^2 + n$ CP^2 is given And hence (*Lemma* to Ex. 60) the locus of P is a ○ Similarly, the proposition may be proved for a figure of any number of sides

THE END

"The book is all, and more than all, it professes to be . The propositions suggested are such as will be found to have most important applications, and the methods of proof are both simple and elegant We know no book which, within so moderate a compass, puts the student in possession of such valuable results

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of the book In most cases the proofs are extraordinarily neat The notes to the Sixth Book are the most satisfactory Feuerbach's Theorem (the nine-points circle touches inscribed and escribed circles) is favoured with two or three proofs, all of which are elegant Dr Hart's extension of it is extremely well proved . We shall have given sufficient commendation to the book when we say that the proofs of these (Malfatti's Problem, and Miquel's Theorem), and equally complex problems, which we used to shudder to attack, even by the powerful weapons of analysis, are easily and triumphantly accomplished by Pure Geometry

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From "MATHESIS," *December*, 1886

"La troisième édition, publice en 1884, et dont il a été rendu compte dans ce journal (t v 1885, p 76–78) fait aujourd'hui place a une quatrième, dont la première partie, renfermant 164 pages, 152 figures et 293 exercices, est la reproduction de six chapitres de la précédente édition, car l'auteur n'a pas eu le temps d'en remanier la substance, mais toute la seconde partie, 58 pages, 23 figures et 181 exercices, intitulée Géométrie récente du triangle, est absolument nouvelle et complétement refondue

"Elle se divise en six sections consacrées aux études suivantes 1° Théorie des points isogonaux et isotomiques, des lignes antiparallèles et des symédianes, 2° theorie des figures directement semblables, 3° cercles de Lemoine et de Tucker, 4° théorie générale d'un système de trois figures semblables, 5° applications particulières de la théorie des figures directement semblables (au cercle de Brocard et au cercle des neuf points), 6° théorie des polygones harmoniques

"Chacune de ces subdivisions, notamment la dernière, est suivei d'exercices intéressants, théorèmes à démontrer ou problèmes à résoudre, parmi lesquels nous devons signaler à l'attention des géomètres les élégantes generalisations dues à MM Casey, G Tarry et J Neuberg dans l'étude des polygones harmoniques

"Nos lecteurs sont déjà familiarisés avec ces nouvelles théories, et ils connaissent aussi l'eminent géomètre qui, depuis trente ans, fait autorité dans l'enseignement des universités de la Grande-Bretagne L'ouvrage est à la hauteur de la réputation du maître, et à tous les degrés d'avancement de leurs études, les élèves y trouveront un guide précieux et instructif Mais cet ouvrage n'est pas seulement utile aux étudiants auxquels il s'addresse,