The Calculus of Functions

of

Several Variables

Section 1.1

Introduction to $\mathbb{R}^n$

Calculus is the study of functional relationships and how related quantities change with each other. In your first exposure to calculus, the primary focus of your attention was on functions involving a single independent variable and a single dependent variable. For such a function f, a single real number input x determines a unique single output value $f(x)$. However, many of the functions of importance both within mathematics itself as well as in the application of mathematics to the rest of the world involve many variables simultaneously. For example, frequently in physics the function which describes the force acting on an object moving in space depends on three variables, the three coordinates which describe the location of the object. If the force function also varies with time, then the force depends on four variables. Moreover, the output of the force function will itself involve three variables, the three coordinate components of the force. Hence the force function is such that it takes three, or four, variables for input and outputs three variables. Far more complicated functions are easy to imagine: the gross national product of a country is a function of thousands of variables with a single variable as output, an airline schedule is a function with thousands of inputs (cities, planes, and people to be scheduled, as well as other variables like fuel costs and the schedules of competing airlines) and perhaps hundreds of outputs (the particular routes flown, along with their times). Although such functions may at first appear to be far more difficult to work with than the functions of single variable calculus, we shall see that we will often be able to reduce problems involving functions of several variables to related problems involving only single variable functions, problems which we may then handle using already familiar techniques.

By definition, a function takes a single input value and associates it with a single output value. Hence, even though in this book the inputs to our functions will often involve several variables, as will the outputs, we will nevertheless want to regard the input and output of a function as single points in some multidimensional space. This is natural in the case of, for example, the force function described above, where the input is a point in three dimensional space, four if we need to use time, but requires some mathematical abstraction if we want to consider the input to the gross national product function as a point in some space of many thousands of dimensions. Because even the geometry of two- and three-dimensional space may be in some respects new to you, we will use this chapter to study the geometry of multidimensional space before proceeding to the study of calculus proper in Chapter 2.

Throughout the book we will let $\mathbb{R}$ denote the set of real numbers.

Definition By *n-dimensional Euclidean space* we mean the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}. \tag{1.1.1}$$

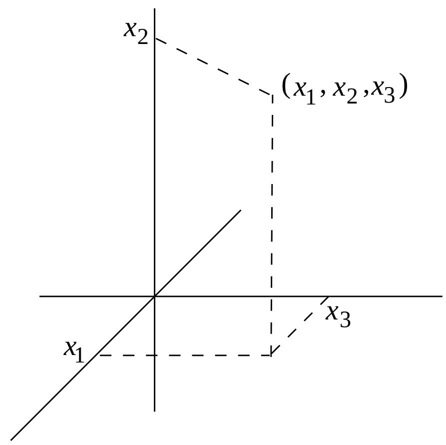


Figure 1.1.1 A point in $\mathbb{R}^3$

That is, $\mathbb{R}^n$ is the space of all ordered n-tuples of real numbers. We will denote a point in this space by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \tag{1.1.2}$$

and, for $i = 1, 2, \ldots, n$, we call x_i the ith *coordinate* of $\mathbf{x}$.

Example When $n = 2$, we have

$$\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\},$$

which is our familiar representation for points in the Cartesian plane. As usual, we will in this case frequently label the coordinates as x and y, or something similar, instead of numbering them as x_1 and x_2.

Example When $n = 3$, we have

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}.$$

Just as we can think of $\mathbb{R}^2$ as a way of assigning coordinates to points in the Euclidean plane, we can think of $\mathbb{R}^3$ as assigning coordinates to three-dimensional Euclidean space. To picture this space, we must imagine three mutually perpendicular axes with the coordinates marked off along the axes as in Figure 1.1.1. Again, we will frequently label the coordinates of a point in $\mathbb{R}^3$ as, for example, x, y, and z, or u, v, and w, rather than using numbered coordinates.

Example If an object moves through space, its location may be specified with four coordinates, three spatial coordinate, say, x, y, and z, and one time coordinate, say t. Thus its location is specified by a point $\mathbf{p} = (x, y, z, t)$ in $\mathbb{R}^4$. Of course, we cannot draw a picture of such a point.

Before beginning our geometric study of $\mathbb{R}^n$, we first need a few basic algebraic definitions.

Definition Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be points in $\mathbb{R}^n$ and let a be a real number. Then we define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \tag{1.1.3}$$

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \tag{1.1.4}$$

and

$$a\mathbf{x} = (ax_1, ax_2, \ldots, ax_n). \tag{1.1.5}$$

Example If $\mathbf{x} = (2, -3, 1)$ and $\mathbf{y} = (-4, 1, -2)$ are two points in $\mathbb{R}^3$, then

$$\mathbf{x} + \mathbf{y} = (-2, -2, -1),$$

$$\mathbf{x} - \mathbf{y} = (6, -4, 3),$$

$$\mathbf{y} - \mathbf{x} = (-6, 4, -3),$$

$$3\mathbf{x} = (6, -9, 3),$$

and

$$-2\mathbf{y} = (8, -2, 4).$$

Notice that we defined addition and subtraction for points in $\mathbb{R}^n$, but we did not define multiplication. In general there is no form of multiplication for such points that is useful for our purpose. Of course, multiplication is defined in the special case $n = 1$ and for the special case $n = 2$ if we consider the points in $\mathbb{R}^2$ as points in the complex plane. We shall see in Section 1.3 that there is also an interesting and useful type of multiplication in $\mathbb{R}^3$. Also note that (1.1.5) does provide a method for multiplying a point in $\mathbb{R}^n$ by a a real number, the result being another point in $\mathbb{R}^n$. In such cases we often refer to the real number as a *scalar* and this multiplication as *scalar multiplication*. We shall provide a geometric interpretation of this form of multiplication shortly.

Geometry of $\mathbb{R}^n$

Recall that if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are two points in $\mathbb{R}^2$, then, using the Pythagorean theorem, the distance from $\mathbf{x}$ to $\mathbf{y}$ is

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}. \tag{1.1.6}$$

This formula is easily generalized to $\mathbb{R}^3$: Suppose $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are two points in $\mathbb{R}^3$. Let $\mathbf{z} = (y_1, y_2, x_3)$. Since the first two coordinates of $\mathbf{y}$ and $\mathbf{z}$ are the same, $\mathbf{y}$ and $\mathbf{z}$ lie on the same vertical line, and so the distance between them is simply

$$|y_3 - x_3|. \tag{1.1.7}$$

Moreover, $\mathbf{x}$ and $\mathbf{z}$ have the same third coordinate, and so lie in the same horizontal plane. Hence the distance between $\mathbf{x}$ and $\mathbf{z}$ is the same as the distance between (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2$, that is,

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}. \tag{1.1.8}$$

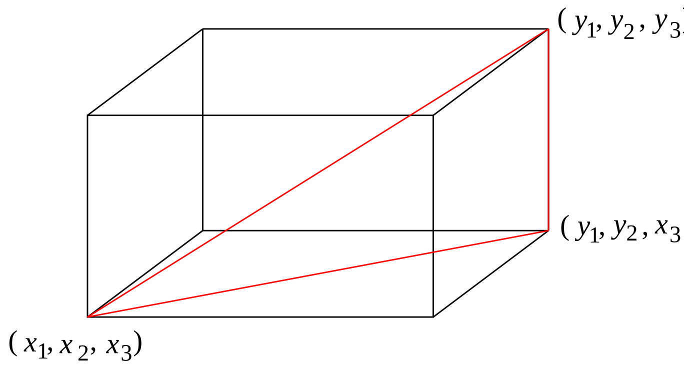


Figure 1.1.2 Distance from $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$

Finally, the points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ form a right triangle with right angle at $\mathbf{z}$. Hence, using the Pythagorean theorem again, the distance from $\mathbf{x}$ to $\mathbf{y}$ is

$$\sqrt{\left(\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}\right)^2 + |y_3 - x_3|^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}.$$

In particular, if we let $\|\mathbf{x}\|$ denote the distance from $\mathbf{x} = (x_1, x_2, x_3)$ to the origin (0, 0, 0) in $\mathbb{R}^3$, then

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{1.1.9}$$

With this notation, the distance from $\mathbf{x}$ to $\mathbf{y}$ is

$$\begin{aligned} \|\mathbf{y} - \mathbf{x}\| &= \|(y_1 - x_1, y_2 - x_2, y_3 - x_3)\| \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}. \end{aligned} \tag{1.1.10}$$

Example If $\mathbf{x} = (1, 2, -3)$ and $\mathbf{y} = (3, -2, 1)$, then the distance from $\mathbf{x}$ to the origin is

$$\|\mathbf{x}\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

and the distance from $\mathbf{x}$ to $\mathbf{y}$ is given by

$$\|\mathbf{y} - \mathbf{x}\| = \|(2, -4, 4)\| = \sqrt{4 + 16 + 16} = 6.$$

Although we do not have any physical analogies to work with when $n > 3$, nevertheless we may generalize (1.1.9) in order to define distance in $\mathbb{R}^n$.

Definition If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a point in $\mathbb{R}^n$, we define the *norm* of $\mathbf{x}$, denoted $\|\mathbf{x}\|$, by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.1.11}$$

For two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we define the distance between $\mathbf{x}$ and $\mathbf{y}$, denoted $d(\mathbf{x}, \mathbf{y})$, by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|. \tag{1.1.12}$$

We will let $\mathbf{0} = (0, 0, \ldots, 0)$ denote the origin in $\mathbb{R}^n$. Then we have

$$\|\mathbf{x}\| = d(\mathbf{x}, \mathbf{0});$$

that is, the norm of $\mathbf{x}$ is the distance from $\mathbf{x}$ to the origin.

Example If $\mathbf{x} = (2, 3, -1, 5)$, a point in $\mathbb{R}^4$, then the distance from $\mathbf{x}$ to the origin is

$$\|\mathbf{x}\| = \sqrt{4 + 9 + 1 + 25} = \sqrt{39}.$$

If $\mathbf{y} = (3, 2, 1, 4)$, then the distance from $\mathbf{x}$ to $\mathbf{y}$ is

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| = \|(1, -1, 2, -1)\| = \sqrt{7}.$$

Note that if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a point in $\mathbb{R}^n$ and a is a scalar, then

$$\begin{aligned} \|a\mathbf{x}\| &= \|(ax_1, ax_2, \ldots, ax_n)\| \\ &= \sqrt{a^2 x_1^2 + a^2 x_2^2 + \cdots + x_n^2} \\ &= |a|\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\ &= |a|\|\mathbf{x}\|. \end{aligned} \tag{1.1.13}$$

That is, the norm of a scalar multiple of $\mathbf{x}$ is just the absolute value of the scalar times the norm of $\mathbf{x}$. In particular, if $\mathbf{x} \neq \mathbf{0}$, then

$$\left\| \frac{1}{\|\mathbf{x}\|}\, \mathbf{x} \right\| = \frac{1}{\|\mathbf{x}\|} \|\mathbf{x}\| = 1.$$

That is,

$$\frac{1}{\|\mathbf{x}\|}\, \mathbf{x}$$

is a unit distance from the origin.

Definition Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be a point in $\mathbb{R}^n$ and let $r > 0$ be a real number. The set of all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ which satisfy the equation

$$(x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2 = r^2 \tag{1.1.14}$$

is called an $(n-1)$-*dimensional sphere* with *radius* r and *center* $\mathbf{p}$, which we denote $S^{n-1}(\mathbf{p}, r)$. The set of all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ which satisfy the inequality

$$(x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2 < r^2 \tag{1.1.15}$$

is called an *open n-dimensional ball* with *radius* r and *center* $\mathbf{p}$, which we denote $B^n(\mathbf{p}, r)$. The set of all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ which satisfy the inequality

$$(x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2 \le r^2 \tag{1.1.16}$$

is called a *closed n-dimensional ball* with *radius* r and *center* $\mathbf{p}$, which we denote $\bar{B}^n(\mathbf{p}, r)$.

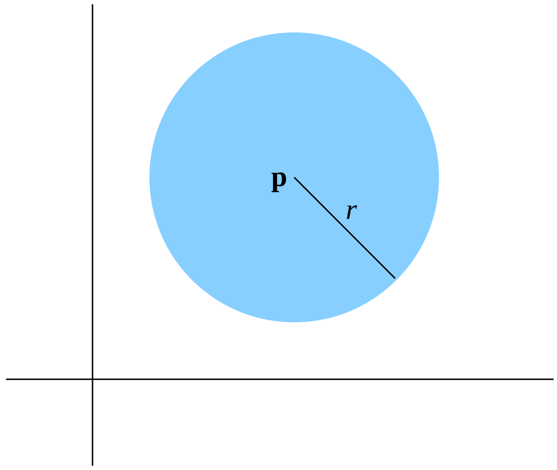


Figure 1.1.3 A closed ball in $\mathbb{R}^2$

A sphere $S^{n-1}(\mathbf{p}, r)$ is the set of all points which lie a fixed distance r from a fixed point $\mathbf{p}$ in $\mathbb{R}^n$. Note that for $n = 1$, $S^0(\mathbf{p}, r)$ consists of only two points, namely, the point $\mathbf{p} - r$ that lies a distance r to the left of $\mathbf{p}$ and the point $\mathbf{p} + r$ that lies a distance r to the right of $\mathbf{p}$; $B^1(\mathbf{p}, r)$ is the open interval $(\mathbf{p} - r, \mathbf{p} + r)$; and $\bar{B}^1(\mathbf{p}, r)$ is the closed interval $[\mathbf{p} - r, \mathbf{p} + r]$. In this sense open and closed balls are natural analogs of open and closed intervals on the real line. For $n = 2$, a sphere is a circle, an open ball is a disk without its enclosing circle, and a closed ball is a disk along with its enclosing circle.

Vectors

Many of the quantities of interest in physics, such as velocities, accelerations, and forces, involve both a magnitude and a direction. For example, we might speak of a force of magnitude 10 newtons acting on an object at the origin in a plane at an angle of $\frac{\pi}{4}$ with the horizontal. It is common to picture such a quantity as an arrow, with length given by the magnitude and with the tip pointing in the specified direction, and to refer to it as a *vector*. Now any point $\mathbf{x} = (x_1, x_2)$, $\mathbf{x} \neq \mathbf{0}$, in $\mathbb{R}^2$ specifies a vector in the plane, namely the vector starting at the origin and ending at $\mathbf{x}$. The *magnitude*, or *length*, of such a vector is $\|\mathbf{x}\|$ and its direction is specified by the angle α that it makes with the horizontal axis or by the angle β that it makes with the vertical axis. Note that

$$\cos(\alpha) = \frac{x_1}{\|\mathbf{x}\|}$$

and

$$\cos(\beta) = \frac{x_2}{\|\mathbf{x}\|}$$

and that, although neither $\cos(\alpha)$ nor $\cos(\beta)$ uniquely determines the direction of the vector by itself, together they completely determine the direction. See Figure 1.1.4.

In general, we may think of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ either as a point in $\mathbb{R}^n$ or as a *vector* in $\mathbb{R}^n$, starting at the origin with *length* $\|\mathbf{x}\|$. If $\mathbf{x} \neq \mathbf{0}$, we say, in analogy with the case in

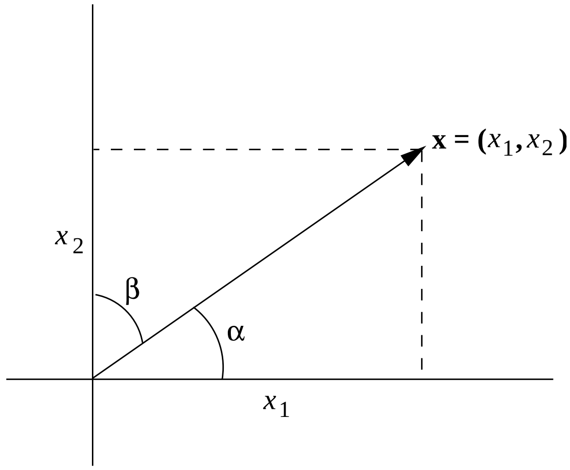


Figure 1.1.4 A vector viewed as an arrow from $\mathbf{0} = (0, 0)$ to $\mathbf{x} = (x_1, x_2)$

$\mathbb{R}^2$, that the direction of $\mathbf{x}$ is the vector

$$\mathbf{u} = \left(\frac{x_1}{\|\mathbf{x}\|}, \frac{x_2}{\|\mathbf{x}\|}, \ldots, \frac{x_n}{\|\mathbf{x}\|} \right) \tag{1.1.17}$$

The coordinates of this vector $\mathbf{u}$ are called the *direction cosines* of $\mathbf{x}$ because we may think of

$$u_k = \frac{x_k}{\|\mathbf{x}\|}$$

as the cosine of the angle between the vector $\mathbf{x}$ and the kth axis for $k = 1, 2, \ldots, n$, an interpretation that will become clearer after our discussion of angles in $\mathbb{R}^n$ in the next section. Alternatively, we may think of $\mathbf{u}$ as a vector of unit length that points in the same direction as $\mathbf{x}$. Any vector of length 1, such as $\mathbf{u}$, is called a *unit vector*. We call $\mathbf{0}$ the zero-vector since it has length 0. Note that $\mathbf{0}$ does not have a direction.

Example The vector $\mathbf{x} = (1, 2, -2, 3)$ in $\mathbb{R}^4$ has length $\|\mathbf{x}\| = \sqrt{18}$ and direction

$$\mathbf{u} = \left(\frac{1}{\sqrt{18}}, \frac{2}{\sqrt{18}}, -\frac{2}{\sqrt{18}}, \frac{3}{\sqrt{18}} \right) = \frac{1}{\sqrt{18}}(1, 2, -2, 3).$$

It is now possible to give geometric meanings to our definitions of scalar multiplication, vector addition, and vector subtraction. First note that if $\mathbf{x} \neq \mathbf{0}$ and $a > 0$, then

$$\|a\mathbf{x}\| = a\|\mathbf{x}\|,$$

so $a\mathbf{x}$ has direction

$$\frac{1}{\|a\mathbf{x}\|} a\mathbf{x} = \frac{1}{\|\mathbf{x}\|}\mathbf{x},$$

the same as $\mathbf{x}$. Hence $a\mathbf{x}$ points in the same direction as $\mathbf{x}$, but with length a times the length of $\mathbf{x}$. If $a < 0$, then

$$\|a\mathbf{x}\| = |a|\|\mathbf{x}\| = -a\|\mathbf{x}\|,$$

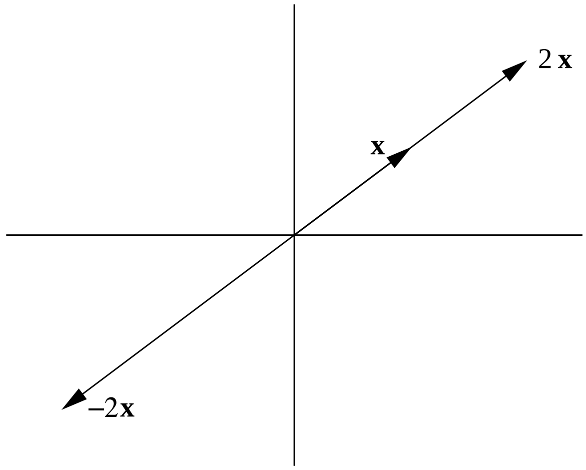


Figure 1.1.5 Examples of scalar multiplication of a vector in $\mathbb{R}^2$

so $a\mathbf{x}$ has direction

$$\frac{1}{\|a\mathbf{x}\|}a\mathbf{x} = -\frac{1}{\|\mathbf{x}\|}\mathbf{x}.$$

Hence, in this case, $a\mathbf{x}$ has the opposite direction of $\mathbf{x}$ with length $|a|$ times the length of $\mathbf{x}$. See Figure 1.1.5 for examples in $\mathbb{R}^2$.

Next consider two vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$ and their sum

$$\mathbf{z} = \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2).$$

Note that the tip of $\mathbf{z}$ is located x_1 units horizontally and x_2 units vertically from the tip of $\mathbf{y}$. Geometrically, the tip of $\mathbf{z}$ is located at the tip of $\mathbf{x}$ if $\mathbf{x}$ were first translated parallel to itself so that its tail now coincided with the tip of $\mathbf{y}$. Equivalently, we can view $\mathbf{z}$ as the diagonal of the parallelogram which has $\mathbf{x}$ and $\mathbf{y}$ for its sides. See Figure 1.1.6 for an example.

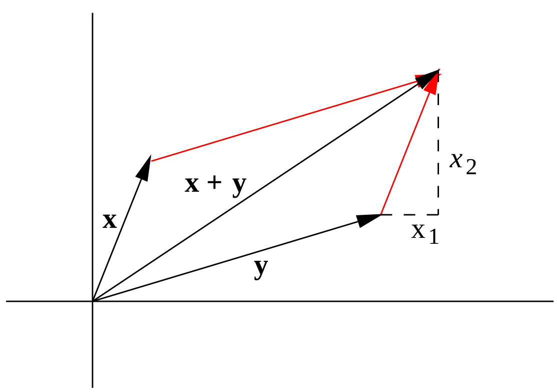


Figure 1.1.6 Example of vector addition in $\mathbb{R}^2$

Finally, consider two vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$ and their difference

$$\mathbf{z} = \mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2).$$

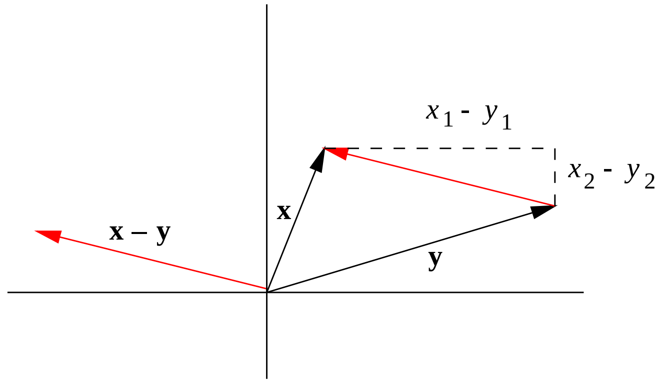


Figure 1.1.7 Example of vector subtraction in $\mathbb{R}^2$

Note that since the coordinates of $\mathbf{z}$ are just the differences in the coordinates of $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{z}$ has the magnitude and direction of an arrow pointing from the tip of $\mathbf{y}$ to the tip of $\mathbf{x}$, as illustrated in Figure 1.1.7. In other words, we may picture $\mathbf{z}$ geometrically by translating an arrow drawn from the tip of $\mathbf{y}$ to the tip of $\mathbf{z}$ parallel to itself until its tail is at the origin.

In the previous discussion it is tempting to think of the arrow from the tip of $\mathbf{y}$ to the tip of $\mathbf{x}$ as really being $\mathbf{x}-\mathbf{y}$, not just a parallel translate of $\mathbf{x}-\mathbf{y}$. In fact, it is convenient and useful to think of parallel translates of a given vector, that is, vectors which have the same direction and magnitude, but with their tails not at the origin, as all being the same vector, just drawn in different places in space. We shall see many instances where viewing vectors in this way significantly helps our understanding.

Before closing this section, we need to call attention to some special vectors.

Definition The vectors

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0,\ldots,0) \\ \mathbf{e}_2 &= (0,1,0,\ldots,0) \\ &\ \ \vdots \\ \mathbf{e}_n &= (0,0,0,\ldots,1) \end{aligned} \tag{1.1.18}$$

in $\mathbb{R}^n$ are called the *standard basis vectors.*

Example In $\mathbb{R}^2$ the standard basis vectors are $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$. Note that if $\mathbf{x} = (x,y)$ is any vector in $\mathbb{R}^2$, then

$$\mathbf{x} = (x,0) + (0,y) = x(1,0) + y(0,1) = x\mathbf{e}_1 + y\mathbf{e}_2.$$

For example, $(2,5) = 2\mathbf{e}_1 + 5\mathbf{e}_2$.

Example In $\mathbb{R}^3$ the standard basis vectors are $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$. Note that if $\mathbf{x} = (x,y,z)$ is any vector in $\mathbb{R}^3$, then

$$\mathbf{x} = (x,0,0) + (0,y,0) + (0,0,z) = x(1,0,0) + y(0,1,0) + z(0,0,1) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

For example, $(1,2,-4) = \mathbf{e}_1 + 2\mathbf{e}_2 - 4\mathbf{e}_3$.

The previous two examples are easily generalized to show that any vector in $\mathbb{R}^n$ may be written as a sum of scalar multiples of the standard basis vectors. Specifically, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then we may write $\mathbf{x}$ as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n. \tag{1.1.19}$$

We say that $\mathbf{x}$ is a *linear combination* of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. It is also important to note that there is only one choice for the scalars in this linear combination. That is, for any vector $\mathbf{x}$ in $\mathbb{R}^n$ there is one and only one way to write $\mathbf{x}$ as a linear combination of the standard basis vectors.

Notes on notation

In this text, we will denote vectors using a plain bold font. This is a common convention, but not the only one used for denoting vectors. Another frequently used convention is to place arrows above a variable which denotes a vector. For example, one might write $\vec{x}$ for what we have been denoting $\mathbf{x}$.

It is also worth noting that in many books the standard basis vectors in $\mathbb{R}^2$ are denoted by $\mathbf{i}$ and $\mathbf{j}$ (or $\vec{i}$ and $\vec{j}$), and the standard basis vectors in $\mathbb{R}^3$ by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (or $\vec{i}$, $\vec{j}$, and $\vec{k}$). Since this notation is not easy to extend to higher dimensions, we will not make much use of it.

Problems

1. Let $\mathbf{x} = (1, 2)$, $\mathbf{y} = (2, 3)$, and $\mathbf{z} = (-2, 4)$. For each of the following, plot the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and the indicated point $\mathbf{w}$.

   (a) $\mathbf{w} = \mathbf{x} + \mathbf{y}$ (b) $\mathbf{w} = 2\mathbf{x} - \mathbf{y}$
   (c) $\mathbf{w} = \mathbf{z} - 2\mathbf{x}$ (d) $\mathbf{w} = 3\mathbf{x} + 2\mathbf{y} - \mathbf{z}$

2. Let $\mathbf{x} = (1, 3, -1)$, $\mathbf{y} = (3, 2, 1)$, and $\mathbf{z} = (-2, 4, -2)$. Compute each of the following.

   (a) $\mathbf{x} + \mathbf{y}$ (b) $\mathbf{x} - \mathbf{z} + 3\mathbf{y}$
   (c) $3\mathbf{z} - 2\mathbf{y}$ (d) $-3\mathbf{x} + 4\mathbf{z}$

3. Let $\mathbf{x} = (1, -1, 2, 3)$, $\mathbf{y} = (-2, 3, 1, -2)$, and $\mathbf{z} = (2, 1, 3, -4)$. Compute each of the following.

   (a) $\mathbf{x} - 2\mathbf{z}$ (b) $\mathbf{y} + \mathbf{x} - 3\mathbf{z}$
   (c) $-3\mathbf{y} - \mathbf{x} + 4\mathbf{z}$ (d) $\mathbf{x} + 3\mathbf{z} - 4\mathbf{y}$

4. Let $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (-2, 3)$. Compute each of the following.

   (a) $\|\mathbf{x}\|$ (b) $\|\mathbf{x} - \mathbf{y}\|$
   (c) $\|3\mathbf{x}\|$ (d) $\|-4\mathbf{y}\|$

5. Let $\mathbf{x} = (2, 3, -1)$, $\mathbf{y} = (2, -1, 5)$, and $\mathbf{z} = (3, -1, -2)$. Compute each of the following.

   (a) $\|\mathbf{x}\|$ (b) $\|\mathbf{x} + 2\mathbf{y}\|$

(c) $\|-5\mathbf{x}\|$ (d) $\|\mathbf{x}+\mathbf{y}+\mathbf{z}\|$

6. Find the distances between the following pairs of points.

(a) $\mathbf{x}=(3,2)$, $\mathbf{y}=(-1,3)$ (b) $\mathbf{x}=(1,2,1)$, $\mathbf{y}=(-2,-1,3)$
(c) $\mathbf{x}=(4,2,1,-1)$, $\mathbf{y}=(1,3,2,-2)$ (d) $\mathbf{z}=(3,-3,0)$, $\mathbf{y}=(-1,2,-5)$
(e) $\mathbf{w}=(1,2,4,-2,3,-1)$, $\mathbf{u}=(3,2,1,-3,2,1)$

7. Draw a picture of the following sets of points in $\mathbb{R}^2$.

(a) $S^1((1,2),1)$ (b) $B^2((1,2),1)$ (c) $\bar{B}^2((1,2),1)$

8. Draw a picture of the following sets of points in $\mathbb{R}$.

(a) $S^0(1,3)$ (b) $B^1(1,3)$ (c) $\bar{B}^1(1,3)$

9. Describe the differences between $S^2((1,2,1),1)$, $B^3((1,2,1),1)$, and $\bar{B}^3((1,2,1),1)$ in $\mathbb{R}^3$.

10. Is the point $(1,4,5)$ in the the open ball $B^3((-1,2,3),4)$?

11. Is the point $(3,2,-1,4,1)$ in the open ball $B^5((1,2,-4,2,3),3)$?

12. Find the length and direction of the following vectors.

(a) $\mathbf{x}=(2,1)$ (b) $\mathbf{z}=(1,1,-1)$
(c) $\mathbf{x}=(-1,2,3)$ (d) $\mathbf{w}=(1,-1,2,-3)$

13. Let $\mathbf{x}=(1,3)$, $\mathbf{y}=(4,1)$, and $\mathbf{z}=(2,-1)$. Plot $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Also, show how to obtain each of the following geometrically.

(a) $\mathbf{w}=\mathbf{x}+\mathbf{y}$ (b) $\mathbf{w}=\mathbf{y}-\mathbf{x}$
(c) $\mathbf{w}=3\mathbf{z}$ (d) $\mathbf{w}=-2\mathbf{z}$
(e) $\mathbf{w}=\frac{1}{2}\mathbf{z}$ (f) $\mathbf{w}=\mathbf{x}+\mathbf{y}+\mathbf{z}$
(g) $\mathbf{w}=\mathbf{x}+3\mathbf{z}$ (h) $\mathbf{w}=\mathbf{x}-\frac{1}{4}\mathbf{y}$

14. Suppose $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)$, and $\mathbf{z}=(z_1,z_2,\ldots,z_n)$ are vectors in $\mathbb{R}^n$ and a, b, and c are scalars. Verify the following.

(a) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ (b) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$
(c) $a(\mathbf{x}+\mathbf{y})=a\mathbf{x}+a\mathbf{y}$ (d) $(a+b)\mathbf{x}=a\mathbf{x}+b\mathbf{x}$
(e) $a(b\mathbf{x})=(ab)\mathbf{x}$ (f) $\mathbf{x}+\mathbf{0}=\mathbf{x}$
(g) $1\mathbf{x}=\mathbf{x}$ (h) $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$, where $-\mathbf{x}=-1\mathbf{x}$

15. Let $\mathbf{u}=(1,1)$ and $\mathbf{v}=(-1,1)$ be vectors in $\mathbb{R}^2$.

(a) Let $\mathbf{x}=(2,1)$. Find scalars a and b such that $\mathbf{x}=a\mathbf{u}+b\mathbf{v}$. Are a and b unique?
(b) Let $\mathbf{x}=(x,y)$ be an arbitrary vector in $\mathbb{R}^2$. Show that there exist unique scalars a and b such that $\mathbf{x}=a\mathbf{u}+b\mathbf{v}$.

(c) The result in (b) shows that $\mathbf{u}$ and $\mathbf{v}$ form a *basis* for $\mathbb{R}^2$ which is different from the standard basis of $\mathbf{e}_1$ and $\mathbf{e}_2$. Show that the vectors $\mathbf{u} = (1, 1)$ and $\mathbf{w} = (-1, -1)$ do not form a basis for $\mathbb{R}^2$. (Hint: Show that there do not exist scalars a and b such that $\mathbf{x} = a\mathbf{u} + \mathbf{w}$ when $\mathbf{x} = (2, 1)$.)

The Calculus of Functions

of

Several Variables

Section 1.2

Angles and the Dot Product

Suppose $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are two vectors in $\mathbb{R}^2$, neither of which is the zero vector $\mathbf{0}$. Let α and β be the angles between $\mathbf{x}$ and $\mathbf{y}$ and the positive horizontal axis, respectively, measured in the counterclockwise direction. Supposing $\alpha \geq \beta$, let $\theta = \alpha - \beta$. Then θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ measured in the counterclockwise direction, as shown in Figure 1.2.1. From the subtraction formula for cosine we have

$$\cos(\theta) = \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta). \tag{1.2.1}$$

Now

$$\cos(\alpha) = \frac{x_1}{\|\mathbf{x}\|},$$

$$\cos(\beta) = \frac{y_1}{\|\mathbf{y}\|},$$

$$\sin(\alpha) = \frac{x_2}{\|\mathbf{x}\|},$$

and

$$\sin(\beta) = \frac{y_2}{\|\mathbf{y}\|}.$$

Thus, we have

$$\cos(\theta) = \frac{x_1 y_1}{\|x\|\|y\|} + \frac{x_2 y_2}{\|x\|\|y\|} = \frac{x_1 y_1 + x_2 y_2}{\|x\|\|y\|}. \tag{1.2.2}$$

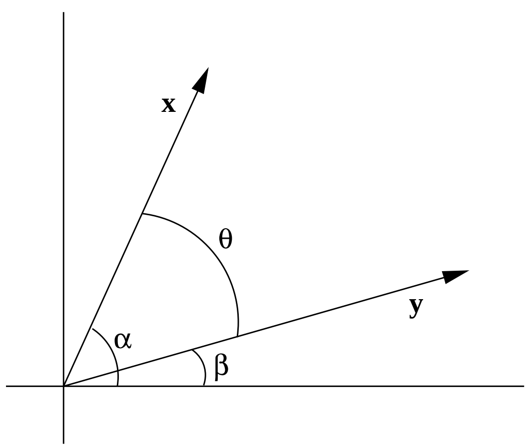


Figure 1.2.1 The angle between two vectors

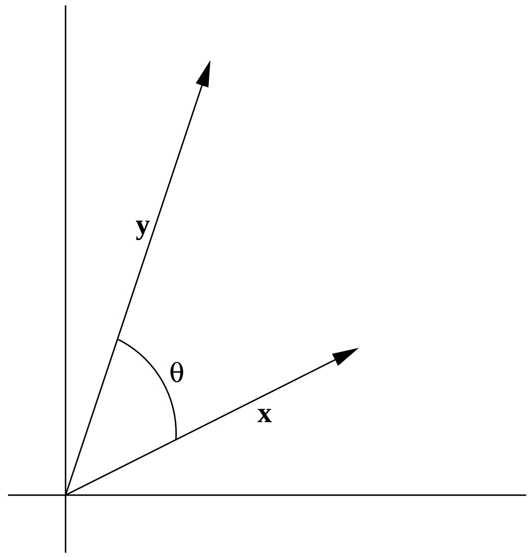


Figure 1.2.2 The angle between $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (1, 3)$

Example Let θ be the smallest angle between $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (1, 3)$, measured in the counterclockwise direction. Then, by (1.2.2), we must have

$$\cos(\theta) = \frac{(2)(1) + (1)(3)}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

Hence

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

See Figure 1.2.2.

With more work it is possible to show that if $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are two vectors in $\mathbb{R}^3$, neither of which is the zero vector $\mathbf{0}$, and θ is the smallest positive angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\cos(\theta) = \frac{x_1y_1 + x_2y_2 + x_3y_3}{\|\mathbf{x}\|\|\mathbf{y}\|} \tag{1.2.3}$$

The term which appears in the numerators in both (1.2.2) and (1.2.3) arises frequently, so we will give it a name.

Definition If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbb{R}^n$, then the *dot product* of $\mathbf{x}$ and $\mathbf{y}$, denoted $\mathbf{x} \cdot \mathbf{y}$, is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n. \tag{1.2.4}$$

Note that the dot product of two vectors is a scalar, not another vector. Because of this, the dot product is also called the *scalar product*. It is also an example of what is called an *inner product* and is often denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.

Example If $\mathbf{x} = (1, 2, -3, -2)$ and $\mathbf{y} = (-1, 2, 3, 5)$, then

$$\mathbf{x} \cdot \mathbf{y} = (1)(-1) + (2)(2) + (-3)(3) + (-2)(5) = -1 + 4 - 9 - 10 = -16.$$

The next proposition lists some useful properties of the dot product.

Proposition For any vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^n$ and scalar α,

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}, \tag{1.2.5}$$

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}, \tag{1.2.6}$$

$$(\alpha\mathbf{x}) \cdot \mathbf{y} = \alpha(\mathbf{x} \cdot \mathbf{y}), \tag{1.2.7}$$

$$\mathbf{0} \cdot \mathbf{x} = 0, \tag{1.2.8}$$

$$\mathbf{x} \cdot \mathbf{x} \geq 0, \tag{1.2.9}$$

$$\mathbf{x} \cdot \mathbf{x} = 0 \text{ only if } \mathbf{x} = \mathbf{0}, \tag{1.2.10}$$

and

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2. \tag{1.2.11}$$

These properties are all easily verifiable using the properties of real numbers and the definition of the dot product and will be left to Problem 9 for you to check.

At this point we can say that if $\mathbf{x}$ and $\mathbf{y}$ are two nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$ and θ is the smallest positive angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}. \tag{1.2.12}$$

We would like to be able to make the same statement about the angle between two vectors in any dimension, but we would first have to define what we mean by the angle between two vectors in $\mathbb{R}^n$ for $n > 3$. The simplest way to do this is to turn things around and use (1.2.12) to define the angle. However, in order for this to work we must first know that

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1,$$

since this is the range of values for the cosine function. This fact follows from the following inequality.

Cauchy-Schwarz Inequality For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|. \tag{1.2.13}$$

To see why this is so, first note that both sides of (1.2.13) are 0 when $y = \mathbf{0}$, and hence are equal in this case. Assuming $\mathbf{x}$ and $\mathbf{y}$ are fixed vectors in $\mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$, let t be a real number and consider the function

$$f(t) = (\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y}). \tag{1.2.14}$$

By (1.2.9), $f(t) \geq 0$ for all t, while from (1.2.6), (1.2.7), and (1.2.11), we see that

$$f(t) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot t\mathbf{y} + t\mathbf{y} \cdot \mathbf{x} + t\mathbf{y} \cdot t\mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y})t + \|\mathbf{y}\|^2 t^2. \tag{1.2.15}$$

Hence f is a quadratic polynomial with at most one root. Since the roots of f are, as given by the quadratic formula,

$$\frac{-2(\mathbf{x} \cdot \mathbf{y}) \pm \sqrt{4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2}}{2\|\mathbf{y}\|^2},$$

it follows that we must have

$$4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0. \tag{1.2.16}$$

Thus

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2, \tag{1.2.17}$$

and so

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|. \tag{1.2.18}$$

Note that $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ if and only if there is some value of t for which $f(t) = 0$, which, by (1.2.8) and (1.2.10), happens if and only if $\mathbf{x} + t\mathbf{y} = \mathbf{0}$, that is, $\mathbf{x} = -t\mathbf{y}$, for some value of t. Moreover, if $\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = 0\mathbf{x}$ for any $\mathbf{x}$ in $\mathbb{R}^n$. Hence, in either case, the Cauchy-Schwarz inequality becomes an equality if and only if either $\mathbf{x}$ is a scalar multiple of $\mathbf{y}$ or $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$.

With the Cauchy-Schwarz inequality we have

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1 \tag{1.2.19}$$

for any nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Thus we may now state the following definition.

Definition If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then we call

$$\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \tag{1.2.20}$$

the *angle between* $\mathbf{x}$ and $\mathbf{y}$.

Example Suppose $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -2, 2)$. Then $\mathbf{x} \cdot \mathbf{y} = 1 - 4 + 6 = 3$, $\|\mathbf{x}\| = \sqrt{14}$, and $\|\mathbf{y}\| = 3$, so if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have

$$\cos(\theta) = \frac{3}{3\sqrt{14}} = \frac{1}{\sqrt{14}}.$$

Hence, rounding to four decimal places,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 1.3002.$$

Example Suppose $\mathbf{x} = (2, -1, 3, 1)$ and $\mathbf{y} = (-2, 3, 1, -4)$. Then $\mathbf{x}\cdot\mathbf{y} = -8$, $\|\mathbf{x}\| = \sqrt{15}$, and $\|\mathbf{y}\| = \sqrt{30}$, so if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have, rounding to four decimal places,

$$\theta = \cos^{-1}\left(\frac{-8}{\sqrt{15}\sqrt{30}}\right) = 1.9575.$$

Example Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and let α_k, $k = 1, 2, \ldots, n$, be the angle between $\mathbf{x}$ and the kth axis. Then α_k is the angle between $\mathbf{x}$ and the standard basis vector $\mathbf{e}_k$. Thus

$$\cos(\alpha_k) = \frac{\mathbf{x}\cdot\mathbf{e}_k}{\|\mathbf{x}\|\|\mathbf{e}_k\|} = \frac{x_k}{\|\mathbf{x}\|}.$$

That is, $\cos(\alpha_1), \cos(\alpha_2), \ldots, \cos(\alpha_n)$ are the direction cosines of $\mathbf{x}$ as defined in Section 1.1. For example, if $\mathbf{x} = (3, 1, 2)$in $\mathbb{R}^3$, then $\|\mathbf{x}\| = \sqrt{14}$ and the direction cosines of $\mathbf{x}$ are

$$\cos(\alpha_1) = \frac{3}{\sqrt{14}},$$

$$\cos(\alpha_2) = \frac{1}{\sqrt{14}},$$

and

$$\cos(\alpha_3) = \frac{2}{\sqrt{14}},$$

giving us, to four decimal places,

$$\alpha_1 = 0.6405,$$

$$\alpha_2 = 1.3002,$$

and

$$\alpha_3 = 1.0069.$$

Note that if $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$ with $\mathbf{x}\cdot\mathbf{y} = 0$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is

$$\cos^{-1}(0) = \frac{\pi}{2}.$$

This is the motivation behind our next definition.

Definition Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to be *orthogonal* (or *perpendicular*), denoted $\mathbf{x} \perp \mathbf{y}$, if $\mathbf{x}\cdot\mathbf{y} = 0$.

It is a convenient convention of mathematics not to restrict the definition of orthogonality to nonzero vectors. Hence it follows from the definition, and (1.2.8), that $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$. Moreover, $\mathbf{0}$ is the only vector in $\mathbb{R}^n$ which has this property, a fact you will be asked to verify in Problem 12.

Example The vectors $\mathbf{x} = (-1, -2)$ and $\mathbf{y} = (1, 2)$ are both orthogonal to $\mathbf{z} = (2, -1)$ in $\mathbb{R}^2$. Note that $\mathbf{y} = -\mathbf{x}$ and, in fact, any scalar multiple of $\mathbf{x}$ is orthogonal to $\mathbf{z}$.

Example In $\mathbb{R}^4$, $\mathbf{x} = (1, -1, 1, -1)$ is orthogonal to $\mathbf{y} = (1, 1, 1, 1)$. As in the previous example, any scalar multiple of $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

Definition We say vectors $\mathbf{x}$ and $\mathbf{y}$ are *parallel* if $\mathbf{x} = \alpha\mathbf{y}$ for some scalar $\alpha \neq 0$.

This definition says that vectors are parallel when one is a nonzero scalar multiple of the other. From our proof of the Cauchy-Schwarz inequality we know that it follows that if $\mathbf{x}$ and $\mathbf{y}$ are parallel, then $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}|$. Thus if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$,

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \pm 1.$$

That is, $\theta = 0$ or $\theta = \pi$. Put another way, $\mathbf{x}$ and $\mathbf{y}$ either point in the same direction or they point in opposite directions.

Example The vectors $\mathbf{x} = (1, -3)$ and $\mathbf{y} = (-2, 6)$ are parallel since $\mathbf{x} = -\frac{1}{2}\mathbf{y}$. Note that $\mathbf{x} \cdot \mathbf{y} = -20$ and $\|\mathbf{x}\|\|\mathbf{y}\| = \sqrt{10}\sqrt{40} = 20$, so $\mathbf{x} \cdot \mathbf{y} = -\|\mathbf{x}\|\|\mathbf{y}\|$. It follows that the angle between $\mathbf{x}$ and $\mathbf{y}$ is π.

Two basic results about triangles in $\mathbb{R}^2$ and $\mathbb{R}^3$ are the triangle inequality (the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side) and the Pythagorean theorem (the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the other side). In terms of vectors in $\mathbb{R}^n$, if we picture a vector $\mathbf{x}$ with its tail at the origin and a vector $\mathbf{y}$ with its tail at the tip of $\mathbf{x}$ as two sides of a triangle, then the remaining side is given by the vector $\mathbf{x} + \mathbf{y}$. Thus the triangle inequality may be stated as follows.

Triangle inequality If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|. \tag{1.2.21}$$

The first step in verifying (1.2.21) is to note that, using (1.2.11) and (1.2.6),

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y} \\ &= \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2. \end{aligned} \tag{1.2.22}$$

Since $\mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$ by the Cauchy-Schwarz inequality, it follows that

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq |\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2,$$

from which we obtain the triangle inequality by taking square roots.

Note that in (1.2.22) we have

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

if and only if $\mathbf{x} \cdot \mathbf{y} = 0$, that is, if and only if $\mathbf{x} \perp \mathbf{y}$. Hence we have the following famous result.

Pythagorean theorem Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are orthogonal if and only if

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2. \tag{1.2.23}$$

Perhaps the most important application of the dot product is in finding the orthogonal projection of one vector onto another. This is illustrated in Figure 1.2.3, where $\mathbf{w}$ represents the projection of $\mathbf{x}$ onto $\mathbf{y}$. The result of the projection is to break $\mathbf{x}$ into the sum of two *components*, $\mathbf{w}$, which is parallel to $\mathbf{y}$, and $\mathbf{x}-\mathbf{w}$, which is orthogonal to $\mathbf{y}$, a procedure which is frequently very useful. To compute $\mathbf{w}$, note that if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\|\mathbf{w}\| = \|\mathbf{x}\||\cos(\theta)| = \|\mathbf{x}\|\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} = \left|\mathbf{x}\cdot\frac{\mathbf{y}}{\|\mathbf{y}\|}\right| = |\mathbf{x}\cdot\mathbf{u}|, \tag{1.2.24}$$

where

$$\mathbf{u} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$$

is the direction of $\mathbf{y}$. Hence $\mathbf{w} = |\mathbf{x}\cdot\mathbf{u}|\mathbf{u}$ when $0 \le \theta \le \frac{\pi}{2}$, which is when $\mathbf{x}\cdot\mathbf{u} > 0$, and $\mathbf{w} = -|\mathbf{x}\cdot\mathbf{u}|\mathbf{u}$ when $\frac{\pi}{2} < \theta \le \pi$, which is when $\mathbf{x}\cdot\mathbf{u} < 0$. Thus, in either case, $\mathbf{w} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}$.

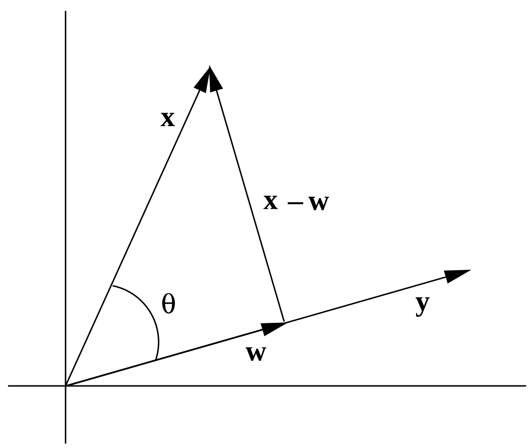


Figure 1.2.3 Orthogonal projection

Definition Given vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{y} \neq \mathbf{0}$, in $\mathbb{R}^n$, the vector

$$\mathbf{w} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}, \tag{1.2.25}$$

where $\mathbf{u}$ is the direction of $\mathbf{y}$, is called the *orthogonal projection*, or simply *projection*, of $\mathbf{x}$ onto $\mathbf{y}$. We also call $\mathbf{w}$ the *component* of $\mathbf{x}$ in the direction of $\mathbf{y}$ and $\mathbf{x}\cdot\mathbf{u}$ the *coordinate* of $\mathbf{x}$ in the direction of $\mathbf{y}$.

In the special case where $\mathbf{y} = \mathbf{e}_k$, the kth standard basic vector, $k = 1, 2, \ldots, n$, we see that the coordinate of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the direction of $\mathbf{y}$ is just $\mathbf{x}\cdot\mathbf{e}_k = x_k$, the kth coordinate of $\mathbf{x}$.

Example Suppose $\mathbf{x} = (1,2,3)$ and $\mathbf{y} = (1,4,0)$. Then the direction of $\mathbf{y}$ is

$$\mathbf{u} = \frac{1}{\sqrt{17}}(1,4,0),$$

so the coordinate of $\mathbf{x}$ in the direction of $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{u} = \frac{1}{\sqrt{17}}(1+8+0) = \frac{9}{\sqrt{17}}.$$

Thus the projection of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\mathbf{w} = \frac{9}{\sqrt{17}}\mathbf{u} = \frac{9}{17}(1,4,0) = \left(\frac{9}{17}, \frac{36}{17}, 0\right).$$

Problems

1. Let $\mathbf{x} = (3,-2)$, $\mathbf{y} = (-2,5)$, and $\mathbf{z} = (4,1)$. Compute each of the following.

   (a) $\mathbf{x} \cdot \mathbf{y}$ (b) $2\mathbf{x} \cdot \mathbf{y}$
   (c) $\mathbf{x} \cdot (3\mathbf{y} - \mathbf{z})$ (d) $-\mathbf{z} \cdot (\mathbf{x} + 5\mathbf{y})$

2. Let $\mathbf{x} = (3,-2,1)$, $\mathbf{y} = (-2,3,5)$, and $\mathbf{z} = (-1,4,1)$. Compute each of the following.

   (a) $\mathbf{x} \cdot \mathbf{y}$ (b) $2\mathbf{x} \cdot \mathbf{y}$
   (c) $\mathbf{x} \cdot (3\mathbf{y} - \mathbf{z})$ (d) $-\mathbf{z} \cdot (\mathbf{x} + 5\mathbf{y})$

3. Let $\mathbf{x} = (3,-2,1,2)$, $\mathbf{y} = (-2,3,4,-5)$, and $\mathbf{z} = (-1,4,1,-2)$. Compute each of the following.

   (a) $\mathbf{x} \cdot \mathbf{y}$ (b) $2\mathbf{x} \cdot \mathbf{y}$
   (c) $\mathbf{x} \cdot (3\mathbf{y} - \mathbf{z})$ (d) $-\mathbf{z} \cdot (\mathbf{x} + 5\mathbf{y})$

4. Find the angles between the following pairs of vectors. First find your answers in radians and then convert to degrees.

   (a) $\mathbf{x} = (1,2)$, $\mathbf{y} = (2,1)$ (b) $\mathbf{z} = (3,1)$, $\mathbf{w} = (-3,1)$
   (c) $\mathbf{x} = (1,1,1)$, $\mathbf{y} = (-1,1,-1)$ (d) $\mathbf{y} = (-1,2,4)$, $\mathbf{z} = (2,3,-1)$
   (e) $\mathbf{x} = (1,2,1,2)$, $\mathbf{y} = (2,1,2,1)$ (f) $\mathbf{x} = (1,2,3,4,5)$, $\mathbf{z} = (5,4,3,2,1)$

5. The three points $(2,1)$, $(1,2)$, and $(-2,1)$ determine a triangle in $\mathbb{R}^2$. Find the measure of its three angles and verify that their sum is π.

6. Given three points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ in $\mathbb{R}^n$, the vectors $\mathbf{q} - \mathbf{p}$, $\mathbf{r} - \mathbf{p}$, and $\mathbf{q} - \mathbf{r}$ describe the sides of the triangle with vertices at $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. For each of the following, find the measure of the three angles of the triangle with vertices at the given points.

   (a) $\mathbf{p} = (1,2,1)$, $\mathbf{q} = (-1,-1,2)$, $\mathbf{r} = (-1,3,-1)$
   (b) $\mathbf{p} = (1,2,1,1)$, $\mathbf{q} = (-1,-1,2,3)$, $\mathbf{r} = (-1,3,-1,2)$

7. For each of the following, find the angles between the given vector and the coordinate axes.

   (a) $\mathbf{x} = (-2, 3)$ (b) $\mathbf{w} = (-1, 2, 1)$

   (c) $\mathbf{y} = (2, 3, 1, -1)$ (d) $\mathbf{x} = (1, 2, 3, 4, 5)$

8. For each of the following, find the coordinate of $\mathbf{x}$ in the direction of $\mathbf{y}$ and the projection $\mathbf{w}$ of $\mathbf{x}$ onto $\mathbf{y}$. In each case verify that $\mathbf{y} \perp (\mathbf{x} - \mathbf{w})$.

   (a) $\mathbf{x} = (-2, 4)$, $\mathbf{y} = (4, 1)$ (b) $\mathbf{x} = (4, 1, 4)$, $\mathbf{y} = (-1, 3, 1)$

   (c) $\mathbf{x} = (-4, -3, 1)$, $\mathbf{y} = (1, -1, 6)$ (d) $\mathbf{x} = (1, 2, 4, -1)$, $\mathbf{y} = (2, -1, 2, 3)$

9. Verify properties (1.2.5) through (1.2.11) of the dot product.

10. If $\mathbf{w}$ is the projection of $\mathbf{x}$ onto $\mathbf{y}$, verify that $\mathbf{y}$ is orthogonal to $\mathbf{x} - \mathbf{w}$.

11. Write $\mathbf{x} = (1, 2, -3)$ as the sum of two vectors, one parallel to $\mathbf{y} = (2, 3, 1)$ and the other orthogonal to $\mathbf{y}$.

12. Suppose $\mathbf{x}$ is a vector with the property that $\mathbf{x} \cdot \mathbf{y} = 0$ for all vectors $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{y} \neq \mathbf{x}$. Show that it follows that $\mathbf{x} = \mathbf{0}$.

Section 1.3

The Cross Product

As we noted in Section 1.1, there is no general way to define multiplication for vectors in $\mathbb{R}^n$, with the product also being a vector of the same dimension, which is useful for our purposes in this book. However, in the special case of $\mathbb{R}^3$ there is a product which we will find useful. One motivation for this product is to consider the following problem: Given two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, not parallel to one another, find a third vector $\mathbf{w} = (w_1, w_2, w_3)$ which is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Thus we want $\mathbf{w} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{y} = 0$, which means we need to solve the equations

$$\begin{aligned} x_1 w_1 + x_2 w_2 + x_3 w_3 &= 0 \\ y_1 w_1 + y_2 w_2 + y_3 w_3 &= 0 \end{aligned} \tag{1.3.1}$$

for w_1, w_2, and w_3. Multiplying the first equation by y_3 and the second by x_3 gives us

$$\begin{aligned} x_1 y_3 w_1 + x_2 y_3 w_2 + x_3 y_3 w_3 &= 0 \\ x_3 y_1 w_1 + x_3 y_2 w_2 + x_3 y_3 w_3 &= 0. \end{aligned} \tag{1.3.2}$$

Subtracting the second equation from the first, we have

$$(x_1 y_3 - x_3 y_1) w_1 + (x_2 y_3 - x_3 y_2) w_2 = 0. \tag{1.3.3}$$

One solution of (1.3.3) is given by setting

$$\begin{aligned} w_1 &= x_2 y_3 - x_3 y_2 \\ w_2 &= -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3. \end{aligned} \tag{1.3.4}$$

Finally, from the first equation in (1.3.1), we now have

$$x_3 w_3 = -x_1 (x_2 y_3 - x_3 y_2) - x_2 (x_3 y_1 - x_1 y_3) = x_1 x_3 y_2 - x_2 x_3 y_1, \tag{1.3.5}$$

from which we obtain the solution

$$w_3 = x_1 y_2 - x_2 y_1. \tag{1.3.6}$$

The choices made in arriving at (1.3.4) and (1.3.6) are not unique, but they are the standard choices which define $\mathbf{w}$ as the *cross* or *vector product* of $\mathbf{x}$ and $\mathbf{y}$.

Definition Given vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, the vector

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) \tag{1.3.7}$$

is called the *cross product*, or *vector product*, of $\mathbf{x}$ and $\mathbf{y}$.

Example If $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -1, 1)$, then

$$\mathbf{x} \times \mathbf{y} = (2 + 3, 3 - 1, -1 - 2) = (5, 2, -3).$$

Note that

$$\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = 5 + 4 - 9 = 0$$

and

$$\mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 5 - 2 - 3 = 0,$$

showing that $\mathbf{x} \perp (\mathbf{x} \times \mathbf{y})$ and $\mathbf{y} \perp (\mathbf{x} \times \mathbf{y})$ as claimed. It is also interesting to note that

$$\mathbf{y} \times \mathbf{x} = (-3 - 2, 1 - 3, 2 + 1) = (-5, -2, 3) = -(\mathbf{x} \times \mathbf{y}).$$

This last calculation holds in general for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$.

Proposition Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbb{R}^3$ and α is any real number. Then

$$\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x}), \tag{1.3.8}$$

$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z}), \tag{1.3.9}$$

$$(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\mathbf{y} \times \mathbf{z}), \tag{1.3.10}$$

$$\alpha(\mathbf{x} \times \mathbf{y}) = (\alpha\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (\alpha\mathbf{y}), \tag{1.3.11}$$

and

$$\mathbf{x} \times \mathbf{0} = \mathbf{0}. \tag{1.3.12}$$

Verification of these properties is straightforward and will be left to Problem 10. Also, notice that

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \tag{1.3.13}$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \tag{1.3.14}$$

and

$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2; \tag{1.3.15}$$

that is, the cross product of two standard basis vectors is either the other standard basis vector or its negative. Moreover, note that in these cases the cross product points in the direction your thumb would point if you were to wrap the fingers of your right hand from the first vector to the second. This is in fact always true and results in what is known as the *right-hand rule* for the orientation of the cross product, as shown in Figure 1.3.1. Hence given two vectors $\mathbf{x}$ and $\mathbf{y}$, we can always determine the direction of $\mathbf{x} \times \mathbf{y}$; to

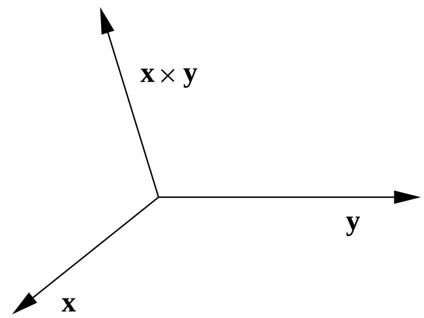


Figure 1.3.1 The right-hand rule

completely identify $\mathbf{x} \times \mathbf{y}$ geometrically, we need only to know its length. Now if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\begin{aligned}
\|\mathbf{x} \times \mathbf{y}\|^2 &= (x_2y_3 - x_3y_2)^2 + (x_3y_1 - x_1y_3)^2 + (x_1y_2 - x_2y_1)^2 \\
&= x_2^2y_3^2 - 2x_2x_3y_2y_3 + x_3^2y_2^2 + x_3^2y_1^2 - 2x_1x_3y_1y_3 + x_1^2y_3^2 + x_1^2y_2^2 \\
&\quad - 2x_1x_2y_1y_2 + x_2^2y_1^2 \\
&= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2) \\
&\quad - (2x_2x_3y_2y_3 + 2x_1x_3y_1y_3 + 2x_1x_2y_1y_2) \\
&= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1y_1 + x_2y_2 + x_3y_3)^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta))^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2(1 - \cos^2(\theta)) \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2\sin^2(\theta).
\end{aligned} \tag{1.3.16}$$

Taking square roots, and noting that $\sin(\theta) \geq 0$ since, by the definition of the angle between two vectors, $0 \leq \theta \leq \pi$, we have the following result.

Proposition If θ is the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, then

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\|\sin(\theta). \tag{1.3.17}$$

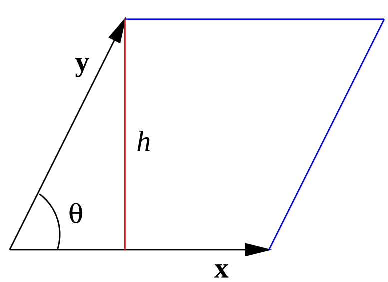


Figure 1.3.2 Height of the parallelogram is $h = \|\mathbf{y}\|\sin(\theta)$

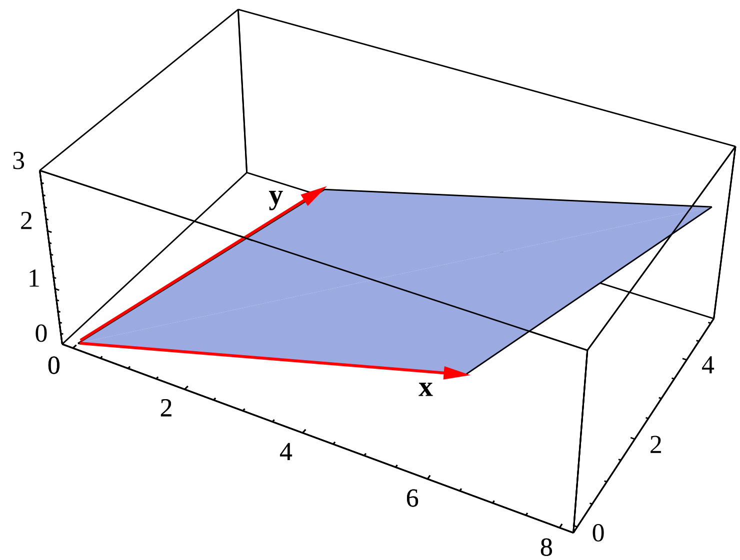


Figure 1.3.3 Parallelogram with vertices at $(0, 0, 0)$, $(6, 1, 1)$, $(8, 5, 2)$, and $(2, 4, 1)$

The last theorem has several interesting consequences. One of these comes from recognizing that if we draw a parallelogram with $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides, as in Figure 1.3.2, then the height of the parallelogram is $\|\mathbf{y}\|\sin(\theta)$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Hence the area of the parallelogram is $\|\mathbf{x}\|\|\mathbf{y}\|\sin(\theta)$, which by (1.3.17) is $\|\mathbf{x} \times \mathbf{y}\|$.

Proposition Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^3$. Then the area of the parallelogram which has $\mathbf{x}$ and $\mathbf{y}$ for adjacent sides is $\|\mathbf{x} \times \mathbf{y}\|$.

Example Consider the parallelogram P with vertices at $(0, 0, 0)$, $(6, 1, 1)$, $(8, 5, 2)$, and $(2, 4, 1)$. Two adjacent sides are specified by the vectors $\mathbf{x} = (6, 1, 1)$ and $\mathbf{y} = (2, 4, 1)$ (see Figure 1.3.3), so the area of P is

$$\|\mathbf{x} \times \mathbf{y}\| = \|(1 - 4, 2 - 6, 24 - 2)\| = \|(-3, -4, 22)\| = \sqrt{509}.$$

See Figure 1.3.4 to see the relationship between $\mathbf{x} \times \mathbf{y}$ and P.

Example Consider the parallelogram P in the plane with vertices at $(1, 1,)$, $(3, 2)$, $(4, 4)$, and $(2, 3)$. Two adjacent sides are given by the vectors from $(1, 1)$ to $(3, 2)$, that is

$$\mathbf{x} = (3, 2) - (1, 1) = (2, 1),$$

and from $(1, 1)$ to $(2, 3)$, that is,

$$\mathbf{y} = (2, 3) - (1, 1) = (1, 2).$$

See Figure 1.3.5. However, since these vectors are in $\mathbb{R}^2$, not in R^3, we cannot compute their cross product. To get around this, we consider the vectors $\mathbf{w} = (2, 1, 0)$ and $\mathbf{v} = (1, 2, 0)$ which are adjacent sides of the same parallelogram viewed as lying in $\mathbb{R}^3$. Then the area of P is given by

$$\|\mathbf{w} \times \mathbf{v}\| = \|(0, 0, 4 - 1)\| = \|(0, 0, 3)\| = 3.$$

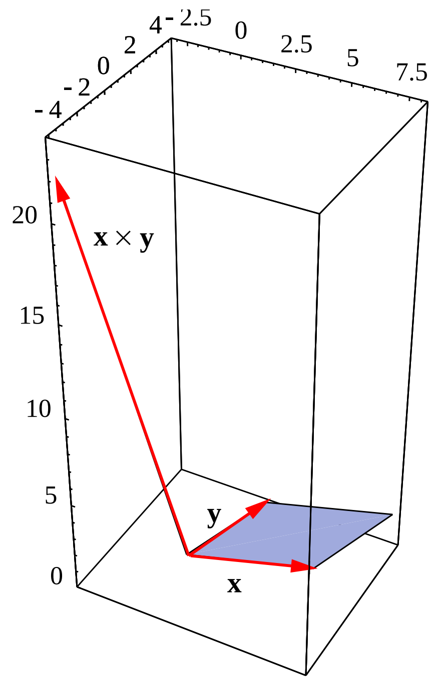


Figure 1.3.4 Parallelogram with adjacent sides $\mathbf{x} = (6, 1, 1)$ and $\mathbf{y} = (2, 4, 1)$

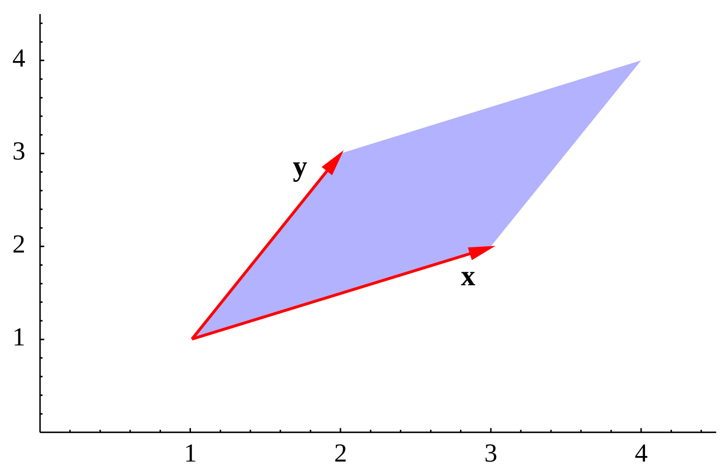


Figure 1.3.5 Parallelogram with vertices at $(1, 1)$, $(3, 2)$, $(4, 4)$, and $(2, 3)$

It is easy to extend the result of the previous theorem to computing the volume of a parallelepiped in $\mathbb{R}^3$. Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are adjacent edges of parallelepiped P, as shown in Figure 1.3.6. Then the volume V of P is $\|\mathbf{x} \times \mathbf{y}\|$, which is the area of the base, multiplied by the height of P, which is the length of the projection of $\mathbf{z}$ onto $\mathbf{x} \times \mathbf{y}$. Since the latter is equal to

$$\left| \mathbf{z} \cdot \frac{\mathbf{x} \times \mathbf{y}}{\|\mathbf{x} \times \mathbf{y}\|} \right|,$$

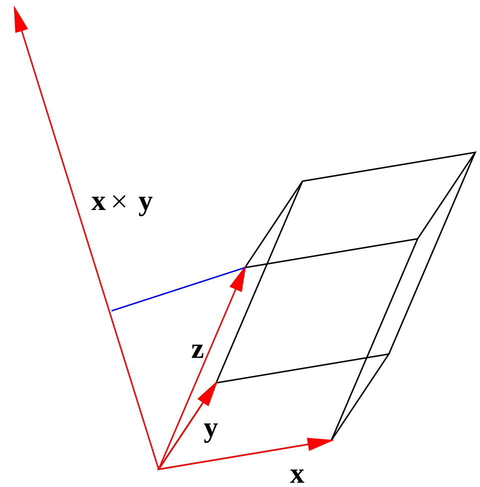


Figure 1.3.6 Parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$

we have

$$V = \|\mathbf{x}\times\mathbf{y}\|\left|\mathbf{z}\cdot\frac{\mathbf{x}\times\mathbf{y}}{\|\mathbf{x}\times\mathbf{y}\|}\right| = |\mathbf{z}\cdot(\mathbf{x}\times\mathbf{y})|. \tag{1.3.18}$$

Proposition The volume of a parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ is $|\mathbf{z}\cdot(\mathbf{x}\times\mathbf{y})|$.

Definition Given three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^3$, the quantity $\mathbf{z}\cdot(\mathbf{x}\times\mathbf{y})$ is called the *scalar triple product* of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}$.

Example Let $\mathbf{x} = (1,4,1)$, $\mathbf{y} = (-3,1,1)$, and $\mathbf{z} = (0,1,5)$ be adjacent edges of parallelepiped P (see Figure 1.3.7). Then

$$\mathbf{x}\times\mathbf{y} = (4-1, -3-1, 1+12) = (3,-4,13),$$

so

$$\mathbf{z}\cdot(\mathbf{x}\times\mathbf{y}) = 0 - 4 + 65 = 61.$$

Hence the volume of P is 61.

The final result of this section follows from (1.3.17) and the fact that the angle between parallel vectors is either 0 or π.

Proposition Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ are parallel if and only if $\mathbf{x}\times\mathbf{y} = \mathbf{0}$.

Note that, in particular, for any vector $\mathbf{x}$ in $\mathbb{R}^3$, $\mathbf{x}\times\mathbf{x} = \mathbf{0}$

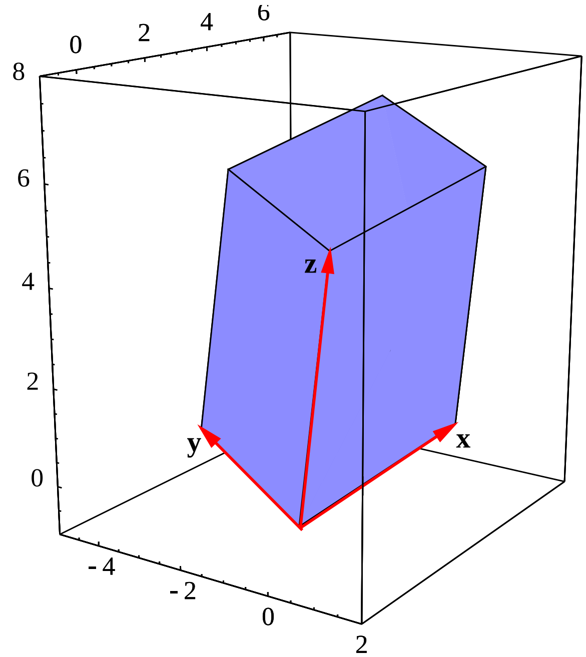


Figure 1.3.7 Parallelepiped with adjacent edges $\mathbf{x} = (1, 4, 1)$, $\mathbf{y} = (-3, 1, 1)$, $\mathbf{z} = (0, 1, 5)$

Problems

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, find $\mathbf{x} \times \mathbf{y}$ and verify that $\mathbf{x} \perp (\mathbf{x} \times \mathbf{y})$ and $\mathbf{y} \perp (\mathbf{x} \times \mathbf{y})$.

   (a) $\mathbf{x} = (1, 2, -1)$, $\mathbf{y} = (-2, 3, -1)$
   (b) $\mathbf{x} = (-2, 1, 4)$, $\mathbf{y} = (3, 1, 2)$
   (c) $\mathbf{x} = (1, 3, -2)$, $\mathbf{y} = (3, 9, 6)$
   (d) $\mathbf{x} = (-1, 4, 1)$, $\mathbf{y} = (3, 2, -1)$

2. Find the area of the parallelogram in $\mathbb{R}^3$ that has the vectors $\mathbf{x} = (2, 3, 1)$ and $\mathbf{y} = (-3, 3, 1)$ for adjacent sides.

3. Find the area of the parallelogram in $\mathbb{R}^2$ that has the vectors $\mathbf{x} = (3, 1)$ and $\mathbf{y} = (1, 4)$ for adjacent sides.

4. Find the area of the parallelogram in $\mathbb{R}^3$ that has vertices at $(1, 1, 1)$, $(2, 3, 2)$, $(-2, 4, 4)$, and $(-3, 2, 3)$.

5. Find the area of the parallelogram in $\mathbb{R}^2$ that has vertices at $(2, -1)$, $(4, -2)$, $(3, 0)$, and $(1, 1)$.

6. Find the area of the triangle in $\mathbb{R}^3$ that has vertices at $(1, 1, 0$, $(2, 3, 1)$, and $(-1, 3, 2)$.

7. Find the area of the triangle in $\mathbb{R}^2$ that has vertices at $(-1, 2)$, $(2, -1)$, and $(1, 3)$.

8. Find the volume of the parallelepiped that has the vectors $\mathbf{x} = (1, 2, 1)$, $\mathbf{y} = (-1, 1, 1)$, and $\mathbf{z} = (-1, -1, 6)$ for adjacent sides.

9. A parallelepiped has base vertices at $(1, 1, 1)$, $(2, 3, 2)$, $(-2, 4, 4)$, and $(-3, 2, 3)$ and top vertices at $(2, 2, 6)$, $(3, 4, 7)$, $(-1, 5, 9)$, and $(-2, 3, 8)$. Find its volume.

10. Verify the properties of the cross product stated in (1.3.8) through (1.3.12).

11. Since $|\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})|$, $|\mathbf{y} \cdot (\mathbf{z} \times \mathbf{x})|$, and $|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$are all equal to the volume of a parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, they should all have the same value. Show that in fact
$$\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}).$$
How do these compare with $\mathbf{z} \cdot (\mathbf{y} \times \mathbf{z})$, $\mathbf{y} \cdot (\mathbf{z} \times \mathbf{x})$, and $\mathbf{x} \cdot (\mathbf{z} \times \mathbf{y})$?

12. Suppose $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^3$. Show directly from the definition of the cross product that $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

13. Show by example that the cross product is not associative. That is, find vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ such that
$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) \neq (\mathbf{x} \times \mathbf{y}) \times \mathbf{z}.$$

The Calculus of Functions

of

Several Variables

Section 1.4

Lines, Planes, and Hyperplanes

In this section we will add to our basic geometric understanding of $\mathbb{R}^n$ by studying lines and planes. If we do this carefully, we shall see that working with lines and planes in $\mathbb{R}^n$ is no more difficult than working with them in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Lines in $\mathbb{R}^n$

We will start with lines. Recall from Section 1.1 that if $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^n$, then, for any scalar t, $t\mathbf{v}$ has the same direction as $\mathbf{v}$ when $t > 0$ and the opposite direction when $t < 0$. Hence the set of points

$$\{t\mathbf{v} : -\infty < t < \infty\}$$

forms a line through the origin. If we now add a vector $\mathbf{p}$ to each of these points, we obtain the set of points

$$\{t\mathbf{v} + \mathbf{p} : -\infty < t < \infty\},$$

which is a line through $\mathbf{p}$ in the direction of $\mathbf{v}$, as illustrated in Figure 1.4.1 for $\mathbb{R}^2$.

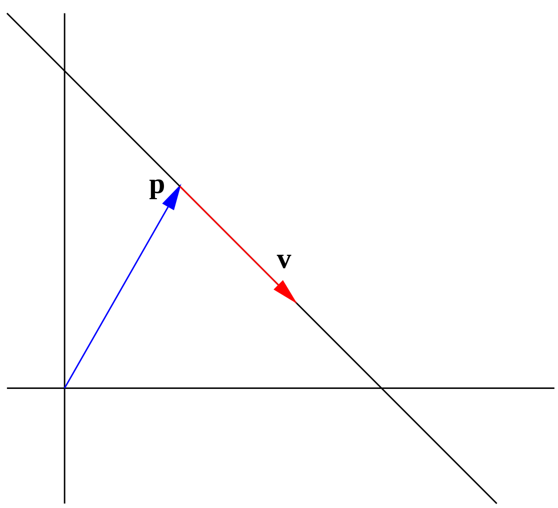


Figure 1.4.1 A line in $\mathbb{R}^2$ through $\mathbf{p}$ in the direction of $\mathbf{v}$

Definition Given a vector $\mathbf{p}$ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, the set of all points $\mathbf{y}$ in $\mathbb{R}^n$ such that

$$\mathbf{y} = t\mathbf{v} + \mathbf{p}, \tag{1.4.1}$$

where $-\infty < t < \infty$, is called the *line* through $\mathbf{p}$ in the direction of $\mathbf{v}$.

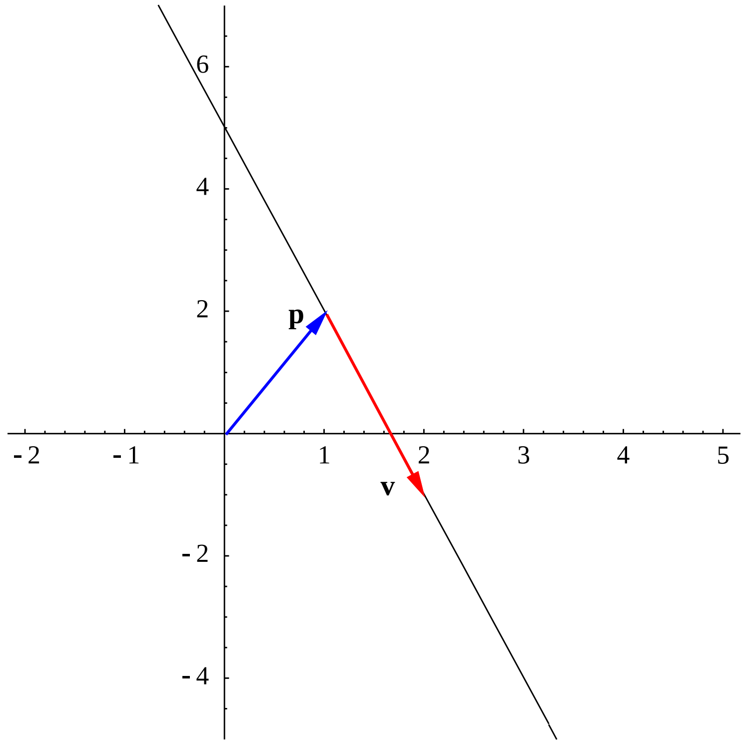


Figure 1.4.2 The line through $\mathbf{p} = (1, 2)$ in the direction of $\mathbf{v} = (1, -3)$

Equation (1.4.1) is called a *vector equation* for the line. If we write $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, then (1.4.1) may be written as

$$(y_1, y_2, \ldots, y_n) = t(v_1, v_2, \ldots, v_n) + (p_1, p_2, \ldots, p_n), \tag{1.4.2}$$

which holds if and only if

$$\begin{aligned} y_1 &= tv_1 + p_1, \\ y_2 &= tv_2 + p_2, \\ \vdots \quad & \quad \vdots \\ y_n &= tv_n + p_n. \end{aligned} \tag{1.4.3}$$

The equations in (1.4.3) are called *parametric equations* for the line.

Example Suppose L is the line in $\mathbb{R}^2$ through $\mathbf{p} = (1, 2)$ in the direction of $\mathbf{v} = (1, -3)$ (see Figure 1.4.2). Then

$$\mathbf{y} = t(1, -3) + (1, 2) = (t + 1, -3t + 2)$$

is a vector equation for L and, if we let $\mathbf{y} = (x, y)$,

$$\begin{aligned} x &= t + 1, \\ y &= -3t + 2 \end{aligned}$$

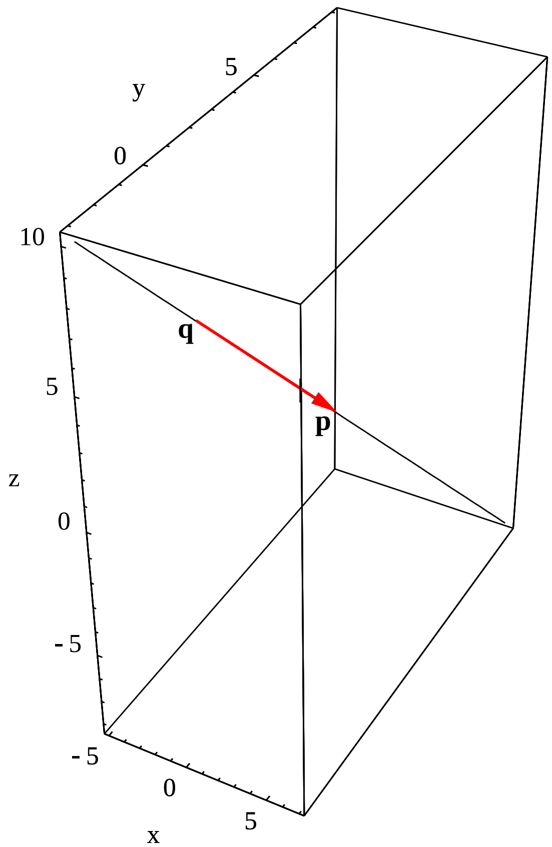


Figure 1.4.3 The line through $\mathbf{p} = (1, 3, 1)$ and $\mathbf{q} = (-1, 1, 4)$

are parametric equations for L. Note that if we solve for t in both of these equations, we have

$$t = x - 1,$$
$$t = \frac{2 - y}{3}.$$

Thus

$$x - 1 = \frac{2 - y}{3},$$

and so

$$y = -3x + 5.$$

Of course, the latter is just the standard slope-intercept form for the equation of a line in $\mathbb{R}^2$.

Example Now suppose we wish to find an equation for the line L in $\mathbb{R}^3$ which passes through the points $\mathbf{p} = (1, 3, 1)$ and $\mathbf{q} = (-1, 1, 4)$ (see Figure 1.4.3). We first note that the vector

$$\mathbf{p} - \mathbf{q} = (2, 2, -3)$$

gives the direction of the line, so

$$\mathbf{y} = t(2, 2, -3) + (1, 3, 1)$$

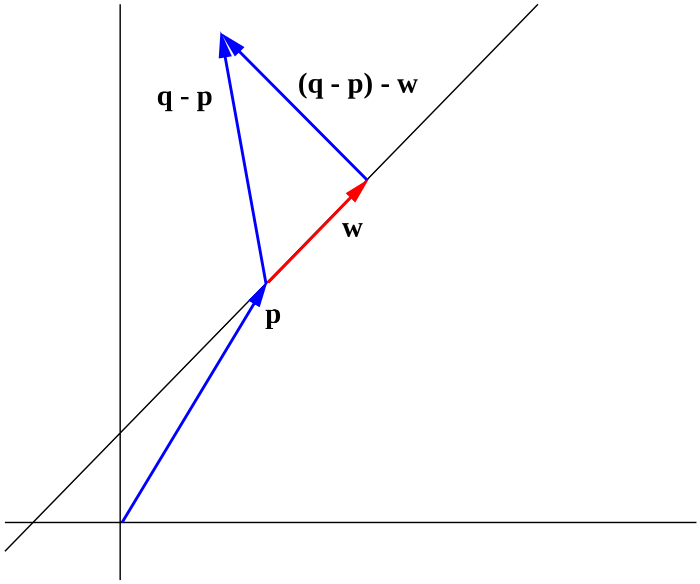


Figure 1.4.4 Distance from a point $\mathbf{q}$ to a line

is a vector equation for L; if we let $\mathbf{y} = (x, y, z)$,

$$\begin{aligned} x &= 2t + 1, \\ y &= 2t + 3, \\ z &= -3t + 1 \end{aligned}$$

are parametric equations for L.

As an application of these ideas, consider the problem of finding the shortest distance from a point $\mathbf{q}$ in $\mathbb{R}^n$ to a line L with equation $\mathbf{y} = t\mathbf{v} + \mathbf{p}$. If we let $\mathbf{w}$ be the projection of $\mathbf{q} - \mathbf{p}$ onto $\mathbf{v}$, then, as we saw in Section 1.2, the vector $(\mathbf{q} - \mathbf{p}) - \mathbf{w}$ is orthogonal to $\mathbf{v}$ and may be pictured with its tail on L and its tip at $\mathbf{q}$. Hence the shortest distance from $\mathbf{q}$ to L is $\|(\mathbf{q} - \mathbf{p}) - \mathbf{w}\|$. See Figure 1.4.4.

Example To find the distance from the point $\mathbf{q} = (2, 2, 4)$ to the line L through the points $\mathbf{p} = (1, 0, 0)$ and $\mathbf{r} = (0, 1, 0)$, we must first find an equation for L. Since the direction of L is given by $\mathbf{v} = \mathbf{r} - \mathbf{p} = (-1, 1, 0)$, a vector equation for L is

$$\mathbf{y} = t(-1, 1, 0) + (1, 0, 0).$$

If we let

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}(-1, 1, 0),$$

then the projection of $\mathbf{q} - \mathbf{p}$ onto $\mathbf{v}$ is

$$\mathbf{w} = ((\mathbf{q} - \mathbf{p}) \cdot \mathbf{u})\mathbf{u} = \left((1, 2, 4) \cdot \frac{1}{\sqrt{2}}(-1, 1, 0)\right) \frac{1}{\sqrt{2}}(-1, 1, 0) = \frac{1}{2}(-1, 1, 0).$$

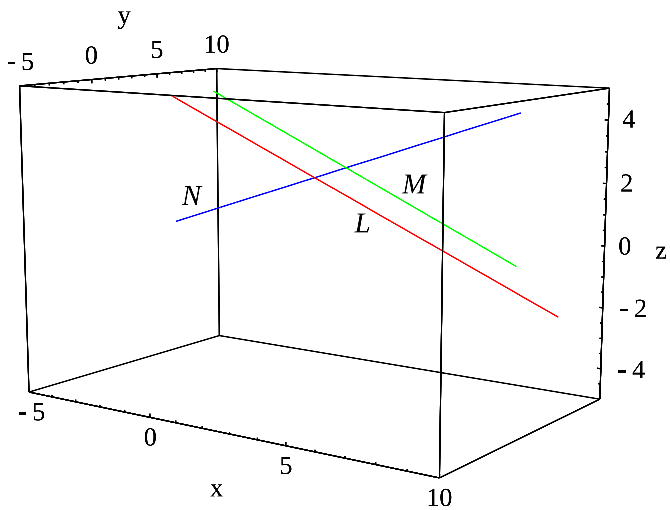


Figure 1.4.5 Parallel (L and M) and perpendicular (L and N) lines

Thus the distance from $\mathbf{q}$ to L is

$$\|(\mathbf{q}-\mathbf{p})-\mathbf{w}\| = \left\|\left(\frac{3}{2},\frac{3}{2},4\right)\right\| = \sqrt{\frac{82}{4}} = \sqrt{20.5}.$$

Definition Suppose L and M are lines in $\mathbb{R}^n$ with equations $\mathbf{y} = t\mathbf{v}+\mathbf{p}$ and $\mathbf{y} = t\mathbf{w}+\mathbf{q}$, respectively. We say L and M are *parallel* if $\mathbf{v}$ and $\mathbf{w}$ are parallel. We say L and M are *perpendicular*, or *orthogonal*, if they intersect and $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Note that, by definition, a line is parallel to itself.

Example The lines L and M in $\mathbb{R}^3$ with equations

$$\mathbf{y} = t(1,2,-1)+(4,1,2)$$

and

$$\mathbf{y} = t(-2,-4,2)+(5,6,1),$$

respectively, are parallel since $(-2,-4,2) = -2(1,2,-1)$, that is, the vectors $(1,2,-1)$ and $(-2,-4,2)$ are parallel. See Figure 1.4.5.

Example The lines L and N in $\mathbb{R}^3$ with equations

$$\mathbf{y} = t(1,2,-1)+(4,1,2)$$

and

$$\mathbf{y} = t(3,-1,1)+(-1,5,-1),$$

respectively, are perpendicular since they intersect at $(5,3,1)$ (when $t=1$ for the first line and $t=2$ for the second line) and $(1,2,-1)$ and $(3,-1,1)$ are orthogonal since

$$(1,2,-1)\cdot(3,-1,1) = 3-2-1 = 0.$$

See Figure 1.4.5.

Planes in $\mathbb{R}^n$

The following definition is the first step in defining a plane.

Definition Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to be *linearly independent* if neither one is a scalar multiple of the other.

Geometrically, $\mathbf{x}$ and $\mathbf{y}$ are linearly independent if they do not lie on the same line through the origin. Notice that for any vector $\mathbf{x}$, $\mathbf{0}$ and $\mathbf{x}$ are not linearly independent, that is, they are *linearly dependent*, since $\mathbf{0} = 0\mathbf{x}$.

Definition Given a vector $\mathbf{p}$ along with linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$, all in $\mathbb{R}^n$, the set of all points $\mathbf{y}$ such that

$$\mathbf{y} = t\mathbf{v} + s\mathbf{w} + \mathbf{p}, \tag{1.4.4}$$

where $-\infty < t < \infty$ and $-\infty < s < \infty$, is called a *plane*.

The intuition here is that a plane should be a two dimensional object, which is guaranteed because of the requirement that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. Also note that if we let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, then (1.4.4) implies that

$$\begin{aligned} y_1 &= tv_1 + sw_1 + p_1, \\ y_2 &= tv_2 + sw_2 + p_2, \\ &\ \ \vdots \qquad\quad \vdots \\ y_n &= tv_n + sw_n + p_n. \end{aligned} \tag{1.4.5}$$

As with lines, (1.4.4) is a *vector equation* for the plane and the equations in (1.4.5) are *parametric equations* for the plane.

Example Suppose we wish to find an equation for the plane P in $\mathbb{R}^3$ which contains the three points $\mathbf{p} = (1, 2, 1)$, $\mathbf{q} = (-1, 3, 2)$, and $\mathbf{r} = (2, 3, -1)$. The first step is to find two linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$ which lie in the plane. Since P must contain the line segments from $\mathbf{p}$ to $\mathbf{q}$ and from $\mathbf{p}$ to $\mathbf{r}$, we can take

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = (-2, 1, 1)$$

and

$$\mathbf{w} = \mathbf{r} - \mathbf{p} = (1, 1, -2).$$

Note that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, a consequence of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ not all lying on the same line. See Figure 1.4.6. We may now write a vector equation for P as

$$\mathbf{y} = t(-2, 1, 1) + s(1, 1, -2) + (1, 2, 1).$$

Note that $\mathbf{y} = \mathbf{p}$ when $t = 0$ and $s = 0$, $\mathbf{y} = \mathbf{q}$ when $t = 1$ and $s = 0$, and $\mathbf{y} = \mathbf{r}$ when $t = 0$ and $s = 1$. If we write $\mathbf{y} = (x, y, z)$, then, expanding the vector equation,

$$(x, y, z) = t(-2, 1, 1) + s(1, 1, -2) + (1, 2, 1) = (-2t + s + 1, t + s + 2, t - 2s + 1),$$

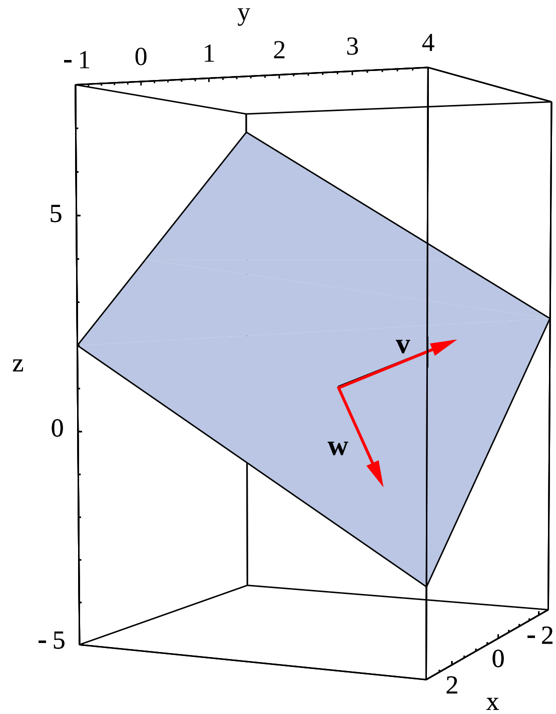


Figure 1.4.6 The plane $\mathbf{y} = t\mathbf{v} + s\mathbf{w} + \mathbf{p}$, with $\mathbf{v} = (-2, 1, 1)$, $\mathbf{w} = (1, 1, -2)$, $\mathbf{p} = (1, 2, 1)$

giving us

$$\begin{aligned} x &= -2t + s + 1, \\ y &= t + s + 2, \\ z &= t - 2s + 1 \end{aligned}$$

for parametric equations for P.

To find the shortest distance from a point $\mathbf{q}$ to a plane P, we first need to consider the problem of finding the projection of a vector onto a plane. To begin, consider the plane P through the origin with equation $\mathbf{y} = t\mathbf{a} + s\mathbf{b}$ where $\|a\| = 1$, $\|b\| = 1$, and $\mathbf{a} \perp \mathbf{b}$. Given a vector $\mathbf{q}$ not in P, let

$$\mathbf{r} = (\mathbf{q} \cdot \mathbf{a})\mathbf{a} + (\mathbf{q} \cdot \mathbf{b})\mathbf{b},$$

the sum of the projections of $\mathbf{q}$ onto $\mathbf{a}$ and onto $\mathbf{b}$. Then

$$\begin{aligned} (\mathbf{q} - \mathbf{r}) \cdot \mathbf{a} &= \mathbf{q} \cdot \mathbf{a} - \mathbf{r} \cdot \mathbf{a} \\ &= \mathbf{q} \cdot \mathbf{a} - (\mathbf{q} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{a}) - (\mathbf{q} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{a}) \\ &= \mathbf{q} \cdot \mathbf{a} - \mathbf{q} \cdot \mathbf{a} = 0, \end{aligned}$$

since $\mathbf{a} \cdot \mathbf{a} = \|a\|^2 = 1$ and $\mathbf{b} \cdot \mathbf{a} = 0$, and, similarly,

$$\begin{aligned} (\mathbf{q} - \mathbf{r}) \cdot \mathbf{b} &= \mathbf{q} \cdot \mathbf{b} - \mathbf{r} \cdot \mathbf{b} \\ &= \mathbf{q} \cdot \mathbf{b} - (\mathbf{q} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{q} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{b}) \\ &= \mathbf{q} \cdot \mathbf{b} - \mathbf{q} \cdot \mathbf{b} = 0. \end{aligned}$$

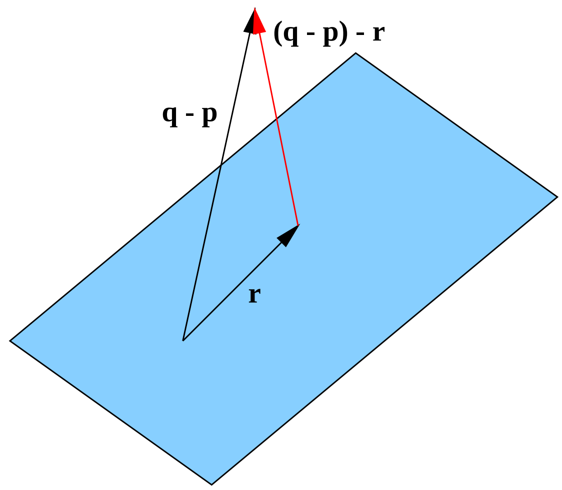


Figure 1.4.7 Distance from a point $\mathbf{q}$ to a plane

It follows that for any $\mathbf{y} = t\mathbf{a} + s\mathbf{b}$ in the plane P,

$$(\mathbf{q} - \mathbf{r}) \cdot \mathbf{y} = (\mathbf{q} - \mathbf{r}) \cdot (t\mathbf{a} + s\mathbf{b}) = t(\mathbf{q} - \mathbf{r}) \cdot \mathbf{a} + s(\mathbf{q} - \mathbf{r}) \cdot \mathbf{b} = 0.$$

That is, $\mathbf{q} - \mathbf{r}$ is orthogonal to every vector in the plane P. For this reason, we call $\mathbf{r}$ the *projection* of $\mathbf{q}$ onto the plane P, and we note that the shortest distance from $\mathbf{q}$ to P is $\|\mathbf{q} - \mathbf{r}\|$.

In the general case, given a point $\mathbf{q}$ and a plane P with equation $\mathbf{y} = t\mathbf{v} + s\mathbf{w} + \mathbf{p}$, we need only find vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \perp \mathbf{b}$, $\|a\| = 1$, $\|b\| = 1$, and the equation $\mathbf{y} = t\mathbf{a} + s\mathbf{b} + \mathbf{p}$ describes the same plane P. You are asked in Problem 29 to verify that if we let $\mathbf{c}$ be the projection of $\mathbf{w}$ onto $\mathbf{v}$, then we may take

$$\mathbf{a} = \frac{1}{\|\mathbf{v}\|}\,\mathbf{v}$$

and

$$\mathbf{b} = \frac{1}{\|\mathbf{w} - \mathbf{c}\|}(\mathbf{w} - \mathbf{c}).$$

If $\mathbf{r}$ is the sum of the projections of $\mathbf{q} - \mathbf{p}$ onto $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{r}$ is the projection of $\mathbf{q} - \mathbf{p}$ onto P and $\|(\mathbf{q} - \mathbf{p}) - \mathbf{r}\|$ is the shortest distance from $\mathbf{q}$ to P. See Figure 1.4.7.

Example To compute the distance from the point $\mathbf{q} = (2, 3, 3)$ to the plane P with equation

$$\mathbf{y} = t(-2, 1, 0) + s(1, -1, 1) + (-1, 2, 1),$$

let $\mathbf{v} = (-2, 1, 0)$, $\mathbf{w} = (1, -1, 1)$, and $\mathbf{p} = (-1, 2, 1)$. Then, using the above notation, we have

$$\mathbf{a} = \frac{1}{\sqrt{5}}(-2, 1, 0),$$

$$\mathbf{c} = (\mathbf{w} \cdot \mathbf{a})\mathbf{a} = -\frac{3}{5}(-2, 1, 0),$$

$$\mathbf{w} - \mathbf{c} = \frac{1}{5}(-1, -2, 5),$$

and

$$\mathbf{b} = \frac{1}{\sqrt{30}}(-1, -2, 5).$$

Since $\mathbf{q} - \mathbf{p} = (3, 1, 2)$, the projection of $\mathbf{q} - \mathbf{p}$ onto P is

$$\mathbf{r} = ((3, 1, 2) \cdot \mathbf{a})\mathbf{a} + ((3, 1, 2) \cdot \mathbf{b})\mathbf{b} = -(-2, 1, 0) + \frac{1}{6}(-1, -2, 5) = \frac{1}{6}(11, -8, 5)$$

and

$$(\mathbf{q} - \mathbf{p}) - \mathbf{r} = \frac{1}{6}(7, 14, 7).$$

Hence the distance from $\mathbf{q}$ to P is

$$\|(\mathbf{q} - \mathbf{p}) - \mathbf{r}\| = \frac{\sqrt{294}}{6} = \frac{7}{\sqrt{6}}.$$

More generally, we say vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are *linearly independent* if no one of them can be written as a sum of scalar multiples of the others. Given a vector $\mathbf{p}$ and linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, we call the set of all points $\mathbf{y}$ such that

$$\mathbf{y} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k + \mathbf{p},$$

where $-\infty < t_j < \infty$, $j = 1, 2, \ldots, k$, a *k-dimensional affine subspace* of $\mathbb{R}^n$. In this terminology, a line is a 1-dimensional affine subspace and a plane is a 2-dimensional affine subspace. In the following, we will be interested primarily in lines and planes and so will not develop the details of the more general situation at this time.

Hyperplanes

Consider the set L of all points $\mathbf{y} = (x, y)$ in $\mathbb{R}^2$ which satisfy the equation

$$ax + by + d = 0, \tag{1.4.6}$$

where a, b, and d are scalars with at least one of a and b not being 0. If, for example, $b \neq 0$, then we can solve for y, obtaining

$$y = -\frac{a}{b}x - \frac{d}{b}. \tag{1.4.7}$$

If we set $x = t$, $-\infty < t < \infty$, then the solutions to (1.4.6) are

$$\mathbf{y} = (x, y) = \left(t, -\frac{a}{b}t - \frac{d}{b}\right) = t\left(1, -\frac{a}{b}\right) + \left(0, -\frac{d}{b}\right). \tag{1.4.8}$$

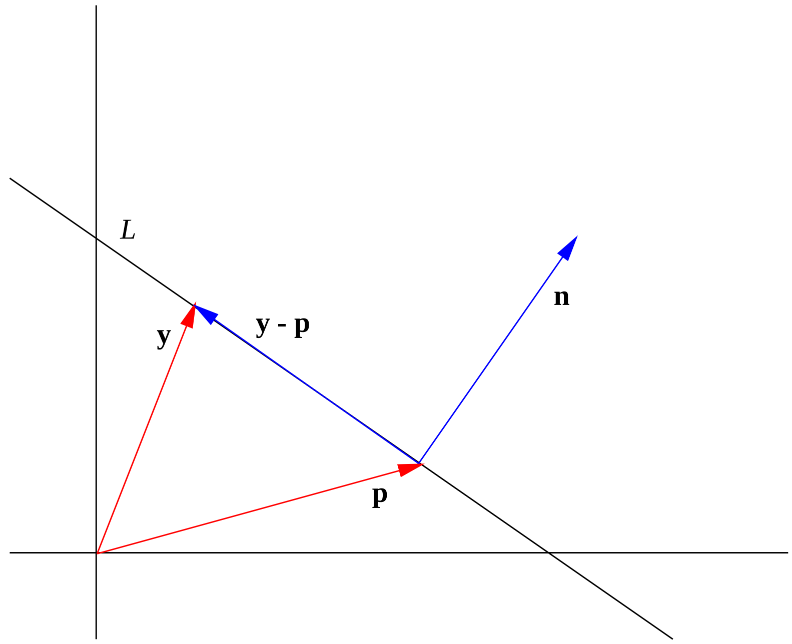


Figure 1.4.8 L is the set of points $\mathbf{y}$ for which $\mathbf{y}-\mathbf{p}$ is orthogonal to $\mathbf{n}$

Thus L is a line through $\left(0, -\frac{d}{b}\right)$ in the direction of $\left(1, -\frac{a}{b}\right)$. A similar calculation shows that if $a \neq 0$, then we can describe L as the line through $\left(-\frac{d}{a}, 0\right)$ in the direction of $\left(-\frac{b}{a}, 1\right)$. Hence in either case L is a line in $\mathbb{R}^2$.

Now let $\mathbf{n} = (a, b)$ and note that (1.4.6) is equivalent to

$$\mathbf{n} \cdot \mathbf{y} + d = 0. \tag{1.4.9}$$

Moreover, if $\mathbf{p} = (p_1, p_2)$ is a point on L, then

$$\mathbf{n} \cdot \mathbf{p} + d = 0, \tag{1.4.10}$$

which implies that $d = -\mathbf{n} \cdot \mathbf{p}$. Thus we may write (1.4.9) as

$$\mathbf{n} \cdot \mathbf{y} - \mathbf{n} \cdot \mathbf{p} = 0,$$

and so we see that (1.4.6) is equivalent to the equation

$$\mathbf{n} \cdot (\mathbf{y} - \mathbf{p}) = 0. \tag{1.4.11}$$

Equation (1.4.11) is a *normal equation* for the line L and $\mathbf{n}$ is a *normal vector* for L. In words, (1.4.11) says that the line L consists of all points in $\mathbb{R}^2$ whose difference with $\mathbf{p}$ is orthogonal to $\mathbf{n}$. See Figure 1.4.8.

Example Suppose L is a line in $\mathbb{R}^2$ with equation

$$2x + 3y = 1.$$

Then a normal vector for L is $\mathbf{n} = (2, 3)$; to find a point on L, we note that when $x = 2$, $y = -1$, so $\mathbf{p} = (2, -1)$ is a point on L. Thus

$$(2, 3) \cdot ((x, y) - (2, -1)) = 0,$$

or, equivalently,

$$(2, 3) \cdot (x - 2, y + 1) = 0,$$

is a normal equation for L. Since $\mathbf{q} = (-1, 1)$ is also a point on L, L has direction $\mathbf{q} - \mathbf{p} = (-3, 2)$. Thus

$$\mathbf{y} = t(-3, 2) + (2, -1)$$

is a vector equation for L. Note that

$$\mathbf{n} \cdot (\mathbf{q} - \mathbf{p}) = (2, 3) \cdot (-3, 2) = 0,$$

so $\mathbf{n}$ is orthogonal to $\mathbf{q} - \mathbf{p}$.

Example If L is a line in $\mathbb{R}^2$ through $\mathbf{p} = (2, 3)$ in the direction of $\mathbf{v} = (-1, 2)$, then $\mathbf{n} = (2, 1)$ is a normal vector for L since $\mathbf{v} \cdot \mathbf{n} = 0$. Thus

$$(2, 1) \cdot (x - 2, y - 3) = 0$$

is a normal equation for L. Multiplying this out, we have

$$2(x - 2) + (y - 3) = 0;$$

that is, L consists of all points (x, y) in $\mathbb{R}^2$ which satisfy

$$2x + y = 7.$$

Now consider the case where P is the set of all points $\mathbf{y} = (x, y, z)$ in $\mathbb{R}^3$ that satisfy the equation

$$ax + by + cz + d = 0, \tag{1.4.12}$$

where a, b, c, and d are scalars with at least one of a, b, and c not being 0. If for example, $a \neq 0$, then we may solve for x to obtain

$$x = -\frac{b}{a}y - \frac{c}{a}z - \frac{d}{a}. \tag{1.4.13}$$

If we set $y = t$, $-\infty < t < \infty$, and $z = s$, $-\infty < s < \infty$, the solutions to (1.4.12) are

$$\begin{aligned} \mathbf{y} &= (x, y, z) \\ &= \left(-\frac{b}{a}t - \frac{c}{a}s - \frac{d}{a}, t, s\right) \\ &= t\left(-\frac{b}{a}, 1, 0\right) + s\left(-\frac{c}{a}, 0, 1\right) + \left(-\frac{d}{a}, 0, 0\right). \end{aligned} \tag{1.4.14}$$

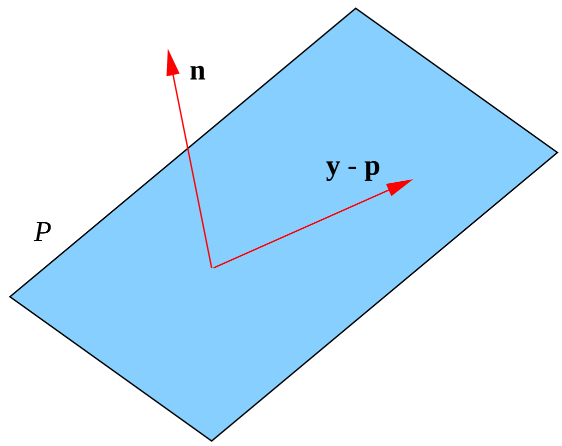


Figure 1.4.9 P is the set of points $\mathbf{y}$ for which $\mathbf{y}-\mathbf{p}$ is orthogonal to $\mathbf{n}$

Thus we see that P is a plane in $\mathbb{R}^3$. In analogy with the case of lines in $\mathbb{R}^2$, if we let $\mathbf{n}=(a,b,c)$ and let $\mathbf{p}=(p_1,p_2,p_3)$ be a point on P, then we have

$$\mathbf{n}\cdot\mathbf{p}+d=ax+by+cz+d=0,$$

from which we see that $\mathbf{n}\cdot\mathbf{p}=-d$, and so we may write (1.4.12) as

$$\mathbf{n}\cdot(\mathbf{y}-\mathbf{p})=0. \tag{1.4.15}$$

We call (1.4.15) a *normal equation* for P and we call $\mathbf{n}$ a *normal vector* for P. In words, (1.4.15) says that the plane P consists of all points in $\mathbb{R}^3$ whose difference with $\mathbf{p}$ is orthogonal to $\mathbf{n}$. See Figure 1.4.9.

Example Let P be the plane in $\mathbb{R}^3$ with vector equation

$$\mathbf{y}=t(2,2,-1)+s(-1,2,1)+(1,1,2).$$

If we let $\mathbf{v}=(2,2,-1)$ and $\mathbf{w}=(-1,2,1)$, then

$$\mathbf{n}=\mathbf{v}\times\mathbf{w}=(4,-1,6)$$

is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Now if $\mathbf{y}$ is on P, then

$$\mathbf{y}=t\mathbf{v}+s\mathbf{w}+\mathbf{p}$$

for some scalars t and s, from which we see that

$$\mathbf{n}\cdot(\mathbf{y}-\mathbf{p})=\mathbf{n}\cdot(t\mathbf{v}+s\mathbf{w})=t(\mathbf{n}\cdot\mathbf{v})+s(\mathbf{n}\cdot\mathbf{w})=0+0=0.$$

That is, $\mathbf{n}$ is a normal vector for P. So, letting $\mathbf{y}=(x,y,z)$,

$$(4,-1,6)\cdot(x-1,y-1,z-2)=0 \tag{1.4.16}$$

is a normal equation for P. Multiplying (1.4.16) out, we see that P consists of all points (x, y, z) in $\mathbb{R}^3$ which satisfy

$$4x - y + 6z = 15.$$

Example Suppose $\mathbf{p} = (1, 2, 1)$, $\mathbf{q} = (-2, -1, 3)$, and $\mathbf{r} = (2, -3, -1)$ are three points on a plane P in $\mathbb{R}^3$. Then

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = (-3, -3, 2)$$

and

$$\mathbf{w} = \mathbf{r} - \mathbf{p} = (1, -5, -2)$$

are vectors lying on P. Thus

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = (16, -4, 18)$$

is a normal vector for P. Hence

$$(16, -4, 18) \cdot (x - 1, y - 2, z - 1) = 0$$

is a normal equation for P. Thus P is the set of all points (x, y, z) in $\mathbb{R}^3$ satisfying

$$16x - 4y + 18y = 26.$$

The following definition generalizes the ideas in the previous examples.

Definition Suppose $\mathbf{n}$ and $\mathbf{p}$ are vectors in $\mathbb{R}^n$ with $\mathbf{n} \neq \mathbf{0}$. The set of all vectors $\mathbf{y}$ in $\mathbb{R}^n$ which satisfy the equation

$$\mathbf{n} \cdot (\mathbf{y} - \mathbf{p}) = 0 \tag{1.4.17}$$

is called a *hyperplane* through the point $\mathbf{p}$. We call $\mathbf{n}$ a *normal vector* for the hyperplane and we call (1.4.17) a *normal equation* for the hyperplane.

In this terminology, a line in $\mathbb{R}^2$ is a hyperplane and a plane in $\mathbb{R}^3$ is a hyperplane. In general, a hyperplane in $\mathbb{R}^n$ is an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. Also, note that if we let $\mathbf{n} = (a_1, a_2, \ldots, a_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, then we may write (1.4.17) as

$$a_1(y_1 - p_1) + a_2(y_2 - p_2) + \cdots + a_n(y_n - p_n) = 0, \tag{1.4.18}$$

or

$$a_1y_1 + a_2y_2 + \cdots + a_ny_n + d = 0 \tag{1.4.19}$$

where $d = -\mathbf{n} \cdot \mathbf{p}$.

Example The set of all points (w, x, y, z) in $\mathbb{R}^4$ which satisfy

$$3w - x + 4y + 2z = 5$$

is a 3-dimensional hyperplane with normal vector $\mathbf{n} = (3, -1, 4, 2)$.

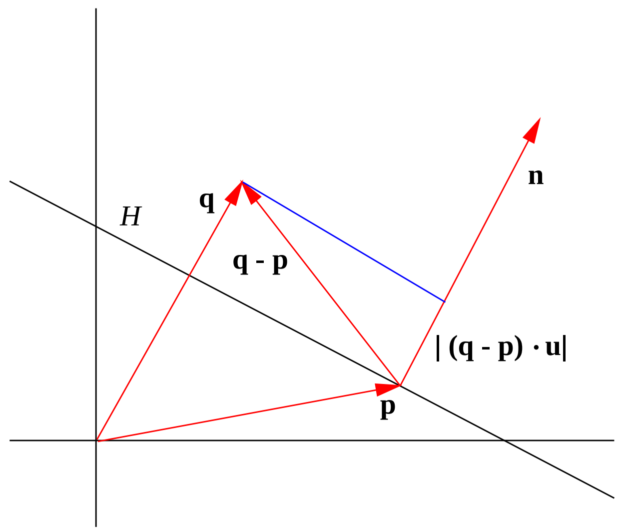


Figure 1.4.10 Distance from a point $\mathbf{q}$ to a hyperplane H

The normal equation description of a hyperplane simplifies a number of geometric calculations. For example, given a hyperplane H through $\mathbf{p}$ with normal vector $\mathbf{n}$ and a point $\mathbf{q}$ in $\mathbb{R}^n$, the distance from $\mathbf{q}$ to H is simply the length of the projection of $\mathbf{q}-\mathbf{p}$ onto $\mathbf{n}$. Thus if $\mathbf{u}$ is the direction of $\mathbf{n}$, then the distance from $\mathbf{q}$ to H is $|(\mathbf{q}-\mathbf{p})\cdot\mathbf{u}|$. See Figure 1.4.10. Moreover, if we let $d = -\mathbf{p}\cdot\mathbf{n}$ as in (1.4.19), then we have

$$|(\mathbf{q}-\mathbf{p})\cdot\mathbf{u}| = |\mathbf{q}\cdot\mathbf{u}-\mathbf{p}\cdot\mathbf{u}| = \frac{\mathbf{q}\cdot\mathbf{n}-\mathbf{p}\cdot\mathbf{n}}{\|\mathbf{n}\|} = \frac{|\mathbf{q}\cdot\mathbf{n}+d|}{\|\mathbf{n}\|}. \tag{1.4.20}$$

Note that, in particular, (1.4.20) may be used to find the distance from a point to a line in $\mathbb{R}^2$ and from a point to a plane in $\mathbb{R}^3$.

Example To find the distance from the point $\mathbf{q} = (2,3,3)$ to the plane P in $\mathbb{R}^3$ with equation

$$x + 2y + z = 4,$$

we first note that $\mathbf{n} = (1,2,1)$ is a normal vector for P. Using (1.4.20) with $d = -4$, we see that the distance from $\mathbf{q}$ to P is

$$\frac{|\mathbf{q}\cdot\mathbf{n}+d|}{\|\mathbf{n}\|} = \frac{|(2,3,3)\cdot(1,2,1)-4|}{\sqrt{6}} = \frac{7}{\sqrt{6}}.$$

Note that this agrees with an earlier example.

We will close this section with a few words about angles between hyperplanes. Note that a hyperplane does not have a unique normal vector. In particular, if $\mathbf{n}$ is a normal vector for a hyperplane H, then $-\mathbf{n}$ is also a normal vector for H. Hence it is always possible to choose the normal vectors required in the following definition.

Definition Let G and H be hyperplanes in $\mathbb{R}^n$ with normal equations

$$\mathbf{m}\cdot(\mathbf{y}-\mathbf{p}) = 0$$

and

$$\mathbf{n} \cdot (\mathbf{y} - \mathbf{q}) = 0,$$

respectively, chosen so that $\mathbf{m} \cdot \mathbf{n} \geq 0$. Then the *angle* between G and H is the angle between $\mathbf{m}$ and $\mathbf{n}$. Moreover, we will say that G and H are *orthogonal* if $\mathbf{m}$ and $\mathbf{n}$ are orthogonal and we will say G and H are *parallel* if $\mathbf{m}$ and $\mathbf{n}$ are parallel.

The effect of the choice of normal vectors in the definition is to make the angle between the two hyperplanes be between 0 and $\frac{\pi}{2}$.

Example To find the angle θ between the two planes in $\mathbb{R}^3$ with equations

$$x + 2y - z = 3$$

and

$$x - 3y - z = 5,$$

we first note that the corresponding normal vectors are $\mathbf{m} = (1, 2, -1)$ and $\mathbf{n} = (1, -3, -1)$. Since $\mathbf{m} \cdot \mathbf{n} = -4$, we will compute the angle between $\mathbf{m}$ and $-\mathbf{n}$. Hence

$$\cos(\theta) = \frac{\mathbf{m} \cdot (-\mathbf{n})}{\|\mathbf{m}\|\|\mathbf{n}\|} = \frac{4}{\sqrt{6}\sqrt{11}} = \frac{4}{\sqrt{66}}.$$

Thus, rounding to four decimal places,

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right) = 1.0560.$$

See Figure 1.4.11.

Example The planes in $\mathbb{R}^3$ with equations

$$3x + y - 2z = 3$$

and

$$6x + 2y - 4z = 13$$

are parallel since their normal vectors are $\mathbf{m} = (3, 1, -2)$ and $\mathbf{n} = (6, 2, -4)$ and $\mathbf{n} = 2\mathbf{m}$.

Problems

1. Find vector and parametric equations for the line in $\mathbb{R}^2$ through $\mathbf{p} = (2, 3)$ in the direction of $\mathbf{v} = (1, -2)$.
2. Find vector and parametric equations for the line in $\mathbb{R}^4$ through $\mathbf{p} = (1, -1, 2, 3)$ in the direction of $\mathbf{v} = (-2, 3, -4, 1)$.
3. Find vector and parametric equations for the lines passing through the following pairs of points.

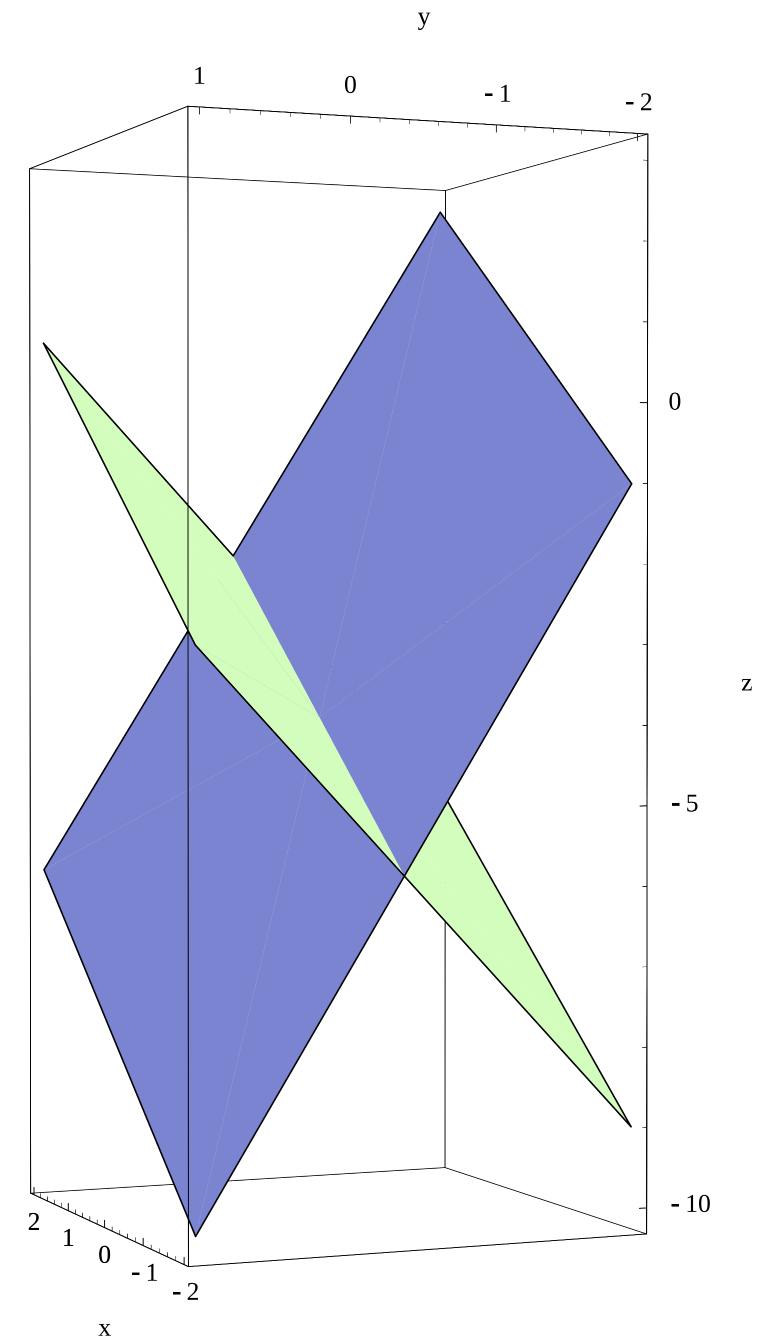


Figure 1.4.11 The planes $x + 2y - z = 3$ and $x - 3y - z = 5$

(a) $\mathbf{p} = (-1, -3)$, $\mathbf{q} = (4, 2)$

(b) $\mathbf{p} = (2, 1, 3)$, $\mathbf{q} = (-1, 2, 1)$

(c) $\mathbf{p} = (3, 2, 1, 4)$, $\mathbf{q} = (2, 0, 4, 1)$

(d) $\mathbf{p} = (4, -3, 2)$, $\mathbf{q} = (1, -2, 4)$

4. Find the distance from the point $\mathbf{q} = (1, 3)$ to the line with vector equation $\mathbf{y} = t(2, 1) + (3, 1)$.

5. Find the distance from the point $\mathbf{q} = (1, 3, -2)$ to the line with vector equation $\mathbf{y} = t(2, -1, 4) + (1, -2, -1)$.

6. Find the distance from the point $\mathbf{r} = (-1, 2, -3)$ to the line through the points $\mathbf{p} = (1, 0, 1)$ and $\mathbf{q} = (0, 2, -1)$.

7. Find the distance from the point $\mathbf{r} = (-1, -2, 2, 4)$ to the line through the points $\mathbf{p} = (2, 1, 1, 2)$ and $\mathbf{q} = (1, 2, -4, 3)$.

8. Find vector and parametric equations for the plane in $\mathbb{R}^3$ which contains the points $\mathbf{p} = (1, 3, -1)$, $\mathbf{q} = (-2, 1, 1)$, and $\mathbf{r} = (2, -3, 2)$.

9. Find vector and parametric equations for the plane in $\mathbb{R}^4$ which contains the points $\mathbf{p} = (2, -3, 4, -1)$, $\mathbf{q} = (-1, 3, 2, -4)$, and $\mathbf{r} = (2, -1, 2, 1)$.

10. Let P be the plane in $\mathbb{R}^3$ with vector equation $\mathbf{y} = t(1, 2, 1) + s(-2, 1, 3) + (1, 0, 1)$. Find the distance from the point $\mathbf{q} = (1, 3, 1)$ to P.

11. Let P be the plane in $\mathbb{R}^4$ with vector equation $\mathbf{y} = t(1, -2, 1, 4) + s(2, 1, 2, 3) + (1, 0, 1, 0)$. Find the distance from the point $\mathbf{q} = (1, 3, 1, 3)$ to P.

12. Find a normal vector and a normal equation for the line in $\mathbb{R}^2$ with vector equation $\mathbf{y} = t(1, 2) + (1, -1)$.

13. Find a normal vector and a normal equation for the line in $\mathbb{R}^2$ with vector equation $\mathbf{y} = t(0, 1) + (2, 0)$.

14. Find a normal vector and a normal equation for the plane in $\mathbb{R}^3$ with vector equation $\mathbf{y} = t(1, 2, 1) + s(3, 1, -1) + (1, -1, 1)$.

15. Find a normal vector and a normal equation for the line in $\mathbb{R}^2$ which passes through the points $\mathbf{p} = (3, 2)$ and $\mathbf{q} = (-1, 3)$.

16. Find a normal vector and a normal equation for the plane in $\mathbb{R}^3$ which passes through the points $\mathbf{p} = (1, 2, -1)$, $\mathbf{q} = (-1, 3, 1)$, and $\mathbf{r} = (2, -2, 2)$.

17. Find the distance from the point $\mathbf{q} = (3, 2)$ in $\mathbb{R}^2$ to the line with equation $x + 2y - 3 = 0$.

18. Find the distance from the point $\mathbf{q} = (1, 2, -1)$ in $\mathbb{R}^3$ to the plane with equation $x + 2y - 3x = 4$.

19. Find the distance from the point $\mathbf{q} = (3, 2, 1, 1)$ in $\mathbb{R}^4$ to the hyperplane with equation $3x + y - 2z + 3w = 15$.

20. Find the angle between the lines in $\mathbb{R}^2$ with equations $3x + y = 4$ and $x - y = 5$.

21. Find the angle between the planes in $\mathbb{R}^3$ with equations $3x - y + 2z = 5$ and $x - 2y + z = 4$.

22. Find the angle between the hyperplanes in $\mathbb{R}^4$ with equations $w + x + y - z = 3$ and $2w - x + 2y + z = 6$.

23. Find an equation for a plane in $\mathbb{R}^3$ orthogonal to the plane with equation $x + 2y - 3z = 4$ and passing through the point $\mathbf{p} = (1, -1, 2)$.

24. Find an equation for the plane in $\mathbb{R}^3$ which is parallel to the plane $x - y + 2z = 6$ and passes through the point $\mathbf{p} = (2, 1, 2)$.

25. Show that if $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbb{R}^n$ with $\mathbf{x} \perp \mathbf{y}$ and $\mathbf{x} \perp \mathbf{z}$, then $\mathbf{x} \perp (a\mathbf{y} + b\mathbf{z})$ for any scalars a and b.

26. Find parametric equations for the line of intersection of the planes in $\mathbb{R}^3$ with equations $x + 2y - 6z = 4$ and $2x - y + z = 2$.

27. Find parametric equations for the plane of intersection of the hyperplanes in $\mathbb{R}^4$ with equations $w - x + y + z = 3$ and $2w + 4x - y + 2z = 8$.

28. Let L be the line in $\mathbb{R}^3$ with vector equation $\mathbf{y} = t(1, 2, -1) + (3, 2, 1)$ and let P be the plane in $\mathbb{R}^3$ with equation $x + 2y - 3z = 8$. Find the point where L intersects P.

29. Let P be the plane in $\mathbb{R}^n$ with vector equation $\mathbf{y} = t\mathbf{v} + s\mathbf{w} + \mathbf{p}$. Let $\mathbf{c}$ be the projection of $\mathbf{w}$ onto $\mathbf{v}$,
$$\mathbf{a} = \frac{1}{\|v\|}\ \mathbf{v},$$
and
$$\mathbf{b} = \frac{1}{\|\mathbf{w} - \mathbf{c}\|}(\mathbf{w} - \mathbf{c}).$$
Show that $\mathbf{y} = t\mathbf{a} + s\mathbf{b} + \mathbf{p}$ is also a vector equation for P.

The Calculus of Functions of Several Variables

Section 1.5

Linear and Affine Functions

One of the central themes of calculus is the approximation of nonlinear functions by linear functions, with the fundamental concept being the derivative of a function. This section will introduce the linear and affine functions which will be key to understanding derivatives in the chapters ahead.

Linear functions

In the following, we will use the notation $f : \mathbb{R}^m \to \mathbb{R}^n$ to indicate a function whose domain is a subset of $\mathbb{R}^m$ and whose range is a subset of $\mathbb{R}^n$. In other words, f takes a vector with m coordinates for input and returns a vector with n coordinates. For example, the function

$$f(x, y, z) = (\sin(x + y), 2x^2 + z)$$

is a function from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Definition We say a function $L : \mathbb{R}^m \to \mathbb{R}^m$ is *linear* if (1) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$,

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}), \tag{1.5.1}$$

and (2) for any vector $\mathbf{x}$ in $\mathbb{R}^m$ and scalar a,

$$L(a\mathbf{x}) = aL(\mathbf{x}). \tag{1.5.2}$$

Example Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x$. Then for any x and y in $\mathbb{R}$,

$$f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y),$$

and for any scalar a,

$$f(ax) = 3ax = af(x).$$

Thus f is linear.

Example Suppose $L : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2).$$

Then if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are vectors in $\mathbb{R}^2$,

$$\begin{aligned} L(\mathbf{x} + \mathbf{y}) &= L(x_1 + y_1, x_2 + y_2) \\ &= (2(x_1 + y_1) + 3(x_2 + y_2), x_1 + y_1 - (x_2 + y_2), 4(x_2 + y_2)) \\ &= (2x_1 + 3x_2, x_1 - x_2, 4x_2) + (2y_1 + 3y_2, y_1 - y_2, 4y_2) \\ &= L(x_1, x_2) + L(y_1, y_2) \\ &= L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

Also, for $\mathbf{x} = (x_1, x_2)$ and any scalar a, we have

$$\begin{aligned}
L(a\mathbf{x}) &= L(ax_1, ax_2) \\
&= (2ax_1 + 3ax_2, ax_1 - ax_2, 4ax_2) \\
&= a(2x_2 + 3x_2, x_1 - x_2, 4x_2) \\
&= aL(\mathbf{x}).
\end{aligned}$$

Thus L is linear.

Now suppose $L : \mathbb{R} \to \mathbb{R}$ is a linear function and let $a = L(1)$. Then for any real number x,

$$L(x) = L(1x) = xL(1) = ax. \tag{1.5.3}$$

Since any function $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = ax$, where a is a scalar, is linear (see Problem 1), it follows that the only functions $L : \mathbb{R} \to \mathbb{R}$ which are linear are those of the form $L(x) = ax$ for some real number a. For example, $f(x) = 5x$ is a linear function, but $g(x) = \sin(x)$ is not.

Next, suppose $L : \mathbb{R}^m \to \mathbb{R}$ is linear and let $a_1 = L(\mathbf{e}_1), a_2 = L(\mathbf{e}_2), \ldots, a_m = L(\mathbf{e}_m)$. If $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is a vector in $\mathbb{R}^m$, then we know that

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m.$$

Thus

$$\begin{aligned}
L(\mathbf{x}) &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m) \\
&= L(x_1\mathbf{e}_1) + L(x_2\mathbf{e}_2) + \cdots + L(x_m\mathbf{e}_m) \\
&= x_1L(\mathbf{e}_1 + x_2L(\mathbf{e}_2) + \cdots + x_mL(\mathbf{e}_m) \\
&= x_1a_1 + x_2a_2 + \cdots + x_ma_m \\
&= \mathbf{a} \cdot \mathbf{x},
\end{aligned} \tag{1.5.4}$$

where $a = (a_1, a_2, \ldots, a_m)$. Since for any vector $\mathbf{a}$ in $\mathbb{R}^m$, the function $L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ is linear (see Problem 1), it follows that the only functions $L : \mathbb{R}^m \to \mathbb{R}$ which are linear are those of the form $L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ for some fixed vector $\mathbf{a}$ in $\mathbb{R}^m$. For example,

$$f(x, y) = (2, -3) \cdot (x, y) = 2x - 3y$$

is a linear function from $\mathbb{R}^2$ to R, but

$$f(x, y, z) = x^2y + \sin(z)$$

is not a linear function from $\mathbb{R}^3$ to R.

Now consider the general case where $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear function. Given a vector $\mathbf{x}$ in $\mathbb{R}^m$, let $L_k(\mathbf{x})$ be the kth coordinate of $L(\mathbf{x})$, $k = 1, 2, \ldots, n$. That is,

$$L(\mathbf{x}) = (L_1(\mathbf{x}), L_2(\mathbf{x}), \ldots, L_n(\mathbf{x})).$$

Since L is linear, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$ we have

$$L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

or, in terms of the coordinate functions,

$$\begin{aligned}(L_1(\mathbf{x}+\mathbf{y}), L_2(\mathbf{x}+\mathbf{y}), \ldots, L_n(\mathbf{x}+\mathbf{y})) &= (L_1(\mathbf{x}), L_2(\mathbf{x}), \ldots, L_n(\mathbf{x})) \\ &\qquad + (L_1(\mathbf{y}), L_2(\mathbf{y}), \ldots, L_n(\mathbf{y})) \\ &= (L_1(\mathbf{x}) + L_1(\mathbf{y}), L_2(\mathbf{x}) + L_2(\mathbf{y}), \\ &\qquad \ldots, L_n(\mathbf{x}) + L_n(\mathbf{y})).\end{aligned}$$

Hence $L_k(\mathbf{x}+\mathbf{y}) = L_k(\mathbf{x}) + L_k(\mathbf{y})$ for $k = 1, 2, \ldots, n$. Similarly, if $\mathbf{x}$ is in $\mathbb{R}^m$ and a is a scalar, then $L(a\mathbf{x}) = aL(\mathbf{x})$, so

$$\begin{aligned}(L_1(a\mathbf{x}), L_2(a\mathbf{x}), \ldots, L_n(a\mathbf{x}) &= a(L_1(\mathbf{x}), L_2(\mathbf{x}), \ldots, L_n(x)) \\ &= (aL_1(\mathbf{x}), aL_2(\mathbf{x}), \ldots, aL_n(x)).\end{aligned}$$

Hence $L_k(a\mathbf{x}) = aL_k(\mathbf{x})$ for $k = 1, 2, \ldots, n$. Thus for each $k = 1, 2, \ldots, n$, $L_k : \mathbb{R}^m \to \mathbb{R}$ is a linear function. It follows from our work above that, for each $k = 1, 2, \ldots, n$, there is a fixed vector $\mathbf{a}_k$ in $\mathbb{R}^m$ such that $L_k(x) = \mathbf{a}_k \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$. Hence we have

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}) \tag{1.5.5}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. Since any function defined as in (1.5.5) is linear (see Problem 1 again), it follows that the only linear functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ must be of this form.

Theorem If $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then there exist vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$ such that

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}) \tag{1.5.6}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$.

Example In a previous example, we showed that the function $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2)$$

is linear. We can see this more easily now by noting that

$$L(x_1, x_2) = ((2, 3) \cdot (x_1, x_2), (1, -1) \cdot (x_1, x_2), (0, 4) \cdot (x_1, x_2)).$$

Example The function

$$f(x, y, z) = (x + y, \sin(x + y + z))$$

is not linear since it cannot be written in the form of (1.5.6). In particular, the function $f_2(x, y, z) = \sin(x+y+z)$ is not linear; from our work above, it follows that f is not linear.

Matrix notation

We will now develop some notation to simplify working with expressions such as (1.5.6). First, we define an $n \times m$ matrix to be to be an array of real numbers with n rows and m columns. For example,

$$M = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$$

is a 3×2 matrix. Next, we will identify a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ in $\mathbb{R}^m$ with the $m \times 1$ matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix},$$

which is called a *column vector*. Now define the product $M\mathbf{x}$ of an $n \times m$ matrix M with an $m \times 1$ column vector $\mathbf{x}$ to be the $n \times 1$ column vector whose kth entry, $k = 1, 2, \ldots, n$, is the dot product of the kth row of M with $\mathbf{x}$. For example,

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+3 \\ 2-1 \\ 0+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix}.$$

In fact, for any vector $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$,

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \\ 4x_2 \end{bmatrix}.$$

In other words, if we let

$$L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2),$$

as in a previous example, then, using column vectors, we could write

$$L(x_1, x_2) = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In general, consider a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}) \tag{1.5.7}$$

for some vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$. If we let M be the $n \times m$ matrix whose kth row is $\mathbf{a}_k$, $k = 1, 2, \ldots, n$, then

$$L(\mathbf{x}) = M\mathbf{x} \tag{1.5.8}$$

for any $\mathbf{x}$ in $\mathbb{R}^m$. Now, from our work above,

$$\mathbf{a}_k = (L_k(\mathbf{e}_1), L_k(\mathbf{e}_2), \ldots, L_k(\mathbf{e}_m)), \tag{1.5.9}$$

which means that the jth column of M is

$$\begin{bmatrix} L_1(\mathbf{e}_j) \\ L_2(\mathbf{e}_j) \\ \vdots \\ L_n(\mathbf{e}_j) \end{bmatrix}, \tag{1.5.10}$$

$j = 1, 2, \ldots, m$. But (1.5.10) is just $L(\mathbf{e}_j)$ written as a column vector. Hence M is the matrix whose columns are given by the column vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_m)$.

Theorem Suppose $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear function and M is the $n \times m$ matrix whose jth column is $L(\mathbf{e}_j)$, $j = 1, 2, \ldots, m$. Then for any vector $\mathbf{x}$ in $\mathbb{R}^m$,

$$L(\mathbf{x}) = M\mathbf{x}. \tag{1.5.11}$$

Example Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$L(x, y, z) = (3x - 2y + z, 4x + y).$$

Then

$$L(\mathbf{e}_1) = L(1, 0, 0) = (3, 4),$$

$$L(\mathbf{e}_2) = L(0, 1, 0) = (-2, 1),$$

and

$$L(\mathbf{e}_3) = L(0, 0, 1) = (1, 0).$$

So if we let

$$M = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix},$$

then

$$L(x, y, z) = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

For example,

$$L(1, -1, 3) = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 + 2 + 3 \\ 4 - 1 + 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}.$$

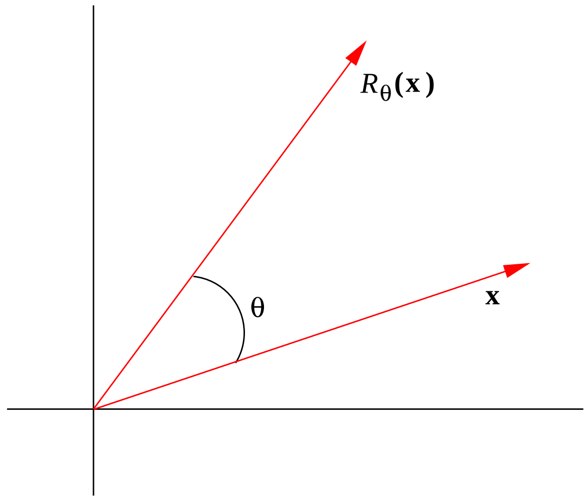


Figure 1.5.1 Rotating a vector in the plane

Example Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates a vector $\mathbf{x}$ in $\mathbb{R}^2$ counterclockwise through an angle θ, as shown in Figure 1.5.1. Geometrically, it seems reasonable that R_θ is a linear function; that is, rotating the vector $\mathbf{x} + \mathbf{y}$ through an angle θ should give the same result as first rotating $\mathbf{x}$ and $\mathbf{y}$ separately through an angle θ and then adding, and rotating a vector $a\mathbf{x}$ through an angle θ should give the same result as first rotating $\mathbf{x}$ through an angle θ and then multiplying by a. Now, from the definition of $\cos(\theta)$ and $\sin(\theta)$,

$$R_\theta(\mathbf{e}_1) = R_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

(see Figure 1.5.2), and, since $\mathbf{e}_2$ is $\mathbf{e}_1$ rotated, counterclockwise, through an angle $\frac{\pi}{2}$,

$$R_\theta(\mathbf{e}_2) = R_{\theta+\frac{\pi}{2}}(\mathbf{e}_1) = \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right) = (-\sin(\theta), \cos(\theta)).$$

Hence

$$R_\theta(x,y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{1.5.12}$$

You are asked in Problem 9 to verify that the linear function defined in (1.5.12) does in fact rotate vectors through an angle θ in the counterclockwise direction. Note that, for example, when $\theta = \frac{\pi}{2}$, we have

$$R_{\frac{\pi}{2}}(x,y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In particular, note that $R_{\frac{\pi}{2}}(1,0) = (0,1)$ and $R_{\frac{\pi}{2}}(0,1) = (-1,0)$; that is, $R_{\frac{\pi}{2}}$ takes $\mathbf{e}_1$ to $\mathbf{e}_2$ and $\mathbf{e}_2$ to $-\mathbf{e}_1$. For another example, if $\theta = \frac{\pi}{6}$, then

$$R_{\frac{\pi}{6}}(x,y) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

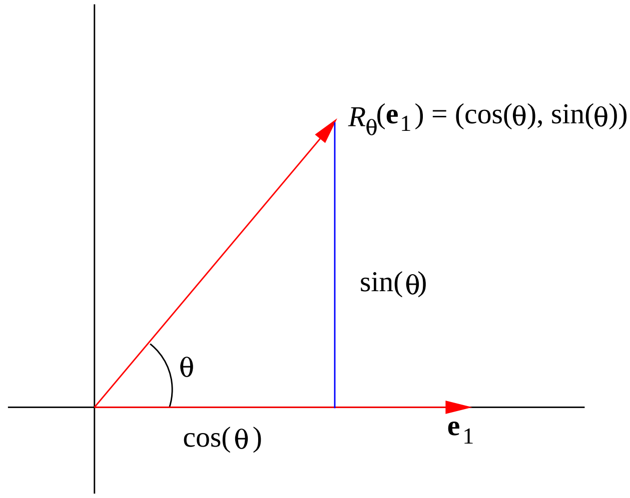


Figure 1.5.2 Rotating $\mathbf{e}_1$ through an angle θ

In particular,

$$R_{\frac{\pi}{6}}(1,2) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-2}{2} \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix}.$$

Affine functions

Definition We say a function $A : \mathbb{R}^m \to \mathbb{R}^n$ is *affine* if there is a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$ and a vector $\mathbf{b}$ in $\mathbb{R}^n$ such that

$$A(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b} \tag{1.5.13}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$.

An affine function is just a linear function plus a translation. From our knowledge of linear functions, it follows that if $A : \mathbb{R}^m \to \mathbb{R}^n$ is affine, then there is an $n \times m$ matrix M and a vector $\mathbf{b}$ in $\mathbb{R}^n$ such that

$$A(\mathbf{x}) = M\mathbf{x} + \mathbf{b} \tag{1.5.14}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is affine, then there are real numbers m and b such that

$$f(x) = mx + b \tag{1.5.15}$$

for all real numbers x.

Example The function

$$A(x, y) = (2x + 3, y - 4x + 1)$$

is an affine function from $\mathbb{R}^2$ to $\mathbb{R}^2$ since we may write it in the form

$$A(x,y) = L(x,y) + (3,1),$$

where L is the linear function

$$L(x,y) = (2x, y - 4x).$$

Note that $L(1,0) = (2,-4)$ and $L(0,1) = (0,1)$, so we may also write A in the form

$$A(x,y) = \begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Example The affine function

$$A(x,y) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

first rotates a vector, counterclockwise, in $\mathbb{R}^2$ through an angle of $\frac{\pi}{4}$ and then translates it by the vector $(1,2)$.

Problems

1. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be vectors in $\mathbb{R}^m$ and define $L : \mathbb{R}^m \to \mathbb{R}^n$ by

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}).$$

   Show that L is linear. What does L look like in the special cases
   (a) $m = n = 1$?
   (b) $n = 1$?
   (c) $m = 1$?

2. For each of the following functions f, find the dimension of the domain space, the dimension of the range space, and state whether the function is linear, affine, or neither.
   (a) $f(x,y) = (3x - y, 4x, x + y)$
   (b) $f(x,y) = (4x + 7y, 5xy)$
   (c) $f(x,y,z) = (3x + z, y - z, y - 2x)$
   (d) $f(x,y,z) = (3x - 4z, x + y + 2z)$
   (e) $f(x,y,z) = \left(3x + 5, y + z, \dfrac{1}{x + y + z}\right)$
   (f) $f(x,y) = 3x + y - 2$
   (g) $f(x) = (x, 3x)$
   (h) $f(w,x,y,z) = (3x, w + x - y + z - 5)$
   (i) $f(x,y) = (\sin(x + y), x + y)$
   (j) $f(x,y) = (x^2 + y^2, x - y, x^2 - y^2)$
   (k) $f(x,y,z) = (3x + 5, y + z, 3x - z + 6, z - 1)$

3. For each of the following linear functions L, find a matrix M such that $L(\mathbf{x}) = M\mathbf{x}$.
   (a) $L(x, y) = (x + y, 2x - 3y)$
   (b) $L(w, x, y, z) = (x, y, z, w)$
   (c) $L(x) = (3x, x, 4x)$
   (d) $L(x) = -5x$
   (e) $L(x, y, z) = 4x - 3y + 2z$
   (f) $L(x, y, z) = (x + y + z, 3x - y, y + 2z)$
   (g) $L(x, y) = (2x, 3y, x + y, x - y, 2x - 3y)$
   (h) $L(x, y) = (x, y)$
   (i) $L(w, x, y, z) = (2w + x - y + 3z, w + 2x - 3z)$

4. For each of the following affine functions A, find a matrix M and a vector $\mathbf{b}$ such that $A(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$.
   (a) $A(x, y) = (3x + 4y - 6, 2x + y - 3)$
   (b) $A(x) = 3x - 4$
   (c) $A(x, y, z) = (3x + y - 4, y - z + 1, 5)$
   (d) $A(w, x, y, z) = (1, 2, 3, 4)$
   (e) $A(x, y, z) = 3x - 4y + z - 1$
   (f) $A(x) = (3x, -x, 2)$
   (g) $A(x_1, x_2, x_3) = (x_1 - x_2 + 1, x_1 - x_3 + 1, x_2 + x_3)$

5. Multiply the following.
   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$
   (b) $\begin{bmatrix} -1 & 2 \\ 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 2 & 1 - 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that maps a vector $\mathbf{x} = (x, y)$ to its reflection across the horizontal axis. Find the matrix M such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

7. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that maps a vector $\mathbf{x} = (x, y)$ to its reflection across the line $y = x$. Find the matrix M such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that maps a vector $\mathbf{x} = (x, y)$ to its reflection across the line $y = -x$. Find the matrix M such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

9. Let R_θ be defined as in (1.5.12).
   (a) Show that for any $\mathbf{x}$ in $\mathbb{R}^2$, $\|R_\theta(\mathbf{x})\| = \|\mathbf{x}\|$.
   (b) For any $\mathbf{x}$ in $\mathbb{R}^2$, let α be the angle between $\mathbf{x}$ and $R_\theta(\mathbf{x})$. Show that $\cos(\alpha) = \cos(\theta)$. Together with (a), this verifies that $R_\theta(\mathbf{x})$ is the rotation of $\mathbf{x}$ through an angle θ.

10. Let $S_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that rotates a vector $\mathbf{x}$ clockwise through an angle θ. Find the matrix M such that $S_\theta(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

11. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, we call the set

$$\{\mathbf{y} : \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^m\}$$

the *image*, or *range*, of f.

(a) Suppose $L : \mathbb{R} \to \mathbb{R}^n$ is linear with $L(1) \neq \mathbf{0}$. Show that the image of L is a line in $\mathbb{R}^n$ which passes through $\mathbf{0}$.

(b) Suppose $L : \mathbb{R}^2 \to \mathbb{R}^n$ is linear and $L(\mathbf{e}_1)$ and $L(\mathbf{e}_2)$ are linearly independent. Show that the image of L is a plane in $\mathbb{R}^n$ which passes through $\mathbf{0}$.

12. Given a function $f : \mathbb{R}^m \to \mathbb{R}$, we call the set

$$\{(x_1, x_2, \ldots, x_m, x_{m+1}) : x_{m+1} = f(x_1, x_2, \ldots, x_m)\}$$

the graph of f. Show that if $L : \mathbb{R}^m \to \mathbb{R}$ is linear, then the graph of L is a hyperplane in $\mathbb{R}^{m+1}$.

The Calculus of Functions

of

Several Variables

Section 1.6

Operations with Matrices

In the previous section we saw the important connection between linear functions and matrices. In this section we will discuss various operations on matrices which we will find useful in our later work with linear functions.

The algebra of matrices

If M is an $n \times m$ matrix with a_{ij} in the ith row and jth column, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then we will write $M = [a_{ij}]$. With this notation the definitions of addition, subtraction, and scalar multiplication for matrices are straightforward.

Definition Suppose $M = [a_{ij}]$ and $N = [b_{ij}]$ are $n \times m$ matrices and c is a real number. Then we define

$$M + N = [a_{ij} + b_{ij}], \tag{1.6.1}$$

$$M - N = [a_{ij} - b_{ij}], \tag{1.6.2}$$

and

$$cM = [ca_{ij}]. \tag{1.6.3}$$

In other words, we define addition, subtraction, and scalar multiplication for matrices by performing these operations on the individual elements of the matrices, in a manner similar to the way we perform these operations on vectors.

Example If

$$M = \begin{bmatrix} 1 & 2 & 3 \\ -5 & 3 & -1 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 3 & 1 & 4 \\ 1 & -3 & 2 \end{bmatrix},$$

then, for example,

$$M + N = \begin{bmatrix} 1+3 & 2+1 & 3+4 \\ -5+1 & 3-3 & -1+2 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 7 \\ -4 & 0 & 1 \end{bmatrix},$$

$$M - N = \begin{bmatrix} 1-3 & 2-1 & 3-4 \\ -5-1 & 3+3 & -1-2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ -6 & 6 & -3 \end{bmatrix},$$

and

$$3M = \begin{bmatrix} 3 & 6 & 9 \\ -15 & 9 & -2 \end{bmatrix}.$$

These operations have natural interpretations in terms of linear functions. Suppose $L : \mathbb{R}^m \to \mathbb{R}^n$ and $K : \mathbb{R}^m \to \mathbb{R}^n$ are linear with $L(\mathbf{x}) = M\mathbf{x}$ and $K(\mathbf{x}) = N\mathbf{x}$ for $n \times m$ matrices M and N. If we define $L + K : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(L + K)(\mathbf{x}) = L(\mathbf{x}) + K(\mathbf{x}), \tag{1.6.4}$$

then

$$(L + K)(\mathbf{e}_j) = L(\mathbf{e}_j) + K(\mathbf{e}_j) \tag{1.6.5}$$

for $j = 1, 2, \ldots, m$. Hence the jth column of the matrix which represents $L+K$ is the sum of the jth columns of M and N. In other words,

$$(L + K)(\mathbf{x}) = (M + N)\mathbf{x} \tag{1.6.6}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. Similarly, if we define $L - K : \mathbb{R}^m \to \mathbb{R}^n$ by

$$(L - K)(\mathbf{x}) = L(\mathbf{x}) - K(\mathbf{x}), \tag{1.6.7}$$

then

$$(L - K)(\mathbf{x}) = (M - N)\mathbf{x}. \tag{1.6.8}$$

If, for any scalar c, we define $cL : \mathbb{R}^m \to \mathbb{R}^n$ by

$$cL(\mathbf{x}) = c(L(\mathbf{x})), \tag{1.6.9}$$

then

$$cL(\mathbf{e}_j) = c(L(\mathbf{e}_j)) \tag{1.6.10}$$

for $j = 1, 2, \ldots, m$. Hence the jth column of the matrix which represents cL is the scalar c times the jth column of M. That is,

$$cL(\mathbf{x}) = (cM)\mathbf{x} \tag{1.6.11}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. In short, the operations of addition, subtraction, and scalar multiplication for matrices corresponds in a natural way with the operations of addition, subtraction, and scalar multiplication for linear functions.

Now consider the case where $L : \mathbb{R}^m \to \mathbb{R}^p$ and $K : \mathbb{R}^p \to \mathbb{R}^n$ are linear functions. Let M be the $p \times m$ matrix such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$ and let N be the $n \times p$ matrix such that $K(\mathbf{x}) = N\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^p$. Since for any $\mathbf{x}$ in $\mathbb{R}^m$, $L(\mathbf{x})$ is in $\mathbb{R}^p$, we can form $K \circ L : \mathbb{R}^m \to \mathbb{R}^n$, the composition of K with L, defined by

$$K \circ L(\mathbf{x}) = K(L(\mathbf{x})). \tag{1.6.12}$$

Now

$$K(L(\mathbf{x})) = N(M\mathbf{x}), \tag{1.6.13}$$

so it would be natural to define NM, the product of the matrices N and M, to be the matrix of $K \circ L$, in which case we would have

$$N(M\mathbf{x}) = (NM)\mathbf{x}. \tag{1.6.14}$$

Thus we want the jth column of NM, $j = 1, 2, \ldots, m$, to be

$$K \circ L(\mathbf{e}_j) = N(L(\mathbf{e}_j)), \tag{1.6.15}$$

which is just the dot product of $L(\mathbf{e}_j)$ with the rows of N. But $L(\mathbf{e}_j)$ is the jth column of M, so the jth column of NM is formed by taking the dot product of the jth column of M with the rows of N. In other words, the entry in the ith row and jth column of NM is the dot product of the ith row of N with the jth column of M. We write this out explicitly in the following definition.

Definition If $N = [a_{ij}]$ is an $n \times p$ matrix and $M = [b_{ij}]$ is a $p \times m$ matrix, then we define the *product* of N and M to be the $n \times m$ matrix $NM = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \tag{1.6.16}$$

$i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Note that NM is an $n \times m$ matrix since $K \circ L : \mathbb{R}^m \to \mathbb{R}^n$. Moreover, the product NM of two matrices N and M is defined only when the number of columns of N is equal to the number of rows of M.

Example If

$$N = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & -2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 2 & -2 & 1 & 3 \\ 1 & 2 & -1 & -2 \end{bmatrix},$$

then

$$\begin{aligned} NM &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 3 \\ 1 & 2 & -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2+2 & -2+4 & 1-2 & 3-4 \\ -2+3 & 2+6 & -1-3 & -3-6 \\ 4-2 & -4-4 & 2+2 & 6+4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 & -1 & -1 \\ 1 & 8 & -4 & -9 \\ 2 & -8 & 4 & 10 \end{bmatrix}. \end{aligned}$$

Note that N is 3×2, M is 2×4, and NM is 3×4. Also, note that it is not possible to form the product in the other order.

Example Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear function defined by

$$L(x, y) = (3x - 2y, x + y, 4y)$$

and let $K : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear function defined by

$$K(x, y, z) = (2x - y + z, x - y - z).$$

Then the matrix for L is

$$M = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ 0 & 4 \end{bmatrix},$$

the matrix for K is

$$N = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix},$$

and the matrix for $K \circ L : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$NM = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 - 1 + 0 & -4 - 1 + 4 \\ 3 - 1 + 0 & -2 - 1 - 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & -7 \end{bmatrix}.$$

In other words,

$$K \circ L(x, y) = \begin{bmatrix} 5 & -1 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x - y \\ 2x - 7y \end{bmatrix}.$$

Note that it in this case it is possible to form the composition in the other order. The matrix for $L \circ K : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$MN = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 6 - 2 & -3 + 2 & 3 + 2 \\ 2 + 1 & -1 - 1 & 1 - 1 \\ 0 + 4 & 0 - 4 & 0 - 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 5 \\ 3 & -2 & 0 \\ 4 & -4 & -4 \end{bmatrix},$$

and so

$$L \circ K(x, y, z) = \begin{bmatrix} 4 & -1 & 5 \\ 3 & -2 & 0 \\ 4 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x - y + 5z \\ 3x - 2y \\ 4x - 4y - 4z \end{bmatrix}.$$

In particular, note that not only is $NM \neq MN$, but in fact NM and MN are not even the same size.

Determinants

The notion of the determinant of a matrix is closely related to the idea of area and volume. To begin our definition, consider the 2×2 matrix

$$M = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

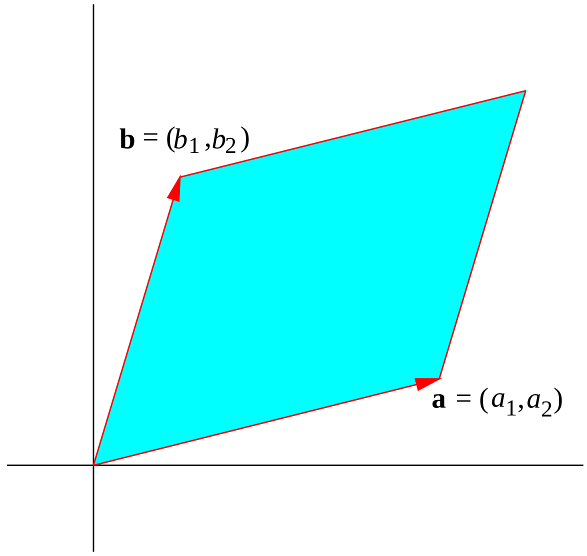


Figure 1.6.1 A parallelogram in $\mathbb{R}^2$ with adjacent sides **a** and **b**

and let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. If P is the parallelogram which has **a** and **b** for adjacent sides and A is the area of P (see Figure 1.6.1), then we saw in Section 1.3 that

$$A = \|(a_1, a_2, 0) \times (b_1, b_2, 0)\| = \|(0, 0, a_1 b_2 - a_2 b_1\| = |a_1 b_2 - a_2 b_1|. \tag{1.6.17}$$

This motivates the following definition.

Definition Given a 2×2 matrix

$$M = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

the *determinant* of M, denoted $\det(M)$, is

$$\det(M) = a_1 b_2 - a_2 b_1. \tag{1.6.18}$$

Hence we have $A = |\det(M)|$. In words, for a 2×2 matrix M, the absolute value of the determinant of M equals the area of the parallelogram which has the rows of M for adjacent sides.

Example We have

$$\det \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} = (1)(5) - (3)(-4) = 5 + 12 = 17.$$

Now consider a 3×3 matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

and let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$. If V is the volume of the parallelepiped P with adjacent edges $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, then, again from Section 1.3,

$$\begin{aligned} V &= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \\ &= |a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)| \\ &= \left| a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \right|. \end{aligned} \tag{1.6.19}$$

Definition Given a 3×3 matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

the *determinant* of M, denoted $\det(M)$, is

$$\det(M) = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}. \tag{1.6.20}$$

Similar to the 2×2 case, we have $V = |\det(M)|$.

Example We have

$$\begin{aligned} \det \begin{bmatrix} 2 & 3 & 9 \\ 2 & 1 & -4 \\ 5 & 1 & -1 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & -4 \\ 1 & -1 \end{bmatrix} - 3 \det \begin{bmatrix} 2 & -4 \\ 5 & -1 \end{bmatrix} + 9 \det \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} \\ &= 2(-1 + 4) - 3(-2 + 20) + 9(2 - 5) \\ &= 6 - 54 - 27 \\ &= -75. \end{aligned}$$

Given an $n \times n$ matrix $M = [a_{ij}]$, let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of M. If for $n = 1$ we first define $\det(M) = a_{11}$ (that is, the determinant of a 1×1 matrix is just the value of its single entry), then we could express, for $n = 2$, the definition of a the determinant of a 2×2 matrix given in (1.6.18) in the form

$$\det(M) = a_{11} \det(M_{11}) - a_{12} \det(M_{12}) = a_{11}a_{22} - a_{12}a_{21}. \tag{1.6.21}$$

Similarly, with $n = 3$, we could express the definition of the determinant of M given in (1.6.20) in the form

$$\det(M) = a_{11} \det(M_{11}) - a_{12} \det(M_{12}) + a_{13} \det(M_{13}). \tag{1.6.22}$$

Following this pattern, we may form a recursive definition for the determinant of an $n \times n$ matrix.

Definition Suppose $M = [a_{ij}]$ is an $n \times n$ matrix and let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column of M, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. For $n = 1$, we define the *determinant* of M, denoted $\det(M)$, by

$$\det(M) = a_{11}. \tag{1.6.23}$$

For $n > 1$, we define the *determinant* of M, denoted $\det(M)$, by

$$\begin{aligned}\det(M) &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(M_{1n}) \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(M_{1j}).\end{aligned} \tag{1.6.24}$$

We call the definition recursive because we have defined the determinant of an $n \times n$ matrix in terms of the determinants of $(n-1)\times(n-1)$ matrices, which in turn are defined in terms of the determinants of $(n-2)\times(n-2)$ matrices, and so on, until we have reduced the problem to computing the determinants of 1×1 matrices.

Example For an example of the determinant of a 4×4 matrix, we have

$$\begin{aligned}\det\begin{bmatrix} 2 & 1 & 3 & 2 \\ 2 & 1 & 4 & 1 \\ -2 & 3 & -1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} &= 2\det\begin{bmatrix} 1 & 4 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} - \det\begin{bmatrix} 2 & 4 & 1 \\ -2 & -1 & 2 \\ 1 & 11 & 1 \end{bmatrix} \\ &\quad + 3\det\begin{bmatrix} 2 & 1 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 1 & 4 \\ -2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \\ &= 2((-1-2) - 4(3-4) + (3+2)) - (2(-1-2) \\ &\quad - 4(-2-2) + (-2+1)) + 3(2(3-4) - (-2-2) \\ &\quad + (-4-3)) - 2(2(3+2) - (-2+1) + 4(-4-3)) \\ &= 2(-3+4+5) - (-6+16-1) + 3(-2+4-7) \\ &\quad - 2(10+1-28) \\ &= 12 - 9 - 15 + 34 \\ &= 22.\end{aligned}$$

The next theorem states that there is nothing special about using the first row of the matrix in the expansion of the determinant specified in (1.6.24), nor is there anything special about expanding along a row instead of a column. The practical effect is that we may compute the determinant of a given matrix expanding along whichever row or column is most convenient. The proof of this theorem would take us too far afield at this point, so we will omit it (but you will be asked to verify the theorem for the special cases $n = 2$ and $n = 3$ in Problem 10).

Theorem Let $M = [a_{ij}]$ be an $n \times n$ matrix and let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of M. Then for any $i = 1, 2, \ldots, n$,

$$\det(M) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}), \tag{1.6.25}$$

and for any $j = 1, 2, \ldots, n$,

$$\det(M) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}), \tag{1.6.26}$$

Example The simplest way to compute the determinant of the matrix

$$M = \begin{bmatrix} 4 & 0 & 3 \\ 2 & 3 & 1 \\ -3 & 0 & -2 \end{bmatrix}$$

is to expand along the second column. Namely,

$$\begin{aligned} \det(M) &= (-1)^{1+2}(0) \det \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} + (-1)^{2+2}(3) \det \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \\ &\quad + (-1)^{3+2}(0) \det \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \\ &= 3(-8+9) \\ &= 3. \end{aligned}$$

You should verify that expanding along the first row, as we did in the definition of the determinant, gives the same result.

In order to return to the problem of computing volumes, we need to define a parallelepiped in $\mathbb{R}^n$. First note that if P is a parallelogram in $\mathbb{R}^2$ with adjacent sides given by the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$P = \{\mathbf{y} : \mathbf{y} = t\mathbf{a} + s\mathbf{b}, 0 \le t \le 1, 0 \le s \le 1\}. \tag{1.6.27}$$

That is, for $0 \le t \le 1$, $t\mathbf{a}$ is a point between $\mathbf{0}$ and $\mathbf{a}$, and for $0 \le s \le 1$, $s\mathbf{b}$ is a point between $\mathbf{0}$ and $\mathbf{b}$; hence $t\mathbf{a} + s\mathbf{b}$ is a point in the parallelogram P. Moreover, every point in P may be expressed in this form. See Figure 1.6.2. The following definition generalizes this characterization of parallelograms.

Definition Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be linearly independent vectors in $\mathbb{R}^n$. We call

$$P = \{\mathbf{y} : y = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n, 0 \le t_i \le 1, i = 1, 2, \ldots, n\} \tag{1.6.28}$$

an *n-dimensional parallelepiped* with adjacent edges $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

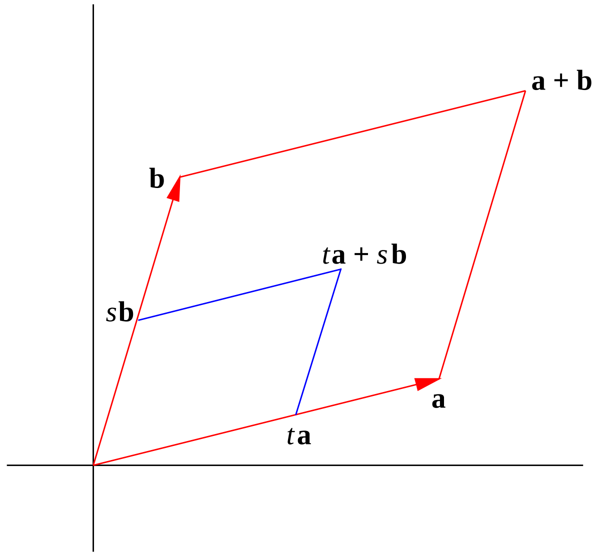


Figure 1.6.2 A parallelogram in $\mathbb{R}^2$ with adjacent sides $\mathbf{a}$ and $\mathbf{b}$

Definition Let P be an n-dimensional parallelepiped with adjacent edges $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and let M be the $n \times n$ matrix which has $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ for its rows. Then the *volume* of P is defined to be $|\det(M)|$.

It may be shown, using (1.6.26) and induction, that if N is the matrix obtained by interchanging the rows and columns of an $n \times n$ matrix M, then $\det(N) = \det(M)$ (see Problem 12). Thus we could have defined M in the previous definition using $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ for columns rather than rows.

Now suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and let M be the $n \times n$ matrix such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Let C be the n-dimensional parallelepiped with adjacent edges $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, the standard basis vectors for $\mathbb{R}^n$. Then C is a 1×1 square when $n = 2$ and a $1 \times 1 \times 1$ cube when $n = 3$. In general, we may think of C as an n-dimensional unit cube. Note that the volume of C is, by definition,

$$\det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = 1.$$

Suppose $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$ are linearly independent and let P be the n-dimensional parallelepiped with adjacent edges $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$. Note that if

$$\mathbf{x} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + \cdots + t_n\mathbf{e}_n,$$

where $0 \le t_k \le 1$ for $k = 1, 2, \ldots, n$, is a point in C, then

$$L(\mathbf{x}) = t_1 L(\mathbf{e}_1) + t_2 L(\mathbf{e}_2) + \cdots + t_n L(\mathbf{e}_n)$$

is a point in P. In fact, L maps the n-dimensional unit cube C exactly onto the n-dimensional parallelepiped P. Since $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$ are the columns of M, it follows that the volume of P equals $|\det(M)|$. In other words, $|\det(M)|$ measures how much L stretches or shrinks the volume of a unit cube.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and M is the $n \times n$ matrix such that $L(\mathbf{x}) = M\mathbf{x}$. If $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$ are linear independent and P is the n-dimensional parallelepiped with adjacent edges $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$, then the volume of P is equal to $|\det(M)|$.

Problems

1. Let $M = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 4 & -1 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 2 & -5 \end{bmatrix}$. Evaluate the following.

   (a) $3M$ (b) $M - N$

   (c) $2M + N$ (d) $2N - 6M$

2. Evaluate the following matrix products.

   (a) $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 1 & 3 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 \\ 0 & 2 & 4 \\ 2 & 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ -2 & 4 \\ 0 & -4 \end{bmatrix}$

3. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ and $K : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by

$$L(x, y, z) = (2x + 3y, y - x + 2z, x + 2y - z)$$

   and

$$K(x, y, z) = (2x + 4y - 3z, x + y + z, 3x - y + 4z).$$

   Find the matrices for the following linear functions.

   (a) $3L$ (b) $L + K$

   (c) $2L - K$ (d) $K + 2L$

   (e) $K \circ L$ (f) $L \circ K$

4. Let $\mathbb{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function which rotates a vector in $\mathbb{R}^2$ counterclockwise through an angle θ. In Section 1.5 we saw that

$$R_\theta(x, y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

   Show that the matrix for $R_\theta \circ R_\alpha$ is the same as the matrix for $R_{\theta+\alpha}$. In other words, show that $R_\theta \circ R_\alpha = R_{\theta+\alpha}$.

5. Compute the determinants of the following matrices.

(a) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 9 \\ 5 & -3 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 & -1 \\ 3 & 1 & 0 \\ 5 & -4 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 3 & -2 & 1 \\ 1 & 4 & -4 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ -3 & 2 & 0 & 1 & 5 \\ 1 & 5 & -2 & 1 & 0 \\ 6 & -5 & 0 & 2 & -4 \end{bmatrix}$

6. Find the area of the parallelogram in $\mathbb{R}^2$ with vertices at $(1,-2)$, $(3,-1)$, $(4,1)$, and $(2,0)$.

7. Find the volume of the parallelepiped in $\mathbb{R}^3$ with bottom vertices at $(1,1,1)$, $(2,3,2)$, $(-1,4,3)$, and $(-2,2,2)$ and top vertices at $(1,0,5)$, $(2,2,6)$, $(-1,3,7)$, and $(-2,1,6)$.

8. Let P be the 4-dimensional parallelepiped with adjacent edges $\mathbf{a}_1 = (2,1,2,1)$, $\mathbf{a}_2 = (-2,0,1,1)$, $\mathbf{a}_3 = (1,1,3,6)$, and $\mathbf{a}_4 = (-3,1,5,0)$. Find the volume of P.

9. Find 2×2 matrices A and B for which $AB \neq BA$.

10. Verify that (1.6.25) and (1.6.26) hold for all 2×2 and 3×3 matrices.

11. An $n \times n$ matrix $M = [a_{ij}]$ is called a *diagonal matrix* if $a_{ij} = 0$ for all $i \neq j$. Show that if M is a diagonal matrix, then $\det(M) = a_{11}a_{22}\cdots a_{nn}$.

12. If M is an $n \times m$ matrix, then the $m \times n$ matrix M^T whose columns are the rows of M is called the *transpose* of M. For example, if

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix},$$

then

$$M^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

(a) Show that for a 2×2 matrix M, $\det(M^T) = \det(M)$.

(b) Show that for a 3×3 matrix M, $\det(M^T) = \det(M)$. (Hint: Using (1.6.26), expand $\det(M)$ along the first row and $\det(M^T)$ along the first column.)

(c) Use induction to show that for any $n \times n$ matrix M, $\det(M^T) = \det(M)$. (Hint: Note that $(M^T)_{ij} = (M_{ji})^T$.)

13. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be vectors in $\mathbb{R}^3$ and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard basis vectors for $\mathbb{R}^3$. Show that applying (1.6.20) to the array

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

yields $\mathbf{x} \times \mathbf{y}$. Discuss what is correct and what is incorrect about the statement

$$\mathbf{x} \times \mathbf{y} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

14. Show that the set of all points $\mathbf{x} = (x, y, z)$ in $\mathbb{R}^3$ which satisfy the equation

$$\det \begin{bmatrix} x & y & z \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix} = 0$$

is a plane passing through the points $(0, 0, 0)$, $(1, 2, -1)$, and $(3, 1, 2)$.

15. Verify directly that if $L : \mathbb{R}^m \to \mathbb{R}^p$ and $K : \mathbb{R}^p \to \mathbb{R}^n$ are linear functions, then $K \circ L : \mathbb{R}^m \to \mathbb{R}^n$ is also a linear function.

The Calculus of Functions

of

Several Variables

Section 2.1

Curves

Now that we have a basic understanding of the geometry of $\mathbb{R}^n$, we are in a position to start the study of calculus of more than one variable. We will break our study into three pieces. In this chapter we will consider functions $f : \mathbb{R} \to \mathbb{R}^n$, in Chapter 3 we will study functions $f : \mathbb{R}^n \to \mathbb{R}$, and finally in Chapter 4 we will consider the general case of functions $f : \mathbb{R}^m \to \mathbb{R}^n$.

Parametrizations of curves

We begin with some terminology and notation. Given a function $f : \mathbb{R} \to \mathbb{R}^n$, let

$$f_k(t) = k\text{th coordinate of } f(t) \tag{2.1.1}$$

for $k = 1, 2, \ldots, n$. We call $f_k : \mathbb{R} \to \mathbb{R}$ the *kth coordinate function* of f. Note that f_k has the same domain as f and that, for any point t in the domain of f,

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t). \tag{2.1.2}$$

If the domain of f is an interval I, then the range of f, that is, the set

$$C = \{\mathbf{x} : \mathbf{x} = f(t) \text{ for some } t \text{ in } I\}, \tag{2.1.3}$$

is called a *curve* with *parametrization* f. The equation $\mathbf{x} = f(t)$, where $\mathbf{x}$ is in $\mathbb{R}^n$, is a *vector equation* for C and, writing $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the equations

$$\begin{aligned} x_1 &= f_1(t), \\ x_2 &= f_2(t), \\ &\ \ \vdots \\ x_n &= f_n(t), \end{aligned} \tag{2.1.4}$$

are *parametric equations* for C.

Example Consider $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (\cos(t), \sin(t))$$

for $0 \le t \le 2\pi$. Then for every value of t, $f(t)$ is a point on the circle C of radius 1 with center at $(0,0)$. Note that $f(0) = (1,0)$, $f\left(\frac{\pi}{2}\right) = (0,1)$, $f(\pi) = (-1,0)$, $f\left(\frac{3\pi}{2}\right) = (0,-1)$,

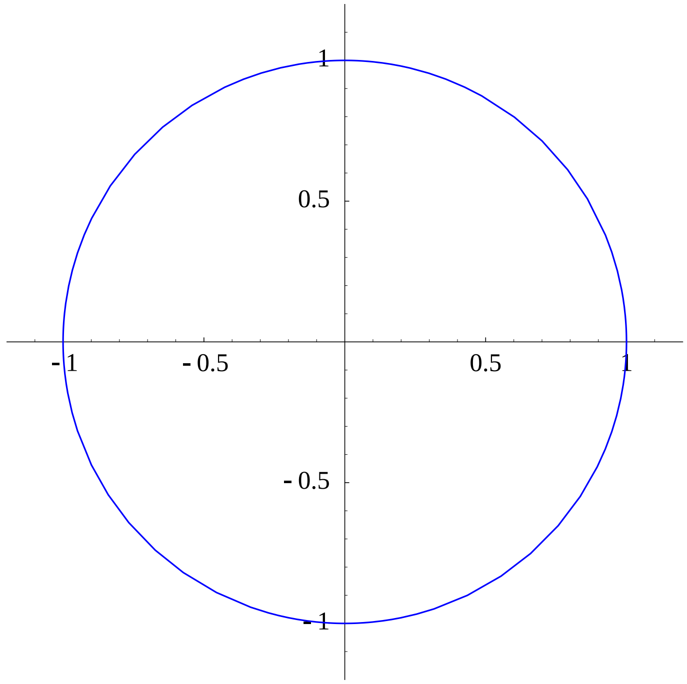


Figure 2.1.1 $f(t) = (\cos(t), \sin(t)$

and $f(2\pi) = (1, 0) = f(0)$. In fact, as t goes from 0 to 2π, $f(t)$ traverses C exactly once in the counterclockwise direction. Thus f is a parametrization of the unit circle C. If we denote a point in $\mathbb{R}^2$ by (x, y), then

$$\begin{aligned} x &= \cos(t), \\ y &= \sin(t), \end{aligned}$$

are parametric equations for C. See Figure 2.1.1. The coordinate functions are

$$\begin{aligned} f_1(t) &= \cos(t), \\ f_2(t) &= \sin(t), \end{aligned}$$

although we frequently write these as simply

$$\begin{aligned} x(t) &= \cos(t), \\ y(t) &= \sin(t). \end{aligned}$$

Example Consider $g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$g(t) = (\sin(2\pi t), \cos(2\pi t))$$

for $0 \leq t \leq 2$. Then g also parametrizes the unit circle C centered at the origin, the same as f in the previous example. However, there is a difference: $g(0) = (0, 1)$, $g\left(\frac{1}{4}\right) = (1, 0)$,

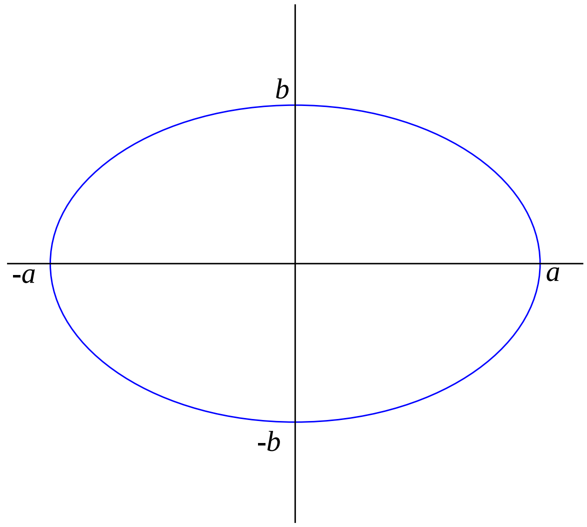


Figure 2.1.2 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$g\left(\frac{1}{2}\right) = (0, -1)$, $g\left(\frac{3}{4}\right) = (-1, 0)$, and $g(1) = (0, 1) = g(0)$, at which point g starts to repeat its values. Hence $g(t)$, starting at $(0, 1)$, traverses C twice in the clockwise direction as t goes from 0 to 2.

Example More generally, suppose a, b, and α are real numbers, with $a > 0$, $b > 0$, and $\alpha \neq 0$, and let

$$x(t) = a\cos(\alpha t),$$
$$y(t) = b\sin(\alpha t).$$

Then

$$\frac{(x(t))^2}{a^2} + \frac{(y(t))^2}{b^2} = \cos^2(\alpha t) + \sin^2(\alpha t) = 1,$$

so $(x(t), y(t))$ is a point on the ellipse E with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

shown in Figure 2.1.2. Thus the function

$$f(t) = (a\cos(\alpha t), b\cos(\alpha t))$$

parametrizes the ellipse E, traversing the complete ellipse as t goes from 0 to $\left|\frac{2\pi}{\alpha}\right|$.

Example Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (t\cos(t), t\sin(t))$$

for $-\infty < t < \infty$. Then for negative values of t, $f(t)$ spirals into the origin as t increases, while for positive values of t, $f(t)$ spirals away from the origin. Part of this curve parametrized by f is shown in Figure 2.1.3.

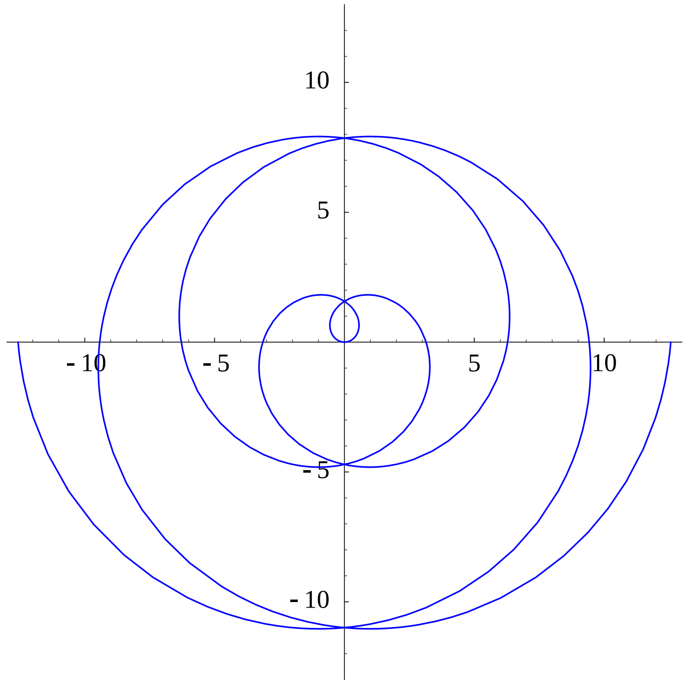


Figure 2.1.3 The spiral $f(t) = (t\cos(t), t\sin(t))$ for $-4\pi \leq t \leq 4\pi$

Example Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (3 - 4t, 2 + 3t)$$

for $-\infty < t < \infty$. Then

$$f(t) = t(-4, 3) + (3, 2),$$

so f is a parametrization of the line through the point $(3, 2)$ in the direction of $(-4, 3)$.

In general, a function $f : \mathbb{R} \to \mathbb{R}^n$ defined by $f(t) = t\mathbf{v} + \mathbf{p}$, where $\mathbf{v} \neq 0$ and $\mathbf{p}$ are vectors in $\mathbb{R}^n$, parametrizes a line in $\mathbb{R}^n$.

Example Suppose $g : \mathbb{R} \to \mathbb{R}^3$ is defined by

$$g(t) = (4\cos(t), 4\sin(t), t)$$

for $-\infty < t < \infty$. If we denote the coordinate functions by

$$\begin{aligned} x(t) &= 4\cos(t), \\ y(t) &= 4\sin(t), \\ z(t) &= t, \end{aligned}$$

then

$$(x(t))^2 + (y(t))^2 = 16\cos^2(t) + 16\sin^2(t) = 16.$$

Hence $g(t)$ always lies on a cylinder of radius 1 centered about the z-axis. As t increases, $g(t)$ rises steadily as it winds around this cylinder, completing one trip around the cylinder

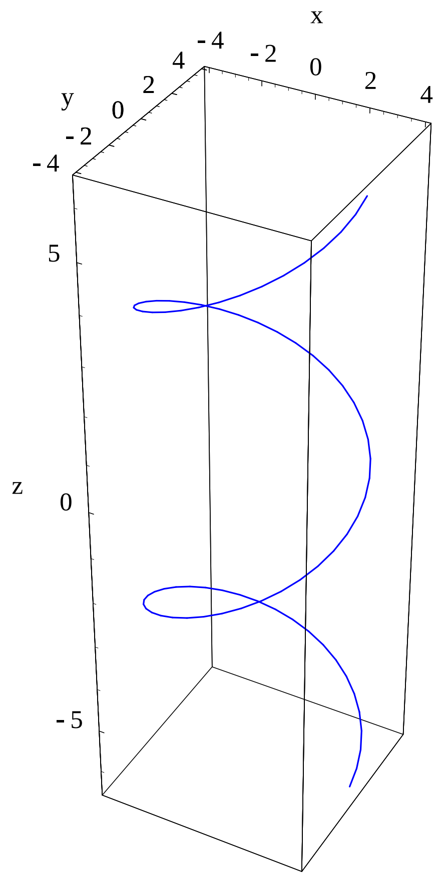


Figure 2.1.4 The helix $f(t) = (4\cos(t), 4\sin(t), t)$, $-2\pi \leq t \leq 2\pi$

over every interval of length 2π. In other words, g parametrizes a helix, part of which is shown in Figure 2.1.4.

Limits in $\mathbb{R}^n$

As was the case in one-variable calculus, limits are fundamental for understanding ideas such as continuity and differentiability. We begin with the definition of the limit of a sequence of points in $\mathbb{R}^m$.

Definition Let $\{\mathbf{x}_n\}$ be a sequence of points in $\mathbb{R}^m$. We say that the *limit* of $\{\mathbf{x}_n\}$ as n approaches infinity is $\mathbf{a}$, written $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$, if for every $\epsilon > 0$ there is a positive integer N such that

$$\|\mathbf{x}_n - \mathbf{a}\| < \epsilon \tag{2.1.5}$$

whenever $n > N$.

Notice that this definition involves only a slight modification of the definition for the limit of a sequence of real numbers, namely, the use of the norm of a vector instead of the

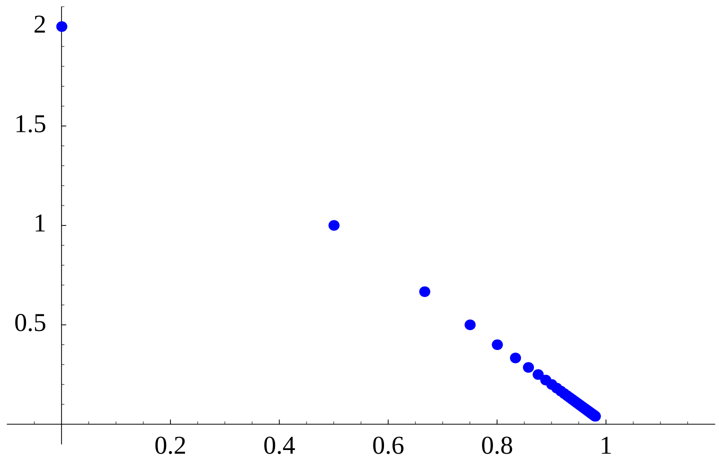


Figure 2.1.5 Points $\left(1-\frac{1}{n}, \frac{2}{n}\right)$ approaching $(1,0)$

absolute value of a real number in (2.1.5). In words, $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if, given any $\epsilon > 0$, we can always find a point in the sequence beyond which all terms of the sequence lie within $B^n(\mathbf{a}, \epsilon)$, the open ball of radius ϵ centered at $\mathbf{a}$.

Example Suppose

$$\mathbf{x}_n = \left(1 - \frac{1}{n}, \frac{2}{n}\right)$$

for $n = 1, 2, 3, \ldots$. Since

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$$

and

$$\lim_{n\to\infty} \frac{2}{n} = 0,$$

we should have

$$\lim_{n\to\infty} \mathbf{x}_n = (1, 0).$$

To verify this, we first note that

$$\|\mathbf{x}_n - (1,0)\| = \left\|\left(-\frac{1}{n}, \frac{2}{n}\right)\right\| = \sqrt{\frac{1}{n^2} + \frac{4}{n^2}} = \frac{\sqrt{5}}{n}.$$

Hence $\|\mathbf{x}_n - (1,0)\| < \epsilon$ whenever $n > \frac{\sqrt{5}}{\epsilon}$. That is, if we let N be any integer greater than or equal to $\frac{\sqrt{5}}{\epsilon}$, then $\|\mathbf{x}_n - (1,0)\| < \epsilon$ whenever $n > N$, verifying that

$$\lim_{n\to\infty} \mathbf{x}_n = (1, 0).$$

See Figure 2.1.5.

Put another way, the definition of the limit of a sequence in $\mathbb{R}^m$ says that a sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^m$ converges to $\mathbf{a}$ in $\mathbb{R}^m$ if and only if the sequence of real numbers $\{\|\mathbf{x}_n - \mathbf{a}\|\}$

converges to 0. That is, $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{a}\| = 0$. Moreover, if we let $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nm})$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, then

$$\|\mathbf{x}_n - \mathbf{a}\| = \sqrt{(x_{n1}-a_1)^2 + (x_{n2}-a_2)^2 + \cdots + (x_{nm}-a_m)^2}, \tag{2.1.6}$$

so $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{a}\| = 0$ if and only if

$$\lim_{n\to\infty} \sqrt{(x_{n1}-a_1)^2 + (x_{n2}-a_2)^2 + \cdots + (x_{nm}-a_m)^2} = 0. \tag{2.1.7}$$

But (2.1.7) can occur only when $\lim_{n\to\infty} (x_{nk} - a_k)^2 = 0$ for $k = 1, 2, \ldots, m$. Hence $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_{nk} = a_k$ for $k = 1, 2, \ldots, m$.

Proposition Suppose $\{\mathbf{x}_n\}$ is a sequence in $\mathbb{R}^m$, $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nm})$, and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$. Then $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_{nk} = a_k$ for $k = 1, 2, \ldots, m$.

This proposition tells us that to compute the limit of a sequence in $\mathbb{R}^m$, we need only compute the limit of each coordinate separately, thus reducing the problem of computing limits in $\mathbb{R}^m$ to the problem of finding limits of sequences of real numbers.

Example If

$$\mathbf{x}_n = \left(\frac{2-n}{n^2}, \sin\left(\frac{1}{n}\right), \cos\left(\frac{3}{n}\right)\right),$$

$n = 1, 2, 3, \ldots$, then

$$\lim_{n\to\infty} \mathbf{x}_n = \left(\lim_{n\to\infty} \frac{2-n}{n^2}, \lim_{n\to\infty} \sin\left(\frac{1}{n}\right), \lim_{n\to\infty} \cos\left(\frac{3}{n}\right)\right) = (0, 0, 1).$$

We may now define the limit of a function $f : \mathbb{R} \to \mathbb{R}^m$ at a real number c. Notice that the definition is identical to the definition of a limit for a real-valued function $f : \mathbb{R} \to \mathbb{R}$.

Definition Let c be a real number, let I be an open interval containing c, and let $J = \{t : t \text{ is in } I, t \neq c\}$. Suppose $f : \mathbb{R} \to \mathbb{R}^m$ is defined for all t in J. Then we say that the *limit* of $f(t)$ as t approaches c is $\mathbf{a}$, denoted $\lim_{t\to c} f(t) = \mathbf{a}$, if for every sequence of real numbers $\{t_n\}$ in J,

$$\lim_{n\to\infty} f(t_n) = \mathbf{a} \tag{2.1.8}$$

whenever $\lim_{n\to\infty} t_n = c$.

As in one-variable calculus, we may define the limit of $f(t)$ as t approaches c from the right, denoted

$$\lim_{t\to c^+} f(t),$$

by restricting to sequences $\{t_n\}$ with $t_n > c$ for $n = 1, 2, 3, \ldots$, and the limit of $f(t)$ as t approaches c from the left, denoted

$$\lim_{t\to c^-} f(t),$$

by restricting to sequences $\{t_n\}$ with $t_n < c$ for $n = 1, 2, 3, \ldots$. Moreover, the following useful proposition follows immediately from our definition and the previous proposition.

Proposition Suppose $f : \mathbb{R} \to \mathbb{R}^m$ with

$$f(t) = (f_1(t), f_2(t), \ldots, f_m(t)).$$

The for any real number c,

$$\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_m(t)). \tag{2.1.9}$$

Hence the problem of computing limits for functions $f : \mathbb{R} \to \mathbb{R}^m$ reduces to the problem of computing limits of the coordinate functions $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, a familiar problem from one-variable calculus. The analogous statements for limits from the right and left also hold.

Example If $f(t) = (t^2 - 1, \sin(t), \cos(t))$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$, then, for example,

$$\lim_{t \to \pi} f(t) = \left(\lim_{t \to \pi} (t^2 - 1), \lim_{t \to \pi} \sin(t), \lim_{t \to \pi} \cos(t) \right) = (\pi^2 - 1, 0, -1).$$

Definitions for continuity also follow the pattern of the related definitions in one-variable calculus.

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^m$. We say f is *continuous at a point* c if

$$\lim_{t \to c} f(t) = f(c). \tag{2.1.10}$$

We say f is *continuous from the right at* c if

$$\lim_{t \to c^+} f(t) = f(c) \tag{2.1.11}$$

and *continuous from the left at* c if

$$\lim_{t \to c^-} f(t) = f(c). \tag{2.1.12}$$

We say f is *continuous* on an open interval (a, b) if f is continuous at every point c in (a, b) and we say f is *continuous* on a closed interval $[a, b]$ if f is continuous on the open interval (a, b), continuous from the right at a, and continuous from the left at b.

If $f(t) = (f_1(t), f_2(t), \ldots, f_m(t))$, then f is continuous at a point c if and only if

$$\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_m(t) = f(c) = (f_1(c), f_2(c), \ldots, f_m(c)),$$

which is true if and only if $\lim\limits_{t \to c} f_k(t) = f_k(c)$ for $k = 1, 2, \ldots, m$. In other words, we have the following useful proposition.

Proposition A function $f : \mathbb{R} \to \mathbb{R}^m$ with $f(t) = (f_1(t), f_2(t), \ldots, f_m(t))$ is continuous at a point c if and only if the coordinate functions $f_1, f_2, \ldots, f_m$ are each continuous at c.

Similar statements hold for continuity from the right and from the left.

Example The function $f : \mathbb{R} \to \mathbb{R}^3$ defined by

$$f(t) = (\sin(t^2), t^3 + 4, \cos(t))$$

is continuous on the interval $(-\infty, \infty)$ since each of its coordinate functions is continuous on $(-\infty, \infty)$.

Problems

1. Plot the curves parametrized by the following functions over the specified intervals I.
   (a) $f(t) = (3t + 1, 2t - 1)$, $I = [-5, 5]$
   (b) $g(t) = (t, t^2)$, $I = [-3, 3]$
   (c) $f(t) = (3\cos(t), 3\sin(t))$, $I = [0, 2\pi]$
   (d) $h(t) = (3\cos(t), 3\sin(t))$, $I = [0, \pi]$
   (e) $f(t) = (4\cos(2t), 2\sin(2t)$, $I = [0, \pi]$
   (f) $g(t) = (-4\cos(t), 2\sin(t))$, $I = [0, \pi]$
   (g) $h(t) = (t\sin(3t), t\cos(3t))$, $I = [-\pi, \pi]$
2. Plot the curves parametrized by the following functions over the specified intervals I.
   (a) $f(t) = (t + 1, 2t - 1, 3t)$, $I = [-4, 4]$
   (b) $g(t) = (\cos(t), t, \sin(t))$, $I = [0, 4\pi]$
   (c) $f(t) = (t\cos(2t), t\sin(2t), t)$, $I = [-10, 10]$
   (d) $h(t) = (\cos(2t), \sin(2t), \sqrt{t})$, $I = [0, 9]$
3. Plot the curves parametrized by the following functions over the specified intervals I.
   (a) $f(t) = (\cos(4\pi t), \sin(5\pi t))$, $I = [-0.5, 0.5]$
   (b) $f(t) = (\cos(6\pi t), \sin(7\pi t))$, $I = [-0.5, 0.5]$
   (c) $h(t) = (\cos^3(t), \sin^3(t))$, $I = [0, 2\pi]$
   (d) $g(t) = (\cos(2\pi t), \sin(2\pi t), \sin(4\pi t))$, $I = [0, 1]$
   (e) $f(t) = (\sin(4t)\cos(t), \sin(4t)\sin(t))$, $I = [0, 2\pi]$
   (f) $h(t) = ((1 + 2\cos(t))\cos(t), (1 + 2\cos(t))\sin(t))$, $I = [0, 2\pi]$
4. Suppose $g : \mathbb{R} \to \mathbb{R}$ and we define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (t, g(t))$. Describe the curve parametrized by f.
5. For each of the following, compute $\lim_{n\to\infty} \mathbf{x}_n$.
   (a) $\mathbf{x}_n = \left(\frac{n+1}{2n+3}, 3 - \frac{1}{n}\right)$
   (b) $\mathbf{x}_n = \left(\sin\left(\frac{n-1}{n}\right), \cos\left(\frac{n-1}{n}\right), \frac{n-1}{n}\right)$

(c) $\mathbf{x}_n = \left(\dfrac{2n-1}{n^2+1}, \dfrac{3n+4}{n+1}, 4 - \dfrac{6}{n^2}, \dfrac{6n+1}{2n^2+5} \right)$

6. Let $f : \mathbb{R} \to \mathbb{R}^3$ be defined by

$$f(t) = \left(\frac{\sin(t)}{t}, \cos(t), 3t^2 \right).$$

Evaluate the following.

(a) $\lim_{t \to \pi} f(t)$

(b) $\lim_{t \to 1} f(t)$

(c) $\lim_{t \to 0} f(t)$

7. Discuss the continuity of each of the following functions.

(a) $f(t) = (t^2 + 1, \cos(2t), \sin(3t))$

(b) $g(t) = (\sqrt{t+1}, \tan(t))$

(c) $f(t) = \left(\dfrac{1}{t^2 - 1}, \sqrt{1 - t^2}, \dfrac{1}{t} \right)$

(d) $g(t) = (\cos(4t), 1 - \sqrt{3t+1}, \sin(5t), \sec(t))$

8. Let $f : \mathbb{R} \to \mathbb{R}^3$ be defined by $f(t) = (t^2, 3t, 2t + 1)$. Find

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

The Calculus of Functions

of

Several Variables

Section 2.2

Best Affine Approximations

In this section we will generalize the basic ideas of the differential calculus of functions $f : \mathbb{R} \to \mathbb{R}$ to functions $f : \mathbb{R} \to \mathbb{R}^n$. Recall that given a function $f : \mathbb{R} \to \mathbb{R}$, we say f is *differentiable* at a point c if there exists an affine function $A : \mathbb{R} \to \mathbb{R}$, $A(x) = m(x-c) + f(c)$, such that

$$\lim_{h\to 0} \frac{f(c+h) - A(c+h)}{h} = 0. \tag{2.2.1}$$

We call A the *best affine approximation* to f at c and m the *derivative* of f at c, denoted $f'(c)$. Moreover, we call the the graph of A, that is, the line with equation

$$y = f'(c)(x-c) + f(c), \tag{2.2.2}$$

the *tangent line* to the graph of f at $(c, f(c))$.

The condition (2.2.1) says that the function $\varphi(h) = f(c+h) - A(c+h)$ is $o(h)$. In general, we say a function $\varphi : \mathbb{R} \to \mathbb{R}$ is $o(h)$ if

$$\lim_{h\to 0} \frac{\varphi(h)}{h} = 0. \tag{2.2.3}$$

Best affine approximations

Generalizing the idea of the best affine approximation to the case of a function $f : \mathbb{R} \to \mathbb{R}^n$ requires only a slight modification of the requirement that $f(c+h) - A(c+h)$ be $o(h)$. Namely, since $f(c+h)-A(c+h)$ is a vector in $\mathbb{R}^n$, we will require that $\|f(c+h)-A(c+h)\|$, instead of $f(c+h)-A(c+h)$, be $o(h)$. If $n = 1$, this will reduce to the one-variable definition since, in that case, $\|f(c+h) - A(c+h)\| = |f(c+h) - A(c+h)|$ and a function $\varphi : \mathbb{R} \to \mathbb{R}$ is $o(h)$ if and only if $|\varphi(h)|$ is $o(h)$.

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$ and c is a point in the domain of f. We call an affine function $A : \mathbb{R} \to \mathbb{R}^n$ the *best affine approximation* to f at c if (1) $A(c) = f(c)$ and (2) $\|R(h)\|$ is $o(h)$, where

$$R(h) = f(c+h) - A(c+h). \tag{2.2.4}$$

Suppose $f : \mathbb{R} \to \mathbb{R}^n$ and $A : \mathbb{R} \to \mathbb{R}^n$ is an affine function for which $A(c) = f(c)$. Since A is affine, there exists a linear function $L : \mathbb{R} \to \mathbb{R}^n$ and a vector $\mathbf{b}$ in $\mathbb{R}^n$ such that $A(t) = L(t) + \mathbf{b}$ for all t in $\mathbb{R}$. Since we have

$$f(c) = A(c) = L(c) + \mathbf{b}, \tag{2.2.5}$$

it follows that $\mathbf{b} = f(c) - L(c)$. Hence, for all t in $\mathbb{R}$,

$$A(t) = L(t) + f(c) - L(c) = L(c - t) + f(c). \tag{2.2.6}$$

Moreover, if $\mathbf{a} = L(1)$, then, from our results in Section 1.5,

$$A(t) = \mathbf{a}(t - c) + f(c). \tag{2.2.7}$$

Hence

$$R(h) = f(c + h) - A(c + h) = f(c + h) - f(c) - \mathbf{a}h, \tag{2.2.8}$$

from which it follows that

$$\begin{aligned}
\lim_{h \to 0^+} \frac{\|R(h)\|}{h} &= \lim_{h \to 0^+} \frac{\|f(c + h) - f(c) - \mathbf{a}h\|}{h} \\
&= \lim_{h \to 0^+} \left\| \frac{f(c + h) - f(c) - \mathbf{a}h}{h} \right\| \\
&= \lim_{h \to 0^+} \left\| \frac{f(c + h) - f(c)}{h} - \mathbf{a} \right\|
\end{aligned} \tag{2.2.9}$$

Thus

$$\lim_{h \to 0^+} \frac{\|R(h)\|}{h} = 0$$

if and only if

$$\lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} = \mathbf{a}.$$

A similar calculation from the left shows that

$$\lim_{h \to 0^-} \frac{\|R(h)\|}{h} = 0$$

if and only if

$$\lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} = \mathbf{a}.$$

Hence

$$\lim_{h \to 0} \frac{\|R(h)\|}{h} = 0 \tag{2.2.10}$$

if and only if

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \mathbf{a}. \tag{2.2.11}$$

That is, A is the best affine approximation to f at c if and only if, for all t in $\mathbb{R}$,

$$A(t) = \mathbf{a}(t - c) + f(c), \tag{2.2.12}$$

where

$$\mathbf{a} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \tag{2.2.13}$$

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$. If

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \tag{2.2.14}$$

exists, then we say f is *differentiable* at c and we call

$$Df(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \tag{2.2.15}$$

the *derivative* of f at c.

Note that (2.2.15) is the same as the formula for the derivative in one-variable calculus. In fact, in the case $n = 1$, (2.2.15) is just the derivative from one-variable calculus. However, if $n > 1$, then $Df(c)$ will be a vector, not a scalar.

The following theorem summarizes our work above.

Theorem Suppose $f : \mathbb{R} \to \mathbb{R}^n$ and c is a point in the domain of f. Then f has a best affine approximation $A : \mathbb{R} \to \mathbb{R}^n$ at c if and only if f is differentiable at c, in which case

$$A(t) = Df(c)(t - c) + f(c). \tag{2.2.16}$$

We saw in Section 2.1 that a limit of a vector-valued function f may be computed by evaluating the limit of each coordinate function separately. This result has an important consequence for computing derivatives. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable at c. If we write

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t),$$

then

$$\begin{aligned}
Df(c) &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \\
&= \lim_{h \to 0} \frac{1}{h}((f_1(c+h), f_2(c+h), \ldots, f_n(c+h) - (f_1(c), f_2(c_), \ldots, f_n(c)) \\
&= \lim_{h \to 0} \left(\frac{f_1(c+h) - f_1(c)}{h}, \frac{f_2(c+h) - f_2(c)}{h}, \ldots, \frac{f_n(c+h) - f_n(c)}{h} \right) \\
&= \left(\lim_{h \to 0} \frac{f_1(c+h) - f_1(c)}{h}, \lim_{h \to 0} \frac{f_2(c+h) - f_2(c)}{h}, \ldots, \lim_{h \to 0} \frac{f_n(c+h) - f_n(c)}{h} \right) \\
&= (f_1'(c), f_2'(c), \ldots, f_n'(c)).
\end{aligned}$$

In words, the derivative of f is the vector whose coordinates are the derivatives of the coordinate functions of f, reducing the problem of differentiating vector-valued functions to the problem of differentiation in single-variable calculus.

Proposition If f is differentiable at c and $f(t) = (f_1(t), f_2(t), \ldots, f_n(t_0))$, then each coordinate function f_k, $k = 1, 2, \ldots, n$, is differentiable at c and

$$Df(c) = (f_1'(c), f_2'(c), \ldots, f_n'(c)). \tag{2.2.17}$$

For an arbitrary point t at which f is differentiable, we will write,

$$Df(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = (f_1'(t), f_2'(t), \ldots, f_n'(t)). \tag{2.2.18}$$

That is, we may think of Df as a vector-valued function itself, with domain being the set of points at which f is differentiable.

Now suppose $f : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve C and is differentiable at c. If $Df(c) \neq \mathbf{0}$, then the best affine approximation

$$A(t) = Df(c)(t - c) + f(c)$$

parametrizes a line, a line which best approximates the curve C for points near $f(c)$. On the other hand, if $Df(c) = \mathbf{0}$, then A is a constant function with range consisting of the single point $f(c)$. These considerations motivate, in part, the following definitions.

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable on (a, b) and $\mathbf{x} = f(t)$ is a parametrization of a curve C for $a < t < b$. If $Df(t)$ is continuous and $Df(t) \neq \mathbf{0}$ for all t in (a, b), then we call f a *smooth* parametrization of C.

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve C in $\mathbb{R}^n$ and let A be the best affine approximation to f at c. If f is smooth on some open interval containing c, then we call the line in $\mathbb{R}^n$ parametrized by A the *tangent line* to C at $f(c)$.

Example Define $f : \mathbb{R} \to \mathbb{R}^n$ by $f(t) = (\cos(t), \sin(t))$ for $-\infty < t < \infty$. Then, as we saw in Section 2.1, f parametrizes the unit circle C centered at the origin. Now

$$Df(t) = (-\sin(t), \cos(t)),$$

so $Df(t)$ is continuous and $\|Df(t)\| = 1$ for all t. Thus f is a smooth parametrization of C. For example,

$$Df\left(\frac{\pi}{6}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

and

$$f\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$

so the best affine approximation to f at $t = \frac{\pi}{6}$ is

$$A(t) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\left(t - \frac{\pi}{6}\right) + \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Figure 2.2.1 shows C along with the tangent line to C at $t = \frac{\pi}{6}$.

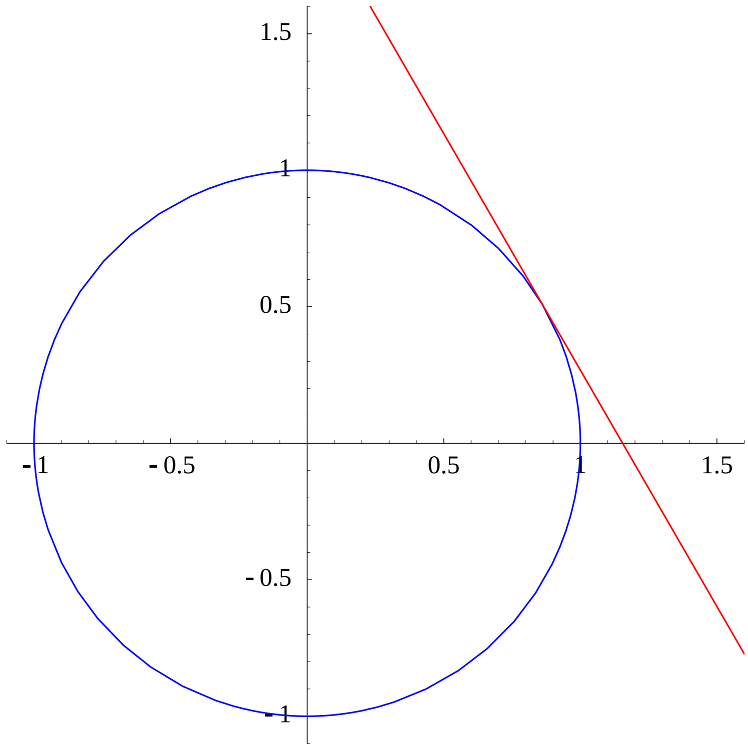


Figure 2.2.1 Unit circle with tangent line at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Example Suppose we define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(t) = (\sin(2\pi t), \cos(2\pi t))$, $-\infty < t < \infty$. Then, as we saw in Section 2.1, g parametrizes the same circle C as f in the previous example. Moreover,

$$Dg(t) = (2\pi\cos(2\pi t), -2\pi\sin(2\pi t))$$

and $\|Dg(t)\| = 1$ for all t, so g is a smooth parametrization of C. However,

$$g\left(\frac{1}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = f\left(\frac{\pi}{6}\right);$$

that is, $g(t)$ is at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ when $t = \frac{1}{6}$, whereas $f(t)$ is at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ when $t = \frac{\pi}{6}$. Moreover,

$$Dg\left(\frac{1}{6}\right) = (\pi, -\pi\sqrt{3}),$$

so the best affine approximation to g at $t = \frac{1}{6}$ is

$$B(t) = (\pi, -\pi\sqrt{3})\left(t - \frac{1}{6}\right) + \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Note that although A, the best affine approximation to f at $t = \frac{\pi}{6}$, and B, the best affine approximation to g at $t = \frac{1}{6}$, are different functions, they parametrize the same line since

$$(\pi, -\pi\sqrt{3}) = -2\pi\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

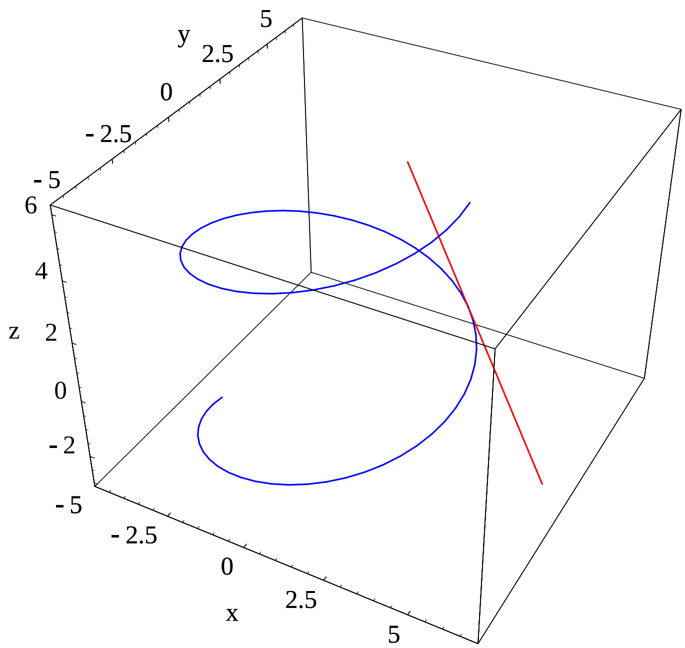


Figure 2.2.2 Helix with tangent line at $\left(2\sqrt{2}, 2\sqrt{2}, \frac{\pi}{4}\right)$

Example Consider the helix C parametrized by $f : \mathbb{R} \to \mathbb{R}^3$ defined by

$$f(t) = (4\cos(t), 4\sin(t), t).$$

Then

$$Df(t) = (-4\sin(t), 4\cos(t), 1).$$

Since Df is continuous and

$$\|Df(t)\| = \sqrt{16\sin^2(t) + 16\cos^2(t) + 1} = \sqrt{17}$$

for all t, f is a smooth parametrization of C. Now, for example,

$$Df\left(\frac{\pi}{4}\right) = \left(-\frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}}, 1\right) = (-2\sqrt{2}, 2\sqrt{2}, 1)$$

and

$$f\left(\frac{\pi}{4}\right) = \left(\frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}}, \frac{\pi}{4}\right) = \left(2\sqrt{2}, 2\sqrt{2}, \frac{\pi}{4}\right),$$

so the best affine approximation to f at $t = \frac{\pi}{4}$ is

$$A(t) = (-2\sqrt{2}, 2\sqrt{2}, 1)\left(t - \frac{\pi}{4}\right) + \left(2\sqrt{2}, 2\sqrt{2}, \frac{\pi}{4}\right).$$

The helix C and the line parametrized by A, namely, the tangent line to C at $t = \frac{\pi}{4}$, are shown in Figure 2.2.2.

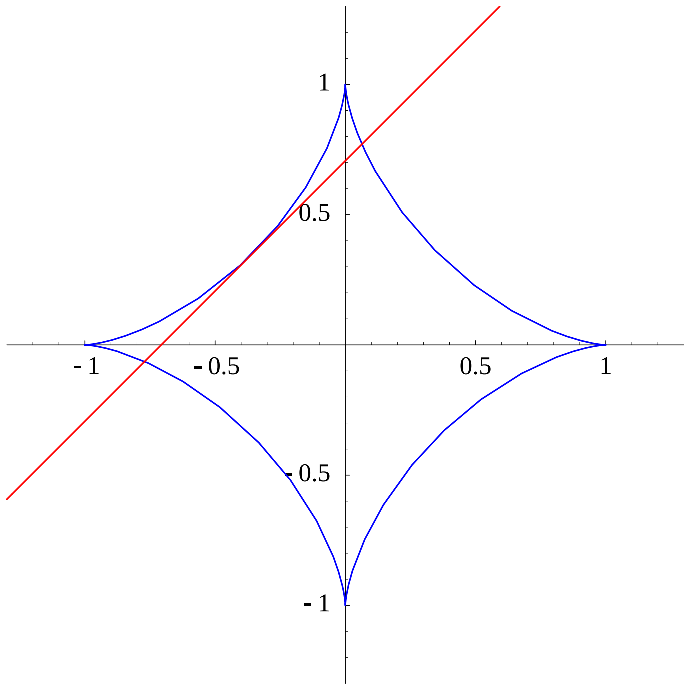


Figure 2.2.3 $h(t) = (\cos^3(t), \sin^3(t))$ with tangent line at $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

Example Let C be the curve in $\mathbb{R}^2$ parametrized by

$$h(t) = (\cos^3(t), \sin^3(t)).$$

Then

$$Dh(t) = (-3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)).$$

Hence Dh is continuous for all t, but h is not a smooth parametrization of C since $Dh(t) = \mathbf{0}$ whenever t is an integer multiple of $\frac{\pi}{2}$. These points correspond to the sharp corners of C at $(1,0)$, $(0,1)$, $(-1,0$, and $(0,-1)$, as shown in Figure 2.2.3. However, h is a smooth parametrization of the four arcs of C which are parametrized by restricting h to the open intervals $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, $\left(\pi, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. Hence, for example, noting that

$$Dh\left(\frac{3\pi}{4}\right) = \left(-\frac{3}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}\right)$$

and

$$h\left(\frac{3\pi}{4}\right) = \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right),$$

we find that the best affine approximation to h at $t = \frac{3\pi}{4}$ is

$$A(t) = \left(-\frac{3}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}\right)\left(t - \frac{3\pi}{4}\right) + \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right).$$

The tangent line parametrized by A is shown in Figure 2.2.3.

Proposition Suppose $f : \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R} \to \mathbb{R}^n$, and $\varphi : \mathbb{R} \to \mathbb{R}$ are all differentiable. Then

$$D(f(t) + g(t)) = Df(t) + Dg(t), \tag{2.2.19}$$

$$D(f(t) - g(t)) = Df(t) - Dg(t), \tag{2.2.20}$$

$$D(\varphi(t)f(t)) = \varphi(t)Df(t) + \varphi'(t)f(t), \tag{2.2.21}$$

$$\frac{d}{dt}(f(t) \cdot g(t)) = f(t) \cdot Dg(t) + Df(t) \cdot g(t), \tag{2.2.22}$$

and

$$D(f(\varphi(t)) = Df(\varphi(t))\varphi'(t)). \tag{2.2.23}$$

Note that all of the statements in this proposition reduce to familiar results from one-variable calculus when $n = 1$. To verify these results, let

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$

and

$$g(t) = (g_1(t), g_2(t), \ldots, g_n(t)).$$

Then

$$\begin{aligned} D(f(t) + g(t)) &= D(f_1(t) + g_1(t), f_2(t) + g_2(t), \ldots, f_n(t) + g_n(t)) \\ &= (f_1'(t) + g_1'(t), f_2'(t) + g_2'(t), \ldots, f_n'(t) + g_n'(t)) \\ &= (f_1'(t), f_2'(t), \ldots, f_n'(t)) + (g_1'(t), g_2'(t), \ldots, g_n'(t)) \\ &= Df(t) + Dg(t), \end{aligned} \tag{2.2.24}$$

verifying (2.2.19). The verification of (2.1.20) is similar. The demonstrations of (2.2.21) and (2.1.22), both of which are generalizations of the product rule from one-variable calculus, follow easily from that result; we will check (2.1.22) here and leave (2.2.21) for Problem 13. Using the product rule, we have

$$\begin{aligned} \frac{d}{dt}(f(t) \cdot g(t)) &= \frac{d}{dt}(f_1(t)g_1(t) + f_2(t)g_2(t) + \cdots + f_n(t)g_n(t)) \\ &= f_1(t)g_1'(t) + f_1'(t)g_1(t) + f_2(t)g_2'(t) + f_2'(t)g_2(t) + \cdots \\ &\qquad + f_n(t)g_n'(t) + f_n'(t)g_n(t) \\ &= f(t) \cdot Dg(t) + Df(t) \cdot g(t). \end{aligned} \tag{2.2.25}$$

Finally, (2.2.23), a generalization of the chain rule from one-variable calculus, follows directly from that result:

$$\begin{aligned} D(f(\varphi(t))) &= D(f_1(\varphi(t)), f_2(\varphi(t)), \ldots, f_n(\varphi(t))) \\ &= (f_1'(\varphi(t))\varphi'(t), f_2'(\varphi(t))\varphi'(t), \ldots, f_n'(\varphi(t))\varphi'(t)) \\ &= Df(\varphi(t))\varphi'(t). \end{aligned} \tag{2.2.26}$$

Reparametrizations

We have seen above that the parametrization of a curve C in $\mathbb{R}^n$ is not unique. For example, we saw that both $f(t) = (\cos(t), \sin(t))$ and $g(t) = (\sin(2\pi t), \cos(2\pi t))$ parametrize the unit circle centered at the origin. However, we also noted that the best affine approximations for the two parametrizations, although distinct functions, nevertheless parametrize the same line at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, the line we have been calling the tangent line. We should suspect that this will be the case in general, that is, the tangent line to a curve C at a particular point should not depend on the particular parametrization of C used in the computation. While avoiding some technicalities, we will provide some justification for these ideas.

Definition Suppose $\mathbf{x} = f(t)$, $a < t < b$, is a smooth parametrization of a curve C in $\mathbb{R}^n$. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ has domain (c, d), range (a, b), and φ' exists and is continuous on (c, d). If $\varphi'(t) \neq 0$ for all t in (c, d), then we call $g(t) = f(\varphi(t))$ a *reparametrization* of f.

Example Let $f(t) = (\cos(t), \sin(t))$ and $g(t) = (\sin(2\pi t), \cos(2\pi t))$. Since

$$\sin(t) = \cos\left(\frac{\pi}{2} - t\right)$$

and

$$\cos(t) = \sin\left(\frac{\pi}{2} - t\right),$$

if follows that

$$g(t) = f\left(\frac{\pi}{2} - 2\pi t\right) = f(\varphi(t)),$$

where

$$\varphi(t) = \frac{\pi}{2} - 2\pi t.$$

That is, g is a reparametrization of f.

Now if $\mathbf{x} = f(t)$, $a < t < b$, is a smooth parametrization of a curve C in $\mathbb{R}^n$ and $g(t) = f(\varphi(t))$, $c < t < d$, is a reparametrization of f, then for any α in (c, d),

$$Dg(\alpha) = D(f(\varphi(\alpha)) = Df(\varphi(\alpha))\varphi'(\alpha). \tag{2.2.27}$$

Hence $Dg(\alpha)$ and $Df(\varphi(\alpha))$ are parallel, the former being the latter multiplied by the scalar $\varphi'(\alpha)$. In other words, the lines parametrized by the best affine approximation to g at $t = \alpha$ and the best affine approximation to f at $t = \varphi(\alpha)$ are the same.

Example In our previous example, we have

$$\varphi'(t) = -2\pi,$$

so, for any α, we should have

$$Dg(\alpha) = -2\pi Df(\varphi(\alpha)).$$

This agrees with our previous calculation using $\alpha = \frac{1}{6}$.

Tangent and normal vectors

If $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C, then, for any t, $Df(t)$ is the direction of the tangent line to C at $f(t)$. Moreover, from our discussion above, if g is a reparametrization of f, say, $g(t) = f(\varphi(t))$, then $Dg(t)$ and $Df(\varphi(t))$ will have the same or opposite direction. In other words, the direction of the tangent line either remains the same or is reversed under reparametrization. On the other hand,

$$\|Dg(t)\| = \|Df(\varphi(t))\||\varphi'(t)|. \tag{2.2.28}$$

As we should expect, although both $Dg(t)$ and $Df(\varphi(T))$ are tangent to the curve at $g(t)$, their lengths do not have to be the same. In Section 2.3 we will discuss how we may think of this in terms of the speed of a particle moving along the curve C, with its position on C at time t given by either $g(t)$ or $f(t)$.

For these and other considerations, it is useful to define a standard tangent vector, unique up to a change in sign.

Definition If $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C, then we call

$$T(t) = \frac{Df(t)}{\|Df(t)\|} \tag{2.2.29}$$

the *unit tangent vector* to C at $f(t)$.

From the preceding, we must keep in mind that the unit tangent vector $T(t)$ is always in reference to some parametrization f of the curve C. Essentially, this is a choice of an orientation for the curve, that is, the direction of motion for a particle whose position at time t is given by $f(t)$.

If $\mathbf{x} = f(t)$, $a < t < b$, is a smooth parametrization of a curve C in $\mathbb{R}^n$, then, by definition, $\|T(t)\| = 1$ for all t in (a, b). Hence

$$T(t) \cdot T(t) = 1 \tag{2.2.30}$$

for all t in (a, b). Differentiating (2.2.30), we have

$$\frac{d}{dt}(T(t) \cdot T(t)) = \frac{d}{dt}1 = 0, \tag{2.2.31}$$

and so, using (2.2.22), we have

$$0 = \frac{d}{dt}(T(t) \cdot T(t)) = T(t) \cdot DT(t) + DT(t) \cdot T(t) = 2DT(t) \cdot T(t) \tag{2.2.32}$$

for all t in (a, b). Thus $T(t) \cdot DT(t) = 0$ for $a < t < b$. In other words, $DT(t)$ is orthogonal to $T(t)$ for all t in (a, b).

Definition If $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C, $T(t)$ is the unit tangent vector to C at $f(t)$, and $DT(t) \neq \mathbf{0}$, then we call

$$N(t) = \frac{DT(t)}{\|DT(t)\|} \tag{2.2.33}$$

the *principal unit normal vector* to C at $f(t)$.

Example Consider the parametrization of the circle in $\mathbb{R}^2$ with radius 2 and center at the origin given by

$$f(t) = (2\cos(4t), 2\sin(4t)).$$

Then

$$Df(t) = (-8\sin(4t), 8\cos(4t))$$

and

$$\|Df(t)\| = \sqrt{64\sin^2(4t) + 64\cos^2(4t)} = 8.$$

Thus the unit tangent vector is

$$T(t) = \frac{Df(t)}{\|Df(t)\|} = (-\sin(4t), \cos(4t)).$$

Moreover,

$$DT(t) = (-4\cos(t), -4\sin(4t)),$$

so

$$\|DT(t)\| = \sqrt{16\cos^2(4t) + 16\sin^2(4)} = 4,$$

and the principal unit normal vector is

$$N(t) = \frac{DT(t)}{\|DT(t)\|} = (-\cos(4t), -\sin(4t)).$$

For example, when $t = \frac{\pi}{24}$ we have

$$f\left(\frac{\pi}{24}\right) = (\sqrt{3}, 1),$$

$$T\left(\frac{\pi}{24}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

and

$$N\left(\frac{\pi}{24}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

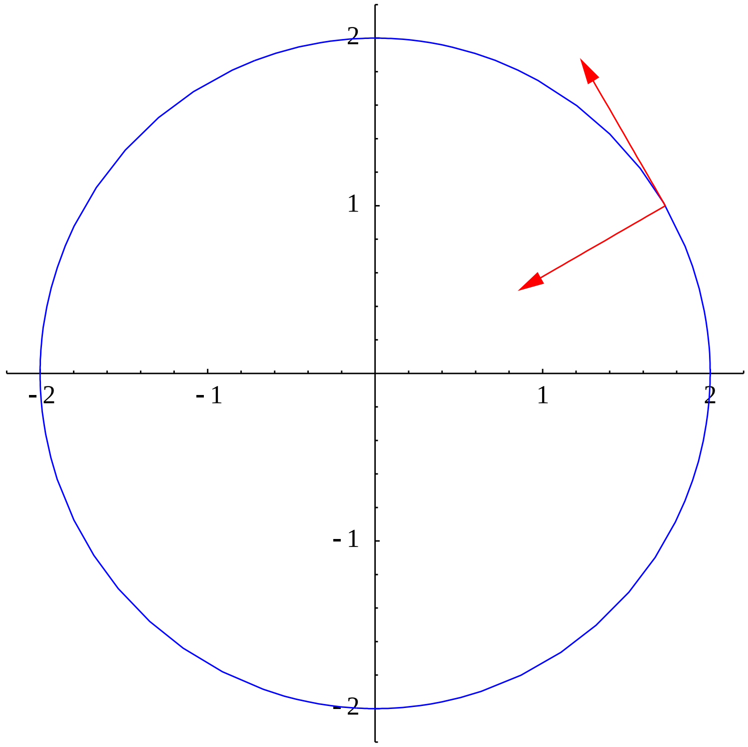


Figure 2.2.4 A circle with unit tangent and normal vectors

Note that, for any value of t, $f(t) \perp T(t)$, $T(t) \perp N(t)$ (as is always the case), and $N(t) = -\frac{1}{2}f(t)$. See Figure 2.2.4.

Example Consider the elliptical helix H parametrized by

$$g(t) = (\cos(t), 2\sin(t), t).$$

Then

$$Dg(t) = (-\sin(t), 2\cos(t), 1),$$

so

$$\begin{aligned}\|Dg(t)\| &= \sqrt{\sin^2(t) + 4\cos^2(t) + 1} \\ &= \sqrt{\sin^2(t) + \cos^2(t) + 3\cos^2(t) + 1} \\ &= \sqrt{2 + 3\cos^2(t)} \\ &= \sqrt{2 + \frac{3}{2}(1 + \cos(2t))} \\ &= \sqrt{\frac{7 + 3\cos(2t)}{2}}.\end{aligned}$$

Hence the unit tangent vector is

$$T(t) = \sqrt{\frac{2}{7 + 3\cos(2t)}}\,(-\sin(t), 2\cos(t), 1).$$

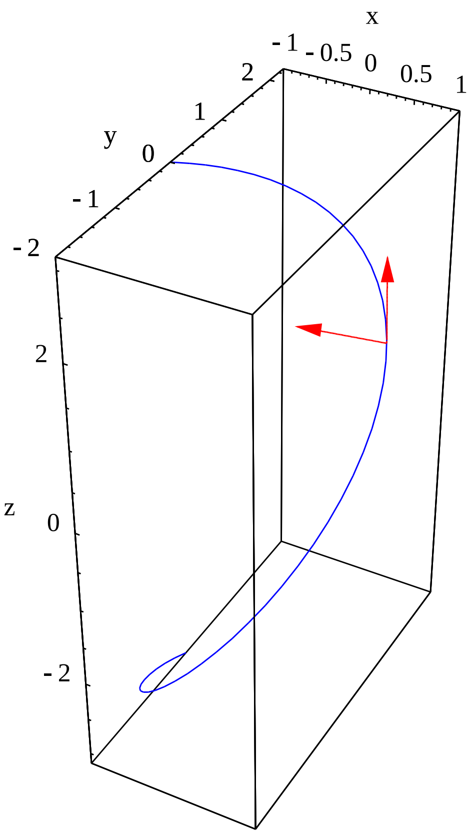


Figure 2.2.5 An elliptical helix with unit tangent and normal vectors

Differentiating using (2.2.21), we have

$$\begin{aligned} DT(t) &= \sqrt{\frac{2}{7+3\cos(2t)}}\,(-\cos(t), -2\sin(t), 0) \\ &\quad + \frac{1}{2}\left(\frac{2}{7+3\cos(2t)}\right)^{-\frac{1}{2}}\left(\frac{12\sin(2t)}{(7+3\cos(2t))^2}\right)(-\sin(t), 2\cos(t), 1) \\ &= \sqrt{\frac{2}{7+3\cos(2t)}}\,(-\cos(t), -2\sin(t), 0) + \frac{3\sqrt{2}\sin(2t)}{(7+3\cos(2t))^{\frac{3}{2}}}\,(-\sin(t), 2\cos(t), 1). \end{aligned}$$

For example, at $t = \frac{\pi}{4}$ we have

$$g\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{\pi}{4}\right),$$

$$T\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{7}}\,(-1, 2, \sqrt{2}),$$

and

$$DT\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{7}}\,(-1, -2, 0) + \frac{3}{7^{\frac{3}{2}}}\,(-1, 2, \sqrt{2}) = \frac{1}{7\sqrt{7}}\,(-10, -8, 3\sqrt{2}).$$

Thus

$$\left\| DT\left(\frac{\pi}{4}\right) \right\| = \frac{1}{7\sqrt{7}}\,\sqrt{100+64+18} = \frac{\sqrt{26}}{7},$$

so the principal unit normal vector at $t = \frac{\pi}{4}$ is

$$N\left(\frac{\pi}{4}\right) = \frac{DT\left(\frac{\pi}{4}\right)}{\left\|DT\left(\frac{\pi}{4}\right)\right\|} = \frac{1}{\sqrt{182}}\,(-10,-8,3\sqrt{2}).$$

See Figure 2.2.5.

As the last example shows, the computations involved in finding the unit tangent vector and the principal unit normal vector can become involved. In fact, that is why we computed the principal unit normal vector only in the particular case $t = \frac{\pi}{4}$ instead of writing out the general formula for $N(t)$. In general these computations can become involved enough that it is often wise to make use of a computer algebra system.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(t) = (t^3, t, 2t+4)$ (b) $g(t) = (3t\cos(2t), 4t\sin(2t))$

   (c) $h(t) = (4t^3-3, \sin(t), e^{-2t})$ (d) $f(t) = (e^{-t}\sin(3t), e^{-t}\cos(3t), te^{-t})$

2. For each of the following, find the best affine approximation to f at the given point.

   (a) $f(t) = (t, t^3)$, $t = 2$ (b) $f(t) = (3\sin(2t), 4\cos(2t))$, $t = \dfrac{\pi}{6}$

   (c) $f(t) = (\cos(t), \sin(t), \cos(2t))$, $t = \dfrac{\pi}{3}$ (d) $f(t) = (2\cos(2t), 3\sin(2t), 3t)$, $t = 0$

3. Let $f(t) = (2\cos(\pi t), 3\sin(\pi t))$ parametrize an ellipse E in $\mathbb{R}^2$. Plot E along with the tangent line at $f\left(\frac{2}{3}\right)$.

4. Let $f(t) = ((1+2\cos(t))\cos(t), (1+2\cos(t))\sin(t))$ parametrize a curve C in $\mathbb{R}^2$. Plot C along with the tangent line at $f\left(\frac{\pi}{6}\right)$.

5. Let $h(t) = \left(\sin(2\pi t), \cos(2\pi t), \frac{t}{2}\right)$ parametrize a circular helix H in $\mathbb{R}^3$. Plot H along with the tangent line at $h\left(\frac{3}{2}\right)$.

6. Let $g(t) = (\cos(\pi t), \sqrt{t}, \sin(\pi t))$ parametrize a curve C in $\mathbb{R}^3$. Plot C along with the tangent line at $g\left(\frac{1}{4}\right)$.

7. Suppose $f : \mathbb{R} \to \mathbb{R}^2$ is defined by $f(t) = (t, \varphi(t))$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable, and let C be the curve in $\mathbb{R}^2$ parametrized by f. Show that the tangent line to C at $f(c)$ is the same as the line tangent to the graph of φ at $(c, \varphi(c)$.

8. Let C be the curve in $\mathbb{R}^2$ parametrized by $f(t) = (t^3, t^6)$, $-\infty < t < \infty$. Is f a smooth parametrization of C? If not, can you find a smooth parametrization of C?

9. Let C be the curve in $\mathbb{R}^2$ parametrized by $f(t) = (t^2, t^2)$, $-\infty < t < \infty$. Show that f is not a smooth parametrization of C. Where is the problem? Plot C and identify the location of the problem.

10. Let $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p}$ be vectors in $\mathbb{R}^n$ and let C be the curve in $\mathbb{R}^n$ parametrized by $f(t) = t\mathbf{v} + \mathbf{p}$. What is the best affine approximation to f at $t = t_0$?

11. For each of the following, find the unit tangent vector and the principal unit normal vector at the indicated point.

    (a) $f(t) = (t, t^2)$, $t = 1$

    (b) $g(t) = (3\sin(2t), 3\cos(2t))$, $t = \dfrac{\pi}{3}$

    (c) $f(t) = (2\cos(t), 4\sin(t))$, $t = \dfrac{\pi}{4}$

    (d) $h(t) = (\cos(\pi t), 2\sin(\pi t))$, $t = \dfrac{3}{4}$

    (e) $g(t) = (\cos(t), \sin(t), t)$, $t = \dfrac{\pi}{3}$

    (f) $f(t) = (2\sin(t), 3\cos(2t), 2t)$, $t = \dfrac{\pi}{4}$

    (g) $f(t) = (\sin(\pi t), -\cos(\pi t), 3t)$, $t = \dfrac{1}{2}$

    (h) $g(t) = (\cos(\pi t^2), \sin(\pi t^2), t^2)$, $t = 1$

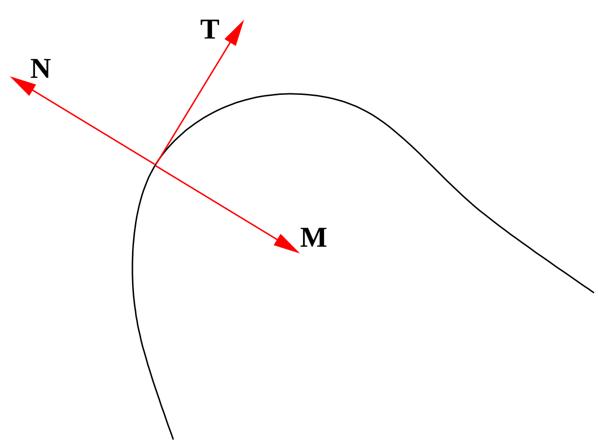
    (i) $f(t) = (t, t^2, t^3)$, $t = 2$

12. Use the fact that $f(t) = (b\cos(t), b\sin(t))$ parametrizes a circle of radius b to show that a radius of a circle is always perpendicular to the tangent line at the point where the radius touches the circle.

13. Verify (2.2.21); that is, show that if $f : \mathbb{R} \to \mathbb{R}^n$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are both differentiable, then
$$D(\varphi(t)f(t)) = \varphi(t)Df(t) + \varphi'(t)f(t).$$

14. Suppose $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}^3$ are both differentiable. Show that
$$D(f(t) \times g(t)) = f(t) \times Dg(t) + Df(t) \times g(t),$$
yet another version of the product rule.

15. The following figure illustrates a curve in $\mathbb{R}^2$ parametrized by some function $f : \mathbb{R} \to \mathbb{R}^2$. If $\mathbf{T}$ is the unit tangent vector at the indicated point on the curve, then either $\mathbf{M}$ or $\mathbf{N}$ is the principal unit normal vector at that point. Which one is it?



The Calculus of Functions

of

Several Variables

Section 2.3

Motion Along a Curve

Velocity and acceleration

Consider a particle moving in space so that its position at time t is given by $\mathbf{x}(t)$. We think of $\mathbf{x}(t)$ as moving along a curve C parametrized by a function f, where $f : \mathbb{R} \to \mathbb{R}^n$. Hence we have $\mathbf{x}(t) = f(t)$, or, more simply, $\mathbf{x} = f(t)$. For us, n will always be 2 or 3, but there are physical situations in which it is reasonable to have larger values of n, and most of what we do in this section will apply to those cases equally well. This is also a good time to introduce the Leibniz notation for a derivative, thus writing

$$\frac{d\mathbf{x}}{dt} = Df(t). \tag{2.3.1}$$

At a given time t_0, the vector $\mathbf{x}(t_0+h) - \mathbf{x}(t_0)$ represents the magnitude and direction of the change of position of the particle along C from time t_0 to time $t_0 + h$, as shown in Figure 2.3.1. Dividing by h, we obtain a vector,

$$\frac{\mathbf{x}(t_0+h) - \mathbf{x}(t_0)}{h}, \tag{2.3.2}$$

with the same direction, but with length approximating the average speed of the particle over the time interval from t_0 to $t_0 + h$. Assuming differentiability and taking the limit as h approaches 0, we have the following definition.

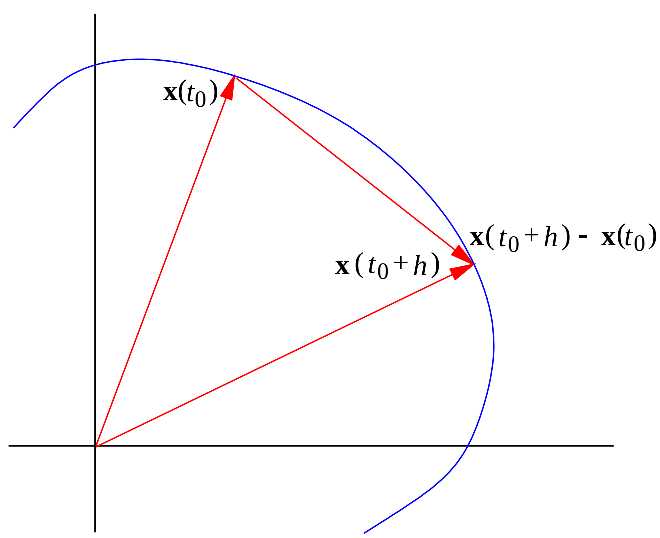


Figure 2.3.1 Motion along a curve C

Definition Suppose $\mathbf{x}(t)$ is the position of a particle at time t moving along a curve C in $\mathbb{R}^n$. We call

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{x}(t) \tag{2.3.3}$$

the *velocity* of the partial at time t and we call

$$s(t) = \|\mathbf{v}(t)\| \tag{2.3.4}$$

the *speed* of the particle at time t. Moreover, we call

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) \tag{2.3.5}$$

the *acceleration* of the particle at time t.

Example Consider a particle moving along an ellipse so that its position at any time t is

$$\mathbf{x} = (2\cos(t), \sin(t)).$$

Then its velocity is

$$\mathbf{v} = (-2\sin(t), \cos(t)),$$

its speed is

$$s = \sqrt{4\sin^2(t) + \cos^2(t)} = \sqrt{3\sin^2(t) + 1},$$

and its acceleration is

$$\mathbf{a} = (-2\cos(t), -\sin(t)).$$

For example, at $t = \frac{\pi}{4}$ we have

$$\mathbf{x}\Big|_{t=\frac{\pi}{4}} = \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}\Big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$s\Big|_{t=\frac{\pi}{4}} = \sqrt{\frac{5}{2}},$$

and

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right).$$

See Figure 2.3.2. Notice that, in this examples, $\mathbf{a} = -\mathbf{x}$ for all values of t.

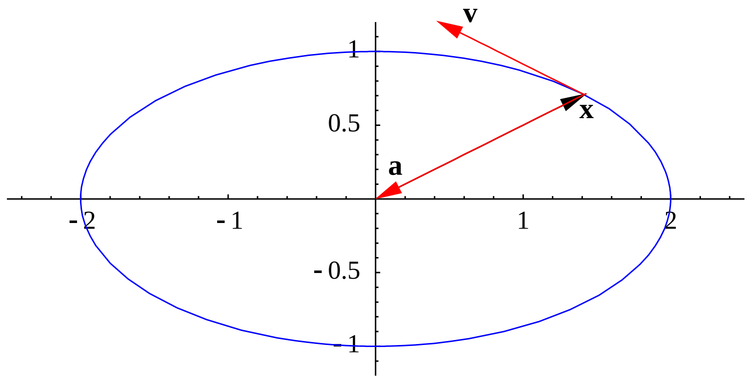


Figure 2.3.2 Position, velocity, and acceleration vectors for motion on an ellipse

Curvature

Suppose $\mathbf{x}$ is the position, $\mathbf{v}$ is the velocity, s is the speed, and $\mathbf{a}$ is the acceleration, at time t, of a particle moving along a curve C. Let $T(t)$ be the unit tangent vector and $N(t)$ be the principal unit normal vector at $\mathbf{x}$. Now

$$T(t) = \frac{\dfrac{d\mathbf{x}}{dt}}{\left\|\dfrac{d\mathbf{x}}{dt}\right\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}}{s}, \tag{2.3.6}$$

so

$$\mathbf{v} = s\|T(t)\|. \tag{2.3.7}$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\, sT(t) = \frac{ds}{dt}\, T(t) + sDT(t) \tag{2.3.8}$$

Since

$$N(t) = \frac{DT(t)}{\|DT(t)\|}, \tag{2.3.9}$$

we have

$$\mathbf{a} = \frac{ds}{dt}\, T(t) + s\|DT(t)\|N(t). \tag{2.3.10}$$

Note that (2.3.10) expresses the acceleration of a particle as the sum of scalar multiples of the unit tangent vector and the principal unit normal vector. That is,

$$\mathbf{a} = a_T T(t) + a_N N(t), \tag{2.3.11}$$

where

$$a_T = \frac{ds}{dt} \tag{2.3.12}$$

and

$$a_N = s\|DT(t)\|. \tag{2.3.13}$$

However, since $T(t)$ and $N(t)$ are orthogonal unit vectors, we also have

$$\begin{aligned}\mathbf{a}\cdot T(t) &= (a_T T(t) + a_N N(t))\cdot T(t)\\ &= a_T(T(t)\cdot T(t)) + a_N(T(t)\cdot N(t))\\ &= a_T\end{aligned} \tag{2.3.14}$$

and

$$\begin{aligned}\mathbf{a}\cdot N(t) &= (a_T T(t) + a_N N(t))\cdot N(t)\\ &= a_T(T(t)\cdot N(t)) + a_N(N(t)\cdot N(t))\\ &= a_N.\end{aligned} \tag{2.3.15}$$

Hence a_T is the coordinate of $\mathbf{a}$ in the direction of $T(t)$ and a_N is the coordinate of $\mathbf{a}$ in the direction of $N(t)$. Thus (2.3.10) writes the acceleration as a sum of its component in the direction of the unit tangent vector and its component in the direction of the principal unit normal vector. In particular, this shows that the acceleration lies in the plane determined by $T(t)$ and $N(t)$. Moreover, a_T is the rate of change of speed, while a_N is the product of the speed s and $\|DT(t)\|$, the magnitude of the rate of change of the unit tangent vector. Since $\|T(t)\| = 1$ for all t, $\|DT(t)\|$ reflects only the rate at which the direction of $T(t)$ is changing; in other words, $\|DT(t)\|$ is a measurement of how fast the direction of the particle moving along the curve C is changing at time t. If we divide this by the speed of the particle, we obtain a standard measurement of the rate of change of direction of C itself.

Definition Given a curve C with smooth parametrization $\mathbf{x} = f(t)$, we call

$$\kappa = \frac{\|DT(t)\|}{s(t)} \tag{2.3.16}$$

the *curvature* of C at $f(t)$.

Using (2.3.16), we can rewrite (2.3.10) as

$$\mathbf{a} = \frac{ds}{dt}\,T(t) + s^2\kappa N(t). \tag{2.3.17}$$

Hence the coordinate of acceleration in the direction of the tangent vector is the rate of change of the speed and the coordinate of acceleration in the direction of the principal normal vector is the square of the speed times the curvature. Thus the greater the speed or the tighter the curve, the larger the size of the normal component of acceleration; the greater the rate at which speed is increasing, the greater the tangential component of acceleration. This is why drivers are advised to slow down while approaching a curve, and then to accelerate while driving through the curve.

Example Suppose a particle moves along a line in $\mathbb{R}^n$ so that its position at any time t is given by

$$\mathbf{x} = t\mathbf{w} + \mathbf{p},$$

where $\mathbf{w} \neq 0$ and $\mathbf{p}$ are vectors in $\mathbb{R}^n$. Then the particle has velocity

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \mathbf{w}$$

and speed $s = \|\mathbf{w}\|$, so the unit tangent vector is

$$T(t) = \frac{\mathbf{v}}{s} = \frac{\mathbf{w}}{\|\mathbf{w}\|}.$$

Hence $T(t)$ is a constant vector, so $DT(t) = \mathbf{0}$ and

$$\kappa = \frac{\|DT(t)\|}{s} = 0$$

for all t. In other words, a line has zero curvature, as we should expect since the tangent vector never changes direction.

Example Consider a particle moving along a circle C in $\mathbb{R}^2$ of radius $r > 0$ and center (a, b), with its position at time given by

$$\mathbf{x} = (r\cos(t) + a, r\sin(t) + b).$$

Then its velocity, speed, and acceleration are

$$\mathbf{v} = (-r\sin(t), r\cos(t)),$$

$$s = \sqrt{r^2 \sin^(t) + r^2\cos^2(t)} = r$$

and

$$\mathbf{a} = (-r\cos(t), -r\sin(t)),$$

respectively. Hence the unit tangent vector is

$$T(t) = \frac{\mathbf{v}}{s} = (-\sin(t), \cos(t)).$$

Thus

$$DT(t) = (-\cos(t), -\sin(t))$$

and

$$\|DT(t)\| = \sqrt{\cos^2(t) + \sin^2(t)} = 1.$$

Hence the curvature of C is, for all t,

$$\kappa = \frac{\|DT(t)\|}{s} = \frac{1}{r}.$$

Thus a circle has constant curvature, namely, the reciprocal of the radius of the circle. In particular, the larger the radius of a circle, the smaller the curvature. Also, note that

$$\frac{ds}{dt} = \frac{d}{dt}r = 0,$$

so, from (2.3.10), we have

$$\mathbf{a} = rN(t),$$

which we can verify directly. That is, the acceleration has a normal component, but no tangential component.

Example Now consider a particle moving along an ellipse E so that its position at any time t is

$$\mathbf{x} = (2\cos(t), \sin(t)).$$

Then, as we saw above, the velocity and speed of the particle are

$$\mathbf{v} = (-2\sin(t), \cos(t))$$

and

$$s = \sqrt{3\sin(^2) + 1},$$

respectively. For purposes of differentiation, it will be helpful to rewrite s as

$$s = \sqrt{\frac{3}{2}(1 - \cos(2t)) + 1} = \sqrt{\frac{5 - 3\cos(2t)}{2}}.$$

Then the unit tangent vector is

$$T(t) = \sqrt{\frac{2}{5 - 3\cos(2t)}}\,(-2\sin(t), \cos(t)).$$

Thus

$$DT(t) = \sqrt{\frac{2}{5 - 3\cos(2t)}}\,(-2\cos(t), -\sin(t)) - \frac{3\sqrt{2}\sin(2t)}{(5 - 3\cos(2t))^{\frac{3}{2}}}(-2\sin(t), \cos(t)).$$

So, for example, at $t = \frac{\pi}{4}$, we have

$$\mathbf{x}\big|_{t=\frac{\pi}{4}} = \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}\big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$s\big|_{t=\frac{\pi}{4}} = \sqrt{\frac{5}{2}},$$

$$T\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{5}}\,(-2, 1),$$

$$DT\left(\frac{\pi}{4}\right) = -\frac{1}{5\sqrt{5}}(4, 8),$$

and

$$\left\| DT\left(\frac{\pi}{4}\right)\right\| = \frac{1}{5\sqrt{5}}\sqrt{16+64} = \frac{4}{5}.$$

Hence the curvature of E at $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$ is

$$\kappa\Big|_{t=\frac{\pi}{4}} = \frac{\frac{4}{5}}{\sqrt{\frac{5}{2}}} = \frac{4\sqrt{2}}{5\sqrt{5}} = 0.05060,$$

where the final numerical value has been rounded to four decimal places. Although the general expression for κ is complicated, it is easily computed and plotted using a computer algebra system, as shown in Figure 2.3.3. Comparing this with the plot of this ellipse in Figure 2.3.2, we can see why the curvature is greatest around $(2, 0)$ and $(-2, 0)$, corresponding to $t = 0$, $t = \pi$, and $t = 2\pi$, and smallest at $(0, 1)$ and $(0, -1)$, corresponding to $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. Finally, as we saw above, the acceleration of the particle is

$$\mathbf{a} = (-2\cos(t), -\sin(t)),$$

so

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right).$$

Now if we write

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = a_T T(t) + a_N N(t),$$

then we may either compute, using (2.3.17),

$$a_T = \frac{ds}{dt}\Big|_{t=\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(5 - 3\cos(2t))^{-\frac{1}{2}}(3\sin(2t)\Big|_{t=\frac{\pi}{4}} = \frac{3}{\sqrt{10}}$$

and

$$a_N = s^2\Big|_{t=\frac{\pi}{4}} k\Big|_{t=\frac{\pi}{4}} = \frac{5}{2}\frac{4\sqrt{2}}{5\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{5}} = \frac{4}{\sqrt{10}},$$

or, using (2.3.14) and (2.3.15),

$$a_T = \mathbf{a}\Big|_{t=\frac{\pi}{4}} \cdot T\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{5}}\,(-2, 1) = \frac{3}{\sqrt{10}}$$

and

$$a_N = \mathbf{a}\Big|_{t=\frac{\pi}{4}} \cdot N\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{4\sqrt{5}}(-4, -8) = \frac{4}{\sqrt{10}}.$$

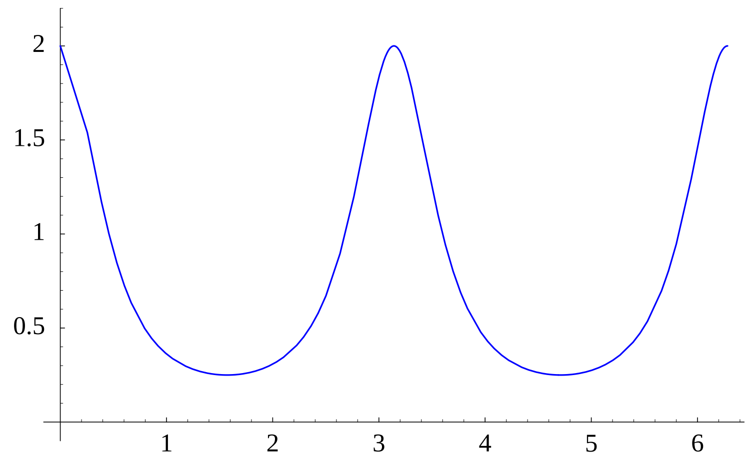


Figure 2.3.3 Curvature of an ellipse

Hence, in either case,

$$\mathbf{a}\big|_{t=\frac{\pi}{4}} = \frac{3}{\sqrt{10}}T\left(\frac{\pi}{4}\right) + \frac{4}{\sqrt{10}}N\left(\frac{\pi}{4}\right).$$

Arc length

Suppose a particle moves along a curve C in $\mathbb{R}^n$ so that its position at time t is given by $\mathbf{x} = f(t)$ and let D be the distance traveled by the particle from time $t = a$ to $t = b$. We will suppose that $s(t) = \|\mathbf{v}(t)\|$ is continuous on $[a, b]$. To approximate D, we divide $[a, b]$ into n subintervals, each of length

$$\Delta t = \frac{b-a}{n},$$

and label the endpoints of the subintervals $a = t_0, t_1, \ldots, t_n = b$. If Δt is small, then the distance the particle travels during the jth subinterval, $j = 1, 2, \ldots, n$, should be, approximately, $s\Delta t$, an approximation which improves as Δt decreases. Hence, for sufficiently small Δt (equivalently, sufficiently large n),

$$\sum_{j=1}^{n} s(t_{j-1})\Delta t \tag{2.3.18}$$

will provide an approximation as close to D as desired. That is, we should define

$$D = \lim_{n\to\infty} \sum_{j=1}^{n} s(t_{j-1})\Delta t. \tag{2.3.19}$$

But (2.3.18) is a Riemann sum (in particular, a left-hand rule sum) which approximates the definite integral

$$\int_a^b s(t)dt. \tag{2.3.20}$$

Hence the limit in (2.3.19) is the value of the definite integral (2.3.20), and so we have the following definition.

Definition Suppose a particle moves along a curve C in $\mathbb{R}^n$ so that its position at time t is given by $\mathbf{x} = f(t)$. Suppose the velocity $\mathbf{v}(t)$ is continuous on the interval $[a, b]$. Then we define the *distance* traveled by the particle from time $t = a$ to time $t = b$ to be

$$\int_a^b \|\mathbf{v}(t)\| dt. \tag{2.3.21}$$

Note that the distance traveled is the length of the curve C if the particle traverses C exactly once. In that case, we call (2.3.21) the *length* of C. In general, for any t such that the interval $[a, t]$ is in the domain of f, we may calculate

$$\sigma(t) = \int_a^t \|\mathbf{v}(u)\| du, \tag{2.3.22}$$

which we call the *arc length function* for C.

Example Consider the helix H parametrized by

$$f(t) = (\cos(t), \sin(t), t).$$

If we let L denote the length of one complete loop of the helix, then a particle traveling along H according to $\mathbf{x} = f(t)$ will traverse this distance as t goes from 0 to 2π. Since

$$\mathbf{v}(t) = (-\sin(t), \cos(t), 1),$$

we have

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}.$$

Hence

$$L = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

Example Suppose a particle moves along a curve C so that its position at time t is given by

$$\mathbf{x} = ((1 + 2\cos(t))\cos(t), (1 + 2\cos(t))\sin(t)).$$

Then C is the curve in Figure 2.3.4, which is called a *limaçon*. The particle will traverse this curve once as t goes from 0 to 2π. Now

$$\mathbf{v} = (-(1 + 2\cos(t))\sin(t) - 2\sin(t)\cos(t), (1 + 2\cos(t))\cos(t) - 2\sin^2(t)),$$

so

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \mathbf{v} \cdot \mathbf{v} \\
&= (1 + 2\cos(t))^2 \sin^2(t) + 4(1 + 2\cos(t))\sin^2(t)\cos(t) + 4\sin^2(t)\cos^2(t) \\
&\qquad + (1 + 2\cos(t))^2 \cos^2(t) - 4(1 + 2\cos(t))\sin^2(t)\cos(t) + 4\sin^4(t) \quad , \\
&= (1 + 2\cos(t))^2(\sin^2(t) + \cos^2(t)) + 4\sin^2(t)\cos^2(t) + 4\sin^4(t) \\
&= (1 + 2\cos(t))^2 + 4\sin^2(t)\cos^2(t) + 4\sin^4(t)
\end{aligned}$$

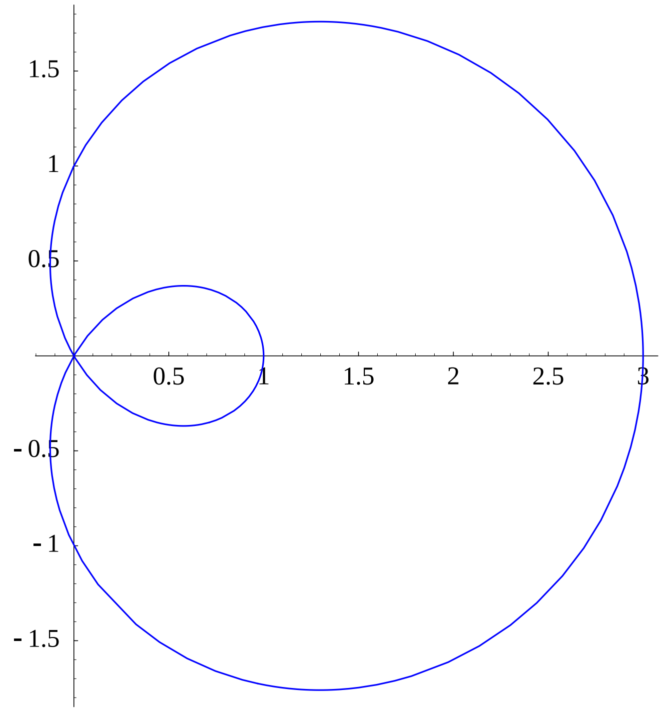


Figure 2.3.4 A limaçon

Hence the length of C is

$$\int_0^{2\pi} \sqrt{(1+2\cos(t))^2 + 4\sin^2(t)\cos^2(t) + 4\sin^4(t)}\, dt = 13.3649,$$

where the integration was performed with a computer and the final result rounded to four decimal places. Note that integrating from 0 to 4π would find the distance the particle travels in going around C twice, namely,

$$\int_0^{4\pi} \sqrt{(1+2\cos(t))^2 + 4\sin^2(t)\cos^2(t) + 4\sin^4(t)}\, dt = 26.7298.$$

Problems

1. For each of the following, suppose a particle is moving along a curve so that its position at time t is given by $\mathbf{x} = f(t)$. Find the velocity and acceleration of the particle.
   (a) $f(t) = (t^2 + 3, \sin(t))$
   (b) $f(t) = (t^2 e^{-2t}, t^3 e^{-2t}, 3t)$
   (c) $f(t) = (\cos(3t^2), \sin(3t^2))$
   (d) $f(t) = (t\cos(t^2), t\sin(t^2), 3t\cos(t^2))$
2. Find the curvature of the following curves at the given point.
   (a) $f(t) = (t, t^2)$, $t = 1$
   (b) $f(t) = (3\cos(t), \sin(t))$, $t = \dfrac{\pi}{4}$

(c) $f(t) = (\cos(t), \sin(t), t)$, $t = \dfrac{\pi}{3}$

(d) $f(t) = (\cos(t), \sin(t), e^{-t})$, $t = 0$

3. Plot the curvature for each of the following curves over the given interval I.
   (a) $f(t) = (t, t^2)$, $I = [-2, 2]$
   (b) $f(t) = (\cos(t), 3\sin(t))$, $I = [0, 2\pi]$
   (c) $g(t) = ((1 + 2\cos(t))\cos(t), (1 + 2\cos(t))\sin(t))$, $I = [0, 2\pi]$
   (d) $h(t) = (2\cos(t), \sin(t), 2t)$, $I = [0, 2\pi]$
   (e) $f(t) = (4\cos(t) + \sin(4t), 4\sin(t) + \sin(4t))$, $I = [0, 2\pi]$

4. For each of the following, suppose a particle is moving along a curve so that its position at time t is given by $\mathbf{x} = f(t)$. Find the coordinates of acceleration in the direction of the unit tangent vector and in the direction of the principal unit normal vector at the specified point. Write the acceleration as a sum of scalar multiples of the unit tangent vector and the principal unit normal vector.
   (a) $f(t) = (\sin(t), \cos(t))$, $t = \dfrac{\pi}{3}$
   (b) $f(t) = (\cos(t), 3\sin(t))$, $t = \dfrac{\pi}{4}$
   (c) $f(t) = (t, t^2)$, $t = 1$
   (d) $f(t) = (\sin(t), \cos(t), t)$, $t = \dfrac{\pi}{3}$

5. Suppose a particle moves along a curve C in $\mathbb{R}^3$ so that its position at time t is given by $\mathbf{x} = f(t)$. Let $\mathbf{v}$, s, and $\mathbf{a}$ denote the velocity, speed, and acceleration of the particle, respectively, and let κ be the curvature of C.
   (a) Using the facts $\mathbf{v} = sT(t)$ and
$$\mathbf{a} = \frac{ds}{dt}\,T(t) + s^2\kappa N(t),$$
   show that
$$\mathbf{v} \times \mathbf{a} = s^3\kappa(T(t) \times N(t)).$$
   (b) Use the result of part (a) to show that
$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

6. Let H be the helix in $\mathbb{R}^3$ parametrized by $f(t) = (\cos(t), \sin(t), t)$. Use the result from Problem 5 to compute the curvature κ of H for any time t.

7. Let C be the elliptical helix in $\mathbb{R}^3$ parametrized by $f(t) = (4\cos(t), 2\sin(t), t)$. Use the result from Problem 5 to compute the curvature κ of C at $t = \frac{\pi}{4}$.

8. Let C be the curve in $\mathbb{R}^2$ which is the graph of the function $\varphi : \mathbb{R} \to \mathbb{R}$. Use the result from Problem 5 to show that the curvature of C at the point $(t, \varphi(t))$ is
$$\kappa = \frac{|\varphi''(t)|}{(1 + (\varphi'(t))^2)^{\frac{3}{2}}}.$$

9. Let P be the graph of $f(t) = t^2$. Use the result from Problem 8 to find the curvature of P at $(1,1)$ and $(2,4)$.

10. Let C be the graph of $f(t) = t^3$. Use the result from Problem 8 to find the curvature of C at $(1,1)$ and $(2,8)$.

11. Let C be the graph of $g(t) = \sin(t)$. Use the result from Problem 8 to find the curvature of C at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

12. For each of the following, suppose a particle is moving along a curve so that its position at time t is given by $\mathbf{x} = f(t)$. Find the distance traveled by the particle over the given time interval.
    (a) $f(t) = (\sin(t), 3\cos(t))$, $I = [0, 2\pi]$
    (b) $f(t) = (\cos(\pi t), \sin(\pi t), 2t)$, $I = [0, 4]$
    (c) $f(t) = (t, t^2)$, $I = [0, 2]$
    (d) $f(t) = (t\cos(t), t\sin(t))$, $I = [0, 2\pi]$
    (e) $f(t) = (\cos(2\pi t), \sin(2\pi t), 3t^2, t)$, $I = [0, 1]$
    (f) $f(t) = (e^{-t}\cos(\pi t), e^{-t}\sin(\pi t))$, $I = [-2, 2]$
    (g) $f(t) = (4\cos(t) + \sin(4t), 4\sin(t) + \sin(4t))$, $I = [0, 2\pi]$

13. Verify that the circumference of a circle of radius r is $2\pi r$.

14. The curve parametrized by
$$f(t) = (\sin(2t)\cos(t), \sin(2t)\sin(t))$$
has four "petals." Find the length of one of these petals.

15. The curve C parametrized by $h(t) = (\cos^3(t), \sin^3(t))$ is called a hypocycloid (see Figure 2.2.3 in Section 2.3). Find the length of C.

16. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and let C be the part of the graph of φ over the interval $[a, b]$. Show that the length of C is
$$\int_a^b \sqrt{1 + (\varphi'(t))^2}\, dt.$$

17. Use the result from Problem 16 to find the length of one arch of the graph of $f(t) = \sin(t)$.

18. Let $h : \mathbb{R} \to \mathbb{R}^n$ parametrize a curve C. We say C is *parametrized by arc length* if $\|Dh(t)\| = 1$ for all t.
    (a) Let σ be the arc length function for C using the parametrization f and let σ^{-1} be its inverse function. Show that the function $g : \mathbb{R} \to \mathbb{R}^n$ defined by $g(u) = f(\sigma^{-1}(u))$ parametrizes C by arc length.

(b) Let C be the circular helix in $\mathbb{R}^3$ with parametrization $f(t) = (\cos(t), \sin(t), t)$. Find a function $g : \mathbb{R} \to \mathbb{R}^n$ which parametrizes C by arc length.

19. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is continuous on the closed interval $[a, b]$ and has coordinate functions $f_1, f_2, \ldots, f_n$. We define the *definite integral* of f over the interval $[a, b]$ to be

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \ldots, \int_a^b f_n(t)dt \right).$$

Show that if a particle moves so its velocity at time t is $\mathbf{v}(t)$, then, assuming $\mathbf{v}$ is a continuous function on an interval $[a, b]$, the position of the particle for any time t in $[a, b]$ is given by

$$\mathbf{x}(t) = \int_a^t \mathbf{v}(s)ds + \mathbf{x}(a).$$

20. Suppose a particle moves along a curve in $\mathbb{R}^3$ so that its velocity at any time t is

$$\mathbf{v}(t) = (\cos(2t), \sin(2t), 3t).$$

If the particle is at $(0, 1, 0)$ when $t = 0$, use Problem 19 to determine its position for any other time t.

21. Suppose a particle moves along a curve in $\mathbb{R}^3$ so that its acceleration at any time t is

$$\mathbf{a}(t) = (\cos(t), \sin(t), 0).$$

If the particle is at $(1, 2, 0)$ with velocity $(0, 1, 1)$ at time $t = 0$, use Problem 19 to determine its position for any other time t.

22. Suppose a projectile is fired from the ground at an angle α with an initial speed v_0, as shown in Figure 2.3.5. Let $\mathbf{x}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ be the position, velocity, and acceleration, respectively, of the projectile at time t.

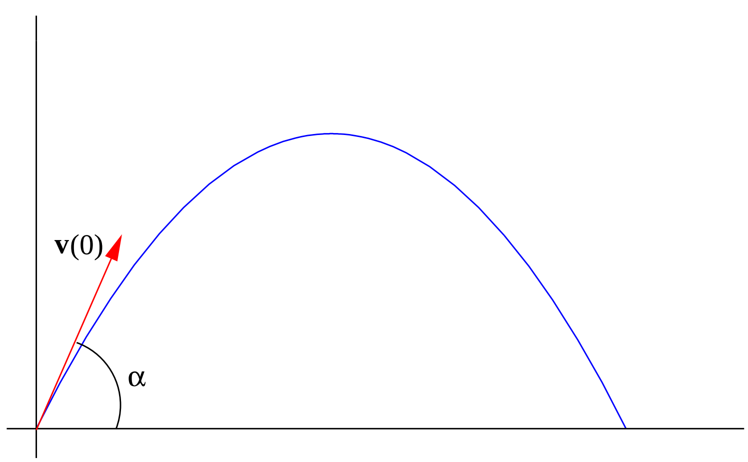


Figure 2.3.5 The path of a projectile

(a) Explain why $\mathbf{x}(0) = (0,0)$, $\mathbf{v}(0) = (v_0\cos(\alpha), v_0\sin(\alpha))$, and $\mathbf{a}(t) = (0, -g)$ for all t, where $g = 9.8$ meters per second per second is the acceleration due to gravity.

(b) Use Problem 19 to find $\mathbf{v}(t)$.

(c) Use Problem 19 to find $\mathbf{x}(t)$.

(d) Show that the curve parametrized by $\mathbf{x}(t)$ is a parabola. That is, let $\mathbf{x}(t) = (x, y)$ and show that $y = ax^2 + bx + c$ for some constants a, b, and c.

(e) Show that the range of the projectile, that is, the horizontal distance traveled, is

$$R = \frac{v_0 \sin(2\alpha)}{g}$$

and conclude that the range is maximized when $\alpha = \frac{\pi}{4}$.

(f) When does the projectile hit the ground?

(g) What is the maximum height reached by the projectile? When does it reach this height?

23. Suppose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are unit vectors in $\mathbb{R}^n$, $m \leq n$, which are mutually orthogonal (that is, $a_i \perp a_j$ when $i \neq j$). If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ with

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m,$$

show that $x_i = \mathbf{x} \cdot \mathbf{a}_i$, $i = 1, 2, \ldots, m$.

The Calculus of Functions

of

Several Variables

Section 3.1

Geometry, Limits, and Continuity

In this chapter we will study functions $f : \mathbb{R}^n \to \mathbb{R}$, functions which take vectors for inputs and give scalars for outputs. For example, the function that takes a point in space for input and gives back the temperature at that point is such a function; the function that reports the gross national product of a country is another such function. Note that the domain space of the first example is three-dimensional, while the domain of the latter has, for most countries, thousands of dimensions. As usual, whenever possible we will state our results for an arbitrary n-dimensional space, although most of our examples will deal with only two or three dimensions.

Level sets and graphs

We begin by considering some geometrical methods for picturing functions of the form $f : \mathbb{R}^n \to \mathbb{R}$.

Definition Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a real number c, we call the set

$$L = \{(x_1, x_2, \ldots, x_n) : f(x_1, x_2, \ldots, x_n) = c\} \tag{3.1.1}$$

a *level set* of f at *level* c. We also call L a *contour* of f. When $n = 2$, we call L a *level curve* of f and when $n = 3$ we call L a *level surface* of f. A plot displaying level sets for several different levels is called a *contour plot.*

Example Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = 2x^2 + y^2.$$

Given a real number c, the set of all points satisfying

$$2x^2 + y^2 = c$$

is a level set of f. For $c < 0$, this set is empty; for $c = 0$, it consists of only the point $(0, 0)$; for any $c > 0$, the level set is an ellipse with center at $(0, 0)$. Hence a contour plot of f, as shown in Figure 3.1.1, consists of concentric ellipses.

Example Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}.$$

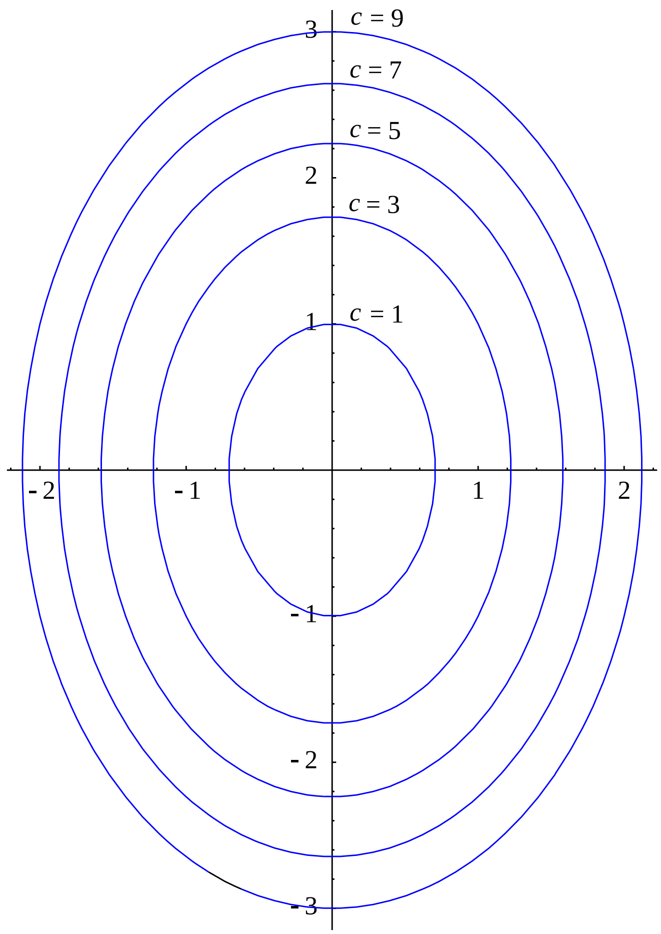


Figure 3.1.1 Level curves $2x^2 + y^2 = c$

For any point (x, y) on the circle of radius $r > 0$ centered at the origin, $f(x, y)$ has the constant value

$$\frac{\sin(r)}{r}.$$

Hence a contour plot of f, like that shown in Figure 3.1.2, consists of concentric circles centered at the origin.

Example Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$f(x, y, z) = x^2 + 2y^2 + 3z^2.$$

The level surface of f with equation

$$x^2 + 2y^2 + 3z^2 = 1$$

is shown in Figure 3.1.3. Note that, for example, fixing a value z_0 of z yields the equation

$$x^2 + y^2 = 1 - 3z_0^2,$$

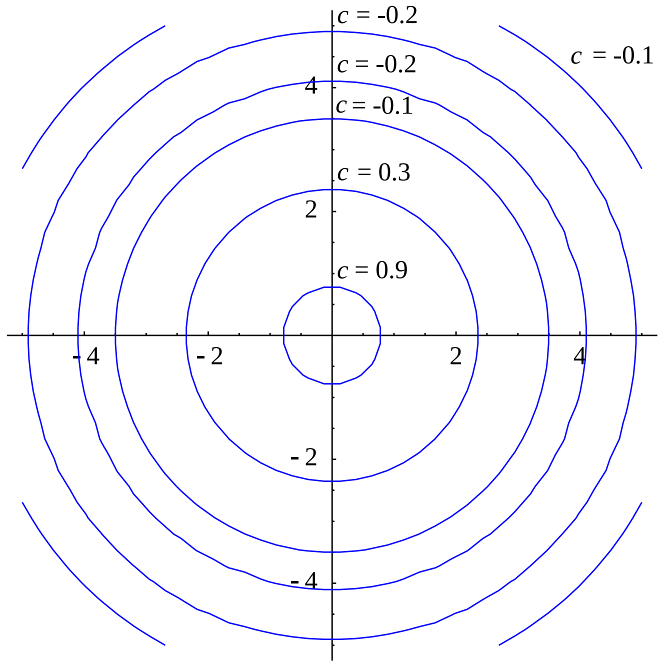


Figure 3.1.2 Level curves $\dfrac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} = c$

the equation of an ellipse. This explains why a slice of the level surface shown in Figure 3.1.3 parallel to the xy-plane is an ellipse. Similarly, slices parallel to the xz-plane and the yz-plane are ellipses, which is why this surface is an example of an *ellipsoid*.

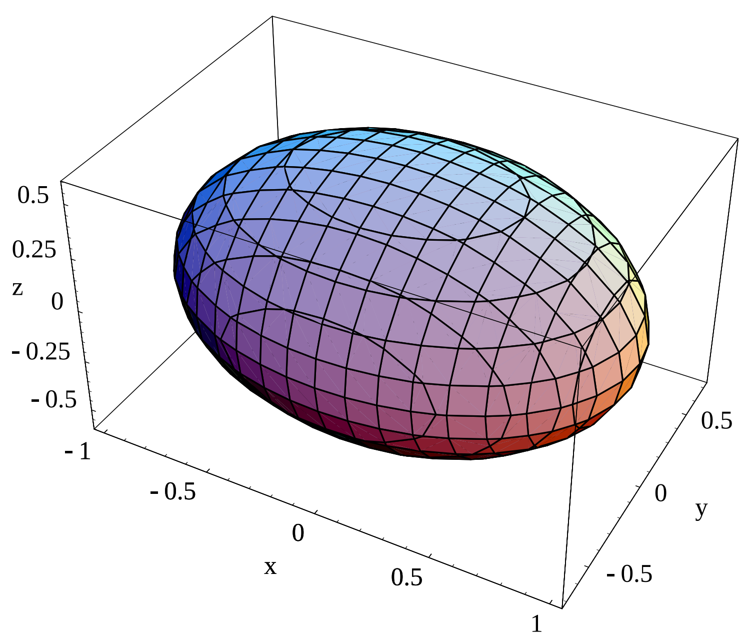


Figure 3.1.3 The level surface $x^2 + 2y^2 + 3z^2 = 1$

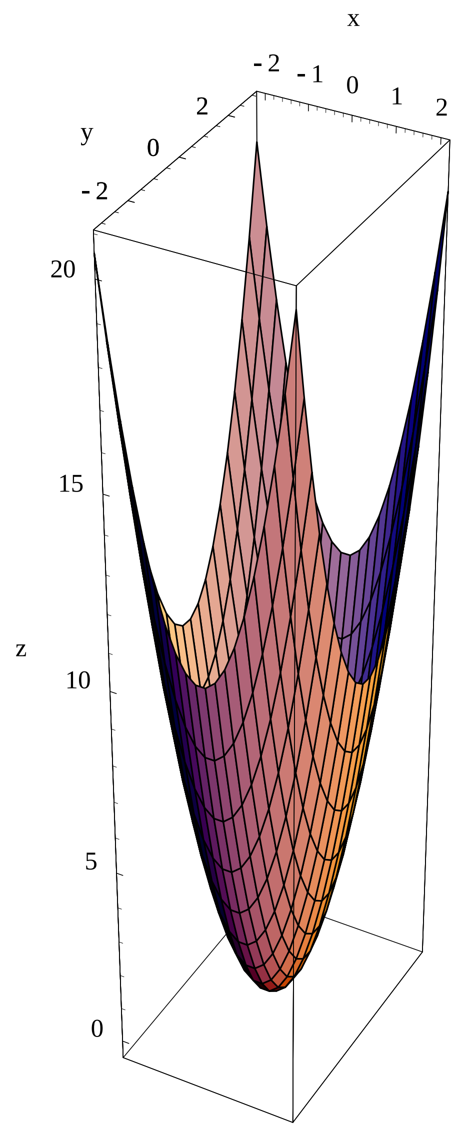


Figure 3.1.4 The paraboloid $z = 2x^2 + y^2$

Definition Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we call the set

$$G = \{(x_1, x_2, \ldots, x_n, x_{n+1}) : x_{n+1} = f(x_1, x_2, \ldots, x_n)\} \tag{3.1.2}$$

the *graph* of f.

Note that the graph G of a function $f : \mathbb{R}^n \to \mathbb{R}$ is in R^{n+1}. As a consequence, we can picture G only if $n = 1$, in which case G is a curve as studied in single-variable calculus, or $n = 2$, in which case G is a surface in $\mathbb{R}^3$.

Example Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 2x^2 + y^2.$$

The graph of f is then the set of all points (x, y, z) in $\mathbb{R}^3$ which satisfy the equation $z = 2x^2 + y^2$. One way to picture the graph of f is to imagine raising the level curves in Figure 3.1.1 to their respective heights above the xy-plane, creating the surface in $\mathbb{R}^3$ shown in Figure 3.1.4. Another way to picture the graph is to consider slices of the graph lying above a grid of lines parallel to the axes in the xy-plane. For example, for a fixed value of x, say x_0, the set of points satisfying the equation $z = 2x_0^2 + y^2$ is a parabola lying above the line $x = x_0$. Similarly, fixing a value y_0 of y yields the parabola $z = 2x^2 + y_0$ lying above the line $y = y_0$. If we draw these parabolas for numerous lines of the form $x = x_0$ and $y = y_0$, we obtain a wire-frame of the graph. The graph shown in Figure 3.1.4 was obtained by filling in the surface patches of a wire-frame mesh, the outline of which is visible on the surface. This surface is an example of a *paraboloid*.

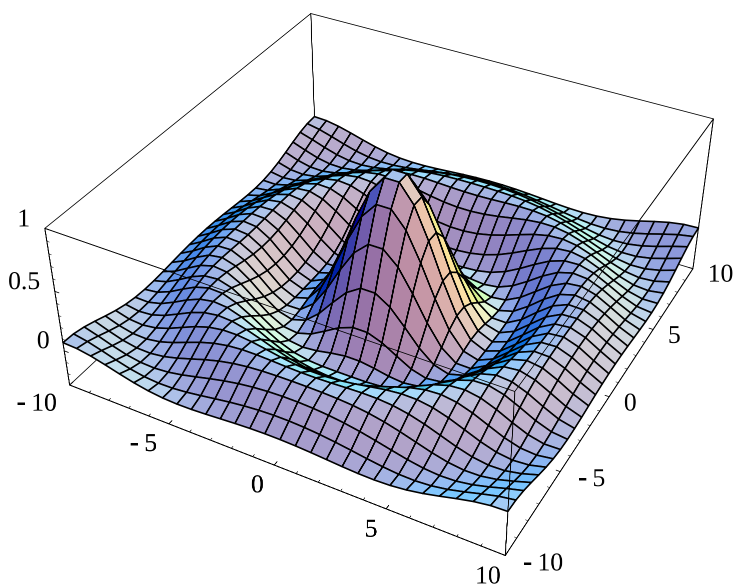


Figure 3.1.5 Graph of $f(x, y) = \dfrac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}$

Example Although the graphs of many functions may be sketched reasonably well by hand using the ideas of the previous example, for most functions a good picture of its graph requires either computer graphics or considerable artistic skill. For example, consider the graph of

$$f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}.$$

Using the contour plot, we can imagine how the graph of f oscillates as we move away from the origin, the level circles of the contour plot rising and falling with the oscillations

of
$$\frac{\sin(r)}{r},$$
where $r = \sqrt{x^2 + y^2}$. Equivalently, the slice of the graph above any line through the origin will be the graph of
$$z = \frac{\sin(r)}{r}.$$
This should give you a good idea what the graph of f looks like, but, nevertheless, most of us could not produce the picture of Figure 3.1.5 without the aid of a computer. Notice that although f is not defined at $(0, 0)$, it appears that $f(x, y)$ approaches 1 as (x, y) approaches 0. This is in fact true, a consequence of the fact that
$$\lim_{r \to 0} \frac{\sin(r)}{r} = 1.$$
We will return to this example after we have introduced limits and continuity.

Limits and continuity

By now the following two definitions should look familiar.

Definition Let $\mathbf{a}$ be a point in $\mathbb{R}^n$ and let O be the set of all points in the open ball of radius $r > 0$ centered at $\mathbf{c}$ except $\mathbf{c}$ itself. That is,
$$O = \{\mathbf{x} : \mathbf{x} \text{ is in } B^n(\mathbf{c}, r), \mathbf{x} \neq \mathbf{c}\}. \tag{3.1.3}$$
Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined for all $\mathbf{x}$ in O. We say the *limit* of $f(\mathbf{x})$ as x approaches $\mathbf{c}$ is L, written $\lim\limits_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x}) = L$, if for every sequence of points $\{\mathbf{x}_m\}$ in O,
$$\lim_{m \to \infty} f(\mathbf{x}_m) = L \tag{3.1.4}$$
whenever $\lim\limits_{m \to \infty} \mathbf{x}_m = \mathbf{c}$.

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined for all $\mathbf{x}$ in some open ball $B^n(\mathbf{c}, r)$, $r > 0$. We say f is *continuous* at $\mathbf{c}$ if
$$\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}). \tag{3.1.5}$$

The following basic properties of limits follow immediately from the analogous properties for limits of sequences.

Proposition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ with
$$\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x}) = L$$
and
$$\lim_{\mathbf{x} \to \mathbf{c}} g(\mathbf{x}) = M.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{c}}(f(\mathbf{x})+g(\mathbf{x})) = L+M, \tag{3.1.6}$$

$$\lim_{\mathbf{x}\to\mathbf{c}}(f(\mathbf{x})-g(\mathbf{x})) = L-M, \tag{3.1.7}$$

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})g(\mathbf{x}) = LM, \tag{3.1.8}$$

$$\lim_{\mathbf{x}\to\mathbf{c}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{L}{M}, \tag{3.1.9}$$

and

$$\lim_{\mathbf{x}\to\mathbf{c}} kf(\mathbf{x}) = kL \tag{3.1.10}$$

for any scalar k.

Now suppose $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$,

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L, \tag{3.1.11}$$

and h is continuous at L. Then for any sequence $\{\mathbf{x}_m\}$ in $\mathbb{R}^n$ with

$$\lim_{m\to\infty} \mathbf{x}_m = \mathbf{c}, \tag{3.1.12}$$

we have

$$\lim_{m\to\infty} f(\mathbf{x}_m) = L, \tag{3.1.13}$$

and so

$$\lim_{m\to\infty} h(f(\mathbf{x}_m)) = h(L) \tag{3.1.14}$$

by the continuity of h at L. Thus we have the following result about compositions of functions.

Proposition If $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$,

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L,$$

and h is continuous at L, then

$$\lim_{\mathbf{x}\to\mathbf{c}} h\circ f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{c}} h(f(\mathbf{x})) = h(L). \tag{3.1.15}$$

Example Suppose we define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x_1, x_2, \ldots, x_n) = x_k,$$

where k is a fixed integer between 1 and n. If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a point in $\mathbb{R}^n$ and $\lim\limits_{m\to\infty} \mathbf{x}_m = \mathbf{a}$, then

$$\lim_{m\to\infty} f(\mathbf{x}_m) = \lim_{m\to\infty} x_{mk} = a_k,$$

where x_{mk} is the kth coordinate of $\mathbf{x}_m$. Thus

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = a_k.$$

This result is a basic building block for the examples that follow. For a particular example, if $f(x, y) = x$, then

$$\lim_{(x,y)\to(2,3)} f(x, y) = \lim_{(x,y)\to(2,3)} x = 2.$$

Example If we define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = xyz,$$

then, using (3.1.8) in combination with the previous example,

$$\begin{aligned}\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) &= \lim_{(x,y,z)\to(a,b,c)} xyz \\ &= (\lim_{(x,y,z)\to(a,b,c)} x)(\lim_{(x,y,z)\to(a,b,c)} y)(\lim_{(x,y,z)\to(a,b,c)} z) \\ &= abc.\end{aligned}$$

for any point (a, b, c) in $\mathbb{R}^3$. For example,

$$\lim_{(x,y,z)\to(3,2,1)} f(x, y, z) = \lim_{(x,y,z)\to(3,2,1)} xyz = (3)(2)(1) = 6.$$

Example Combining the previous examples with (3.1.6), (3.1.7), (3.1.8), and (3.1.10), we have

$$\begin{aligned}\lim_{(x,y,z)\to(2,1,3)} (xy^2 + 3xyz - 6xz) &= (\lim_{(x,y,z)\to(2,1,3)} x)(\lim_{(x,y,z)\to(2,1,3)} y)(\lim_{(x,y,z)\to(2,1,3)} y) \\ &\quad + 3(\lim_{(x,y,z)\to(2,1,3)} x)(\lim_{(x,y,z)\to(2,1,3)} y)(\lim_{(x,y,z)\to(2,1,3)} z) \\ &\quad - 6(\lim_{(x,y,z)\to(2,1,3)} x)(\lim_{(x,y,z)\to(2,1,3)} z) \\ &= (2)(1)(1) + (3)(2)(1)(3) - (6)(2)(3) \\ &= -16.\end{aligned}$$

The last three examples are all examples of polynomials in several variables. In general, a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x_1, x_2, \ldots, x_n) = ax_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where a is a scalar and $i_1, i_2, \ldots, i_n$ are nonnegative integers, is called a *monomial*. A function which is a sum of monomials is called a *polynomial*. The following proposition is a consequence of the previous examples and (3.1.6), (3.1.7), (3.1.8), and (3.1.10).

Proposition If $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, then for any point $\mathbf{c}$ in $\mathbb{R}^n$,

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}). \tag{3.1.16}$$

In other words, f is continuous at every point $\mathbf{c}$ in $\mathbb{R}^n$.

If g and h are both polynomials, then we call the function

$$f(\mathbf{x}) = \frac{g(\mathbf{x})}{h(\mathbf{x})} \tag{3.1.17}$$

a *rational function*. The next proposition is a consequence of the previous theorem and (3.1.9).

Proposition If is a rational function defined at $\mathbf{c}$, then

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}). \tag{3.1.18}$$

In other words, f is continuous at every point $\mathbf{c}$ in its domain.

Example Since

$$f(x,y,z) = \frac{x^2y + 3xyz^2}{4x^2 + 3z^2}$$

is a rational function, we have, for example,

$$\lim_{(x,y,z)\to(2,1,3)} f(x,y,z) = \lim_{(x,y,z)\to(2,1,3)} \frac{x^2y + 3xyz^2}{4x^2 + 3z^2} = \frac{4+54}{16+27} = \frac{58}{43}.$$

Example Combining (3.1.18) with (3.1.15), we have

$$\begin{aligned}\lim_{(x,y,z)\to(1,2,1)} \log\left(\frac{1}{x^2+y^2+z^2}\right) &= \log\left(\lim_{(x,y,z)\to(1,2,1)} \frac{1}{x^2+y^2+z^2}\right)\\ &= \log\left(\frac{1}{6}\right)\\ &= -\log(6).\end{aligned}$$

From the continuity of the square root function and our result above about the continuity of polynomials, we may conclude that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots, x_n) = \|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

is a continuous function. This fact is useful in computing some limits, particularly in combination with the fact that for any point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \geq \sqrt{x_k^2} = |x_k| \tag{3.1.19}$$

for any $k = 1, 2, \ldots, n$.

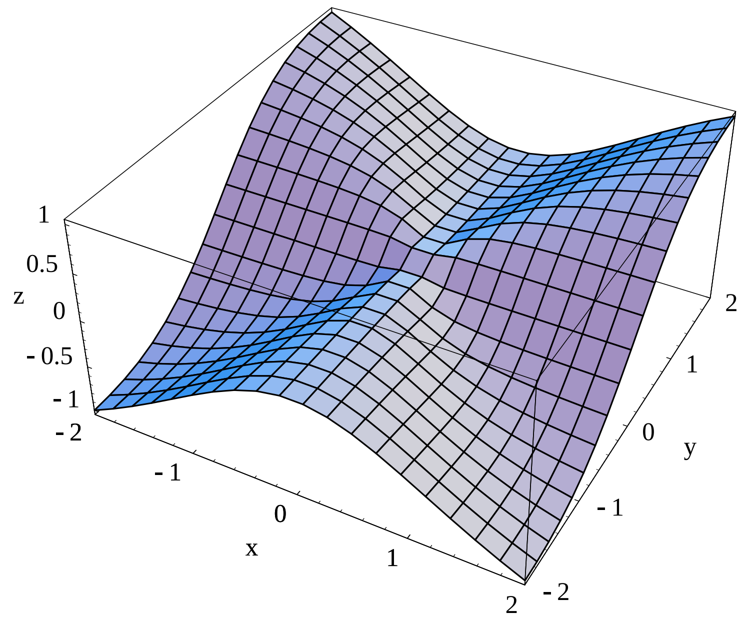


Figure 3.1.6 Graph of $f(x,y)=\dfrac{x^2y}{x^2+y^2}$

Example Suppose $f:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$f(x,y)=\frac{x^2y}{x^2+y^2}.$$

Although f is a rational function, we cannot use (3.1.18) to compute

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

since f is not defined at $(0,0)$. However, if we let $\mathbf{x}=(x,y)$, then, using (3.1.19),

$$|f(x,y)|=\left|\frac{x^2y}{x^2+y^2}\right|=\frac{|x|^2|y|}{|x^2+y^2|}=\frac{|x|^2|y|}{\|\mathbf{x}\|^2}\le\frac{\|\mathbf{x}\|^2\|\mathbf{x}\|}{\|\mathbf{x}\|^2}=\|\mathbf{x}\|.$$

Now

$$\lim_{(x,y)\to(0,0)}\|\mathbf{x}\|=0,$$

so

$$\lim_{(x,y)\to(0,0)}|f(x,y)|=0.$$

Hence

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}=0.$$

See Figure 3.1.6.

Recall that for a function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\lim_{t \to c} \varphi(t) = L$$

if and only if both

$$\lim_{t \to c^-} \varphi(t) = L$$

and

$$\lim_{t \to c^+} \varphi(t) = L.$$

In particular, if the one-sided limits do not agree, we may conclude that the limit does not exist. Similar reasoning may be applied to a function $f : \mathbb{R}^n \to \mathbb{R}$, the difference being that there are infinitely many different curves along which the variable $\mathbf{x}$ might approach a given point $\mathbf{c}$ in $\mathbb{R}^n$, as opposed to only the two directions of approach in $\mathbb{R}$. As a consequence, it is not possible to establish the existence of a limit with this type of argument. Nevertheless, finding two ways to approach $\mathbf{c}$ which yield different limiting values is sufficient to show that the limit does not exist.

Example Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(x, y) = \frac{xy}{x^2 + y^2}.$$

If we define $\alpha : \mathbb{R}^2 \to \mathbb{R}$ by $\alpha(t) = (t, 0)$, then

$$\lim_{t \to 0} \alpha(t) = \lim_{t \to 0}(t, 0) = (0, 0)$$

and

$$\lim_{t \to 0} g(\alpha(t)) = \lim_{t \to 0} f(t, 0) = \lim_{t \to 0} \frac{0}{t^2} = 0.$$

Now α is a parametrization of the x-axis, so the previous limit computation says that $g(x, y)$ approaches 0 as (x, y) approaches $(0, 0)$ along the x-axis. However, if we define $\beta : \mathbb{R} \to \mathbb{R}^2$ by $\beta(t) = (t, t)$, then β parametrizes the line $x = y$,

$$\lim_{t \to 0} \beta(t) = \lim_{t \to 0}(t, t) = (0, 0),$$

and

$$\lim_{t \to 0} g(\beta(t)) = \lim_{t \to 0} f(t, t) = \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2}.$$

Hence $g(x, y)$ approaches $\frac{1}{2}$ as (x, y) approaches $(0, 0)$ along the line $x = y$. Since these two limits are different, we may conclude that $g(x, y)$ does not have a limit as (x, y) approaches $(0, 0)$. Note that g in this example and f in the previous example are very similar functions, although our limit calculations show that their behavior around $(0, 0)$ differs significantly. In particular, f has a limit as (x, y) approaches $(0, 0)$, whereas g does not. This may be

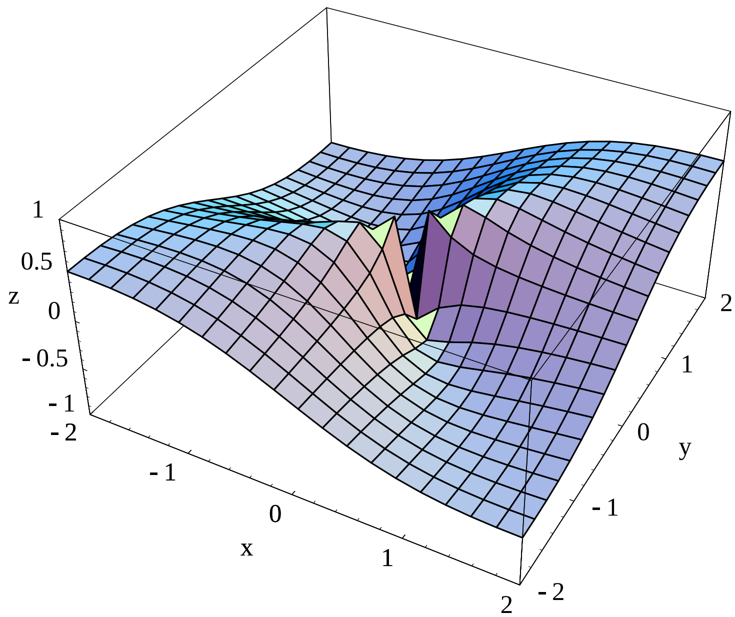


Figure 3.1.7 Graph of $g(x,y) = \dfrac{xy}{x^2+y^2}$

seen by comparing the graph of g in Figure 3.1.7, which has a tear at the origin, with that of f in Figure 3.1.6.

The next proposition lists some basic properties of continuous functions, all of which follow immediately from the similar list of properties of limits.

Proposition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are both continuous at $\mathbf{c}$. Then the functions with values at $\mathbf{x}$ given by

$$f(\mathbf{x}) + g(\mathbf{x}), \tag{3.1.20}$$

$$f(\mathbf{x}) - g(\mathbf{x}), \tag{3.1.21}$$

$$f(\mathbf{x})g(\mathbf{x}), \tag{3.1.22}$$

$$\frac{f(\mathbf{x})}{g(\mathbf{x})} \tag{3.1.23}$$

(provided $g(\mathbf{c}) \neq 0$), and

$$kf(\mathbf{x}), \tag{3.1.24}$$

where k is any scalar, are all continuous at $\mathbf{c}$.

From the result above about the limit of a composition of two functions, we have the following proposition.

Proposition If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{c}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous at $f(\mathbf{c})$, then $\varphi \circ f$ is continuous at $\mathbf{c}$.

Example Since the function $\varphi(t) = \sin(t)$ is continuous for all t and the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

is continuous at all points (x, y, z) in $\mathbb{R}^3$, the function

$$g(x, y, z) = \sin(\sqrt{x^2 + y^2 + z^2})$$

is continuous at all points (x, y, z) in $\mathbb{R}^3$.

Example Since the function

$$h(x, y) = \sin(\sqrt{x^2 + y^2})$$

is continuous for all (x, y) in $\mathbb{R}^2$ (same argument as in the previous example) and the function

$$g(x, y) = \sqrt{x^2 + y^2}$$

is continuous for all (x, y) in $\mathbb{R}^2$, the function

$$f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}$$

is, using (3.1.23), continuous at every point $(x, y) \neq (0, 0)$ in $\mathbb{R}^2$. Moreover, if we let $\mathbf{x} = (x, y)$, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|} = \lim_{r\to 0} \frac{\sin(r)}{r} = 1.$$

Hence the discontinuity at $(0, 0)$ is *removable*. That is, if we define

$$g(x, y) = \begin{cases} \dfrac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 1, & \text{if } (x, y) = (0, 0), \end{cases}$$

then g is continuous for all (x, y) in $\mathbb{R}^2$.

Open and closed sets

In single-variable calculus we talk about a function being continuous not just at a point, but on an open interval, meaning that the function is continuous at every point in the open interval. Similarly, we need to generalize the definition of continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$ from that of continuity at a point in $\mathbb{R}^n$ to the idea of a function being continuous on a set in $\mathbb{R}^n$. Now the condition for a function f to be continuous at a point $\mathbf{c}$ requires that f be defined on some open ball containing $\mathbf{c}$. Hence, in order to say that f is continuous at every point in some set U, it is necessary that, given any point $\mathbf{u}$ in U, f

be defined on some open ball containing $\mathbf{u}$. This provides the motivation for the following definition.

Definition We say a set of points U in $\mathbb{R}^n$ is *open* if whenever $\mathbf{u}$ is a point in U, there exists a real number $r > 0$ such that the open ball $B^n(\mathbf{u}, r)$ lies entirely within U. We say a set of points C in $\mathbb{R}^n$ is *closed* if the set of all points in $\mathbb{R}^n$ which do not lie in C form an open set.

Example $\mathbb{R}^n$ is itself an open set.

Example Any open ball in $\mathbb{R}^n$ is an open set. In particular, any open interval in $\mathbb{R}$ is an open set. To see why, consider an open ball $B^n(\mathbf{a}, r)$ in $\mathbb{R}^n$. Given a point $\mathbf{y}$ in $B^n(\mathbf{a}, r)$, let s be the smaller of $\|\mathbf{y}-\mathbf{a}\|$ (the distance from $\mathbf{y}$ to the center of the ball) and $r - \|\mathbf{y}-\mathbf{a}\|$ (the distance from $\mathbf{y}$ to the edge of the ball). Then $B^n(\mathbf{y}, s)$ is an open ball which lies entirely within $B^n(\mathbf{a}, r)$. Hence $B^n(\mathbf{a}, r)$ is an open set.

Example Any closed ball in $\mathbb{R}^n$ is a closed set. In particular, any closed interval in $\mathbb{R}$ is a closed set. To see why, consider a closed ball $\bar{B}^n(\mathbf{a}, r)$. Given a point $\mathbf{y}$ not in $\bar{B}^n(\mathbf{a}, r)$, let $s = \|\mathbf{y} - \mathbf{a}\| - r$, the distance from $\mathbf{y}$ to the edge of $\bar{B}^n(\mathbf{a}, r)$. Then $B^n(\mathbf{y}, s)$ is an open ball which lies entirely outside of $\bar{B}^n(\mathbf{x}, r)$. Hence $\bar{B}^n(\mathbf{x}, r)$ is a closed set.

Example Given real numbers $a_1 < b_1, a_2 < b_2, \ldots, a_n < b_n$, we call the set

$$U = \{(x_1, x_2, \ldots, x_n) : a_i < x_i < b_i, i = 1, 2, \ldots, n\}$$

an *open rectangle* in $\mathbb{R}^n$ and the set

$$C = \{(x_1, x_2, \ldots, x_n) : a_i \le x_i \le b_i, i = 1, 2, \ldots, n\}$$

a *closed rectangle* in $\mathbb{R}^n$. An argument similar to that in the previous example shows that U is an open set and C is a closed set.

Definition We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous on an open set U* if f is continuous at every point u in U.

Example The function

$$f(x, y, z) = \frac{3xyz - 6x}{x^2 + y^2 + z^2 + 1}$$

is continuous on $\mathbb{R}^3$.

Example The functions

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0), \end{cases}$$

and

$$g(x, y) = \begin{cases} \dfrac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 1, & \text{if } (x, y) = (0, 0), \end{cases}$$

are, from our work in previous examples, continuous on $\mathbb{R}^2$.

Example The function

$$g(x,y) = \frac{xy}{x^2+y^2}$$

is continuous on the open set

$$U = \{(x,y) : (x,y) \neq (0,0)\}.$$

Note that in this case it is not possible to define g at $(0,0)$ in such a way that the resulting function is continuous at $(0,0)$, a consequence of our work above showing that g does not have a limit as (x,y) approaches $(0,0)$.

Example The function

$$f(x,y) = \log(xy)$$

is continuous on the open set

$$U = \{(x,y) : x > 0 \text{ and } y > 0\}.$$

Problems

1. Plot the graph and a contour plot for each of the following functions. Do your plots over regions large enough to illustrate the behavior of the function.
   - (a) $f(x,y) = x^2 + 4y^2$
   - (b) $f(x,y) = x^2 - y^2$
   - (c) $f(x,y) = 4y^2 - 2x^2$
   - (d) $h(x,y) = \sin(x)\cos(y)$
   - (e) $f(x,y) = \sin(x+y)$
   - (f) $g(x,y) = \sin(x^2+y^2)$
   - (g) $g(x,y) = \sin(x^2-y^2)$
   - (h) $h(x,y) = xe^{-\sqrt{x^2+y^2}}$
   - (i) $f(x,y) = \dfrac{1}{2\pi}e^{-\frac{1}{2\pi}(x^2+y^2)}$
   - (j) $f(x,y) = \sin(\pi\sin(x) + y)$
   - (k) $h(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$
   - (l) $g(x,y) = \log(\sqrt{x^2+y^2})$
2. For each of the following, plot the contour surface $f(x,y,z) = c$ for the specified value of c.
   - (a) $f(x,y,z) = x^2+y^2+z^2$, $c = 4$
   - (b) $f(x,y,z) = x^2+4y^2+2z^2$, $c = 7$
   - (c) $f(x,y,z) = x^2+y^2-z^2$, $c = 1$
   - (d) $f(x,y,z) = x^2-y^2+z^2$, $c = 1$
3. Evaluate the following limits.
   - (a) $\displaystyle\lim_{(x,y)\to(2,1)} (3xy + x^2y + 4y)$
   - (b) $\displaystyle\lim_{(x,y,z)\to(1,2,1)} \frac{3xyz}{2xy^2+4z}$
   - (c) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\cos(3xy)}{\sqrt{x^2+1}}$
   - (d) $\displaystyle\lim_{(x,y,z)\to(2,1,3)} ye^{2x-3y+z}$

4. For each of the following, either find the specified limit or explain why the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x+y^2}$ (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

   (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-e^{-(x^2+y^2)}}{x^2+y^2}$ (f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

5. Let $f(x,y) = \dfrac{x^2y}{x^4+4y^2}$.

   (a) Define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (t,0)$. Show that $\lim_{t\to 0} f(\alpha(t)) = 0$.

   (b) Define $\beta : \mathbb{R} \to \mathbb{R}^2$ by $\beta(t) = (0,t)$. Show that $\lim_{t\to 0} f(\beta(t)) = 0$.

   (c) Show that for any real number m, if we define $\gamma : \mathbb{R} \to \mathbb{R}^2$ by $\gamma(t) = (t, mt)$, then $\lim_{t\to 0} f(\gamma(t)) = 0$.

   (d) Define $\delta : \mathbb{R} \to \mathbb{R}^2$ by $\delta(t) = (t,t^2)$. Show that $\lim_{t\to 0} f(\delta(t)) = \dfrac{1}{5}$.

   (e) What can you conclude about $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+4y^2}$?

   (f) Plot the graph of f and explain your results in terms of the graph.

6. Discuss the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{1-e^{-\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0), \\ 1, & \text{if } (x,y) = (0,0). \end{cases}$$

7. Discuss the continuity of the function

$$g(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4}, & \text{if } (x,y) \neq (0,0), \\ 1, & \text{if } (x,y) = (0,0). \end{cases}$$

8. For each of the following, decide whether the given set is open, closed, neither open nor closed, or both open and closed.

   (a) $(3,10)$ in $\mathbb{R}$

   (b) $[-2,5]$ in $\mathbb{R}$

   (c) $\{(x,y) : x^2+y^2 < 4\}$ in $\mathbb{R}^2$

   (d) $\{(x,y) : x^2+y^2 > 4\}$ in $\mathbb{R}^2$

(e) $\{(x,y) : x^2 + y^2 \leq 4\}$ in $\mathbb{R}^2$
(f) $\{(x,y) : x^2 + y^2 = 4\}$ in $\mathbb{R}^2$
(g) $\{(x,y,z) : -1 < x < 1, -2 < y < 3, 2 < z < 5\}$ in $\mathbb{R}^3$
(h) $\{(x,y) : -3 < x \leq 4, -2 \leq y < 1\}$ in $\mathbb{R}^2$

9. Give an example of a subset of $\mathbb{R}$ which is neither open nor closed.
10. Is it possible for a subset of $\mathbb{R}^2$ to be both open and closed? Explain.

The Calculus of Functions

of

Several Variables

Section 3.2

Directional Derivatives and the Gradient

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, the derivative at a point c, that is,

$$\varphi'(c) = \lim_{h \to 0} \frac{\varphi(c+h) - \varphi(c)}{h}, \tag{3.2.1}$$

is the slope of the best affine approximation to φ at c. We may also regard it as the slope of the graph of φ at $(c, \varphi(c))$, or as the instantaneous rate of change of $\varphi(x)$ with respect to x when $x = c$. As a prelude to finding the best affine approximations for a function $f : \mathbb{R}^n \to \mathbb{R}$, we will first discuss how to generalize (3.2.1) to this setting using the ideas of slopes and rates of change for our motivation.

Directional derivatives

Example Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 4 - 2x^2 - y^2,$$

the graph of which is pictured in Figure 3.2.1. If we imagine a bug moving along this surface, then the slope of the path encountered by the bug will depend both on the bug's position and the direction in which it is moving. For example, if the bug is above the point $(1, 1)$ in the xy-plane, moving in the direction of the vector $\mathbf{v} = (-1, -1)$ will cause it to head directly towards the top of the graph, and thus have a steep rate of ascent, whereas moving in the direction of $-\mathbf{v} = (1, 1)$ would cause it to descend at a fast rate. These two possibilities are illustrated by the red curve on the surface in Figure 3.2.1. For another example, heading around the surface above the ellipse

$$2x^2 + y^2 = 3$$

in the xy-plane, which from $(1, 1)$ means heading initially in the direction of the vector $\mathbf{w} = (-1, 2)$, would lead the bug around the side of the hill with no change in elevation, and hence a slope of 0. This possibility is illustrated by the green curve on the surface in Figure 3.2.1. Thus in order to talk about the slope of the graph of f at a point, we must specify a direction as well. For example, suppose the bug moves in the direction of $\mathbf{v}$. If we let

$$\mathbf{u} = -\frac{1}{\sqrt{2}}\,(1, 1),$$

the direction of $\mathbf{v}$, then, letting $\mathbf{c} = (1, 1)$,

$$\frac{f(\mathbf{c} + h\mathbf{u}) - f(\mathbf{c})}{h}$$

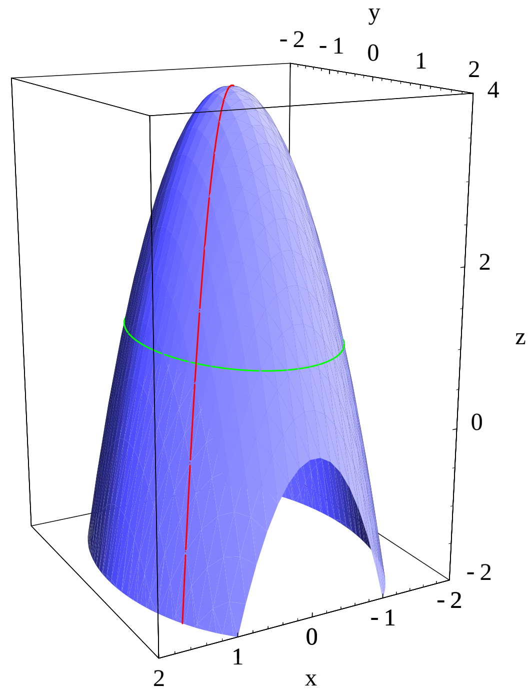


Figure 3.2.1 Graph of $f(x) = 4 - 2x^2 - y^2$

would, for any $h > 0$, represent an approximation to the slope of the graph of f at $(1, 1)$ in the direction of $\mathbf{u}$. As in single-variable calculus, we should expect that taking the limit as h approaches 0 should give us the exact slope at $(1, 1)$ in the direction of $\mathbf{u}$. Now

$$\begin{aligned} f(\mathbf{c} + h\mathbf{u}) - f(\mathbf{c}) &= f\left(1 - \frac{h}{\sqrt{2}}, 1 - \frac{h}{\sqrt{2}}\right) - f(1,1) \\ &= 4 - 2\left(1 - \frac{h}{\sqrt{2}}\right)^2 - \left(1 - \frac{h}{\sqrt{2}}\right)^2 - 1 \\ &= 3 - 3\left(1 - \sqrt{2}h + \frac{h^2}{2}\right) \\ &= 3\sqrt{2}h - \frac{3h^2}{2} \\ &= h\left(3\sqrt{2} - \frac{3h}{2}\right), \end{aligned}$$

so

$$\lim_{h \to 0} \frac{f(\mathbf{c} + h\mathbf{u}) - f(\mathbf{c})}{h} = \lim_{h \to 0} \left(3\sqrt{2} - \frac{3h}{2}\right) = 3\sqrt{2}.$$

Hence the graph of f has a slope of $3\sqrt{2}$ if we start above $(1, 1)$ and head in the direction of $\mathbf{u}$; similar computations would show that the slope in the direction of $-\mathbf{u}$ is $-3\sqrt{2}$ and the slope in the direction of

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{5}}\,(-1, 2)$$

is 0.

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on an open ball about a point $\mathbf{c}$. Given a unit vector $\mathbf{u}$, we call

$$D_{\mathbf{u}}f(\mathbf{c}) = \lim_{h \to 0} \frac{f(\mathbf{c} + h\mathbf{u}) - f(\mathbf{c})}{h}, \tag{3.2.2}$$

provided the limit exists, the *directional derivative* of f in the direction of $\mathbf{u}$ at $\mathbf{c}$.

Example From our work above, if $f(x, y) = 4 - 2x^2 - y^2$ and

$$u = -\frac{1}{\sqrt{2}}(1, 1),$$

then $D_{\mathbf{u}}f(1, 1) = 3\sqrt{2}$.

Directional derivatives in the direction of the standard basis vectors will be of special importance.

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on an open ball about a point $\mathbf{c}$. If we consider f as a function of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and let $\mathbf{e}_k$ be the kth standard basis vector, $k = 1, 2, \ldots, n$, then we call $D_{\mathbf{e}_k}f(\mathbf{c})$, if it exists, the *partial derivative* of f with respect to x_k at $\mathbf{c}$.

Notations for the partial derivative of f with respect to x_k at an arbitrary point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ include $D_{x_k}f(x_1, x_2, \ldots, x_n)$, $f_{x_k}(x_1, x_2, \ldots, x_n)$, and

$$\frac{\partial}{\partial x_k} f(x_1, x_2, \ldots, x_n).$$

Now suppose $f : \mathbb{R}^n \to \mathbb{R}$ and, for fixed $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(t, x_2, \ldots, x_n).$$

Then

$$\begin{aligned} f_{x_1}(x_1, x_2, \ldots, x_n) &= \lim_{h \to 0} \frac{f((x_1, x_2, \ldots, x_n) + h\mathbf{e}_1) - f(x_1, x_2, \ldots, x_n)}{h} \\ &= \lim_{h \to 0} \frac{f((x_1, x_2, \ldots, x_n) + (h, 0, \ldots, 0)) - f(x_1, x_2, \ldots, x_n)}{h} \\ &= \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h} \\ &= \lim_{h \to 0} \frac{g(x_1 + h) - g(x_1)}{h} \\ &= g'(x_1). \end{aligned} \tag{3.2.3}$$

In other words, we may compute the partial derivative $f_{x_1}(x_1, x_2, \ldots, x_n)$ by treating $x_2, x_3, \ldots, x_n$ as constants and differentiating with respect to x_1 as we would in single-variable calculus. The same statement holds for any coordinate: To find the partial derivative with respect to x_k, treat the other coordinates as constants and differentiate as if the function depended only on x_k.

Example If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = 3x^2 - 4xy^2,$$

then, treating y as a constant and differentiating with respect to x,

$$f_x(x, y) = 6x - 4y^2$$

and, treating x as a constant and differentiating with respect to y,

$$f_y(x, y) = -8xy.$$

Example If $f : \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$f(w, x, y, z) = -\log(w^2 + x^2 + y^2 + z^2),$$

then

$$\frac{\partial}{\partial w} f(w, z, y, z) = -\frac{2w}{w^2 + x^2 + y^2 + z^2},$$

$$\frac{\partial}{\partial x} f(w, z, y, z) = -\frac{2x}{w^2 + x^2 + y^2 + z^2},$$

$$\frac{\partial}{\partial y} f(w, z, y, z) = -\frac{2y}{w^2 + x^2 + y^2 + z^2},$$

and

$$\frac{\partial}{\partial z} f(w, z, y, z) = -\frac{2z}{w^2 + x^2 + y^2 + z^2}.$$

Example Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

We saw in Section 3.1 that $\displaystyle\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist; in particular, g is not continuous at $(0, 0)$. However,

$$\frac{\partial}{\partial x} g(0, 0) = \lim_{h\to 0} \frac{g((0, 0) + h(1, 0)) - g(0, 0)}{h} = \lim_{h\to 0} \frac{g(h, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$\frac{\partial}{\partial y}g(0,0) = \lim_{h\to 0}\frac{g((0,0)+h(0,1))-g(0,0)}{h} = \lim_{h\to 0}\frac{g(0,h)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

This shows that it is possible for a function to have partial derivatives at a point without being continuous at that point. However, we shall see in Section 3.3 that this function is not differentiable at $(0,0)$; that is, f does not have a best affine approximation at $(0,0)$.

The gradient

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on an open ball containing the point $\mathbf{c}$ and $\frac{\partial}{\partial x_k}f(\mathbf{c})$ exists for $k = 1, 2, \ldots, n$. We call the vector

$$\nabla f(\mathbf{c}) = \left(\frac{\partial}{\partial x_1}f(\mathbf{c}), \frac{\partial}{\partial x_2}f(\mathbf{c}), \ldots, \frac{\partial}{\partial x_n}f(\mathbf{c})\right) \tag{3.2.4}$$

the *gradient* of f at $\mathbf{c}$.

Example If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = 3x^2 - 4xy^2,$$

then

$$\nabla f(x,y) = (6x - 4y^2, -8xy).$$

Thus, for example, $\nabla f(2,-1) = (8,16)$.

Example If $f : \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$f(w,x,y,z) = -\log(w^2 + x^2 + y^2 + z^2),$$

then

$$\nabla f(w,x,y,z) = -\frac{2}{w^2+x^2+y^2+z^2}(w,x,y,z).$$

Thus, for example,

$$\nabla f(1,2,2,1) = -\frac{1}{5}(1,2,2,1).$$

Notice that if $f : \mathbb{R}^n \to \mathbb{R}$, then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$; that is, we may view the gradient as a function which takes an n-dimensional vector for input and returns another n-dimensional vector. We call a function of this type a *vector field*.

Definition We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on an open set U if f is continuous on U and, for $k = 1, 2, \ldots, n$, $\frac{\partial f}{\partial x_k}$ is continuous on U.

Now suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 on some open ball containing the point $\mathbf{c} = (c_1, c_2)$. Let $\mathbf{u} = (u_1, u_2)$ be a unit vector and suppose we wish to compute the directional derivative $D_{\mathbf{u}}f(\mathbf{c})$. From the definition, we have

$$\begin{aligned} D_{\mathbf{u}}f(\mathbf{c}) &= \lim_{h \to 0} \frac{f(\mathbf{c} + h\mathbf{u}) - f(\mathbf{c})}{h} \\ &= \lim_{h \to 0} \frac{f(c_1 + hu_1, c_2 + hu_2) - f(c_1, c_2)}{h} \\ &= \lim_{h \to 0} \frac{f(c_1 + hu_1, c_2 + hu_2) - f(c_1 + hu_1, c_2) + f(c_1 + hu_1, c_2) - f(c_1, c_2)}{h} \\ &= \lim_{h \to 0} \left(\frac{f(c_1 + hu_1, c_2 + hu_2) - f(c_1 + hu_1, c_2)}{h} + \frac{f(c_1 + hu_1, c_2) - f(c_1, c_2)}{h} \right). \end{aligned}$$

For a fixed value of $h \neq 0$, define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = f(c_1 + hu_1, c_2 + t). \tag{3.2.5}$$

Note that φ is differentiable with

$$\begin{aligned} \varphi'(t) &= \lim_{s \to 0} \frac{\varphi(t + s) - \varphi(t)}{s} \\ &= \lim_{s \to 0} \frac{f(c_1 + hu_1, c_2 + t + s) - f(c_1 + hu_1, c_2 + t)}{s} \\ &= \frac{\partial}{\partial y} f(c_1 + hu_1, c_2 + t). \end{aligned} \tag{3.2.6}$$

Hence if we define $\alpha : \mathbb{R} \to \mathbb{R}$ by

$$\alpha(t) = \varphi(u_2 t) = f(c_1 + hu_1, c_2 + tu_2), \tag{3.2.7}$$

then α is differentiable with

$$\alpha'(t) = u_2 \varphi'(u_2 t) = u_2 \frac{\partial}{\partial y} f(c_1 + hu_1, c_2 + tu_2). \tag{3.2.8}$$

By the Mean Value Theorem from single-variable calculus, there exists a number a between 0 and h such that

$$\frac{\alpha(h) - \alpha(0)}{h} = \alpha'(a). \tag{3.2.9}$$

Putting (3.2.7) and (3.2.8) into (3.2.9), we have

$$\frac{f(c_1 + hu_1, c_2 + hu_2) - f(c_1 + hu_2, c_2)}{h} = u_2 \frac{\partial}{\partial y} f(c_1 + hu_1, c_2 + au_2). \tag{3.2.10}$$

Similarly, if we define $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\beta(t) = f(c_1 + tu_1, c_2), \tag{3.2.11}$$

then β is differentiable,

$$\beta'(t) = u_1 \frac{\partial}{\partial x} f(c_1 + tu_1, c_2), \tag{3.2.12}$$

and, using the Mean Value Theorem again, there exists a number b between 0 and h such that

$$\frac{f(c_1 + hu_1, c_2) - f(c_1, c_2)}{h} = \frac{\beta(h) - \beta(0)}{h} = \beta'(b) = u_1 \frac{\partial}{\partial x} f(c_1 + bu_1, c_2). \tag{3.2.13}$$

Putting (3.2.10) and (3.2.13) into our expression for $D_{\mathbf{u}} f(\mathbf{c})$ above, we have

$$D_{\mathbf{u}} f(\mathbf{c}) = \lim_{h \to 0} \left(u_2 \frac{\partial}{\partial y} f(c_1 + hu_1, c_2 + au_2) + u_1 \frac{\partial}{\partial x} f(c_1 + bu_1, c_2) \right). \tag{3.2.14}$$

Now both a and b approach 0 as h approaches 0 and both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are assumed to be continuous, so evaluating the limit in (3.2.14) gives us

$$D_{\mathbf{u}} f(\mathbf{c}) = u_2 \frac{\partial}{\partial y} f(c_1, c_2) + u_1 \frac{\partial}{\partial x} f(c_1, c_2) = \nabla f(\mathbf{c}) \cdot \mathbf{u}. \tag{3.2.15}$$

A straightforward generalization of (3.2.15) to the case of a function $f : \mathbb{R}^n \to \mathbb{R}$ gives us the following theorem.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on an open ball containing the point $\mathbf{c}$. Then for any unit vector $\mathbf{u}$, $D_{\mathbf{u}} f(\mathbf{c})$ exists and

$$D_{\mathbf{u}} f(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{u}. \tag{3.2.16}$$

Example If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = 4 - 2x^2 - y^2,$$

then

$$\nabla f(x, y) = (-4x, -2y).$$

If

$$\mathbf{u} = -\frac{1}{\sqrt{2}}(1, 1),$$

then

$$D_{\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = (-4, -2) \cdot \left(-\frac{1}{\sqrt{2}}(1, 1) \right) = \frac{6}{\sqrt{2}} = 3\sqrt{2},$$

as we saw in this first example of this section. Note also that

$$D_{-\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot (-\mathbf{u}) = (-4, -2) \cdot \left(\frac{1}{\sqrt{2}}(1, 1) \right) = -\frac{6}{\sqrt{2}} = -3\sqrt{2}$$

and, if

$$\mathbf{w} = \frac{1}{\sqrt{5}}(-1, 2),$$

$$D_{\mathbf{w}} f(1,1) = \nabla f(1,1) \cdot (\mathbf{w}) = (-4,-2) \cdot \left(\frac{1}{\sqrt{5}}(-1,2)\right) = 0,$$

as claimed earlier.

Example Suppose the temperature at a point in a metal cube is given by

$$T(x, y, z) = 80 - 20xe^{-\frac{1}{20}(x^2+y^2+z^2)},$$

where the center of the cube is taken to be at $(0, 0, 0)$. Then we have

$$\frac{\partial}{\partial x}T(x,y,z) = 2x^2 e^{-\frac{1}{20}(x^2+y^2+z^2)} - 20e^{-\frac{1}{20}(x^2+y^2+z^2)},$$

$$\frac{\partial}{\partial y}T(x,y,z) = 2xye^{-\frac{1}{20}(x^2+y^2+z^2)},$$

and

$$\frac{\partial}{\partial z}T(x,y,z) = 2xze^{-\frac{1}{20}(x^2+y^2+z^2)},$$

so

$$\nabla T(x,y,z) = e^{-\frac{1}{20}(x^2+y^2+z^2)}(2x^2 - 20, 2xy, 2xz).$$

Hence, for example, the rate of change of temperature at the origin in the direction of the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}(1, -1, 1)$$

is

$$D_{\mathbf{u}} T(0,0,0) = \nabla T(0,0,0) \cdot \mathbf{u} = (-20, 0, 0) \cdot \left(\frac{1}{\sqrt{3}}(1,-1,1)\right) = -\frac{20}{\sqrt{3}}.$$

An application of the Cauchy-Schwarz inequality to (3.2.16) shows us that

$$|D_{\mathbf{u}} f(\mathbf{c})| = |\nabla f(\mathbf{c}) \cdot \mathbf{u}| \leq \|\nabla f(\mathbf{c})\| \|\mathbf{u}\| = \|\nabla f(\mathbf{c})\|. \tag{3.2.17}$$

Thus the magnitude of the rate of change of f in any direction at a given point never exceeds the length of the gradient vector at that point. Moreover, in our discussion of the Cauchy-Schwarz inequality we saw that we have equality in (3.2.17) if and only if $\mathbf{u}$ is parallel to $\nabla f(\mathbf{c})$. Indeed, supposing $\nabla f(\mathbf{c}) \neq \mathbf{0}$, when

$$\mathbf{u} = \frac{\nabla f(\mathbf{c})}{\|\nabla f(\mathbf{c})\|},$$

we have

$$D_{\mathbf{u}}f(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{u} = \frac{\nabla f(\mathbf{c}) \cdot \nabla f(\mathbf{c})}{\|\nabla f(\mathbf{c})\|} = \frac{\|\nabla f(\mathbf{c})\|^2}{\|\nabla f(\mathbf{c})\|} = \|\nabla f(\mathbf{c})\| \tag{3.2.18}$$

and

$$D_{-\mathbf{u}}f(\mathbf{c}) = -\|\nabla f(\mathbf{c})\|. \tag{3.2.19}$$

Hence we have the following result.

Proposition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on an open ball containing the point $\mathbf{c}$. Then $D_{\mathbf{u}}f(\mathbf{c})$ has a maximum value of $\|\nabla f(\mathbf{c})\|$ when $\mathbf{u}$ is the direction of $\nabla f(\mathbf{c})$ and a minimum value of $-\|\nabla f(\mathbf{c})\|$ when $\mathbf{u}$ is the direction of $-\nabla f(\mathbf{c})$.

In other words, the gradient vector points in the direction of the maximum rate of increase of the function and the negative of the gradient vector points in the direction of the maximum rate of decrease of the function. Moreover, the length of the gradient vector tells us the rate of increase in the direction of maximum increase and its negative tells us the rate of decrease in the direction of maximum decrease.

Example As we saw above, if $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = 4 - 2x^2 - y^2,$$

then

$$\nabla f(x, y,) = (-4x, -2y).$$

Thus $\nabla f(1, 1) = (-4, -2)$. Hence if a bug standing above $(1, 1)$ on the graph of f wants to head in the direction of most rapid ascent, it should move in the direction of the unit vector

$$\mathbf{u} = \frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = -\frac{1}{\sqrt{5}}(2, 1).$$

If the bug wants to head in the direction of most rapid descent, it should move in the direction of the unit vector

$$-\mathbf{u} = \frac{1}{\sqrt{5}}(2, 1).$$

Moreover,

$$D_{\mathbf{u}}f(1, 1) = \|\nabla f(1, 1)\| = \sqrt{20}$$

and

$$D_{-\mathbf{u}}f(1, 1) = -\|\nabla f(1, 1)\| = -\sqrt{20}.$$

Figure 3.2.2 shows scaled values of $\nabla f(x, y)$ plotted for a grid of points (x, y). The vectors are scaled so that they fit in the plot, without overlap, yet still show their relative magnitudes. This is another good geometric way to view the behavior of the function. Supposing our bug were placed on the side of the graph above $(1, 1)$ and that it headed up the hill in such a manner that it always chose the direction of steepest ascent, we can see that it would head more quickly toward the y-axis than toward the x-axis. More explicitly,

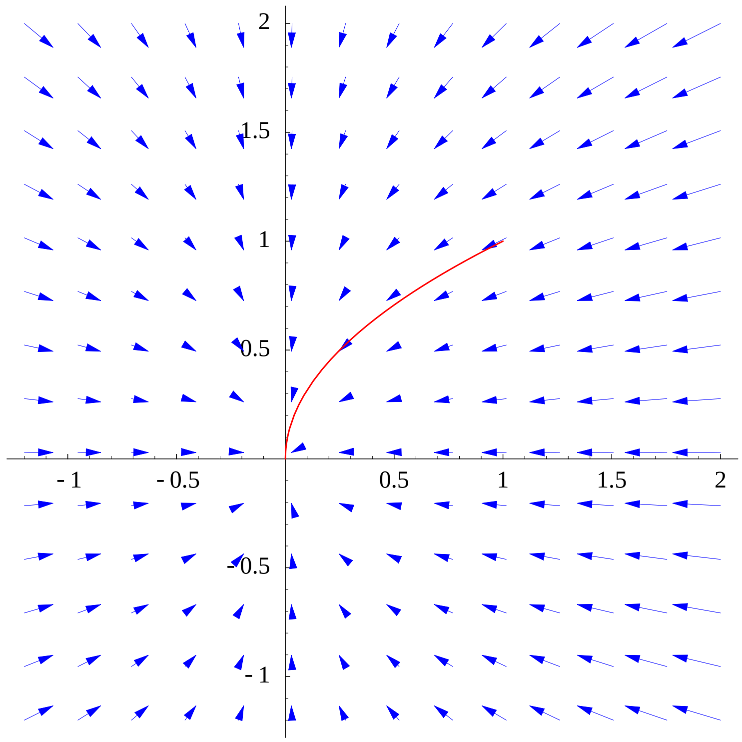


Figure 3.2.2 Scaled gradient vectors for $f(x, y) = 4 - 2x^2 - y^2$

if C is the shadow of the path of the bug in the xy-plane, then the slope of C at any point (x, y) would be

$$\frac{dy}{dx} = \frac{-2y}{-4x} = \frac{y}{2x}.$$

Hence

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x}.$$

If we integrate both sides of this equality, we have

$$\int \frac{1}{y}\frac{dy}{dx}\, dx = \int \frac{1}{2x}\, dx.$$

Thus

$$\log|y| = \frac{1}{2}\log|x| + c$$

for some constant c, from which we have

$$e^{\log|y|} = e^{\frac{1}{2}\log|x|+c}.$$

It follows that

$$y = k\sqrt{|x|},$$

where $k = \pm e^c$. Since $y = 1$ when $x = 1$, $k = 1$ and we see that C is the graph of $y = \sqrt{x}$. Figure 3.2.2 shows C along with the plot of the gradient vectors of f, while Figure 3.2.3 shows the actual path of the bug on the graph of f.

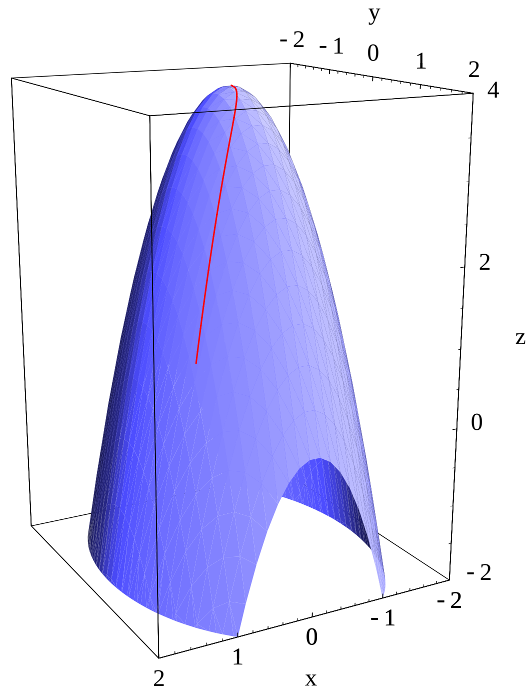


Figure 3.2.3 Graph $f(x, y) = 4 - 2x^2 - y^2$ with path of most rapid ascent from $(1, 1, 1)$

Example For a two-dimensional version of the temperature example discussed above, consider a metal plate heated so that its temperature at (x, y) is given by

$$T(x, y) = 80 - 20xe^{-\frac{1}{20}(x^2+y^2)}.$$

Then

$$\nabla T(x, y) = e^{-\frac{1}{20}(x^2+y^2)}(2x^2 - 20, 2xy),$$

so, for example,

$$\nabla T(0, 0) = (-20, 0).$$

Thus at the origin the temperature is increasing most rapidly in the direction of $u = (-1, 0)$ and decreasing most rapidly in the direction of $(1, 0)$. Moreover,

$$D_{\mathbf{u}}T(0, 0) = \|\nabla f(0, 0)\| = 20$$

and

$$D_{-\mathbf{u}}T(0, 0) = -\|\nabla f(0, 0)\| = 20.$$

Note that

$$D_{-\mathbf{u}}T(0, 0) = \frac{\partial}{\partial x}T(0, 0)$$

and

$$D_{\mathbf{u}}T(0, 0) = -\frac{\partial}{\partial x}T(0, 0).$$

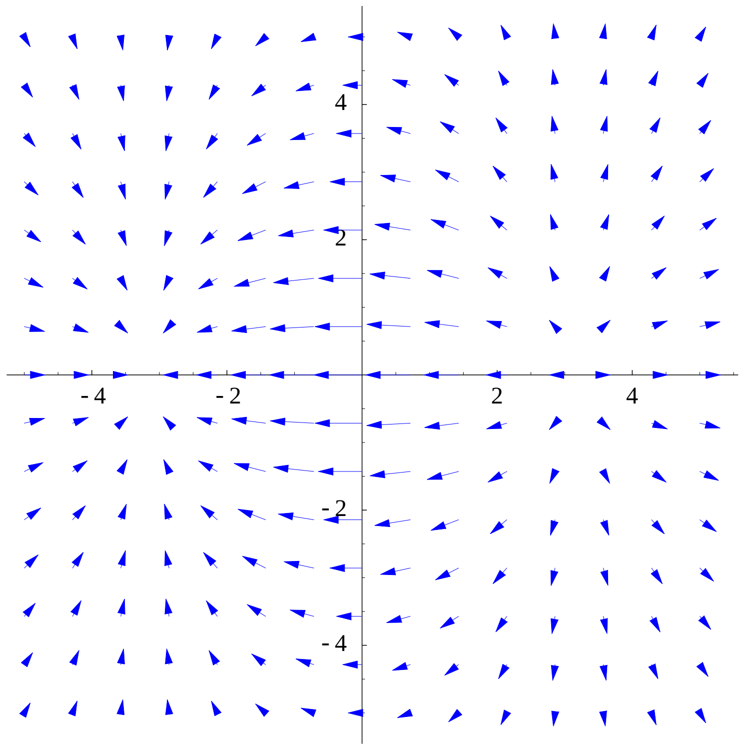


Figure 3.2.4 Scaled gradient vectors for $T(x,y) = 80 - 20xe^{-\frac{1}{20}(x^2+y^2)}$

Figure 3.2.4 is a plot of scaled gradient vectors for this temperature function. From the plot it is easy to see which direction a bug placed on this metal plate would have to choose in order to warm up as rapidly as possible. It should also be clear that the temperature has a relative maximum around $(-3, 0)$ and a relative minimum around $(3, 0)$; these points are, in fact, exactly $(-\sqrt{10}, 0)$ and $(\sqrt{10}, 0)$, the points where $\nabla T(x, y) = (0, 0)$. We will consider the problem of finding maximum and minimum values of functions of more than one variable in Section 3.5.

Problems

1. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = 3x^2 + 2y^2.$$

Let

$$\mathbf{u} = \frac{1}{\sqrt{5}}(1, 2).$$

Find $D_{\mathbf{u}} f(3, 1)$ directly from the definition (3.2.2).

2. For each of the following functions, find the partial derivatives with respect to each variable.

(a) $f(x,y) = \dfrac{4x}{x^2+y^2}$

(b) $g(x,y) = 4xy^2e^{-y^2}$

(c) $f(x,y,z) = 3x^2y^3z^4 - 13x^2y$

(d) $h(x,y,z) = 4xze^{-\frac{1}{x^2+y^2+z^2}}$

(e) $g(w,x,y,z) = \sin(\sqrt{w^2+x^2+2y^2+3z^2})$

3. Find the gradient of each of the following functions.

(a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

(b) $g(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

(c) $f(w,x,y,z) = \tan^{-1}(4w+3x+5y+z)$

4. Find $D_{\mathbf{u}}f(\mathbf{c})$ for each of the following.

(a) $f(x,y) = 3x^2+5y^2$, $\mathbf{u} = \dfrac{1}{\sqrt{13}}(3,-2)$, $\mathbf{c} = (-2,1)$

(b) $f(x,y) = x^2-2y^2$, $\mathbf{u} = \dfrac{1}{\sqrt{5}}(-1,2)$, $\mathbf{c} = (-2,3)$

(c) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, $\mathbf{u} = \dfrac{1}{\sqrt{6}}(1,2,1)$, $\mathbf{c} = (-2,2,1)$

5. For each of the following, find the directional derivative of f at the point $\mathbf{c}$ in the direction of the specified vector $\mathbf{w}$.

(a) $f(x,y) = 3x^2y$, $\mathbf{w} = (2,3)$, $\mathbf{c} = (-2,1)$

(b) $f(x,y,z) = \log(x^2+2y^2+z^2)$, $\mathbf{w} = (-1,2,3)$, $\mathbf{c} = (2,1,1)$

(c) $f(t,x,y,z) = tx^2yz^2$, $\mathbf{w} = (1,-1,2,3)$, $\mathbf{c} = (2,1,-1,2)$

6. A metal plate is heated so that its temperature at a point (x,y) is

$$T(x,y) = 50y^2e^{-\frac{1}{5}(x^2+y^2)}.$$

A bug is placed at the point $(2,1)$.

(a) The bug heads toward the point $(1,-2)$. What is the rate of change of temperature in this direction?

(b) In what direction should the bug head in order to warm up at the fastest rate? What is the rate of change of temperature in this direction?

(c) In what direction should the bug head in order to cool off at the fastest rate? What is the rate of change of temperature in this direction?

(d) Make a plot of the gradient vectors and discuss what it tells you about the temperatures on the plate.

7. A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Discuss the behavior of a heat seeking bug placed on a metal plate heated so that the temperature at (x,y) is given by

$$T(x,y) = 100 - 40xye^{-\frac{1}{10}(x^2+y^2)}.$$

8. Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

We saw above that both partial derivatives of g exist at $(0,0)$, although g is not continuous at $(0,0)$.

(a) Show that neither $\frac{\partial g}{\partial x}$ nor $\frac{\partial g}{\partial y}$ is continuous at $(0,0)$.

(b) Let

$$\mathbf{u} = \frac{1}{\sqrt{2}}(1,1).$$

Show that $D_{\mathbf{u}}g(0,0)$ does not exist. In particular, $D_{\mathbf{u}}g(0,0) \neq \nabla g(0,0) \cdot \mathbf{u}$.

9. Suppose the price of a certain commodity, call it commodity A, is x dollars per unit and the price of another commodity, B, is y dollars per unit. Moreover, suppose that $d_A(x,y)$ represents the number of units of A that will be sold at these prices and $d_B(x,y)$ represents the number of units of B that will be sold at these prices. These functions are known as the *demand functions* for A and B.

(a) Explain why it is reasonable to assume that

$$\frac{\partial}{\partial x} d_A(x,y) < 0$$

and

$$\frac{\partial}{\partial y} d_B(x,y) < 0$$

for all (x,y).

(b) Suppose the two commodities are competitive. For example, they might be two different brands of the same product. In this case, what would be reasonable assumptions for the signs of

$$\frac{\partial}{\partial y} d_A(x,y)$$

and

$$\frac{\partial}{\partial x} d_B(x,y)?$$

(c) Suppose the two commodities complement each other. For example, commodity A might be a computer and commodity B a type of software. In this case, what would be reasonable assumptions for the signs of

$$\frac{\partial}{\partial y} d_A(x,y)$$

and

$$\frac{\partial}{\partial x} d_B(x,y)?$$

10. Suppose $P(x_1, x_2, \ldots, x_n)$ represents the total production per week of a certain factory as a function of x_1, the number of workers, and other variables, such as the size of the supply inventory, the number of hours the assembly lines run per week, and so on. Show that average productivity

$$\frac{P(x_1, x_2, \ldots, x_n)}{x_1}$$

increases as x_1 increases if and only if

$$\frac{\partial}{\partial x_1} P(x_1, x_2, \ldots, x_n) > \frac{P(x_1, x_2, \ldots, x_n)}{x_1}.$$

11. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on an open ball about the point $\mathbf{c}$.
    (a) Given a unit vector $\mathbf{u}$, what is the relationship between $D_{\mathbf{u}} f(\mathbf{c})$ and $D_{-\mathbf{u}} f(\mathbf{c})$?
    (b) Is it possible that $D_{\mathbf{u}} f(\mathbf{c}) > 0$ for every unit vector $\mathbf{u}$?

The Calculus of Functions

of

Several Variables

Section 3.3

Best Affine Approximations

Best affine approximations

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $\mathbf{c}$, we wish to find the affine function $A : \mathbb{R}^n \to \mathbb{R}$ which best approximates f for points close to $\mathbf{c}$. As before, best will mean that the remainder function,

$$R(\mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h}), \tag{3.3.1}$$

approaches 0 at a sufficiently fast rate. In this context, since $R(\mathbf{h})$ is a scalar and $\mathbf{h}$ is a vector, sufficiently fast will mean that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|} = 0. \tag{3.3.2}$$

Generalizing our previous notation, we will say that a function $R : \mathbb{R}^n \to \mathbb{R}$ satisfying (3.3.2) is $o(\mathbf{h})$. Note that if $n = 1$ this extended definition of $o(\mathbf{h})$ is equivalent to the definition given in Section 2.2.

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on an open ball containing the point $\mathbf{c}$. We call an affine function $A : \mathbb{R}^n \to \mathbb{R}$ the *best affine approximation* to f at $\mathbf{c}$ if (1) $A(\mathbf{c}) = f(\mathbf{c})$ and (2) $R(\mathbf{h})$ is $o(\mathbf{h})$, where

$$R(\mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h}). \tag{3.3.3}$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and suppose $A : \mathbb{R}^n \to \mathbb{R}$ is the best affine approximation to f at $\mathbf{c}$. Since A is affine, there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ and a scalar b such that

$$A(\mathbf{x}) = L(\mathbf{x}) + b \tag{3.3.4}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$. Since $A(\mathbf{c}) = f(\mathbf{c})$, we have

$$f(\mathbf{c}) = L(\mathbf{c}) + b, \tag{3.3.5}$$

which implies that

$$b = f(\mathbf{c}) - L(\mathbf{c}). \tag{3.3.6}$$

Hence

$$A(\mathbf{x}) = L(\mathbf{x}) + f(\mathbf{c}) - L(\mathbf{c}) = L(\mathbf{x} - \mathbf{c}) + f(\mathbf{c}) \tag{3.3.7}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$. Moreover, if we let

$$\mathbf{a} = (L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)), \tag{3.3.8}$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are, as usual, the standard basis vectors for $\mathbb{R}^n$, then, from our results in Section 1.5,

$$L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} \tag{3.3.9}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$. Hence

$$A(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{c}) + f(\mathbf{c}), \tag{3.3.10}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$, and we see that A is completely determined by the vector $\mathbf{a}$

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on an open ball containing the point $\mathbf{c}$. If f has a best affine approximation at $\mathbf{c}$, then we say f is *differentiable* at $\mathbf{c}$. Moreover, if the best affine approximation to f at $\mathbf{c}$ is given by

$$A(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{c}) + f(\mathbf{c}), \tag{3.3.11}$$

then we call $\mathbf{a}$ the *derivative* of f at $\mathbf{c}$ and write $Df(\mathbf{c}) = \mathbf{a}$.

Now suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{c}$ with best affine approximation A and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) = Df(\mathbf{c})$. Since

$$R(\mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - \mathbf{a} \cdot \mathbf{h} - f(\mathbf{c}) \tag{3.3.12}$$

is $o(\mathbf{h})$, we must have

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|} = 0. \tag{3.3.13}$$

In particular, for $k = 1, 2, \ldots, n$, if we let $\mathbf{h} = t\mathbf{e}_k$, then $\mathbf{h}$ approaches $\mathbf{0}$ as t approaches 0, so

$$0 = \lim_{t \to 0} \frac{R(t\mathbf{e}_k)}{\|t\mathbf{e}_k\|} = \lim_{t \to 0} \frac{f(\mathbf{c} + t\mathbf{e}_k) - t(\mathbf{a} \cdot \mathbf{e}_k) - f(\mathbf{c})}{|t|} = \lim_{t \to 0} \frac{f(\mathbf{c} + t\mathbf{e}_k) - ta_k - f(\mathbf{c})}{|t|}$$

First considering $t > 0$, we have

$$0 = \lim_{t \to 0^+} \frac{f(\mathbf{c} + t\mathbf{e}_k) - ta_k - f(\mathbf{c})}{t} = \lim_{t \to 0^+} \left(\frac{f(\mathbf{c} + t\mathbf{e}_k) - f(\mathbf{c})}{t} - a_k \right), \tag{3.3.14}$$

implying that

$$a_k = \lim_{t \to 0^+} \frac{f(\mathbf{c} + t\mathbf{e}_k) - f(\mathbf{c})}{t}. \tag{3.3.15}$$

With $t < 0$, we have

$$0 = \lim_{t \to 0^-} \frac{f(\mathbf{c} + t\mathbf{e}_k) - ta_k - f(\mathbf{c})}{-t} = - \lim_{t \to 0^-} \left(\frac{f(\mathbf{c} + t\mathbf{e}_k) - f(\mathbf{c})}{t} - a_k \right), \tag{3.3.16}$$

implying that

$$a_k = \lim_{t \to 0^-} \frac{f(\mathbf{c} + t\mathbf{e}_k) - f(\mathbf{c})}{t}. \tag{3.3.17}$$

Hence

$$a_k = \lim_{t \to 0} \frac{f(\mathbf{c} + t\mathbf{e}_k) - f(\mathbf{c})}{t} = \frac{\partial}{\partial x_k} f(\mathbf{c}). \tag{3.3.18}$$

Thus we have shown that

$$\mathbf{a} = \left(\frac{\partial}{\partial x_1} f(\mathbf{c}), \frac{\partial}{\partial x_2} f(\mathbf{c}), \ldots, \frac{\partial}{\partial x_n} f(\mathbf{c}) \right) = \nabla f(\mathbf{c}). \tag{3.3.19}$$

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{c}$, then

$$Df(\mathbf{c}) = \nabla f(\mathbf{c}) \tag{3.3.20}$$

It now follows that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{c}$, then the best affine approximation to f at $\mathbf{c}$ is

$$A(\mathbf{x}) = \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}) - f(\mathbf{c}). \tag{3.3.21}$$

However, the converse does not hold: it is possible for $\nabla f(\mathbf{c})$ to exist even when f is not differentiable at $\mathbf{c}$. Before looking at an example, note that if f is differentiable at $\mathbf{c}$ and A is the best affine approximation to f at $\mathbf{c}$, then, since $R(\mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h})$ is $o(\mathbf{h})$,

$$\lim_{\mathbf{h} \to \mathbf{0}} (f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h})) = \lim_{\mathbf{h} \to \mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|} \|\mathbf{h}\| = 0\|\mathbf{0}\| = 0. \tag{3.3.22}$$

Now A is continuous at $\mathbf{c}$, so it follows that

$$\lim_{\mathbf{h} \to \mathbf{0}} f(\mathbf{c} + \mathbf{h}) = \lim_{\mathbf{h} \to \mathbf{0}} A(\mathbf{c} + \mathbf{h}) = A(\mathbf{c}) = f(\mathbf{c}). \tag{3.3.23}$$

In other words, f is continuous at $\mathbf{c}$.

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{c}$, then f is continuous at $\mathbf{c}$.

Example Consider the function

$$g(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

In Section 3.1 we showed that g is not continuous at $(0, 0)$ and in Section 3.2 we saw that $\nabla g(0, 0) = (0, 0)$. Since g is not continuous at $(0, 0)$, it now follows, from the previous theorem, that g is not differentiable at $(0, 0)$, even though the gradient exists at that point. From the graph of g in Figure 3.3.1 (originally seen in Figure 3.1.7), we can see

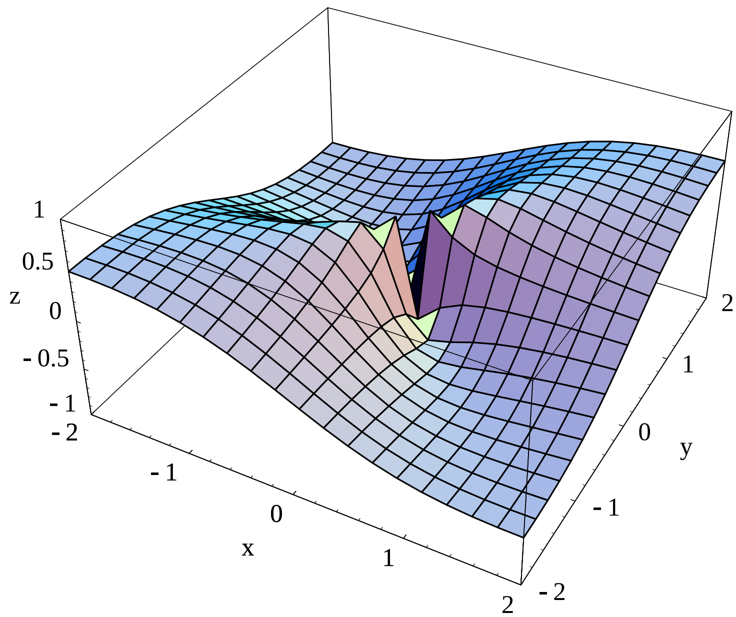


Figure 3.3.1 The graph of a nondifferentiable function

that the fact that g is not differentiable, in fact, not even continuous, at the origin shows up geometrically as a tear in the surface.

From this example we see that the differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{c}$ requires more than just the existence of the gradient of f at $\mathbf{c}$. It turns out that continuity of the partial derivatives of f on an open ball containing $\mathbf{c}$ suffices to show that f is differentiable at $\mathbf{c}$. Note that the partial derivatives of g in the previous example are not continuous (see Problem 8 of Section 3.2).

So we will now assume that $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on some open ball containing $\mathbf{c}$. If we define an affine function $A : \mathbb{R}^n \to \mathbb{R}$ by

$$A(\mathbf{x}) = \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}) + f(\mathbf{c}), \tag{3.3.24}$$

then the remainder function is

$$R(\mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot \mathbf{h}. \tag{3.3.25}$$

We need to show that $R(\mathbf{h})$ is $o(\mathbf{h})$. Toward that end, for a fixed $\mathbf{h} \neq \mathbf{0}$, define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = f(\mathbf{c} + t\mathbf{h}). \tag{3.3.26}$$

We first note that φ is differentiable with

$$\begin{aligned}\varphi'(t) &= \lim_{s \to 0} \frac{\varphi(t+s) - \varphi(t)}{s} \\ &= \lim_{s \to 0} \frac{f(\mathbf{c} + (t+s)\mathbf{h}) - f(\mathbf{c} + t\mathbf{h})}{s}\end{aligned}$$

$$
\begin{aligned}
&= \|\mathbf{h}\| \lim_{s \to 0} \frac{f\left(\mathbf{c} + t\mathbf{h} + s\|\mathbf{h}\|\frac{\mathbf{h}}{\|\mathbf{h}\|}\right) - f(\mathbf{c} + t\mathbf{h})}{s\|\mathbf{h}\|} \\
&= \|\mathbf{h}\| D_{\frac{\mathbf{h}}{\|\mathbf{h}\|}} f(\mathbf{c} + t\mathbf{h}) \\
&= \|\mathbf{h}\| \left(\nabla f(\mathbf{c} + t\mathbf{h}) \cdot \frac{\mathbf{h}}{\|\mathbf{h}\|}\right) \\
&= \nabla f(\mathbf{c} + t\mathbf{h}) \cdot \mathbf{h}.
\end{aligned} \tag{3.3.27}
$$

From the Mean Value Theorem of single-variable calculus, it follows that there exists a number s between 0 and 1 such that

$$\varphi'(s) = \varphi(1) - \varphi(0) = f(\mathbf{c} + \mathbf{h}) - f(\mathbf{c}). \tag{3.3.28}$$

Hence we may write

$$R(\mathbf{h}) = \nabla f(\mathbf{c} + s\mathbf{h}) \cdot \mathbf{h} - \nabla f(\mathbf{c}) \cdot \mathbf{h} = (\nabla f(\mathbf{c} + s\mathbf{h}) - \nabla f(\mathbf{c})) \cdot \mathbf{h}. \tag{3.3.29}$$

Applying the Cauchy-Schwarz inequality to (3.3.29),

$$|R(\mathbf{h})| \leq \|\nabla f(\mathbf{c} + s\mathbf{h}) - \nabla f(\mathbf{c})\| \|\mathbf{h}\|, \tag{3.3.30}$$

and so

$$\frac{|R(\mathbf{h})|}{\|\mathbf{h}\|} \leq \|\nabla f(\mathbf{c} + s\mathbf{h}) - \nabla f(\mathbf{c})\|. \tag{3.3.31}$$

Now the partial derivatives of f are continuous, so

$$
\begin{aligned}
\lim_{\mathbf{h} \to \mathbf{0}} \|\nabla f(\mathbf{c} + s\mathbf{h}) - \nabla f(\mathbf{c})\| &= \|\nabla f(\mathbf{c} + s\mathbf{0}) - \nabla f(\mathbf{c})\| \\
&= \|\nabla f(\mathbf{c}) - \nabla f(\mathbf{c})\| \\
&= 0.
\end{aligned} \tag{3.3.32}
$$

Hence

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|} = 0. \tag{3.3.33}$$

That is, $R(\mathbf{h})$ is $o(\mathbf{h})$ and A is the best affine approximation to f at $\mathbf{c}$. Thus we have the following fundamental theorem.

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on an open ball containing the point $\mathbf{c}$, then f is differentiable at $\mathbf{c}$.

Example Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = 4 - 2x^2 - y^2.$$

To find the best affine approximation to f at $(1, 1)$, we first compute

$$\nabla f(x, y) = (-4x, -2y).$$

Thus $\nabla f(1,1) = (-4,-2)$ and $f(1,1) = 1$, so the best affine approximation is

$$A(x,y) = (-4,-2)\cdot(x-1,y-1)+1.$$

Simplifying, we have

$$A(x,y) = -4x - 2y + 7.$$

Example Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}.$$

Then

$$\nabla f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z).$$

Thus, for example, the best affine approximation to f at $(2,1,2)$ is

$$\begin{aligned} A(x,y,z) &= \nabla f(2,1,2)\cdot(x-2,y-1,z-2)+f(2,1,2) \\ &= \frac{1}{3}(2,1,2)\cdot(x-2,y-1,z-2)+3 \\ &= \frac{2}{3}(x-2)+\frac{1}{3}(y-1)+\frac{2}{3}(z-2)+3 \\ &= \frac{2}{3}x+\frac{1}{3}y+\frac{2}{3}z. \end{aligned}$$

Now suppose we let (x,y,z) be the lengths of the three sides of a solid block, in which case $f(x,y,z)$ represents the length of the diagonal of the box. Moreover, suppose we measure the sides of the block and find them to have lengths $x = 2+\epsilon_x$, $y = 1+\epsilon_y$, and $z = 2+\epsilon_z$, where $|\epsilon_x| \le h$, $|\epsilon_y| \le h$, and $|\epsilon_z| \le h$ for some positive number h representing the limit of the accuracy of our measuring device. We now estimate the diagonal of the box to be

$$f(2,1,2) = 3$$

with an error of

$$\begin{aligned} |f(2+\epsilon_x,1+\epsilon_y,2+\epsilon_z)-f(2,1,2)| &\approx |A(2+\epsilon_x,1+\epsilon_y,2+\epsilon_z)-3| \\ &= \left|\frac{2}{3}\epsilon_x+\frac{1}{3}\epsilon_y+\frac{2}{3}\epsilon_z\right| \\ &\le \frac{2}{3}|\epsilon_x|+\frac{1}{3}|\epsilon_y|+\frac{2}{3}|\epsilon_z| \\ &\le h\left(\frac{2}{3}+\frac{1}{3}+\frac{2}{3}\right) \\ &= \frac{5}{3}h. \end{aligned}$$

That is, we expect our error in estimating the diagonal of the block to be no more that $\frac{5}{3}$ times the maximum error in our measurements of the sides of the block. For example, if the error in our length measurements is off by no more than ± 0.1 centimeters, then our estimate of the diagonal of the box is off by no more than ± 0.17 centimeters.

Note that if $A : \mathbb{R}^n \to \mathbb{R}$ is the best affine approximation to $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, then the graph of A is the set of all points $(x_1, x_2, \ldots, x_n, z)$ in $\mathbb{R}^{n+1}$ satisfying

$$z = \nabla f(\mathbf{c}) \cdot (x_1 - c_1, x_2 - c_2, \ldots, x_n - c_n) + f(\mathbf{c}). \tag{3.3.34}$$

Letting

$$\mathbf{n} = \left(\frac{\partial}{\partial x_1} f(\mathbf{c}), \frac{\partial}{\partial x_2} f(\mathbf{c}), \ldots, \frac{\partial}{\partial x_n} f(\mathbf{c}), -1 \right), \tag{3.3.35}$$

we may describe the graph of A as the set of all points in $\mathbb{R}^{n+1}$ satisfying

$$\mathbf{n} \cdot (x_1 - c_1, x_2 - c_2, \ldots, x_n - c_n, z - f(\mathbf{c})) = 0. \tag{3.3.36}$$

Thus the graph of A is a hyperplane in $\mathbb{R}^{n+1}$ passing through the point $(c_1, c_2, \ldots, c_n, f(\mathbf{c}))$ (a point on the graph of f) with normal vector $\mathbf{n}$.

Definition If $A : \mathbb{R}^n \to \mathbb{R}$ is the best affine approximation to $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, then we call the graph of A the *tangent hyperplane* to the graph of f at $(c_1, c_2, \ldots, c_n, f(\mathbf{c}))$.

Example We saw above that the best affine approximation to

$$f(x, y) = 4 - 2x^2 - y^2$$

at $(1, 1)$ is

$$A(x, y) = 7 - 4x - 2y.$$

Hence the equation of the tangent plane to the graph of f at is

$$z = 7 - 4x - 2y,$$

or

$$4x + 2y + z = 7.$$

Note that the vector $\mathbf{n} = (4, 2, 1)$ is normal to the tangent plane, and hence normal to the graph of f at $(1, 1, 1)$. The graph of f along with the tangent plane at $(1, 1, 1)$ is shown in Figure 3.3.2.

The chain rule

Suppose $\varphi : \mathbb{R} \to \mathbb{R}^n$ is differentiable at a point c and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at the point $\varphi(c)$. Then the composition of f and φ is a function $f \circ \varphi : \mathbb{R} \to \mathbb{R}$. To compute the derivative of $f \circ \varphi$ at c, we must evaluate

$$(f \circ \varphi)'(c) = \lim_{h \to 0} \frac{f \circ \varphi(c + h) - f \circ \varphi(c)}{h} = \lim_{h \to 0} \frac{f(\varphi(c + h)) - f(\varphi(c))}{h}. \tag{3.3.37}$$

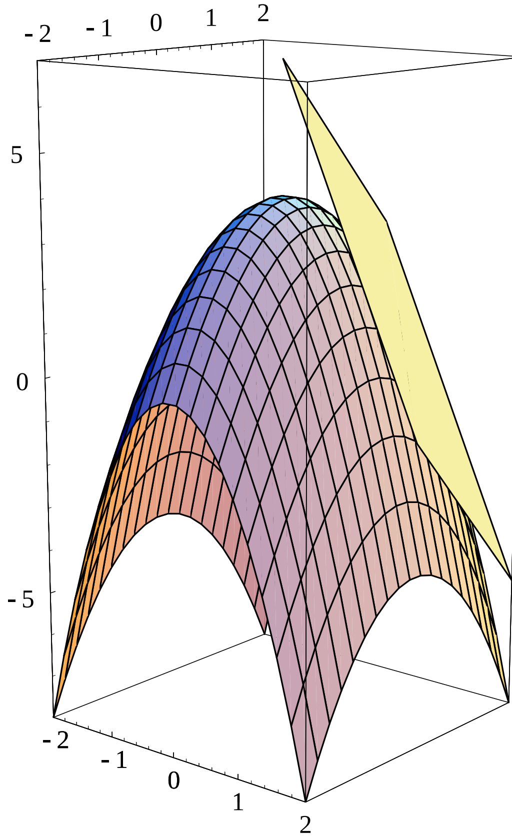


Figure 3.3.2 A plane tangent to the graph of $f(x, y) = 4 - 2x^2 - y^2$

Let A be the best affine approximation to f at $\mathbf{a} = \varphi(c)$ and let $\mathbf{k} = \varphi(c+h) - \varphi(c)$. Then

$$f(\varphi(c+h)) = f(\mathbf{a}+\mathbf{k}) = A(\mathbf{a}+\mathbf{k}) + R(\mathbf{k}), \tag{3.3.38}$$

where $R(\mathbf{k})$ is $o(\mathbf{k})$. Now

$$A(\mathbf{a}+\mathbf{k}) = \nabla f(\mathbf{a}) \cdot \mathbf{k} + f(\mathbf{a}), \tag{3.3.39}$$

so

$$\begin{aligned} f(\varphi(c+h)) - f(\varphi(c)) &= f(\mathbf{a}+\mathbf{k}) - f(\mathbf{a}) \\ &= \nabla f(\mathbf{a}) \cdot \mathbf{k} + R(\mathbf{k}) \\ &= \nabla f(\mathbf{a}) \cdot (\varphi(c+h) - \varphi(c)) + R(\mathbf{k}). \end{aligned} \tag{3.3.40}$$

Substituting (3.3.40) into (3.3.37),we have

$$\begin{aligned} (f \circ \varphi)'(c) &= \lim_{h \to 0} \frac{\nabla f(\mathbf{a}) \cdot (\varphi(c+h) - \varphi(c)) + R(\mathbf{k})}{h} \\ &= \lim_{h \to 0} \nabla f(\mathbf{a}) \cdot \frac{\varphi(c+h) - \varphi(c)}{h} + \lim_{h \to 0} \frac{R(\mathbf{k})}{h} \\ &= \nabla f(\mathbf{a}) \cdot D\varphi(\mathbf{c}) + \lim_{h \to 0} \frac{R(\mathbf{k})}{h}. \end{aligned} \tag{3.3.41}$$

Now $R(\mathbf{k})$ is $o(\mathbf{k})$, so

$$\lim_{\mathbf{k}\to\mathbf{0}} \frac{R(\mathbf{k})}{\|\mathbf{k}\|} = 0,$$

from which it follows that, for any given $\epsilon > 0$, we have

$$\frac{|R(\mathbf{k})|}{\|\mathbf{k}\|} < \epsilon \tag{3.3.42}$$

for sufficiently small $\mathbf{k} \neq 0$. Since $R(\mathbf{0}) = 0$, it follows that

$$|R(\mathbf{k})| < \epsilon\|\mathbf{k}\| \tag{3.3.43}$$

for all $\mathbf{k}$ sufficiently small. Moreover, φ is continuous at c, so we may choose h small enough to guarantee that

$$\mathbf{k} = \varphi(c+h) - \varphi(h)$$

is small enough for (3.3.43) to hold. Hence for sufficiently small $h \neq 0$,

$$\frac{|R(\mathbf{k})|}{h} < \frac{\epsilon\|\mathbf{k}\|}{h}. \tag{3.3.44}$$

Now

$$\lim_{h\to 0} \frac{\|\mathbf{k}\|}{h} = \lim_{h\to 0} \frac{\|\varphi(c+h) - \varphi(c)\|}{h} = \|D\varphi(c)\| \tag{3.3.45}$$

and the choice of ϵ was arbitrary, so it follows that

$$\lim_{h\to\mathbf{0}} \frac{R(\mathbf{k})}{h} = 0. \tag{3.3.46}$$

Hence

$$(f\circ\varphi)'(c) = \nabla f(\mathbf{a})\cdot D\varphi(c). \tag{3.3.47}$$

This is a version of the chain rule.

Theorem Suppose $\varphi : \mathbb{R} \to \mathbb{R}^n$ is differentiable at c and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\varphi(c)$. Then

$$(f\circ\varphi)'(c) = \nabla f(\varphi(c))\cdot D\varphi(c). \tag{3.3.48}$$

If we imagine a particle moving along the curve C parametrized by φ, with velocity $\mathbf{v}(t)$ and unit tangent vector $T(t)$ at time t, then (3.3.48) says that the rate of change of f along C at $\varphi(c)$ is

$$\nabla f(\varphi(c))\cdot\mathbf{v}(c) = \|\mathbf{v}(c)\|\nabla f(\varphi(c))\cdot T(c) = \|\mathbf{v}(c)\|D_{T(c)}f(\varphi(c)). \tag{3.3.49}$$

In other words, the rate of change of f along C is the rate of change of f in the direction of $T(t)$ multiplied by the speed of the particle moving along the curve.

Example Suppose that the temperature at a point (x, y, z) inside a cubical region of space is given by

$$T(x, y, z) = 80 - 20xe^{-\frac{1}{20}(x^2+y^2+z^2)}.$$

Moreover, suppose a bug flies through this region along the elliptical helix parametrized by

$$\varphi(t) = (\cos(\pi t), 2\sin(\pi t), t).$$

Then

$$\nabla T(x, y, z) = e^{-\frac{1}{20}(x^2+y^2+z^2)}(2x^2 - 20, 2xy, 2xz)$$

and

$$D\varphi(t) = (-\pi\sin(\pi t), 2\pi\cos(\pi t), 1).$$

Hence, for example, if we want to know the rate of change of temperature for the bug at $t = \frac{1}{3}$, we would evaluate

$$D\varphi\left(\frac{1}{3}\right) = \left(-\frac{\sqrt{3}\pi}{2}, \pi, 1\right)$$

and

$$\nabla T\left(\varphi\left(\frac{1}{3}\right)\right) = \nabla T\left(\frac{1}{2}, \sqrt{3}, \frac{1}{3}\right) = e^{-\frac{121}{720}}\left(-\frac{39}{2}, \sqrt{3}, \frac{1}{3}\right),$$

so

$$\begin{aligned}(T\circ\varphi)'\left(\frac{1}{3}\right) &= e^{-\frac{121}{720}}\left(-\frac{39}{2}, \sqrt{3}, \frac{1}{3}\right)\cdot\left(-\frac{\sqrt{3}\pi}{2}, \pi, 1\right)\\ &= e^{-\frac{121}{720}}\left(\frac{39\pi\sqrt{3}}{4} + \sqrt{3}\pi + \frac{1}{3}\right)\\ &= 49.73,\end{aligned}$$

where the final value has been rounded to two decimal places. Hence at that moment the temperature for the bug is increasing at rate of 49.73° per second. We could also express this as

$$\left.\frac{dT}{dt}\right|_{t=\frac{1}{3}} = 49.73°.$$

For an alternative formulation of the chain rule, suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $x_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are all differentiable and let $w = f(x_1, x_2, \ldots, x_n)$. If x_1, x_2, $\ldots$, x_n are all functions of t, then, by the chain rule,

$$\begin{aligned}\frac{dw}{dt} &= \left(\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \ldots, \frac{\partial w}{\partial x_n}\right)\cdot\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right)\\ &= \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.\end{aligned} \tag{3.3.50}$$

Example Suppose the dimensions of a box are increasing so that its length, width, and height at time t are, in centimeters,

$$x = 3t,$$
$$y = t^2,$$

and

$$z = t^3,$$

respectively. Since the volume of the box is

$$V = xyz,$$

the rate of change of the volume is

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} = 3yz + 2xzt + 3xyt^2.$$

Hence, for example, at $t = 2$ we have $x = 6$, $y = 4$, and $z = 8$, so

$$\left.\frac{dV}{dt}\right|_{t=2} = 96 + 192 + 288 = 576 \text{ cm}^3/\text{sec}.$$

The gradient and level sets

Now consider a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $\mathbf{a}$ on the level set S specified by $f(\mathbf{x}) = c$ for some scalar c. Suppose $\varphi : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C which lies entirely on S and passes through $\mathbf{a}$. Let $\varphi(b) = \mathbf{a}$. Then the composition of f and φ is a constant function; that is,

$$g(t) = f \circ \varphi(t) = f(\varphi(t)) = c \qquad (3.3.51)$$

for all values of t. Thus, using the chain rule,

$$0 = g'(b) = \nabla f(\varphi(b)) \cdot D\varphi(b) = \nabla f(\mathbf{a}) \cdot D\varphi(b). \qquad (3.3.52)$$

Hence

$$\nabla f(\mathbf{a}) \perp D\varphi(b). \qquad (3.3.53)$$

Now $D\varphi(b)$ is tangent to C at $\mathbf{a}$; moreover, since (3.3.53) holds for any curve in S passing through $\mathbf{a}$, $\nabla f(\mathbf{a})$ is orthogonal to every vector tangent to S. In other words, $\nabla f(\mathbf{a})$ is normal to the hyperplane tangent to S at $\mathbf{a}$. Thus we have the following theorem.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable on an open ball containing the point $\mathbf{a}$, and let S be the set of all points in $\mathbb{R}^n$ such that $f(\mathbf{x}) = f(\mathbf{a})$. If $\nabla f(\mathbf{a}) \neq \mathbf{0}$, then the hyperplane with equation

$$\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0 \qquad (3.3.54)$$

is tangent to S at $\mathbf{a}$.

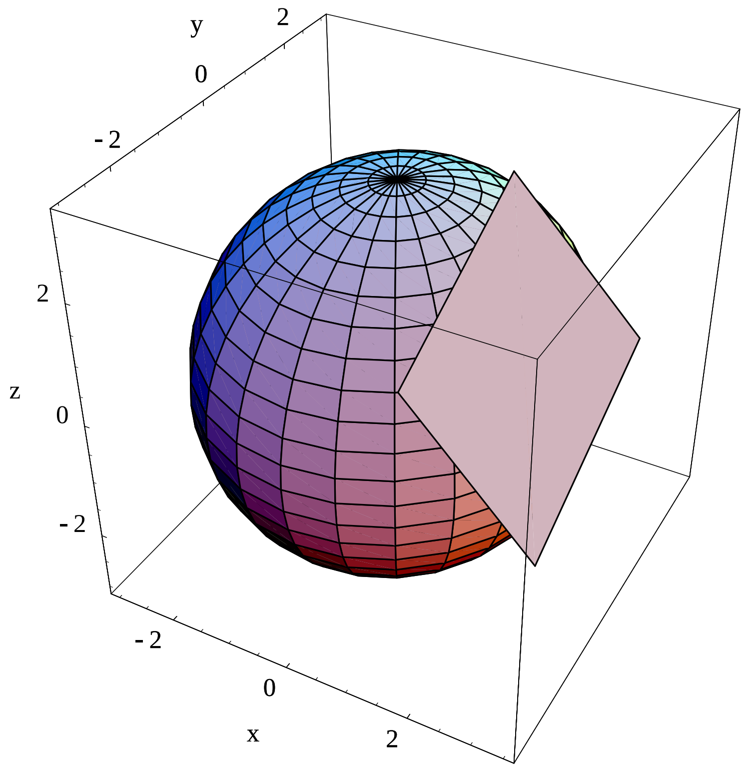


Figure 3.3.3 Sphere with tangent plane

For $n = 2$, the hyperplane described by (3.3.54) will be a tangent line to a curve; for $n = 3$, it will be a tangent plane to a surface.

Example The set of all points S in $\mathbb{R}^3$ satisfying

$$x^2 + y^2 + z^2 = 9$$

is a sphere with radius 3 centered at the origin. We will find an equation for the plane tangent to S at $(2, -1, 2)$. First note that S is a level surface for the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Now

$$\nabla f(x, y, z) = (2x, 2y, 2z),$$

so

$$\nabla f(2, -1, 2) = (4, -2, 4).$$

Thus an equation for the tangent plane is

$$(4, -2, 4) \cdot (x - 2, y + 1, z - 2) = 0,$$

or

$$4x - 2y + 4z = 18.$$

See Figure 3.3.3.

Problems

1. For each of the following, find the best affine approximation to the given function at the specified point $\mathbf{c}$.
   (a) $f(x,y) = 3x^2 + 4y^2 - 2$, $\mathbf{c} = (2,1)$
   (b) $g(x,y) = y^2 - x^2$, $\mathbf{c} = (1,-2)$
   (c) $g(x,y) = y^2 - x^2$, $\mathbf{c} = (0,0)$
   (d) $f(x,y,z) = -\log(x^2 + y^2 + z^2)$, $\mathbf{c} = (1,0,0)$
   (e) $h(w,x,y,z) = w^2 + x^2 + 3y^2 = 2z^2$, $\mathbf{c} = (1,2,-2,1)$
2. For each of the following, find the equation of the plane tangent to the graph of f for the given point $\mathbf{c}$. Plot the graph and the tangent plane together.
   (a) $f(x,y) = 4x^2 + y^2$, $\mathbf{c} = (1,-1)$
   (b) $f(x,y) = \sqrt{9 - x^2 - y^2}$, $\mathbf{c} = (2,1)$
   (c) $f(x,y) = 9 - x^2 - y^2$, $\mathbf{c} = (2,-2)$
   (d) $f(x,y) = 3y^2 - x^2$, $\mathbf{c} = (1,-1)$
3. Suppose $A : \mathbb{R}^n \to \mathbb{R}$ is the best affine approximation to $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{c}$. Explain why $|\nabla f(\mathbf{c}) \cdot \mathbf{h}|$ is a good approximation for $|f(\mathbf{c} + \mathbf{h}) - f(\mathbf{c})|$ when $\|\mathbf{h}\|$ is small. That is, explain why $|\nabla f(\mathbf{c}) \cdot \mathbf{h}|$ is a good approximation for the error in approximating $f(\mathbf{c}+\mathbf{h})$ by $f(\mathbf{c})$.
4. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x,y,z) = xyz$.
   (a) Find the best affine approximation to f at $(3,2,4)$.
   (b) Suppose x, y, and z represent the length, width, and height of a box. Suppose you measure the length to be $3 \pm h$ centimeters, the width to be $2 \pm h$ centimeters, and the height to be $4 \pm h$ centimeters. Use the best affine approximation from (a) to approximate the maximum error you would make in computing the volume of the box from these measurements.
5. A metal plate is heated so that its temperature at a point (x,y) is
$$T(x,y) = 50y^2 e^{-\frac{1}{5}(x^2+y^2)}.$$
   A bug moves along the ellipse parametrized by
$$\alpha(t) = (\cos(t), 2\sin(t)).$$
   Find the rate of change of temperature for the bug at times $t = 0$, $t = \frac{\pi}{4}$, and $t = \frac{\pi}{2}$.
6. Let x, y, and z be the length, width, and height, respectively, of a box. Suppose the box is increasing in size so that when $x = 3$ centimeters, $y = 2$ centimeters, and $z = 5$ centimeters, the length is increasing at rate of 2 centimeters per second, the width at a rate of 4 centimeters per second, and the height at a rate of 3 centimeters per second.
   (a) Find the rate of change of the volume of the box at this time.
   (b) Find the rate of change of the length of the diagonal of the box at this time.

7. Suppose $w = -\log(x^2 + y^2 + z^2)$ and $(x, y, z) = (4t, \sin(t), \cos(t)$. Find
$$\left.\frac{dw}{dt}\right|_{t=\frac{\pi}{3}}.$$

8. The kinetic energy K of an object of mass m moving in a straight line with velocity v is
$$K = \frac{1}{2}mv^2.$$
If, at time $t = t_0$, $m = 2000$ kilograms, $v = 50$ meters per second, m is decreasing at a rate of 2 kilograms per second, and v is increasing at a rate of 1.5 meters per second per second, find
$$\left.\frac{dK}{dt}\right|_{t=t_0}.$$

9. Each of the following equations specifies some curve in $\mathbb{R}^2$. In each case, find an equation for the line tangent to the curve at the given point $\mathbf{a}$.
   (a) $x^2 + y^2 = 5$, $\mathbf{a} = (2, 1)$
   (b) $2x^2 + 4y^2 = 18$, $\mathbf{a} = (1, -2)$
   (c) $y^2 - x = 0$, $\mathbf{a} = (4, -2)$
   (d) $y^2 - x^2 = 5$, $\mathbf{a} = (-2, 3)$

10. Each of the following equations specifies some surface in $\mathbb{R}^3$. In each case, find an equation for the plane tangent to the surface at the given point $\mathbf{a}$.
   (a) $x^2 + y^2 + z^2 = 14$, $\mathbf{a} = (2, 1, -3)$
   (b) $x^2 + 3y^2 + 2z^2 = 9$, $\mathbf{a} = (2, -1, 1)$
   (c) $x^2 + y^2 - z^2 = 1$, $\mathbf{a} = (1, 2, 2)$
   (d) $xyz = 6$, $\mathbf{a} = (1, 2, 3)$

11. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), $f(a, b) = c$, and $\frac{\partial}{\partial y}f(a, b) \neq 0$. Let C be the level curve of f with equation $f(x, y) = c$. Show that
$$y = -\frac{\frac{\partial}{\partial x}f(a, b)}{\frac{\partial}{\partial y}f(a, b)}(x - a) + b$$
is an equation for the line tangent to C at (a, b).

Section 3.4

Second-Order Approximations

In one-variable calculus, Taylor polynomials provide a natural way to extend best affine approximations to higher-order polynomial approximations. It is possible to generalize these ideas to scalar-valued functions of two or more variables, but the theory rapidly becomes involved and technical. In this section we will be content merely to point the way with a discussion of second-degree Taylor polynomials. Even at this level, it is best to leave full explanations for a course in advanced calculus.

Higher-order derivatives

The first step is to introduce higher order derivatives. If $f : \mathbb{R}^n \to \mathbb{R}$ has partial derivatives which exist on an open set U, then, for any $i = 1, 2, 3, \ldots, n$, $\frac{\partial f}{\partial x_i}$ is itself a function from $\mathbb{R}^n$ to $\mathbb{R}$. The partial derivatives of $\frac{\partial f}{\partial x_i}$, if they exist, are called *second-order partial derivatives* of f. We may denote the partial derivative of $\frac{\partial f}{\partial x_i}$ with respect to x_j, $j = 1, 2, 3, \ldots$, evaluated at a point $\mathbf{x}$, by either $\frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{x})$, or $f_{x_i x_j}(\mathbf{x})$, or $D_{x_i x_j} f(\mathbf{x})$. Note the order in which the variables are written; it is possible that differentiating first with respect to x_i and second with respect x_j will yield a different result than if the order were reversed. If $j = i$, we will write $\frac{\partial^2}{\partial x_i^2} f(\mathbf{x})$ for $\frac{\partial^2}{\partial x_i \partial x_i} f(\mathbf{x})$. It is, of course, possible to extend this notation to third, fourth, and higher-order derivatives.

Example Suppose $f(x, y) = x^2 y - 3x \sin(2y)$. Then

$$f_x(x, y) = 2xy - 3\sin(2y)$$

and

$$f_y(x, y) = x^2 - 6x\cos(2y),$$

so

$$f_{xx}(x, y) = 2y,$$

$$f_{xy}(x, y) = 2x - 6\cos(2y),$$

$$f_{yy}(x, y) = 12x\sin(2y),$$

and

$$f_{yx}(x, y) = 2x - 6\cos(2y).$$

Note that, in this example, $f_{xy}(x, y) = f_{yx}(x, y)$. For an example of a third-order derivative,

$$f_{yxy}(x, y) = 12\sin(2y).$$

Example Suppose $w = xy^2z^3 - 4xy\log(z)$. Then, for example,

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}(y^2z^3 - 4y\log(z)) = 2yz^3 - 4\log(z)$$

and

$$\frac{\partial^2 w}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial z}\left(3xy^2z^2 - \frac{4xy}{z}\right) = 6xy^2z + \frac{4xy}{z^2}.$$

Also,

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}(2xyz^3 - 4x\log(z)) = 2yz^3 - 4\log(z),$$

and so

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}.$$

In both of our examples we have seen instances where mixed second partial derivatives, that is, second-order partial derivatives with respect to two different variables, taken in different orders are equal. This is not always the case, but does follow if we assume that both of the mixed partial derivatives in question are continuous.

Definition We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open set U if $f_{x_j x_i}$ is continuous on U for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Theorem If f is C^2 on an open ball containing a point $\mathbf{c}$, then

$$\frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{c}) = \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{c})$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Although we have the tools to verify this result, we will leave the justification for a more advanced course.

We shall see that it is convenient to use a matrix to arrange the second partial derivatives of a function f. If $f : \mathbb{R}^n \to \mathbb{R}$, there are n^2 second partial derivatives and this matrix will be $n \times n$.

Definition Suppose the second-order partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ all exist at the point $\mathbf{c}$. We call the $n \times n$ matrix

$$Hf(\mathbf{c}) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(\mathbf{c}) & \frac{\partial^2}{\partial x_2 \partial x_1} f(\mathbf{c}) & \frac{\partial^2}{\partial x_3 \partial x_1} f(\mathbf{c}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f(\mathbf{c}) \\ \frac{\partial^2}{\partial x_1 \partial x_2} f(\mathbf{c}) & \frac{\partial^2}{\partial x_2^2} f(\mathbf{c}) & \frac{\partial^2}{\partial x_3 \partial x_2} f(\mathbf{c}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} f(\mathbf{c}) \\ \frac{\partial^2}{\partial x_1 \partial x_3} f(\mathbf{c}) & \frac{\partial^2}{\partial x_2 \partial x_3} f(\mathbf{c}) & \frac{\partial^2}{\partial x_3^2} f(\mathbf{c}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_3} f(\mathbf{c}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_n} f(\mathbf{c}) & \frac{\partial^2}{\partial x_2 \partial x_n} f(\mathbf{c}) & \frac{\partial^2}{\partial x_3 \partial x_n} f(\mathbf{c}) & \cdots & \frac{\partial^2}{\partial x_n^2} f(\mathbf{c}) \end{bmatrix} \tag{3.4.1}$$

the *Hessian* of f at $\mathbf{c}$.

Put another way, the Hessian of f at $\mathbf{c}$ is the $n \times n$ matrix whose ith row is $\nabla f_{x_i}(\mathbf{c})$.

Example Suppose $f(x,y) = x^2y - 3x\sin(2y)$. Then, using our results from above,

$$Hf(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} 2y & 2x - 6\cos(y) \\ 2x - 6\cos(2y) & 12x\sin(2y) \end{bmatrix}.$$

Thus, for example,

$$Hf(2,0) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}.$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open ball $B^2(\mathbf{c}, r)$ and let $\mathbf{h} = (h_1, h_2)$ be a point with $\|\mathbf{h}\| < r$. If we define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = f(\mathbf{c} + t\mathbf{h})$, then $\varphi(0) = f(\mathbf{c})$ and $\varphi(1) = f(\mathbf{c}+\mathbf{h})$. From the one-variable calculus version of Taylor's theorem, we know that

$$\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2}\varphi''(s), \tag{3.4.2}$$

where s is a real number between 0 and 1. Using the chain rule, we have

$$\varphi'(t) = \nabla f(\mathbf{c} + t\mathbf{h}) \cdot \frac{d}{dt}(\mathbf{c} + t\mathbf{h}) = \nabla f(\mathbf{c} + t\mathbf{h}) \cdot \mathbf{h} = f_x(\mathbf{c} + t\mathbf{h})h_1 + f_y(\mathbf{c} + t\mathbf{h})h_2 \tag{3.4.3}$$

and

$$\begin{aligned} \varphi''(t) &= h_1 \nabla f_x(\mathbf{c} + t\mathbf{h}) \cdot \mathbf{h} + h_2 \nabla f_y(\mathbf{c} + t\mathbf{h}) \cdot \mathbf{h} \\ &= (h_1 \nabla f_x(\mathbf{c} + t\mathbf{h}) + h_2 \nabla f_y(\mathbf{c} + t\mathbf{h}) \cdot \mathbf{h} \\ &= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} f_{xx}(\mathbf{c} = t\mathbf{h}) & f_{xy}(\mathbf{c} + t\mathbf{h}) \\ f_{yx}(\mathbf{c} + t\mathbf{h}) & f_{yy}(\mathbf{c} + t\mathbf{h}) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \mathbf{h}^T Hf(\mathbf{c} + t\mathbf{h})\mathbf{h}, \end{aligned} \tag{3.4.4}$$

where we have used the notation

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

and

$$\mathbf{h}^T = \begin{bmatrix} h_1 & h_2 \end{bmatrix},$$

the latter being called the *transpose* of $\mathbf{h}$ (see Problem 12 of Section 1.6). Hence

$$\varphi'(0) = \nabla f(\mathbf{c}) \cdot \mathbf{h} \tag{3.4.5}$$

and

$$\varphi''(s) = \frac{1}{2}\mathbf{h}^T Hf(c + s\mathbf{h})\mathbf{h}, \tag{3.4.6}$$

so, substituting into (3.4.2), we have

$$f(\mathbf{c} + \mathbf{h}) = \varphi(1) = f(\mathbf{c}) + \nabla f(\mathbf{c}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^T Hf(\mathbf{c} + s\mathbf{h})\mathbf{h}. \tag{3.4.7}$$

This result, a version of Taylor's theorem, is easily generalized to higher dimensions.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open ball $B^n(\mathbf{c}, r)$ and let $\mathbf{h}$ be a point with $\|\mathbf{h}\| < r$. Then there exists a real number s between 0 and 1 such that

$$f(\mathbf{c}+\mathbf{h}) = f(\mathbf{c}) + \nabla f(\mathbf{c}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^T Hf(\mathbf{c}+s\mathbf{h})\mathbf{h}. \tag{3.4.8}$$

If we let $\mathbf{x} = \mathbf{c} + \mathbf{h}$ and evaluate the Hessian at $\mathbf{c}$, (3.4.8) becomes a polynomial approximation for f.

Definition If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open ball about the point $\mathbf{c}$, then we call

$$P_2(\mathbf{x}) = f(\mathbf{c}) + \nabla f(\mathbf{c}) \cdot (\mathbf{x}-\mathbf{c}) + \frac{1}{2}(\mathbf{x}-\mathbf{c})^T Hf(\mathbf{c})(\mathbf{x}-\mathbf{c}) \tag{3.4.9}$$

the *second-order Taylor polynomial* for f at $\mathbf{c}$.

Example To find the second-order Taylor polynomial for $f(x,y) = e^{-2x+y}$ at $(0,0)$, we compute

$$\nabla f(x,y) = (-2e^{-2x+y}, e^{-2x+y})$$

and

$$Hf(x,y) = \begin{bmatrix} 4e^{-2x+y} & -2e^{-2x+y} \\ -2e^{-2x+y} & e^{-2x+y} \end{bmatrix},$$

from which it follows that

$$\nabla f(0,0) = (-2,1)$$

and

$$Hf(0,0) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} P_2(x,y) &= f(0,0) + \nabla f(0,0) \cdot (x,y) + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} Hf(0,0) \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 1 + (-2,1)\cdot(x,y) + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= 1 - 2x + y = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 4x-2y \\ -2x+y \end{bmatrix} \\ &= 1 - 2x + y + \frac{1}{2}(4x^2 - 2xy - 2xy + y^2) \\ &= 1 - 2x + y + 2x^2 - 2xy + \frac{1}{2}y^2. \end{aligned}$$

Symmetric matrices

Note that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2 on an open ball about the point $\mathbf{c}$, then the entry in the ith row and jth column of $Hf(\mathbf{c})$ is equal to the entry in the jth row and ith column of $Hf(\mathbf{c})$ since

$$\frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{c}) = \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{c}).$$

Definition We call a matrix $M = [a_{ij}]$ with the property that $a_{ij} = a_{ji}$ for all $i \neq j$ a *symmetric matrix.*

Example The matrices

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & -7 \end{bmatrix}$$

are both symmetric, while the matrices

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ -2 & 4 & -6 \end{bmatrix}$$

are not symmetric.

Example The Hessian of any C^2 scalar valued function is a symmetric matrix. For example, the Hessian of $f(x, y) = e^{-2x+y}$, namely,

$$Hf(x,y) = \begin{bmatrix} 4e^{-2x+y} & -2e^{-2x+y} \\ -2e^{-2x+y} & e^{-2x+y} \end{bmatrix},$$

is symmetric for any value of (x, y).

Given an $n \times n$ symmetric matrix M, the function $q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$q(\mathbf{x}) = \mathbf{x}^T M \mathbf{x}$$

is a quadratic polynomial. When M is the Hessian of some function f, this is the form of the quadratic term in the second-order Taylor polynomial for f. In the next section it will be important to be able to determine when this term is positive for all $\mathbf{x} \neq \mathbf{0}$ or negative for all $\mathbf{x} \neq \mathbf{0}$.

Definition Let M be an $n \times n$ symmetric matrix and define $q : \mathbb{R}^n \to \mathbb{R}$ by

$$q(\mathbf{x}) = \mathbf{x}^T M \mathbf{x}.$$

We say M is *positive definite* if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, *negative definite* if $q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, and *indefinite* if there exists an $\mathbf{x} \neq 0$ for which $q(\mathbf{x}) > 0$ and an $\mathbf{x} \neq \mathbf{0}$ for which $q(\mathbf{x}) < 0$. Otherwise, we say M is *nondefinite.*

In general it is not easy to determine to which of these categories a given symmetric matrix belongs. However, the important special case of 2×2 matrices is straightforward. Consider

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and let

$$q(x,y) = [\,x \quad y\,]\, M \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2. \tag{3.4.10}$$

If $a \neq 0$, then we may complete the square in (3.4.10) to obtain

$$\begin{aligned} q(x,y) &= a\left(x^2 + \frac{2b}{a}xy\right) + cy^2 \\ &= a\left(\left(x + \frac{b}{a}y\right)^2 - \frac{b^2}{a^2}y^2\right) + cy^2 \\ &= a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2 \\ &= a\left(x + \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a}y^2 \\ &= a\left(x + \frac{b}{a}y\right)^2 + \frac{\det(M)}{a}y^2 \qquad (3.4.11). \end{aligned}$$

Now suppose $\det(M) > 0$. Then from (3.4.11) we see that $q(x,y) > 0$ for all $(x,y) \neq (0,0)$ if $a > 0$ and $q(x,y) < 0$ for all $(x,y) \neq (0,0)$ if $a < 0$. That is, M is positive definite if $a > 0$ and negative definite if $a < 0$. If $\det(M) < 0$, then $q(1,0)$ and $q\left(-\frac{b}{a}, 1\right)$ will have opposite signs, and so M is indefinite. Finally, suppose $\det(M) = 0$. Then

$$q(x,y) = a\left(x + \frac{b}{a}y\right)^2,$$

so $q(x,y) = 0$ when $x = -\frac{b}{a}y$. Moreover, $q(x,y)$ has the same sign as a for all other values of (x,y). Hence in this case M is nondefinite.

Similar analyses for the case $a = 0$ give us the following result.

Theorem Suppose

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

If $\det(M) > 0$, then M is positive definite if $a > 0$ and negative definite if $a < 0$. If $\det(M) < 0$, then M is indefinite. If $\det(M) = 0$, then M is nondefinite.

Example The matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

is positive definite since $\det(M) = 5 > 0$ and $2 > 0$.

Example The matrix

$$M = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$$

is negative definite since $\det(M) = 7 > 0$ and $-2 < 0$.

Example The matrix

$$M = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$$

is indefinite since $\det(M) = -7 < 0$.

Example The matrix

$$M = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

is nondefinite since $\det(M) = 0$.

In the next section we will see how these ideas help us identify local extreme values for scalar valued functions of two variables.

Problems

1. Let $f(x, y) = x^3y^2 - 4x^2e^{-3y}$. Find the following.
   (a) $\dfrac{\partial^2}{\partial x \partial y} f(x, y)$
   (b) $\dfrac{\partial^2}{\partial y \partial x} f(x, y)$
   (c) $\dfrac{\partial^2}{\partial x^2} f(x, y)$
   (d) $\dfrac{\partial^3}{\partial x \partial y \partial x} f(x, y)$
   (e) $\dfrac{\partial^3}{\partial x \partial y^2} f(x, y)$
   (f) $\dfrac{\partial^3}{\partial y^3} f(x, y)$
   (g) $f_{yy}(x, y)$
   (h) $f_{yxy}(x, y)$
2. Let $f(x, y, z) = \dfrac{xy}{x^2 + y^2 + z^2}$. Find the following.
   (a) $\dfrac{\partial^2}{\partial z \partial x} f(x, y, z)$
   (b) $\dfrac{\partial^2}{\partial y \partial z} f(x, y, z)$
   (c) $\dfrac{\partial^2}{\partial z^2} f(x, y, z)$
   (d) $\dfrac{\partial^3}{\partial x \partial y \partial z} f(x, y, z)$
   (e) $f_{zyx}(x, y, z)$
   (f) $f_{yyy}(x, y, z)$
3. Find the Hessian of each of the following functions.
   (a) $f(x, y) = 3x^2y - 4xy^3$
   (b) $g(x, y) = 4e^{-x} \cos(3y)$
   (c) $g(x, y, z) = 4xy^2z^3$
   (d) $f(x, y, z) = -\log(x^2 + y^2 + z^2)$
4. Find the second-order Taylor polynomial for each of the following at the point $\mathbf{c}$.
   (a) $f(x, y) = xe^{-y}$, $\mathbf{c} = (0, 0)$
   (b) $g(x, y) = x \sin(x + y)$, $\mathbf{c} = (0, 0)$

(c) $f(x,y) = \dfrac{1}{x+y}$, $\mathbf{c} = (1,1)$

(d) $g(x,y,z) = e^{x-2y+3z}$, $\mathbf{c} = (0,0,0)$

5. Classify each of the following symmetric 2×2 matrices as either positive definite, negative definite, indefinite, or nondefinite.

(a) $\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$

6. Let M be an $n \times n$ symmetric nondefinite matrix and define $q : \mathbb{R}^n \to \mathbb{R}$ by

$$q(\mathbf{x}) = \mathbf{x}^T M \mathbf{x}.$$

Explain why (1) there exists a vector $\mathbf{a} \neq \mathbf{0}$ such that $q(\mathbf{a}) = 0$ and (2) either $q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$ or $q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

7. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open ball $B^n(\mathbf{c}, r)$, $\nabla f(\mathbf{c}) = \mathbf{0}$, and $Hf(\mathbf{x})$ is positive definite for all $\mathbf{x}$ in $B^n(\mathbf{c}, r)$. Show that $f(\mathbf{c}) < f(\mathbf{x})$ for all $\mathbf{x}$ in $B^n(\mathbf{c}, r)$. What would happen if $Hf(\mathbf{x})$ were negative definite for all $\mathbf{x}$ in $B^n(\mathbf{c}, r)$? What does this say in the case $n = 1$?

8. Let

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that $f_x(0,y) = -y$ for all y.

(b) Show that $f_y(x,0) = x$ for all x.

(c) Show that $f_{yx}(0,0) \neq f_{xy}(0,0)$.

(d) Is f C^2?

The Calculus of Functions

of

Several Variables

Section 3.5

Extreme Values

After a few preliminary results and definitions, we will apply our work from the previous sections to the problem of finding maximum and minimum values of scalar-valued functions of several variables. The story here parallels to a great extent the story from one-variable calculus, with the inevitable twists and turns due to the presence of additional variables. We will begin with a definition very similar to the analogous definition for functions of a single variable.

The Extreme Value Theorem

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on a set S. We say f has a *maximum value* of M at $\mathbf{c}$ if $f(\mathbf{c}) = M$ and $M \geq f(\mathbf{x})$ for all $\mathbf{x}$ in S. We say f has a *minimum value* of m at $\mathbf{c}$ if $f(\mathbf{c}) = m$ and $m \leq f(\mathbf{x})$ for all $\mathbf{x}$ in S.

The maximum and minimum values of the previous definition are sometimes referred to as *global* maximum and minimum values in order to distinguish them from the *local* maximum and minimum values of the next definition.

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined on a open set U. We say f has a *local maximum value* of M at $\mathbf{c}$ if $f(\mathbf{c}) = M$ and $M \geq f(\mathbf{x})$ for all $\mathbf{x}$ in $B^n(\mathbf{c}, r)$ for some $r > 0$. We say f has a *local minimum value* of m at $\mathbf{c}$ if $f(\mathbf{c}) = m$ and $m \leq f(\mathbf{x})$ for all $\mathbf{x}$ in $B^n(\mathbf{c}, r)$ for some $r > 0$.

We will say *extreme value*, or *global extreme value*, when referring to a value of f which is either a global maximum or a global minimum value, and *local extreme value* when referring to a value which is either a local maximum or a local minimum value.

In one-variable calculus, the Extreme Value Theorem, the statement that every continuous function on a finite closed interval has a maximum and a minimum value, was extremely useful in searching for extreme values. There is a similar result for our current situation, but first we need the following definition.

Definition We say a set S in $\mathbb{R}^n$ is *bounded* if there exists an $r > 0$ such that S is contained in the open ball $B^n(\mathbf{0}, r)$.

Equivalently, a set S is bounded as long as there is a fixed distance r such that no point in S is farther away from the origin than r.

Example Any open or closed ball in $\mathbb{R}^n$ is a bounded set.

Example The infinite rectangle

$$\{(x, y) : 1 < x < 3, -\infty < y < \infty\}$$

is not bounded.

Extreme Value Theorem Suppose $f:\mathbb{R}^n \to \mathbb{R}$ is continuous on an open set U. If S is a closed and bounded subset of U, than f has a maximum value and a minimum value on S.

We leave the justification of this theorem for a more advanced course.

Our work now is to find criteria for locating candidates for points where local extreme values might occur, and then to classify these points once we have found them. To begin, suppose we know $f:\mathbb{R}^n \to \mathbb{R}$ is differentiable on an open set U and that it has a local extreme value at $\mathbf{c}$. Then for any unit vector $\mathbf{u}$, the function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(t) = f(\mathbf{c}+t\mathbf{u})$ must have an extreme value at $t=0$. Hence, from a result in one-variable calculus, we must have

$$0 = g'(0) = D_{\mathbf{u}}f(\mathbf{c}) = \nabla f(\mathbf{c})\cdot\mathbf{u}.$$

Since $\mathbf{u}$ was an arbitrary unit vector in $\mathbb{R}^n$, we have, in particular,

$$0 = \nabla f(\mathbf{c})\cdot\mathbf{e}_k = \frac{\partial}{\partial x_i} f(\mathbf{c})$$

for $i = 1, 2, \cdots, n$. That is, we must have $\nabla f(\mathbf{c}) = \mathbf{0}$. Note that, by itself, $\nabla f(\mathbf{c}) = \mathbf{0}$ only says that the slope of the graph of f is 0 in the direction of the standard basis vectors, but this in fact implies that the slope is 0 in all directions because $D_{\mathbf{u}}f(\mathbf{c}) = \nabla f(\mathbf{c})\cdot\mathbf{u}$ for any unit vector $\mathbf{u}$.

Theorem If $f:\mathbb{R}^n \to \mathbb{R}$ is differentiable on an open set U and has a local extreme value at $\mathbf{c}$, then $\nabla f(\mathbf{c}) = \mathbf{0}$.

Definition If $f:\mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{c}$ and $\nabla f(\mathbf{c}) = \mathbf{0}$, then we call $\mathbf{c}$ a *critical point* of f. We call a point $\mathbf{c}$ at which f is not differentiable a *singular point* of f.

Recall that to find the extreme values of a continuous function $f:\mathbb{R}\to\mathbb{R}$ on a closed interval, we need only to evaluate f at all critical and singular points inside the interval as well as at the endpoints of the interval, and then inspect these values to identify the largest and smallest. The story is similar in the situation of a function $f:\mathbb{R}^n\to\mathbb{R}$ which is defined on a closed and bounded set S and is continuous on some open set containing S, except instead of having endpoints to consider, we have the entire *boundary* of S to consider.

Definition Let S be a set in $\mathbb{R}^n$. We call a point $\mathbf{a}$ in $\mathbb{R}^n$ a *boundary point* of S if for every $r > 0$, the open ball $B^n(\mathbf{a}, r)$ contains both points in S and points outside of S. We call the set of all boundary points of S the *boundary* of S.

Example The boundary of the closed set

$$\bar{B}^2((0,0),3) = \{(x,y) : x^2 + y^2 \le 9\}$$

is the circle

$$S^1((0,0),3) = \{(x,y) : x^2 + y^2 = 9\}.$$

Example In general, the boundary of the closed ball $\bar{B}^n(\mathbf{a}, r)$ is the sphere $S^{n-1}(\mathbf{a}, r)$.

Example The boundary of the closed rectangle

$$R = \{(x,y) : 1 \le x \le 3, 2 \le y \le 5\}$$

consists of the line segments from $(1,2)$ to $(3,2)$, $(3,2)$ to $(3,5)$, $(3,5)$ to $(1,5)$, and $(1,5)$ to $(1,2)$.

Example Suppose we wish to find the global extreme values for the function $f(x,y) = x^2 + y^2$ on the closed set

$$D = \{(x,y) : x^2 + 4y^2 \le 4\}.$$

We first find all the critical and singular points. Now

$$\nabla f(x,y) = (2x, 2y),$$

so

$$\nabla f(x,y) = (0,0)$$

if and only if

$$\begin{aligned} 2x &= 0, \\ 2y &= 0. \end{aligned}$$

Hence the only critical point is $(0,0)$. There are no singular points, but we must consider the boundary of S, the ellipse

$$B = \{(x,y) : x^2 + 4y^2 = 4\}.$$

Now we may use

$$\varphi(t) = (2\cos(t), \sin(t)),$$

$0 \le t \le 2\pi$, to parametrize B. It follows that any extreme value of f occurring on B will also be an extreme value of

$$\begin{aligned} g(t) &= f(\varphi(t)) \\ &= f(2\cos(t), \sin(t)) \\ &= 4\cos^2(t) + \sin^2(t) \\ &= 4\cos^2(t) + (1 - \cos^2(t)) \\ &= 3\cos^2(t) + 1 \end{aligned}$$

on the closed interval $[0, 2\pi]$. Now

$$g'(t) = -6\cos(t)\sin(t),$$

so the critical points of g occur at points t in $(0, 2\pi)$ where either $\cos(t) = 0$ or $\sin(t) = 0$. Hence the critical points of g are $t = \frac{\pi}{2}$, $t = \pi$, and $t = \frac{3\pi}{2}$. Moreover, we need to consider the endpoints $t = 0$ and $t = 2\pi$. Hence we have four more candidates for the location

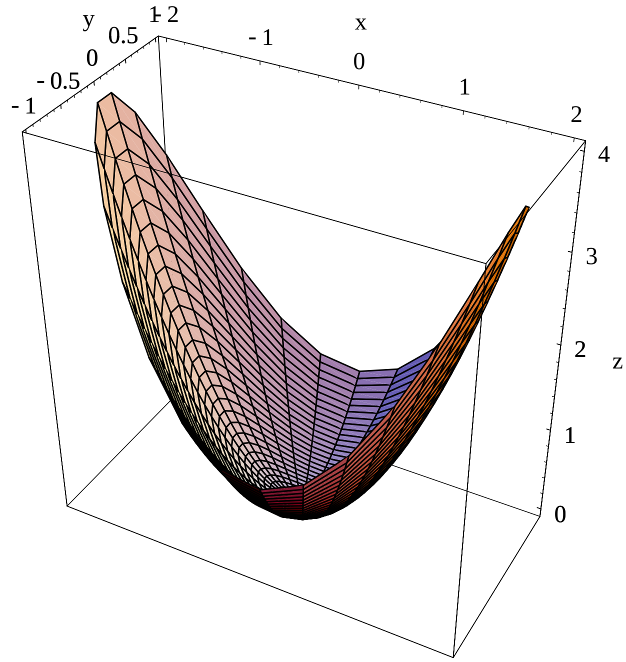


Figure 3.5.1 Graph of $f(x,y) = x^2 + y^2$ on $D = \{(x,y) : x^2 + 4y^2 \leq 4\}$

of extreme values, namely, $\varphi(0) = \varphi(2\pi) = (2,0)$, $\varphi\left(\frac{\pi}{2}\right) = (0,1)$, $\varphi(\pi) = (-2,0)$, and $\varphi\left(\frac{3\pi}{2}\right) = (0,-1)$. Evaluating f at these five points, we have

$$f(0,0) = 0,$$

$$f(2,0) = 4,$$

$$f(0,1) = 1,$$

$$f(-2,0) = 4,$$

and

$$f(0,-1) = 1.$$

Comparing these values, we see that f has a maximum value of 4 at $(2,0)$ and $(-2,0)$ and a minimum value of 0 at $(0,0)$. See Figure 3.5.1 for the graph of f on the set D.

As the previous example shows, dealing with the boundary of a region can require a significant amount of work. In this example we were helped by the fact that the boundary was one-dimensional and was easily parametrized. This is not always the case. For example, the boundary of the closed ball $\bar{B}^3((0,0,0),1)$ in $\mathbb{R}^3$ is the sphere $S^2((0,0,0),1)$ with equation

$$x^2 + y^2 + z^2 = 1,$$

a two-dimensional surface. We shall see in Chapter 4 that it is possible to parametrize such surfaces, but that would still leave us with a two-dimensional problem. We will return to this problem later in this section when we present a much more elegant solution based on our knowledge of level sets and gradient vectors.

Finding local extrema

For now we will turn our attention to identifying local extreme values. Recall from one-variable calculus that one of the most useful ways to identify a local extreme value is through the second derivative test. That is, if c is a critical point of $\varphi : \mathbb{R} \to \mathbb{R}$, then $\varphi''(c) > 0$ implies that φ has a local minimum at c and $\varphi''(c) < 0$ implies φ has a local maximum at c. Taylor's theorem provides an easy way to see why this is so. For example, suppose c is a critical point of φ, φ'' is continuous on an open interval containing c, and $\varphi''(c) > 0$. Then there is an interval $I = (c - r, c + r)$, $r > 0$, such that φ'' is continuous on I and $\varphi''(t) > 0$ for all t in I. By Taylor's theorem, for any h with $|h| < r$, there is a number s between c and $c + h$ such that

$$\varphi(c+h) = \varphi(c) + \varphi'(c)h + \frac{1}{2}\varphi''(s)h^2 = \varphi(c) + \frac{1}{2}\varphi''(c)h^2 > \varphi(c), \tag{3.5.1}$$

where we have used the fact that $\varphi'(c) = 0$ since c is a critical point of φ. Hence $\varphi(c)$ is a local minimum value of φ.

Similar considerations lead to a second derivative test for a function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose $\mathbf{c}$ is a critical point of f, f is C^2 on an open set containing $\mathbf{c}$, and $Hf(\mathbf{c})$ is positive definite. Let $B^n(\mathbf{c}, r)$, $r > 0$, be an open ball on which f is C^2 and $Hf(\mathbf{c})$ is positive definite. Then, by the version of Taylor's theorem in Section 3.4, for any $\mathbf{h}$ with $\|\mathbf{h}\| < r$, there is a number s between 0 and 1 such that

$$f(\mathbf{c}+\mathbf{h}) = f(\mathbf{c}) + \nabla f(\mathbf{c}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^T Hf(\mathbf{c}+s\mathbf{h})\mathbf{h} = f(\mathbf{c}) + \frac{1}{2}\mathbf{h}^T Hf(\mathbf{c}+s\mathbf{h})\mathbf{h} > f(\mathbf{c}), \tag{3.5.2}$$

where $\nabla f(\mathbf{c}) = \mathbf{0}$ since $\mathbf{c}$ is a critical point of f, and the final inequality follows from the assumption that $Hf(\mathbf{x})$ is positive definite for x in $B^n(\mathbf{c}, r)$. Hence $f(\mathbf{c})$ is a local minimum value of f. The same argument shows that if $Hf(\mathbf{c})$ is negative definite, then $f(\mathbf{c})$ is a local maximum value of f. If $Hf(\mathbf{c})$ is indefinite, then there will be arbitrarily small $\mathbf{h}$ for which

$$\frac{1}{2}\mathbf{h}^T Hf(\mathbf{c}+s\mathbf{h})\mathbf{h} > 0$$

and arbitrarily small $\mathbf{h}$ for which

$$\frac{1}{2}\mathbf{h}^T Hf(\mathbf{c}+s\mathbf{h})\mathbf{h} < 0.$$

Hence there will be arbitrarily small $\mathbf{h}$ for which $f(\mathbf{c} + \mathbf{h}) > f(\mathbf{c})$ and arbitrarily small $\mathbf{h}$ for which $f(\mathbf{c} + \mathbf{h}) < f(\mathbf{c})$. In this case, $f(\mathbf{c})$ is neither a local minimum nor a local maximum. In this case, we call $\mathbf{c}$ a *saddle point*. Finally, if $Hf(\mathbf{c})$ is nondefinite, then we do not have enough information to classify the critical point. We may now state the second derivative test.

Second derivative test Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open set U. If $\mathbf{c}$ is a critical point of f in U, then $f(\mathbf{c})$is a local minimum value of f if $Hf(\mathbf{c})$ is positive definite, $f(\mathbf{c})$ is a local maximum value of f if $Hf(\mathbf{c})$ is negative definite, and $\mathbf{c}$ is a saddle point if $Hf(\mathbf{c})$ is indefinite. If $Hf(\mathbf{c})$ is nondefinite, then more information is needed in order to classify $\mathbf{c}$.

The next example gives an indication for the source of the term saddle point.

Example To find the local extreme values of $f(x, y) = x^2 - y^2$, we begin by finding

$$\nabla f(x, y) = (2x, -2y).$$

Now

$$\nabla f(x, y) = (0, 0)$$

if and only if

$$\begin{aligned} 2x &= 0, \\ -2y &= 0, \end{aligned}$$

which occurs if and only if $x = 0$ and $y = 0$. Thus f has the single critical point $(0, 0)$. Now

$$Hf(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

so

$$Hf(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Thus

$$\det(Hf(0, 0)) = (2)(-2) = -4 < 0.$$

Hence $Hf(0, 0)$ is indefinite and so, by the second derivative test, $(0, 0)$ is a saddle point. Looking at the graph of f in Figure 3.5.2, we can see the reason for this: since $f(x, 0) = x^2$ and $f(0, y) = -y^2$, the slice of the graph of f above the x-axis is a parabola opening upward while the slice of the graph of f above the y-axis is a parabola opening downward.

Example Consider $f(x, y) = xye^{-(x^2+y^2)}$. Then

$$\nabla f(x, y) = e^{-(x^2+y^2)}(y - 2x^2y, x - 2xy^2).$$

Hence, since $e^{-(x^2+y^2)} > 0$ for all (x, y),

$$\nabla f(x, y) = (0, 0)$$

if and only if

$$\begin{aligned} y - 2x^2y &= 0, \\ x - 2xy^2 &= 0, \end{aligned}$$

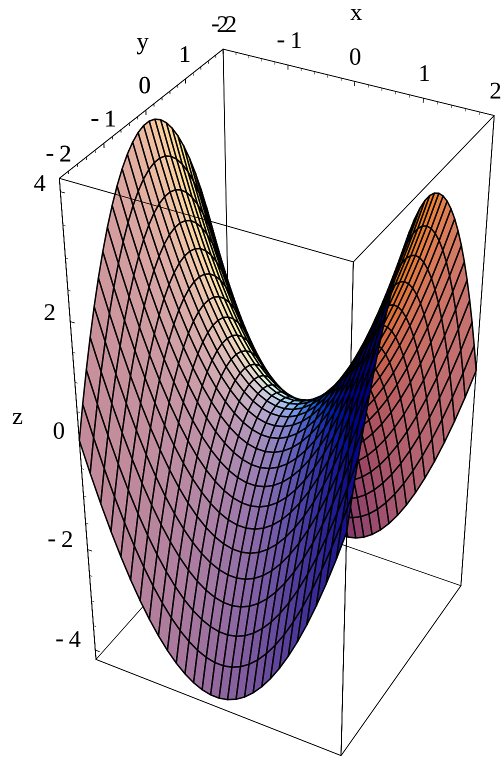


Figure 3.5.2 Graph of $f(x,y) = x^2 - y^2$

which occurs if and only if

$$\begin{aligned} y(1-2x^2) &= 0, \\ x(1-2y^2) &= 0. \end{aligned}$$

Now the first equation is satisfied if either $y = 0$ or $1 - 2x^2 = 0$. If $y = 0$, then the second equation becomes $x = 0$, so $(0,0)$ is a critical point. If $1 - 2x^2 = 0$, then either $x = -\frac{1}{\sqrt{2}}$ or $x = \frac{1}{\sqrt{2}}$. For either of these values of x, the second equation is satisfied if and only if $1 - 2y^2 = 0$, that is, $y = -\frac{1}{\sqrt{2}}$ or $y = \frac{1}{\sqrt{2}}$. Hence we have four more critical points: $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Now

$$Hf(x,y) = e^{-(x^2+y^2)} \begin{bmatrix} 4x^3y - 6xy & 4x^2y^2 - 2x^2 - 2y^2 + 1 \\ 4x^2y^2 - 2x^2 - 2y^2 + 1 & 4y^3x - 6xy \end{bmatrix},$$

so

$$Hf(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$Hf\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = Hf\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = e^{-1} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

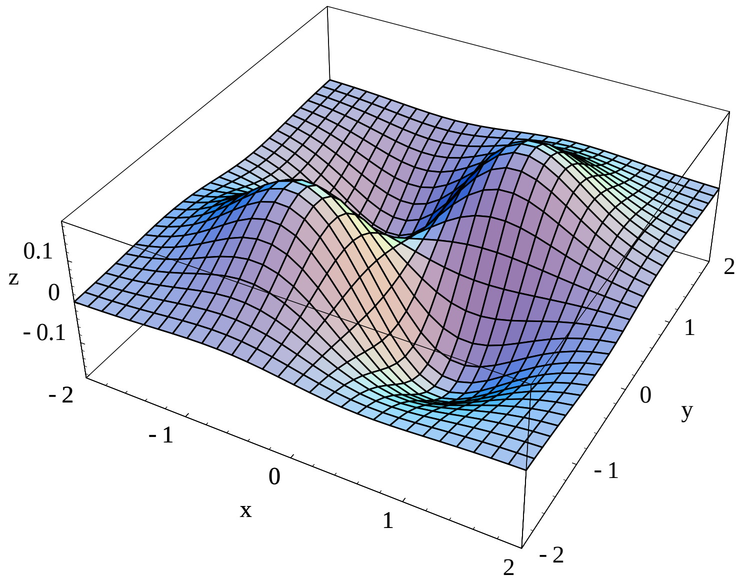


Figure 3.5.3 Graph of $f(x,y) = xye^{-(x^2+y^2)}$

and

$$Hf\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = Hf\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = e^{-1}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since

$$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0,$$

$$\det\begin{bmatrix} -2e^{-1} & 0 \\ 0 & -2e^{-1} \end{bmatrix} = 4e^{-2} > 0,$$

and

$$\det\begin{bmatrix} 2e^{-1} & 0 \\ 0 & 2e^{-1} \end{bmatrix} = 4e^{-2} > 0,$$

we see that $Hf(0,0)$ is indefinite, $Hf\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $Hf\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are negative definite, and $Hf\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $Hf\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are positive definite. Thus $(0,0)$ is a saddle point of f, f has local maximums of $\frac{1}{2}e^{-1}$ at both $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and local minimums of $-\frac{1}{2}e^{-1}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.See Figure 3.5.3.

Finding global extrema

The graph of $f(x,y) = xye^{-(x^2+y^2)}$ in Figure 3.5.3 suggests that local extreme values found in the previous example are in fact global extreme values for f on all of $\mathbb{R}^2$. We may verify that this in fact the case as follows. First note that, since

$$\lim_{r\to\infty} r^2 e^{-r^2} = 0,$$

we may choose R large enough so that

$$r^2 e^{-r^2} < \frac{1}{2}e^{-1}$$

whenever $r \geq R$. Now for any point (x, y) with $\|(x, y)\| = r \geq R$ we have

$$|f(x, y)| = |xye^{-(x^2+y^2)}| = |x||y|e^{-(x^2+y^2)} \leq r^2 e^{-r^2} < \frac{1}{2}e^{-1}.$$

Hence $f(x, y)$ is between $-\frac{1}{2}e^{-1}$ and $\frac{1}{2}e^{-1}$ for all points (x, y) outside of the closed disk $D = \bar{B}^2((0, 0), R)$. Moreover, since $f(x, y)$ is between $-\frac{1}{2}e^{-1}$ and $\frac{1}{2}e^{-1}$ for all points (x, y) on the boundary of D, f has a minimum value of $-\frac{1}{2}e^{-1}$ and a maximum value of $\frac{1}{2}e^{-1}$ on D. Hence these values are actually the global extreme values of f on all of $\mathbb{R}^2$.

Example A farmer wishes to build a rectangular storage bin, without a top, with a volume of 500 cubic meters using the least amount of material possible. If we let x and y be the dimensions of the base of the bin and z be the height, all measured in meters, then the farmer wishes to minimize the surface area of the bin, given by

$$S = xy + 2xz + 2yz, \tag{3.5.3}$$

subject to the constraint on the volume, namely,

$$500 = xyz.$$

Solving for z in the latter expression and substituting in to (3.5.3), we have

$$S = xy + 2x\left(\frac{500}{xy}\right) + 2y\left(\frac{500}{xy}\right) = xy + \frac{1000}{y} + \frac{1000}{x}.$$

This is the function we need to minimize on the infinite open rectangle

$$R = \{(x, y) : x > 0, y > 0\}.$$

Now

$$\frac{\partial S}{\partial x} = y - \frac{1000}{x^2}$$

and

$$\frac{\partial S}{\partial y} = x - \frac{1000}{y^2}$$

so to find the critical points of S we need to solve

$$y - \frac{1000}{x^2} = 0,$$
$$x - \frac{1000}{y^2} = 0.$$

Solving for y in the first of these, we have

$$y = \frac{1000}{x^2},$$

which, when substituted into the second, gives us

$$x - \frac{x^4}{1000} = 0.$$

Hence we want

$$x\left(1 - \frac{x^3}{1000}\right) = 0,$$

from which it follows that either $x = 0$ or $x = 10$. Since the first of these will not give us a point in R, we have $x = 10$ and

$$y = \frac{1000}{10^2} = 10.$$

Thus the only critical point is $(10, 10)$. Now

$$HS(x,y) = \begin{bmatrix} \dfrac{2000}{x^3} & 1 \\ 1 & \dfrac{2000}{y^3} \end{bmatrix},$$

so

$$HS(10,10) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Thus

$$\det(HS(10,10)) = 3,$$

and so $HS(10,10)$ is positive definite. This shows that S has a local minimum of

$$S\Big|_{x=10,y=10} = (10)(10) + \frac{1000}{10} + \frac{1000}{10} = 300$$

at $(x,y) = (10,10)$. To show that this is actually the global minimum value of S, we proceed as follows. Let D be the closed rectangle

$$D = \{(x,y) : 1 \le x \le 400, 1 \le y \le 400\}.$$

Now if $0 < x \le 1$, then

$$\frac{1000}{x} \ge 1000,$$

and so $S > 300$. Similarly, if $0 < y \le 1$, then $S > 300$. Moreover, if $x \ge 400$ and $y \ge 1$, then $xy \ge 400$, and so $S > 300$. Similarly, if $y \ge 400$ and $x \ge 1$, then $S > 300$. Hence $S > 300$ for all (x,y) outside of D and for all (x,y) on the boundary of D. Hence S has a

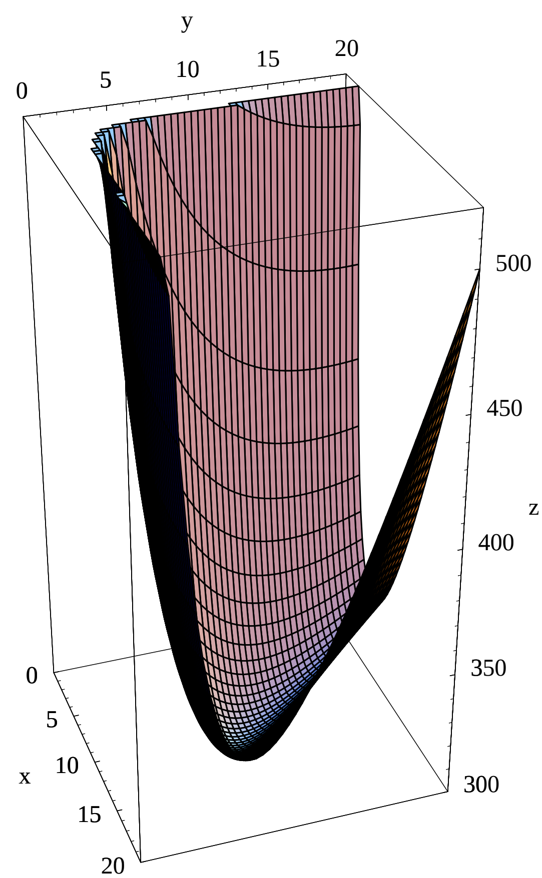


Figure 3.5.4 Graph of $S = xy + \dfrac{1000}{x} + \dfrac{1000}{y}$

global minimum of 300 on D, which, from the preceding observations, must in fact be the global minimum of S on all of R. See the graph of S in Figure 3.5.4. Finally, when $x = 10$ and $y = 10$, we have

$$z = \frac{500}{(10)(10)} = 5,$$

so the farmer should build her bin to have a base of 10 meters by 10 meters and a height of 5 meters.

Lagrange multipliers

This last example has much in common with our first example in that they both involve finding extreme values of a function restricted to a lower-dimensional subset. In our first example, we had to find the extreme values of $f(x, y) = x^2 + y^2$ restricted to the one-dimensional ellipse with equation $x^2 + 4y^2 = 4$; in the example we just finished, we had to find the minimum value of $S = xy + 2xz + 2yz$, a function of three variables, restricted to the two-dimensional surface defined by the equation $xyz = 500$. Although they were similar, we approached these problems somewhat differently. In the first, we parametrized the ellipse and then maximized the composition of f with this parametrization; in the latter, we solved for z in terms of x and y and then substituted into the formula for S to make S effectively a function of two variables. Now we will describe a general approach which applies to both situations. Often, but not always, this method is easier to apply then the other two techniques. In practice, one tries to select the method that will yield an answer with the least resistance.

For the general case, consider two differentiable functions, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, and suppose we wish to find the extreme values of f on the level set S of g determined by the constraint $g(\mathbf{x}) = 0$. If f has an extreme value at a point $\mathbf{c}$ on S, then $f(\mathbf{c})$ must be an extreme value of f along any curve passing through $\mathbf{c}$. Thus if $\varphi : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve in S with $\varphi(b) = \mathbf{c}$, then the function $h(t) = f(\varphi(t))$ has an extreme value at b. Hence

$$0 = h'(b) = \nabla f(\varphi(b)) \cdot D\varphi(b) = \nabla f(\mathbf{c}) \cdot D\varphi(b). \tag{3.5.4}$$

Since (3.5.4) holds for any curve in S through $\mathbf{c}$ and $D\varphi(b)$ is tangent to the given curve at $\mathbf{c}$, it follows that $\nabla f(\mathbf{c})$ is orthogonal to the tangent hyperplane to S at $\mathbf{c}$. But S is a level set of g, so we know from our work in Section 3.3 that the vector $\nabla g(\mathbf{c})$, provided it is nonzero, is a normal vector for the tangent hyperplane to S at $\mathbf{c}$. Hence $\nabla f(\mathbf{c})$ and $\nabla g(\mathbf{c})$ must be parallel. That is, there must exist a scalar λ such that

$$\nabla f(\mathbf{c}) = \lambda \nabla f(\mathbf{c}). \tag{3.5.5}$$

The idea now is that in looking for extreme values, we need only consider points $\mathbf{c}$ for which both $g(\mathbf{c}) = 0$ and $\nabla f(\mathbf{c}) = \lambda \nabla g(\mathbf{c})$ for some scalar λ. The scalar λ is known as a *Lagrange multiplier*, and this method for finding extreme values subject to a constraining equation is known as the *method of Lagrange multipliers*.

Example Suppose that the temperature at a point (x, y, z) on the unit sphere $S = S^2((0,0,0),1)$ is given by

$$T(x, y, z) = 30 + 5(x + z).$$

To find the extreme values of T, we first define

$$g(x, y, z) = x^2 + y^2 + z^2 - 1,$$

thus making S the level surface of g specified by $g(x, y, z) = 0$. Now

$$\nabla f(x, y, z) = (5, 0, 5)$$

and

$$\nabla g(x, y, z) = (2x, 2y, 2z).$$

The candidates for the locations of extreme values will be solutions of the equations

$$\begin{aligned} \nabla f(x, y, z) &= \lambda g(x, y, z), \\ g(x, y, z) &= 0, \end{aligned}$$

that is,

$$(5, 0, 5) = \lambda(2x, 2y, 2z),$$
$$x^2 + y^2 + z^2 - 1 = 0.$$

Hence we need to solve the following system of four equation in four unknowns:

$$5 = 2\lambda x,$$
$$0 = 2\lambda y,$$
$$5 = 2\lambda z,$$
$$x^2 + y^2 + z^2 = 1.$$

Now $5 = 2\lambda x$ implies that $\lambda \neq 0$, and so $0 = 2\lambda y$ implies that $y = 0$. Moreover, $5 = 2\lambda x$ and $5 = 2\lambda z$ imply that $2\lambda x = 2\lambda z$, from which it follows, since $\lambda \neq 0$, that $x = z$. Substituting these results into the final equation, we have

$$1 = x^2 + y^2 + z^2 = x^2 + 0 + x^2 = 2x^2.$$

Thus $x = -\frac{1}{\sqrt{2}}$ or $x = \frac{1}{\sqrt{2}}$, and we have two solutions for our equations,

$$\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

and

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

At this point, since T is continuous and S is closed and bounded, we need only evaluate T at these points and compare their values. Now

$$T\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = 30 - 5\sqrt{2} = 22.93$$

and

$$T\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = 30 + 5\sqrt{2} = 37.07,$$

where the final values have been rounded to two decimal places, so the maximum temperature on the sphere is 37.07 at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and the minimum temperature is 22.93 at $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$.

Example Suppose the farmer in our earlier example is faced with the opposite problem: Given 300 square meters of material, what are the dimensions of the rectangular bin, without a top, that holds the largest volume? If we again let x and y be the dimensions of the base of the bin and z be its height, then we want to maximize

$$V = xyz$$

on the region where $x > 0$, $y > 0$, and $z > 0$, subject to the constraint that

$$xy + 2xz + 2yz = 300.$$

If we let

$$g(x, y, z) = xy + 2xz + 2yz - 300,$$

then our problem is to maximize V subject to the constraint $g(x, y, z) = 0$. Now

$$\nabla V = (yz, xz, xy)$$

and

$$\nabla g(x, y, z) = (y + 2z, x + 2z, 2x + 2y),$$

so the system of equations

$$\begin{aligned} \nabla V &= \lambda \nabla g(x, y, z), \\ g(x, y, z) &= 0, \end{aligned}$$

becomes the system

$$yz = \lambda(y + 2z), \tag{3.5.6}$$

$$xz = \lambda(x + 2z), \tag{3.5.7}$$

$$xy = \lambda(2x + 2y), \tag{3.5.8}$$

$$xy + 2xz + 2yz = 300. \tag{3.5.9}$$

Equations (3.5.6) and (3.5.7) imply that

$$\lambda = \frac{yx}{y + 2z}$$

and

$$\lambda = \frac{xz}{x + 2z,}$$

so

$$\frac{yz}{y + 2z} = \frac{xz}{x + 2z},$$

that is,

$$\frac{y}{y + 2z} = \frac{x}{x + 2z}.$$

Hence

$$xy + 2yz = xy + 2xz.$$

Thus $2yz = 2xz$, so $x = y$. Substituting this result into (3.5.8) gives us $x^2 = 4\lambda x$, from which it follows that $x = 4\lambda$. Substituting into (3.5.7), we have

$$4\lambda z = \lambda(4\lambda + 2z) = 4\lambda^2 + 2\lambda z.$$

Hence $2\lambda z = 4\lambda^2$, so $z = 2\lambda$. Putting $x = 4\lambda$, $y = 4\lambda$, and $z = 2\lambda$ into (3.5.9) yields the equation

$$16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 300.$$

Thus $48\lambda^2 = 300$, so

$$\lambda = \pm\sqrt{\frac{300}{48}} = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}.$$

Now x, y, and z are all positive, so we must have $\lambda = \frac{5}{2}$, giving us $x = 10$, $y = 10$, and $z = 5$. To show that we have the location of the maximum value of V, let

$$S = \{(x, y, z) : g(x, y, z) = 0, x > 0, y > 0, z > 0\}$$

and let D be that part of S for which $1 \leq x \leq 150$, $1 \leq y \leq 150$, and $1 \leq z \leq 150$. Note that if (x, y, z) lies on S, then

$$300 = xy + 2xz + 2yz$$

and so $xy \leq 300$, $xz \leq 150$, and $yz \leq 150$. Moreover,

$$z = \frac{300 - xy}{2x + 2y}.$$

Now if either $x \geq 150$ or $y \geq 150$, then

$$z \leq \frac{300}{300} \leq 1,$$

so

$$V = xyz \leq (300)(1) = 300.$$

If $x \leq 1$,

$$V = xyz \leq (1)(150) = 150$$

and, similarly, if $y \leq 1$,

$$V = yxz \leq (1)(150) = 150.$$

Thus if (x, y, z) is either on the boundary of D or outside of D, then $V \leq 300$. Since

$$V\Big|_{(x,y,z)=(10,10,5)} = 500,$$

it follows that the global maximum of V on S must occur inside D. In fact, this maximum value must be 500 cubic meters, occurring when $x = 10$ meters, $y = 10$ meters, and $z = 5$ meters.

Problems

1. Find the maximum and minimum values of $f(x, y) = xy$ on the set $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

2. Find the maximum and minimum values of $f(x,y) = 8 - x^2 - y^2$ on the set $D = \{(x,y) : x^2 + 9y^2 \leq 9\}$.
3. Find the maximum and minimum values of $f(x,y) = x^2 + 3xy + y^2$ on the set $D = \{(x,y) : x^2 + y^2 \leq 4\}$.
4. Find all local extreme values of $f(x,y) = xe^{-(x^2+y^2)}$.
5. Find all local extreme values of $g(x,y) = x^2e^{-(x^2+y^2)}$.
6. Find all local extreme values of $g(x,y) = \dfrac{1}{1 + x^2 + y^2}$.
7. Find all local extreme values of $f(x,y) = 4xy - 2x^2 - y^4$.
8. Find all local extreme values of $h(x,y) = 2x^4 + y^4 - x^2 - 2y^2$.
9. Find all local extreme values of $f(x,y,z) = x^2 + y^2 + z^2$.
10. Find all local extreme values of $g(x,y,z) = x^2 + y^2 - z^2$.
11. A farmer wishes to build a rectangular bin, with a top, to hold a volume of 1000 cubic meters. Find the dimensions of the bin that will minimize the amount of material needed in its construction.
12. A farmer wishes to build a rectangular bin, with a top, using 600 square meters of material. Find the dimensions of the bin that will maximize the volume.
13. Find the extreme values of $f(x,y,z) = x + y + z$ on the sphere with equation $x^2 + y^2 + z^2 = 1$.
14. Find the minimum distance in $\mathbb{R}^2$ from the origin to the line with equation $3x + 2y = 4$.
15. Find the minimum distance in $\mathbb{R}^3$ from the origin to the plane with equation $2x + 4y + z = 6$.
16. Find the minimum distance in $\mathbb{R}^2$ from the origin to the curve with equation $xy = 1$.
17. The ellipsoid with equation $x^2 + 2y^2 + z^2 = 4$ is heated so that its temperature at (x,y,z) is given by $T(x,y,z) = 70 + 10(x - z)$. Find the hottest and coldest points on the ellipsoid.
18. Suppose an airline requires that the sum of the length, width, and height of carry-on luggage cannot exceed 45 inches (assuming the luggage is in the shape of a rectangular box). Find the dimensions of a piece of carry-on luggage that has the maximum volume.
19. Let $f(x,y) = (y - 4x^2)(y - x^2)$.
    (a) Verify that $(0,0)$ is a critical point of f.
    (b) Show that $Hf(0,0)$ is nondefinite.
    (c) Show that along any line through the origin, f has a local minimum at $(0,0)$.
    (d) Find a curve through the origin such that, along the curve, f has a local maximum at $(0,0)$. Note that this shows that $(0,0)$ is a saddle point.

20. Let $f(x,y) = (x-y)^2$. Find all critical points of f and categorize them according as they are either saddle points or the location of local extreme values. Is the second derivative test useful in this case?

21. Let $g(x,y) = \sin(x^2+y^2)$. Find all critical points of g. Which critical points are the location of local maximums? Local minimums? Are there any saddle points?

22. What does a plot of the gradient vectors look like around a saddle point of a function $f : \mathbb{R}^2 \to \mathbb{R}$? You might look at some examples, like $f(x,y) = x^2 - y^2$, $f(x,y) = xy$, or even $f(x,y) = xye^{-(x^2+y^2)}$.

23. Given n points (x_1,y_1), (x_2,y_2), $\dots$, (x_n,y_n) in $\mathbb{R}^2$, the line with equation $y = mx+b$ which minimizes

$$L(m,b) = \sum_{i=1}^{n} (y_1 - (mx_i + b))^2$$

is called the *least squares line.*

(a) Give a geometric interpretation for $L(m,b)$.

(b) Show that the parameters of the least squares line are

$$m = \frac{n\displaystyle\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\displaystyle\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

and

$$b = \bar{y} - m\bar{x},$$

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

and

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

24. The following table is taken from a report prepared in the 1960's to study the effect of leaks of radioactive waste from storage bins at the nuclear facilities at Hanford, Washington, on the cancer rates in nine Oregon counties which border the Columbia River. The table gives an index of exposure, which takes into account such things as distance from the Hanford facilities and the distance of the population from the river, along with the cancer mortality rate per 100,000 people.

County	**Index of Exposure**	**Cancer Mortality Rate**
Umatilla	2.49	147.1
Morrow	2.57	130.1
Gilliam	3.41	129.9
Sherman	1.25	113.5
Wasco	1.62	137.5
Hood River	3.83	162.3
Portland	11.64	207.5
Columbia	6.41	177.9
Clatsop	8.34	210.3

Using Problem 22, find the least squares line for this data (let the index of exposure be the x data). Plot the points along with the line.

Section 3.6

Definite Integrals

We will first define the definite integral for a function $f : \mathbb{R}^2 \to \mathbb{R}$ and later indicate how the definition may be extended to functions of three or more variables.

Cartesian products

We will find the following notation useful. Given two sets of real numbers A and B, we define the *Cartesian product* of A and B to be the set

$$A \times B = \{(x, y) : x \in A, y \in B\}. \tag{3.6.1}$$

For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, then

$$A \times B = \{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}.$$

In particular, if $a < b$, $c < d$, $A = [a, b]$, and $B = [c, d]$, then $A \times B = [a, b] \times [c, d]$ is the closed rectangle

$$\{(x, y) : a \leq x \leq b, c \leq y \leq d\},$$

as shown in Figure 3.6.1.

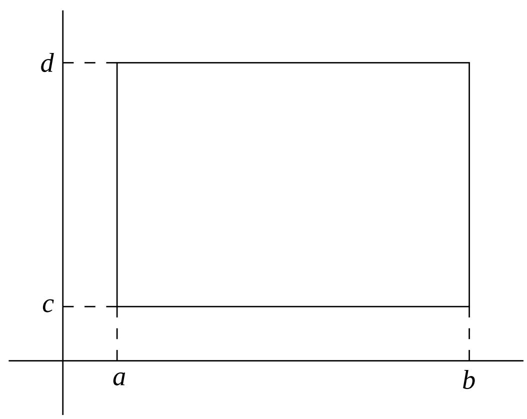


Figure 3.6.1 The closed rectangle $[a, b] \times [c, d]$

More generally, given real numbers $a_i < b_i$, $i = 1, 2, 3, \ldots, n$, we may write

$$[a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$$

for the closed rectangle

$$\{(x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$$

and

$$(a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n)$$

for the open rectangle

$$\{(x_1, x_2, \ldots, x_n) : a_i < x_i < b_i, i = 1, 2, \ldots, n\}.$$

Definite integrals on rectangles

Given $a < b$ and $c < d$, let

$$D = [a, b] \times [c, d]$$

and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined on all of D. Moreover, we suppose f is *bounded* on D, that is, there exist constants m and M such that $m \leq f(x, y) \leq M$ for all (x, y) in D. In particular, the Extreme Value Theorem implies that f is bounded on D if f is continuous on D. Our definition of the definite integral of f over the rectangle D will follow the definition from one-variable calculus. Given positive integers m and n, we let P be a *partition* of $[a, b]$ into m intervals, that is, a set $P = \{x_0, x_1, \ldots, x_m\}$ where

$$a = x_0 < x_1 < \cdots < x_m = b, \tag{3.6.2}$$

and we let Q be a partition of $[c, d]$ into n intervals, that is, a set $Q = \{y_0, y_1, \ldots, y_n\}$ where

$$a = y_0 < y_1 < \cdots < y_n = b. \tag{3.6.3}$$

We will let $P \times Q$ denote the partition of D into mn rectangles

$$D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \tag{3.6.4}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Note that D_{ij} has area $\Delta x_i \Delta y_j$, where

$$\Delta x_i = x_i - x_{i-1} \tag{3.6.5}$$

and

$$\Delta y_j = y_j - y_{j-1}. \tag{3.6.6}$$

An example is shown in Figure 3.6.2.

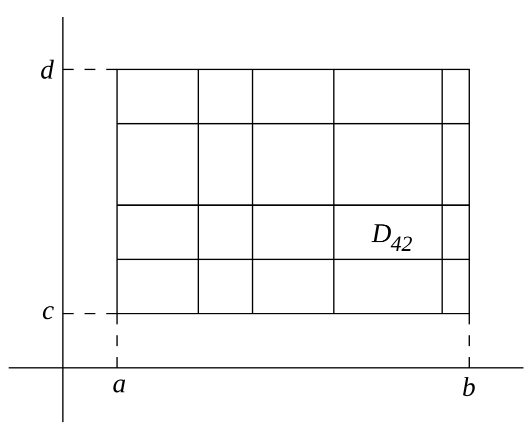


Figure 3.6.2 A partition of a rectangle $[a, b] \times [c, d]$

Now let m_{ij} be the largest real number with the property that $m_{ij} \leq f(x, y)$ for all (x, y) in D_{ij} and M_{ij} be the smallest real number with the property that $f(x, y) \leq M_{ij}$ for all (x, y) in D_{ij}. Note that if f is continuous on D, then m_{ij} is simply the minimum value of f on D_{ij} and M_{ij} is the maximum value of f on D_{ij}, both of which are guaranteed to exist by the Extreme Value Theorem. If f is not continuous, our assumption that f is bounded nevertheless guarantees the existence of the m_{ij} and M_{ij}, although the justification for this statement lies beyond the scope of this book.

We may now define the *lower sum*, $L(f, P \times Q)$, for f with respect to the partition $P \times Q$ by

$$L(f, P \times Q) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \Delta x_i y_j \tag{3.6.7}$$

and the *upper sum*, $U(f, P \times Q)$, for f with respect to the partition $P \times Q$ by

$$U(f, P \times Q) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} \Delta x_i y_j. \tag{3.6.8}$$

Geometrically, if $f(x, y) \geq 0$ for all (x, y) in D and V is the volume of the region which lies beneath the graph of f and above the rectangle D, then $L(f, P \times Q)$ and $U(f, P \times Q)$ represent lower and upper bounds, respectively, for V. (See Figure 3.6.3 for an example of one term of a lower sum). Moreover, we should expect that these bounds can be made arbitrarily close to V using sufficiently fine partitions P and Q. In part this implies that we may characterize V as the only real number which lies between $L(f, P \times Q)$ and $U(f, P \times Q)$ for all choices of partitions P and Q. This is the basis for the following definition.

Definition Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded on the rectangle $D = [a, b] \times [c, d]$. With the notation as above, we say f is *integrable* on D if there exists a unique real number I such that

$$L(f, P \times Q) \leq I \leq U(f, P \times Q) \tag{3.6.9}$$

for all partitions P of $[a, b]$ and Q of $[c, d]$. If f is integrable on D, we call I the *definite integral* of f on D, which we denote

$$I = \int\int_D f(x, y) dx dy. \tag{3.6.10}$$

Geometrically, if $f(x, y) \geq 0$ for all (x, y) in D, we may think of the definite integral of f on D as the volume of the region in R^3 which lies beneath the graph of f and above the rectangle D. Other interpretations include total mass of the rectangle D (if $f(x, y)$ represents the density of mass at the point (x, y)) and total electric charge of the rectangle D (if $f(x, y)$ represents the charge density at the point (x, y)).

Example Suppose $f(x, y) = x^2 + y^2$ and $D = [0, 1] \times [0, 3]$. If we let

$$P = \{0, \frac{1}{2}, 1\}$$

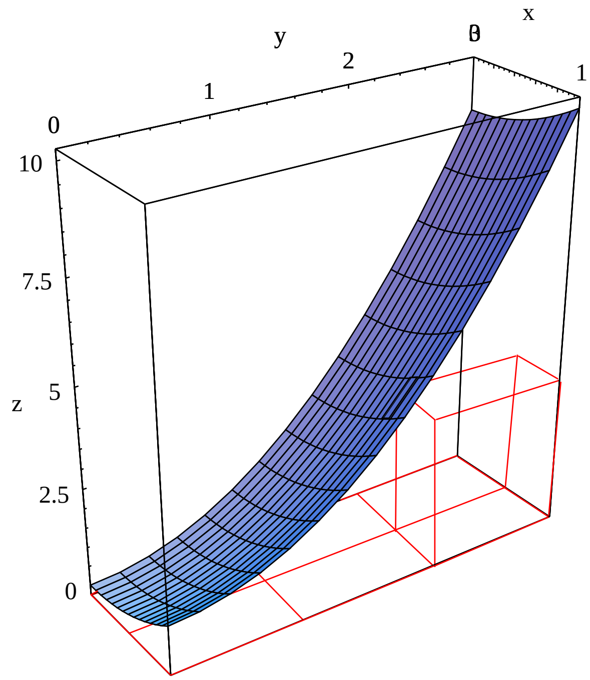


Figure 3.6.3 Graph of $f(x, y) = x^2 + y^2$ showing one term of a lower sum

and

$$Q = \{0, 1, 2, 3\},$$

then the minimum value of f on each rectangle of the partition $P \times Q$ occurs at the lower left-hand corner of the rectangle and the maximum value of f occurs at the upper right-hand corner of the rectangle. See Figure 3.6.3 for a picture of one term of the lower sum. Hence

$$\begin{aligned}
L(f, P \times Q) &= f(0,0) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 0\right) \times \frac{1}{2} \times 1 + f(0,1) \times \frac{1}{2} \times 1 \\
&\qquad + f\left(\frac{1}{2}, 1\right) \times \frac{1}{2} \times 1 + f(0,2) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 2\right) \times \frac{1}{2} \times 1 \\
&= 0 + \frac{1}{8} + \frac{1}{2} + \frac{5}{8} + 2 + \frac{17}{8} \\
&= \frac{43}{8} = 5.375
\end{aligned}$$

and

$$\begin{aligned}
U(f, P \times Q) &= f\left(\frac{1}{2}, 1\right) \times \frac{1}{2} \times 1 + f(1,1) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 2\right) \times \frac{1}{2} \times 1 \\
&\qquad + f(1,2) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 3\right) \times \frac{1}{2} \times 1 + f(1,3) \times \frac{1}{2} \times 1
\end{aligned}$$

$$= \frac{5}{8} + 1 + \frac{17}{8} + \frac{5}{2} + \frac{37}{8} + 5$$
$$= \frac{127}{8} = 15.875.$$

We will see below that the continuity of f implies that f is integrable on D, so we may conclude that

$$5.375 \leq \int\int (x^2 + y^2) dx dy \leq 15.875.$$

Example Suppose k is a constant and $f(x, y) = k$ for all (x, y) in the rectangle $D = [a, b] \times [c, d]$. The for any partitions $P = \{x_0, x_1, \ldots, x_m\}$ of $[a, b]$ and $Q = \{y_0, y_1, \ldots, y_n\}$ of $[c, d]$, $m_{ij} = k = M_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Hence

$$\begin{aligned} L(f, P \times Q) = U(f, P \times Q) &= \sum_{i=1}^{m} \sum_{j=1}^{n} k \Delta x_i \Delta y_j \\ &= k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta x_i \Delta y_j \\ &= k \times (\text{area of } D) \\ &= k(b-a)(d-c). \end{aligned}$$

Hence f is integrable and

$$\int\int_D f(x, y) dx dy = \int\int_D k dx dy = k(b-a)(d-c).$$

Of course, geometrically this result is saying that the volume of a box with height k and base D is k times the area of D. In particular, if $k = 1$ we see that

$$\int\int_D dx dy = \text{area of } D.$$

Example If $D = [1, 2] \times [-1, 3]$, then

$$\int_D 5 dx dy = 5(2-1)(3+1) = 20.$$

The properties of the definite integral stated in the following proposition follow easily from the definition, although we will omit the somewhat technical details.

Proposition Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are both integrable on the rectangle $D = [a, b] \times [c, d]$ and k is a scalar constant. Then

$$\int\int_D (f(x, y) + g(x, y)) dx dy = \int\int_D f(x, y) dx dy + \int\int_D g(x, y) dx dy, \tag{3.6.11}$$

$$\int\int_D kf(x,y)dxdy = k\int\int_D f(x,y)dxdy, \tag{3.6.12}$$

and, if $f(x,y) \leq g(x,y)$ for all (x,y) in D,

$$\int\int_D f(x,y)dxdy \leq \int\int_D g(x,y)dxdy. \tag{3.6.13}$$

Our definition does not provide a practical method for determining whether a given function is integrable or not. A complete characterization of integrability is beyond the scope of this text, but we shall find one simple condition very useful: if f is continuous on an open set containing the rectangle D, then f is integrable on D. Although we will not attempt a full proof of this result, the outline is as follows. If f is continuous on $D = [a,b]\times[c,d]$ and we are given any $\epsilon > 0$, then it is possible to find partitions P of $[a,b]$ and Q of $[c,d]$ sufficiently fine to guarantee that if (x,y) and (u,v) are points in the same rectangle D_{ij} of the partition $P\times Q$ of D, then

$$|f(x,y) - f(u,v)| < \frac{\epsilon}{(b-a)(d-c)}. \tag{3.6.14}$$

(Note that this is not a direct consequence of the continuity of f, but follows from a slightly deeper property of continuous functions on closed bounded sets known as *uniform continuity.*) It follows that if m_{ij} is the minimum value and M_{ij} is the maximum value of f on D_{ij}, then

$$\begin{aligned} U(f,P\times Q) - L(f,P\times Q) &= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\Delta x_i\Delta y_j - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\Delta x_i\Delta y_j \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}(M_{ij}-m_{ij})\Delta x_i\Delta y_j \\ &< \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\epsilon}{(b-a)(d-c)}\Delta x_i\Delta y_j \\ &= \frac{\epsilon}{(b-a)(d-c)}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta x_i\Delta y_j \\ &= \frac{\epsilon}{(b-a)(d-c)}(b-a)(d-c) \\ &= \epsilon. \end{aligned} \tag{3.6.15}$$

It now follows that we may find upper and lower sums which are arbitrarily close, from which follows the integrability of f.

Theorem If f is continuous on an open set containing the rectangle D, then f is integrable on D.

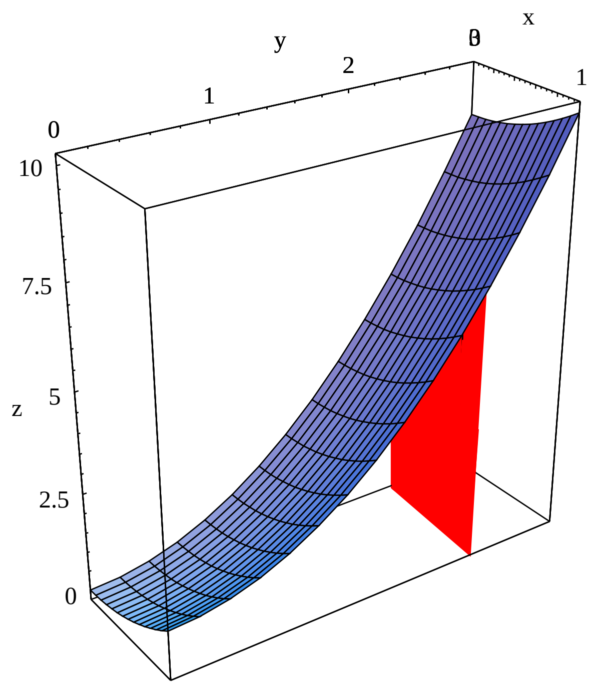


Figure 3.6.4 A slice of the region beneath $f(x, y) = x^2 + y^2$ with area $\alpha(2)$

Example If $f(x, y) = x^2 + y^2$, then f is continuous on all of R^2. Hence f is integrable on $D = [0, 1] \times [0, 3]$.

Iterated integrals

Now suppose we have a rectangle $D = [a, b] \times [c, d]$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) \geq 0$ for all (x, y) in D. Let

$$B = \{(x, y, z) : (x, y) \in D, 0 \leq z \leq f(x, y)\}. \tag{3.6.16}$$

Then B is the region in $\mathbb{R}^3$ bounded below by D and above by the graph of f. If we let V be the volume of B, then

$$V = \int\int_D f(x, y)dxdy. \tag{3.6.17}$$

However, there is another approach to finding V. If, for every $c \leq y \leq d$, we let

$$\alpha(y) = \int_a^b f(x, y)dx, \tag{3.6.18}$$

then $\alpha(y)$ is the area of a slice of B cut by a plane orthogonal to both the xy-plane and the yz-plane and passing through the point $(0, y, 0)$ on the y-axis (see Figure 3.6.4 for an example). If we let the partition $Q = \{y_0, y_1, \ldots, y_n\}$ divide $[c, d]$ into n intervals of equal length Δy, then we may approximate V by

$$\sum_{j=1}^{n} \alpha(y_i)\Delta y. \tag{3.6.19}$$

That is, we may approximate V by slicing B into slabs of thickness Δy perpendicular to the yz-plane, and then summing approximations to the volume of each slab. As n increases, this approximation should converge to V; at the same time, since (3.6.19) is a right-hand rule approximation to the definite integral of α over $[c, d]$, the sum should converge to

$$\int_c^d \alpha(y)dy$$

as n increases. That is, we should have

$$V = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha(y_i)\Delta y = \int_c^d \alpha(y)dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy. \tag{3.6.20}$$

Note that the expression on the right-hand side of (3.6.20) is not the definite integral of f over D, but rather two successive integrals of one variable. Also, we could have reversed our order and first integrated with respect to y and then integrated the result with respect to x.

Definition Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined on a rectangle $D = [a, b] \times [c, d]$. The *iterated integrals* of f over D are

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy \tag{3.6.21}$$

and

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx. \tag{3.6.22}$$

In the situation of the preceding paragraph, we should expect the iterated integrals in (3.6.21) and (3.6.22) to be equal since they should both equal V, the volume of the region B. Moreover, since we also know that

$$V = \int\int_D f(x,y)dxdy,$$

the iterated integrals should both be equal to the definite integral of f over D. These statements may in fact be verified as long as f is integrable on D and the iterated integrals exist. In this case, iterated integrals provide a method of evaluating double integrals in terms of integrals of a single variable (for which we may use the Fundamental Theorem of Calculus).

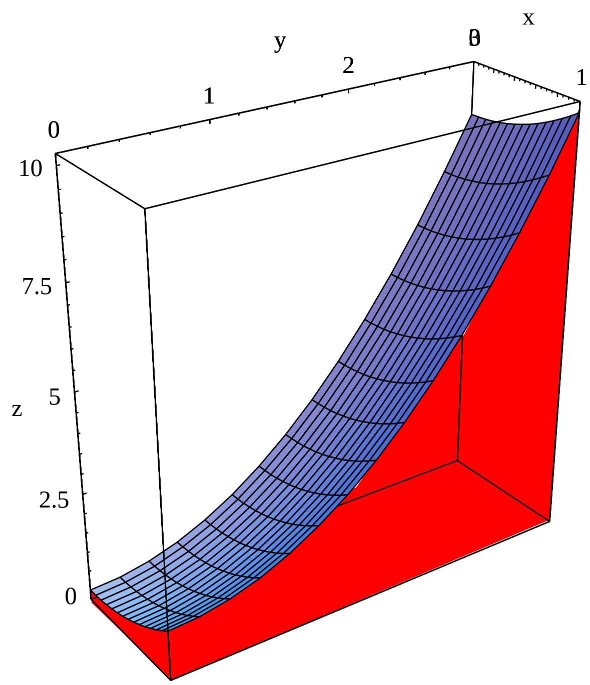


Figure 3.6.5 Region beneath $f(x,y) = x^2 + y^2$ over the rectangle $[0,1] \times [0,3]$

Fubini's Theorem (for rectangles) Suppose f is integrable over the rectangle $D = [a,b] \times [c,d]$. If

$$\int_c^d \int_a^b f(x,y)dxdy$$

exists, then

$$\int\int_D f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dxdy. \tag{3.6.23}$$

If

$$\int_a^b \int_c^d f(x,y)dydx$$

exists, then

$$\int\int_D f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dxdy. \tag{3.6.24}$$

Example To find the volume V of the region beneath the graph of $f(x,y) = x^2 + y^2$ and over the rectangle $D = [0,1] \times [0,3]$ (as shown in Figure 3.6.5), we compute

$$\begin{aligned} V &= \int\int_D (x^2 + y^2)dxdy \\ &= \int_0^3 \int_0^1 (x^2 + y^2)dxdy \end{aligned}$$

$$
\begin{aligned}
&= \int_0^3 \left(\frac{x^3}{3} + xy^2 \right) \Big|_0^1 dy \\
&= \int_0^3 \left(\frac{1}{3} + y^2 \right) dy \\
&= \left(\frac{y}{3} + \frac{y^3}{3} \right) \Big|_0^3 \\
&= 1 + 9 \\
&= 10.
\end{aligned}
$$

We could also compute the iterated integral in the other order:

$$
\begin{aligned}
V &= \int\int_D (x^2 + y^2) dx dy \\
&= \int_0^1 \int_0^3 (x^2 + y^2) dy dx \\
&= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right) \Big|_0^3 dx \\
&= \int_0^1 (3x^2 + 9) dx \\
&= (x^3 + 9y) \big|_0^1 \\
&= 1 + 9 \\
&= 10.
\end{aligned}
$$

Example If $D = [1, 2] \times [0, 1]$, then

$$
\int\int_D x^2 y dx dy = \int_1^2 \int_0^1 x^2 y dy dx = \int_1^2 \frac{x^2 y^2}{2} \Big|_0^1 dx = \int_1^2 \frac{x^2}{2}\, dx = \frac{x^3}{6} \Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.
$$

Definite integrals on other regions

Integrals over intervals suffice for most applications of functions of a single variable. However, for functions of two variables it is important to consider integrals on regions other than rectangles. To extend our definition, consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined on a bounded region D. Let D^* be a rectangle containing D and, for any (x, y) in D^*, define

$$
f^*(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in D, \\ 0, & \text{if } (x, y) \notin D. \end{cases} \tag{3.6.25}
$$

In other words, f^* is identical to f on D and 0 at all points of D^* outside of D. Now if f^* is integrable on D^*, and since the the region where f^* is 0 should contribute nothing

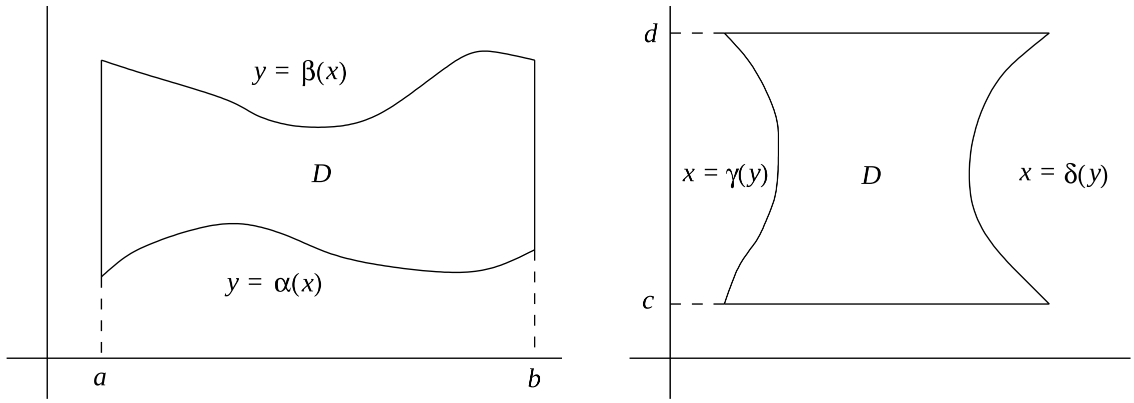


Figure 3.6.6 Regions of Type I and Type II

to the value of the integral, it is reasonable to define the integral of f over D to be equal to the integral of f^* over D^*.

Definition Suppose f is defined on a bounded region D of $\mathbb{R}^2$ and let D^* be any rectangle containing D. Define f^* as in (3.6.25). We say f is integrable on D if f^* is integrable on D^*, in which case we define

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f^*(x,y)dxdy. \tag{3.6.26}$$

Note that the integrability of f on a region D depends not only on the nature of f, but on the region D as well. In particular, even if f is continuous on an open set containing D, it may still turn out that f is not integrable on D because of the complicated nature of the boundary of D. Fortunately, there are two basic types of regions which occur frequently and to which our previous theorems generalize.

Definition We say a region D in $\mathbb{R}^2$ is of *Type I* if there exist real numbers $a < b$ and continuous functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ such that $\alpha(x) \leq \beta(x)$ for all x in $[a, b]$ and

$$D = \{(x,y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}. \tag{3.6.27}$$

We say a region D in R^2 is of *Type II* if there exist real numbers $c < d$ and continuous functions $\gamma : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R} \to \mathbb{R}$ such that $\gamma(y) \leq \delta(y)$ for all y in $[c, d]$ and

$$D = \{(x,y) : c \leq y \leq d, \gamma(y) \leq x \leq \delta(y)\}. \tag{3.6.28}$$

Figure 3.6.6 shows typical examples of regions of Type I and Type II.

Example If D is the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$, then

$$D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

Hence D is a Type I region with $\alpha(x) = 0$ and $\beta(x) = x$. Note that we also have

$$D = \{(x,y) : 0 \leq y \leq 1, y \leq x \leq 1\},$$

so D is also a Type II region with $\gamma(y) = y$ and $\delta(y) = 1$. See Figure 3.6.7.

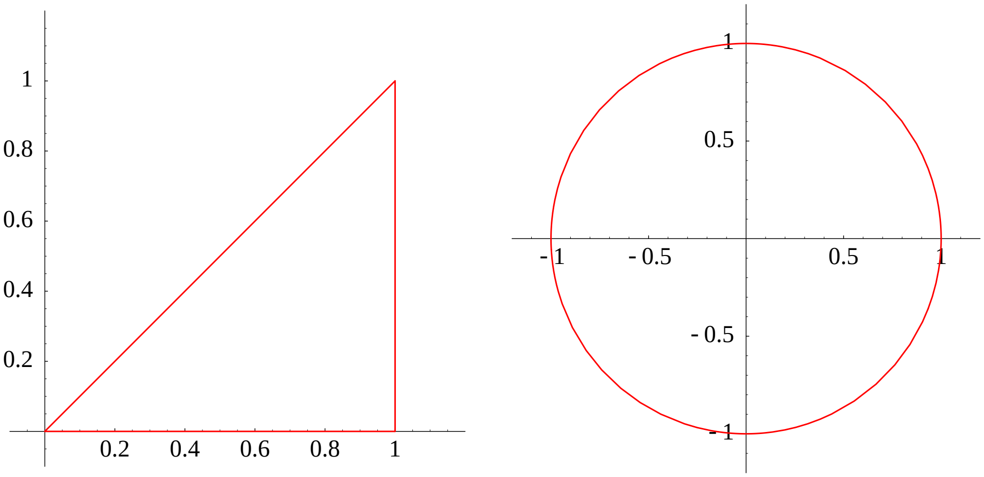


Figure 3.6.7 Two regions which are of both Type I and Type II

Example The closed disk

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$

is both a region of Type I, with

$$D = \{(x, y) : -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\},$$

and a region of Type II, with

$$D = \{(x, y) : -1 \leq y \leq 1, -\sqrt{1-y^2} \leq x \leq \sqrt{1-x^2}\}.$$

See Figure 3.6.7.

Example Let D be the region which lies beneath the graph of $y = x^2$ and above the interval $[-1, 1]$ on the x-axis. Then

$$D = \{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq x^2\},$$

so D is a region of Type I. However, D is not a region of Type II. See Figure 3.6.8.

Theorem If D is a region of Type I or a region of Type II and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on an open set containing D, then f is integrable on D.

Fubini's Theorem (for regions of Type I and Type II) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable on the region D. If D is a region of Type I with

$$D = \{(x, y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}$$

and the iterated integral

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x, y) dy dx$$

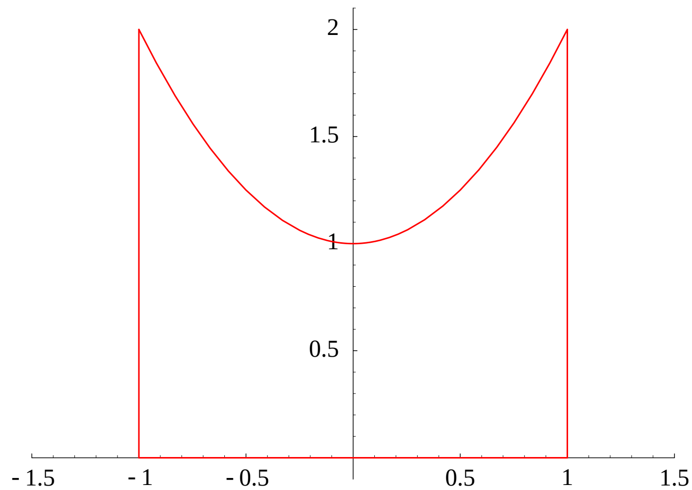


Figure 3.6.8 A region which is of Type I but not of Type II

exists, then

$$\int\int_D f(x,y)dxdy = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)dydx. \tag{3.6.29}$$

If D is a region of Type II with

$$D = \{(x,y) : c \leq y \leq d, \gamma(y) \leq x \leq \delta(y)\}$$

and the iterated integral

$$\int_c^d \int_{\gamma(y)}^{\delta(y)} f(x,y)dxdy$$

exists, then

$$\int\int_D f(x,y)dxdy = \int_c^d \int_{\gamma(y)}^{\delta(y)} f(x,y)dydx. \tag{3.6.30}$$

Example Let D be the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$, as in the example above. Expressing D as a region of Type I, we have

$$\int\int_D xydxdy = \int_0^1 \int_0^x xydydx = \int_0^1 \left.\frac{xy^2}{2}\right|_0^x dx = \int_0^1 \frac{x^3}{2}\,dx = \left.\frac{x^4}{8}\right|_0^1 = \frac{1}{8}.$$

Since D is also a region of Type II, we may evaluate the integral in the other order as well, obtaining

$$\int\int_D xydxdy = \int_0^1 \int_y^1 xydxdy = \int_0^1 \left.\frac{x^2y}{2}\right|_y^1 dy = \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2}\right) dy = \left.\left(\frac{y^2}{4} - \frac{y^4}{8}\right)\right|_0^1 = \frac{1}{8}.$$

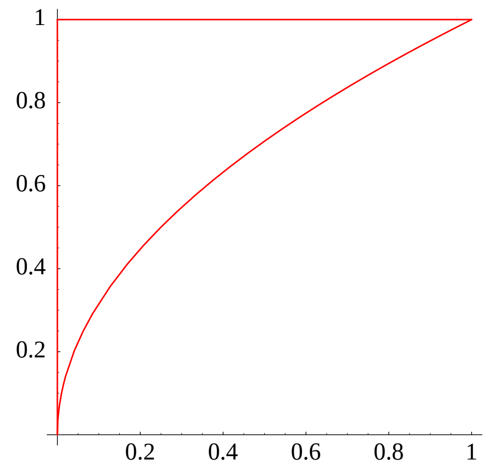


Figure 3.6.9 The region $D = \{(x, y) : 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\}$

In the last example the choice of integration was not too important, with the first order being perhaps slightly easier than the second. However, there are times when the choice of the order of integration has a significant effect on the ease of integration.

Example Let $D = \{(x, y) : 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\}$ (see Figure 3.6.9). Since D is both of Type I and of Type II, we may evaluate

$$\int\int_D e^{-y^3} dx dy$$

either as

$$\int_0^1 \int_{\sqrt{x}}^1 e^{-y^3} dy dx$$

or as

$$\int_0^1 \int_0^{y^2} e^{-y^3} dx dy.$$

The first of these two iterated integrals requires integrating $g(y) = e^{-y^3}$; however, we may evaluate the second easily:

$$\begin{aligned} \int\int_D e^{-y^3} dx dy &= \int_0^1 \int_0^{y^2} e^{-y^3} dx dy \\ &= \int_0^1 x e^{-y^3} \Big|_0^{y^2} dy \\ &= \int_0^1 y^2 e^{-y^3} dy \\ &= -\frac{1}{3} e^{-y^3} \Big|_0^1 \\ &= \frac{1}{3}(1 - e^{-1}). \end{aligned}$$

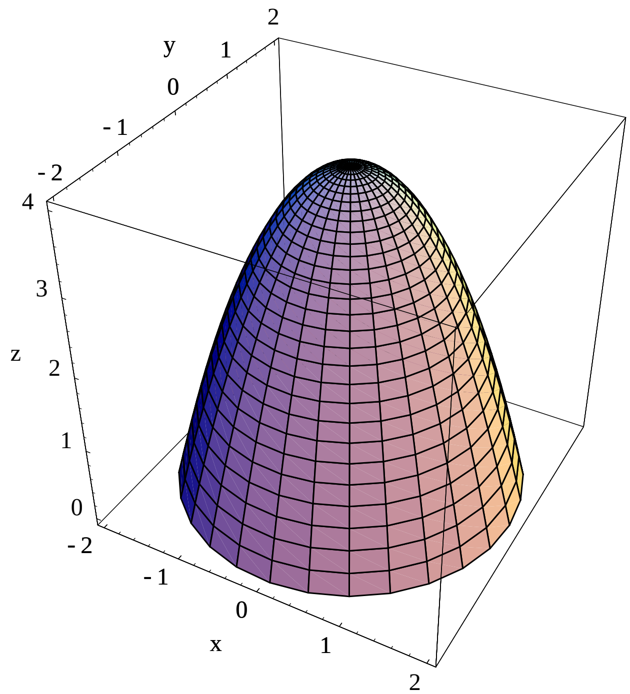


Figure 3.6.10 Region bounded by $z = 4 - x^2 - y^2$ and the xy-plane

Example Let V be the volume of the region lying below the paraboloid P with equation $z = 4 - x^2 - y^2$ and above the xy-plane (see Figure 3.6.10). Since the surface P intersects the xy-plane when

$$4 - x^2 - y^2 = 0,$$

that is, when

$$x^2 + y^2 = 4,$$

V is the volume of the region bounded above by the graph of $f(x,y) = 4 - x^2 - y^2$ and below by the region

$$D = \{(x,y) : x^2 + y^2 \leq 4\}.$$

If we describe D as a Type I region, namely,

$$D = \{(x,y) : -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\},$$

then we may compute

$$\begin{aligned} V &= \int\int_D (4 - x^2 - y^2)dxdy \\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)dydx \\ &= \int_{-2}^{2} \left(4y - x^2y - \frac{y^3}{3}\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{-2}^{2} \left(8\sqrt{4-x^2} - 2x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{\frac{3}{2}} \right) dx \\
&= 2\int_{-2}^{2} \left((4-x^2)\sqrt{4-x^2} - \frac{1}{3}(4-x^2)^{\frac{3}{2}} \right) dx \\
&= \frac{4}{3}\int_{-2}^{2} (4-x^2)^{\frac{3}{2}} dx.
\end{aligned}
$$

Using the substitution $x = 2\sin(\theta)$, we have $dx = 2\cos(\theta)d\theta$, and so

$$
\begin{aligned}
V &= \frac{4}{3}\int_{-2}^{2} (4-x^2)^{\frac{3}{2}} dx \\
&= \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 - 4\sin^2(\theta))^{\frac{3}{2}} 2\cos(\theta)d\theta \\
&= \frac{64}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta)d\theta \\
&= \frac{64}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta \\
&= \frac{16}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\cos(2\theta) + \cos^2(2\theta))d\theta \\
&= \frac{16}{3}\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(4\theta)}{2}\, d\theta \right) \\
&= \frac{16}{3}\left(\pi + \frac{\theta}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{8}\sin(4\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) \\
&= \frac{16}{3}\left(\pi + \frac{\pi}{2} \right) \\
&= 8\pi.
\end{aligned}
$$

Integrals of functions of three or more variables

We will now sketch how to extend the definition of the definite integral to higher dimensions. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is bounded on an n-dimensional closed rectangle

$$D = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n].$$

Let P_1, P_2, $\ldots$, P_n partition the intervals $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$, $[a_n, b_n]$ into m_1, m_2, $\ldots$, m_n, respectively, intervals, and let $P_1 \times P_2 \times \cdots \times P_n$ represent the corresponding partition of D into $m_1 m_2 \cdots m_n$ n-dimensional closed rectangles $D_{i_1 i_2 \cdots i_n}$. If $m_{i_1 i_2 \cdots i_n}$ is the largest real number such that $m_{i_1 i_2 \cdots i_n} \le f(\mathbf{x})$ for all $\mathbf{x}$ in $D_{i_1 i_2 \cdots i_n}$ and $M_{i_1 i_2 \cdots i_n}$ is the smallest real number such that $f(\mathbf{x}) \leq M_{i_1 i_2 \cdots i_n}$ for all $\mathbf{x}$ in $D_{i_1 i_2 \cdots i_n}$, then we may define the lower sum

$$L(f, P_1 \times P_2 \times \cdots \times P_n) = \sum_{i_1=1}^{m_1} \sum_{i_2}^{m_2} \cdots \sum_{i_n=1}^{m_n} m_{i_1 i_2 \cdots i_n} \Delta x_{1i_1} \Delta x_{2i_2} \cdots \Delta x_{ni_n} \tag{3.6.31}$$

and the upper sum

$$U(f, P_1 \times P_2 \times \cdots \times P_n) = \sum_{i_1=1}^{m_1} \sum_{i_2}^{m_2} \cdots \sum_{i_n=1}^{m_n} M_{i_1 i_2 \cdots i_n} \Delta x_{1i_1} \Delta x_{2i_2} \cdots \Delta x_{ni_n}, \tag{3.6.32}$$

where Δx_{jk} is the length of the kth interval of the partition P_j. We then say f is *integrable* on D if there exists a unique real number I with the property that

$$L(f, P_1 \times P_2 \times \cdots \times P_n) \leq I \leq U(f, P_1 \times P_2 \times \cdots \times P_n) \tag{3.6.33}$$

for all choices of partitions P_1, P_2, ..., P_n and we write

$$I = \int \cdots \int \int_D f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n, \tag{3.6.34}$$

or

$$I = \int \cdots \int \int_D f(\mathbf{x}) d\mathbf{x}, \tag{3.6.35}$$

for the *definite integral* of f on D.

We may now generalize the definition of the integral to more general regions in the same manner as above. Moreover, our integrability theorem and Fubini's theorem, with appropriate changes, hold as well. When $n = 3$, we may interpret

$$\int \int \int_D f(x, y, z) dx dy dz \tag{3.6.36}$$

to be the total mass of D if $f(x, y, z)$ represents the density of mass at (x, y, z), or the total electric charge of D if $f(x, y, z)$ represents the electric charge density at (x, y, z). For any value of n we may interpret

$$\int \cdots \int \int_D dx_1 dx_2 \cdots dx_n \tag{3.6.37}$$

to be the n-dimensional volume of D. We will not go into further details, preferring to illustrate with examples.

Example Suppose D is the closed rectangle

$$\begin{aligned} D &= \{(x, y, z, t) : 0 \leq x \leq 1, -1 \leq y \leq 1, -2 \leq z \leq 2, 0 \leq t \leq 2\} \\ &= [0, 1] \times [-1, 1] \times [-2, 2] \times [0, 2]. \end{aligned}$$

Then

$$\begin{aligned}
\int\int\int\int_D (x^2+y^2+z^2-t^2)dxdydzdt &= \int_0^1\int_{-1}^1\int_{-2}^2\int_0^2 (x^2+y^2+z^2-t^2)dtdzdydx \\
&= \int_0^1\int_{-1}^1\int_{-2}^2 \left(x^2t+y^2t+z^2t-\frac{t^3}{3}\right)\Big|_0^2 dzdydx \\
&= \int_0^1\int_{-1}^1\int_{-2}^2 \left(2x^2+2y^2+2z^2-\frac{8}{3}\right) dzdxdy \\
&= \int_0^1\int_{-1}^1 \left(2x^2z+2y^2z+\frac{2z^3}{3}-\frac{8z}{3}\right)\Big|_{-2}^2 dydx \\
&= \int_0^1\int_{-1}^1 \left(8x^2+8y^2+\frac{32}{3}-\frac{32}{3}\right) dydx \\
&= \int_0^1 \left(8x^2y+\frac{8y^2}{3}\right)\Big|_{-1}^1 dx \\
&= \int_0^1 \left(16x^2+\frac{16}{3}\right) dx \\
&= \left(\frac{16x^3}{3}+\frac{16x}{3}\right)\Big|_0^1 \\
&= \frac{32}{3}.
\end{aligned}$$

Example Let D be the region in $\mathbb{R}^3$ bounded by the the three coordinate planes and the plane P with equation $z=1-x-y$ (see Figure 3.6.11). Suppose we wish to evaluate

$$\int\int\int_D xyzdxdydz.$$

Note that the side of D which lies in the xy-plane, that is, the plane $z=0$, is a triangle with vertices at $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$. Or, strictly in terms of x and y coordinates, we may describe this face as the triangle in the first quadrant bounded by the line $y=1-x$ (see Figure 3.6.11). Hence x varies from 0 to 1, and, for each value of x, y varies from 0 to $1-x$. Finally, once we have fixed a values for x and y, z varies from 0 up to P, that is, to $1-x-y$. Hence we have

$$\begin{aligned}
\int\int\int_D xyzdxdydz &= \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyzdzdydx \\
&= \int_0^1\int_0^{1-x} \frac{xyz^2}{2}\Big|_0^{1-x-y} dydx \\
&= \int_0^1\int_0^{1-x} \frac{xy(1-x-y)^2}{2} dydx
\end{aligned}$$

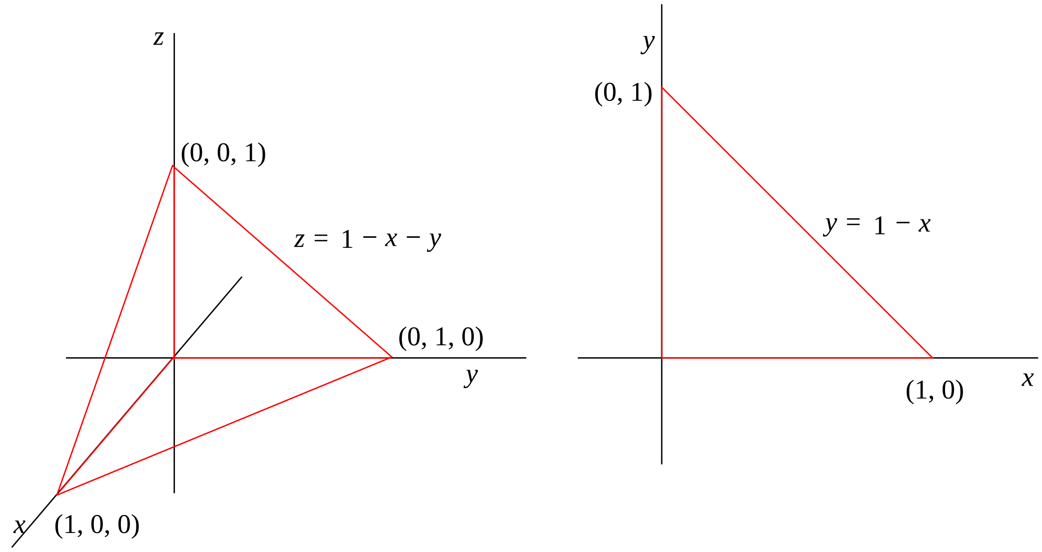


Figure 3.6.11 Region bounded by the coordinate planes and the plane $z = 1 - x - y$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 \int_0^{1-x} (xy - 2x^2y + x^3y + 2x^2y^2 + xy^3)dydx \\
&= \frac{1}{2}\int_0^1 \left(\frac{xy^2}{2} - 2x^2y^2 + \frac{x^3y^2}{2} + \frac{2x^2y^3}{3} + \frac{xy^4}{4}\right)\Bigg|_0^{1-x} dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{3x}{4} - \frac{10x^2}{3} + \frac{9x^3}{2} - 2x^4 + \frac{x^5}{12}\right) dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{3x^2}{8} - \frac{10x^3}{9} + \frac{9x^4}{8} - \frac{2x^5}{5} + \frac{x^6}{72}\right)\Bigg|_0^1 \\
&= \frac{1}{2}\left(\frac{3}{8} - \frac{10}{9} + \frac{9}{8} - \frac{2}{5} + \frac{1}{72}\right) \\
&= \frac{1}{720}.
\end{aligned}
$$

Example Let V be the volume of the region D in $\mathbb{R}^3$ bounded by the paraboloids with equations $z = 10 - x^2 - y^2$ and $z = x^2 + y^2 - 8$ (see Figure 3.6.12). We will find V by evaluating

$$V = \int\int\int_D dxdydz.$$

To set up an iterated integral, we first note that the paraboloid $z = 10 - x^2 - y^2$ opens downward about the z-axis and the paraboloid $z = x^2 + y^2 - 8$ opens upward about the z axis. The two paraboloids intersect when

$$10 - x^2 - y^2 = x^2 + y^2 - 8,$$

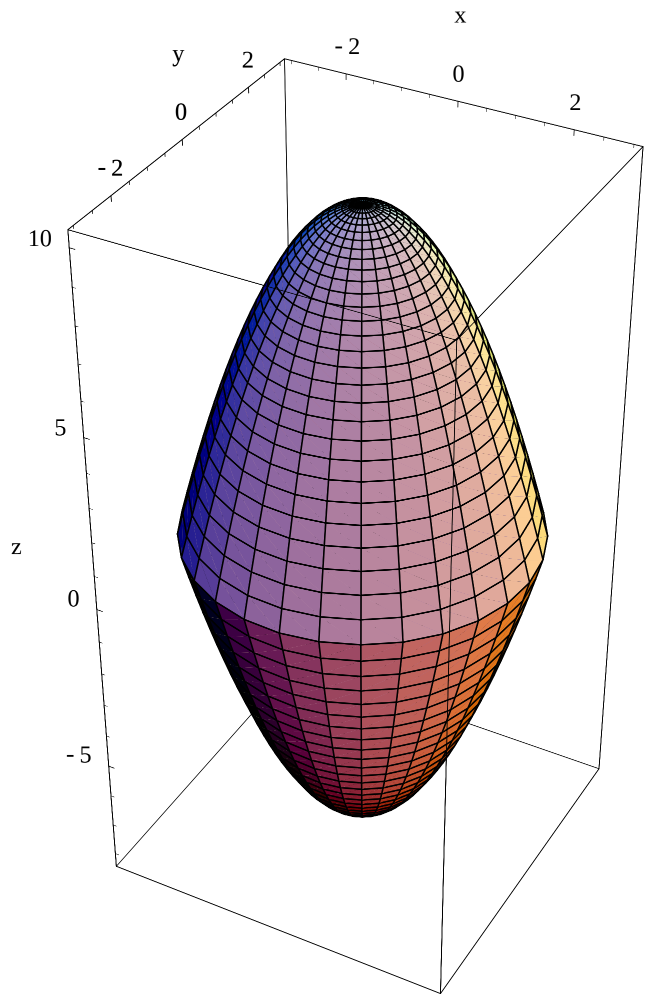


Figure 3.6.12 Region bounded by $z = 10 - x^2 - y^2$ and $z = x^2 + y^2 - 8$

that is, when

$$x^2 + y^2 = 9.$$

Now we may describe the region in the xy-plane described by $x^2 + y^2 \leq 9$ as the set of points (x, y) for which $-3 \leq x \leq 3$ and, for every such fixed x,

$$-\sqrt{3 - x^2} \leq y \leq \sqrt{3 - x^2}.$$

Moreover, once we have fixed x and y so that (x, y) is inside the circle $x^2 + y^2 = 9$, then (x, y, z) is in D provided $x^2 + y^2 - 8 \leq z \leq 10 - x^2 - y^2$. Hence we have

$$
\begin{aligned}
V &= \int\int\int_D dxdydz \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2-8}^{10-x^2-y^2} dzdydx \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_{x^2+y^2-8}^{10-x^2-y^2} dydx \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (18-2x^2-2y^2)dydx \\
&= \int_{-3}^{3}\left(18y-2x^2y-\frac{2y^3}{3}\right)\Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx \\
&= \int_{-3}^{3}\left(36\sqrt{9-x^2}-4x^2\sqrt{9-x^2}-\frac{4}{3}(9-x^2)^{\frac{3}{2}}\right)dx \\
&= \int_{-3}^{3}\sqrt{9-x^2}\left(36-4x^2-\frac{4}{3}(9-x^2)\right)dx \\
&= \frac{8}{3}\int_{-3}^{3}(9-x^2)^{\frac{3}{2}}dx.
\end{aligned}
$$

Using the substitution $x = 3\sin(\theta)$, we have $dx = 3\cos(\theta)d\theta$, and so

$$
\begin{aligned}
V &= \frac{8}{3}\int_{-3}^{3}(9-x^2)^{\frac{3}{2}}dx \\
&= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(9-9\sin^2(x))^{\frac{3}{2}}(3\cos(\theta))d\theta \\
&= 216\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4(\theta)d\theta \\
&= 216\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta \\
&= 54\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos(2\theta)+\cos^2(2\theta))d\theta \\
&= 54\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos(4\theta)}{2}\,d\theta\right) \\
&= 54\pi+27\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{27}{4}\sin(4\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 81\pi.
\end{aligned}
$$

Problems

1. Evaluate each of the following iterated integrals.

   (a) $\int_1^3 \int_0^2 3xy^2 dydx$ (b) $\int_0^{\frac{\pi}{2}} \int_0^{\pi} 4x \sin(x+y) dydx$

   (c) $\int_{-2}^2 \int_{-1}^1 (4 - x^2y^2) dxdy$ (d) $\int_0^2 \int_0^1 e^{x+y} dxdy$

2. Evaluate the following definite integrals over the given rectangles.

   (a) $\int\int_D (y^2 - 2xy) dxdy$, $D = [0,2] \times [0,1]$ (b) $\int\int_D \frac{1}{(x+y)^2} dxdy$, $D = [1,2] \times [1,3]$

   (c) $\int\int_D ye^{-x} dxdy$, $D = [0,1] \times [0,2]$ (d) $\int\int_D \frac{1}{2x+y} dxdy$, $D = [1,2] \times [0,1]$

3. For each of the following, evaluate the iterated integrals and sketch the region of integration.

   (a) $\int_0^2 \int_0^y (xy^2 - x^2) dxdy$ (b) $\int_0^1 \int_{x^4}^{x^2} (x^2 + y^2) dydx$

   (c) $\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) dydx$ (d) $\int_0^1 \int_0^{y^2} xye^{-x-y} dxdy$

4. Find the volume of the region beneath the graph of $f(x,y) = 2 + x^2 + y^2$ and above the rectangle $D = [-1,1] \times [-2,2]$.

5. Find the volume of the region beneath the graph of $f(x,y) = 4 - x^2 + y^2$ and above the region $D = \{(x,y) : 0 \leq x \leq 2, -x \leq y \leq x\}$. Sketch the region D.

6. Evaluate $\int\int_D xydxdy$, where D is the region bounded by the x-axis, the y-axis, and the line $y = 2 - x$.

7. Evaluate $\int\int_D e^{-x^2} dxdy$ where $D = \{(x,y) : 0 \leq y \leq 1, y \leq x \leq 1\}$.

8. Find the volume of the region in $\mathbb{R}^3$ described by $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 4 - 2y - 4x$.

9. Find the volume of the region in $\mathbb{R}^3$ lying above the xy-plane and below the surface with equation $z = 16 - x^2 - y^2$.

10. Find the volume of the region in $\mathbb{R}^3$ lying above the xy-plane and below the surface with equation $z = 4 - 2x^2 - y^2$.

11. Evaluate each of the following iterated integrals.

    (a) $\int_1^2 \int_0^3 \int_{-2}^2 (4 - x^2 - z^2) dydxdz$ (b) $\int_{-2}^3 \int_{-1}^2 \int_0^2 3xyzdxdydz$

    (c) $\int_0^4 \int_0^x \int_0^{x+y} (x^2 - yz) dzdydx$ (d) $\int_0^1 \int_0^x \int_0^{x+y} \int_0^{x+y+z} wdwdzdydx$

12. Find the volume of the region in $\mathbb{R}^3$ bounded by the paraboloids with equations $z = 3 - x^2 - y^2$ and $z = x^2 + y^2 - 5$.

13. Evaluate $\int\int\int_D xy\,dx\,dy\,dz$, where D is the region bounded by the xy-plane, the yz-plane, the xz-plane, and the plane with equation $z = 4 - x - y$.

14. If $f(x, y, z)$ represents the density of mass at the point (x, y, z) of an object occupying a region D in $\mathbb{R}^3$, then
$$\int\int\int_D f(x, y, z)dxdydz$$
is the total mass of the object and the point $(\bar{x}, \bar{y}, \bar{z})$, where
$$\bar{x} = \frac{1}{m}\int\int\int_D xf(x, y, z)dxdydz,$$
$$\bar{y} = \frac{1}{m}\int\int\int_D yf(x, y, z)dxdydz,$$
and
$$\bar{z} = \frac{1}{m}\int\int\int_D zf(x, y, z)dxdydz,$$
is called the *center of mass* of the object. Suppose D is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $z = 4 - x - 2y$.
    (a) Find the total mass and center of mass for an object occupying the region D with mass density given by $f(x, y, z) = 1$.
    (b) Find the total mass and center of mass for an object occupying the region D with mass density given by $f(x, y, z) = z$.

15. If X and Y are points chosen at random from the interval $[0, 1]$, then the probability that (X, Y) lies in a subset D of the unit square $[0, 1] \times [0, 1]$ is $\int\int_D dxdy$.
    (a) Find the probability that $X \leq Y$.
    (b) Find the probability that $X + Y \leq 1$.
    (c) Find the probability that $XY \geq \frac{1}{2}$.

16. If X, Y, and Z are points chosen at random from the interval $[0, 1]$, then the probability that (X, Y, Z) lies in a subset D of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ is $\int\int\int_D dxdydz$.
    (a) Find the probability that $X \leq Y \leq Z$.
    (b) Find the probability that $X + Y + Z \leq 1$.

The Calculus of Functions

of

Several Variables

Section 3.7

Change of Variables in Integrals

One of the basic techniques for evaluating an integral in one-variable calculus is substitution, replacing one variable with another in such a way that the resulting integral is of a simpler form. Although slightly more subtle in the case of two or more variables, a similar idea provides a powerful technique for evaluating definite integrals.

Linear change of variables

We will present the main idea through an example. Let

$$D = \{(x, y) : 9x^2 + 4y^2 \leq 36\},$$

the region inside the ellipse which intersects the x-axis at $(-2, 0)$ and $(2, 0)$ and the y-axis at $(0, -3)$ and $(0, 3)$. To find the area of D, we evaluate

$$\int\int_D dxdy = \int_{-2}^{2} \int_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} dydx = \int_{-2}^{2} 3\sqrt{4 - x^2}\, dx = 6\pi,$$

where the final integral may be evaluated using the substitution $x = 2\sin(\theta)$ or by noting that

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

is one-half of the area of a circle of radius 2. Alternatively, suppose we write the equation of the ellipse as

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

and make the substitution $x = 2u$ and $y = 3v$. Then $u = \frac{x}{2}$ and $v = \frac{y}{3}$, so if (x, y) is a point in D, then

$$u^2 + v^2 = \frac{x^2}{4} + \frac{y^2}{9} \leq 1.$$

That is, if (x, y) is a point in D, then (u, v) is a point in the unit disk

$$E = \{(u, v) : u^2 + v^2 \leq 1\}.$$

Conversely, if (u, v) is a point in E, then

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4u^2}{4} + \frac{9v^2}{9} = u^2 + v^2 \leq 1,$$

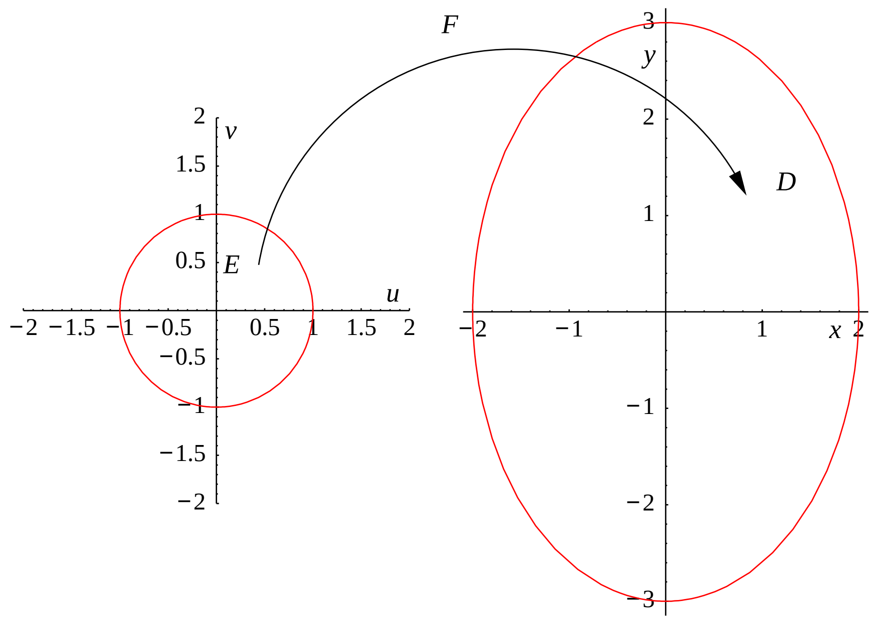


Figure 3.7.1 F maps E onto D

so (x, y) is a point in D. Thus the function $F(u, v) = (2u, 3v)$ takes the region E, a closed disk of radius 1, and stretches it onto the region D (as shown in Figure 3.7.1). However, note that even though every point in E corresponds to exactly one point in D, and, conversely, every point in D corresponds to exactly on point in E, nevertheless E and D do not have the same area. To see how F changes area, consider what it does to the unit square S with sides $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. The area of S is 1, but F maps S onto a rectangle R with sides

$$F(1, 0) = (2, 0)$$

and

$$F(0, 1) = (0, 3)$$

and area 6. This a special case of a general fact we saw in Section 1.6: the linear function F, with associated matrix

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

maps the unit square S onto a parallelogram R with area

$$|\det(M)| = 6.$$

The important fact for us here is that 1 unit of area in the uv-plane corresponds to 6 units of area in the xy-plane. Hence the area of D will be 6 times the area of E. That is,

$$\int\int_D dxdy = \int\int_E |\det(M)| dudv = \int\int_E 6dudv = 6\int\int_E dudv = 6\pi,$$

where the final integral is simply the area inside a circle of radius 1.

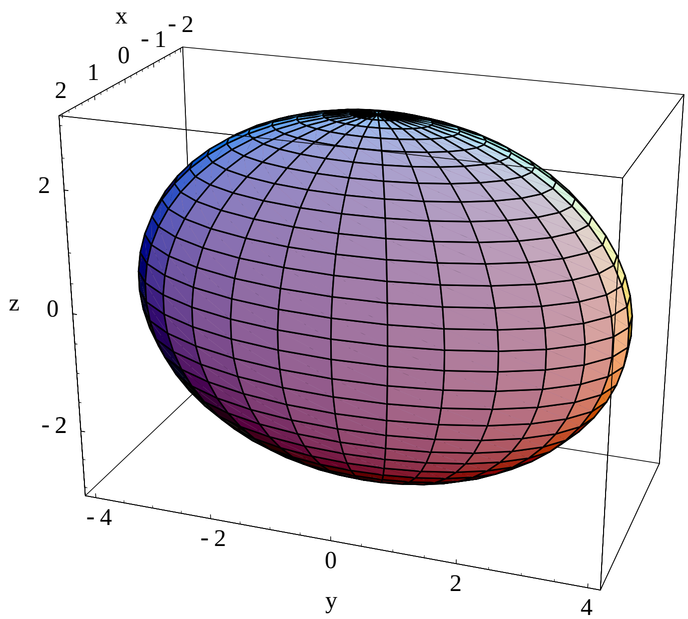


Figure 3.7.2 The ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$

These ideas provide the background for a proof of the following theorem.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on a an open set U containing the closed bounded set D. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function, M is an $n \times n$ matrix such that $F(\mathbf{u}) = M\mathbf{u}$, and $\det(M) \neq 0$. If F maps the region E onto the region D and we define the change of variables

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

then

$$\begin{aligned} &\int\int\cdots\int_D f(x_1,x_2,\ldots,x_n)dx_1dx_2\cdots dx_n \\ &\qquad = \int\int\cdots\int_E f(F(u_1,u_2,\ldots,u_n))|\det(M)|du_1du_2\cdots du_n. \end{aligned} \tag{3.7.1}$$

Example Let D be the region in $\mathbb{R}^3$ bounded by the ellipsoid with equation

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

See Figure 3.7.2. If we make the change of variables $x = 2u$, $y = 4v$, and $z = 3w$, that is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

then, for any (x, y, z) in D, we have

$$u^2 + v^2 + w^2 = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1.$$

That is, if (x, y, z) lies in D, then the corresponding (u, v, w) lies in the closed unit ball $E = \bar{B}^3((0, 0, 0), 1)$. Conversely, if (u, v, w) lies in E, then

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = \frac{4u^2}{4} + \frac{16v^2}{16} + \frac{9w^2}{9} = u^2 + v^2 + w^2 \leq 1,$$

so (x, y, z) lies in D. Hence, the change of variables $F(u, v, w) = (2u, 4v, 3w)$ maps E onto D. Now

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 24,$$

so if V is the volume of D, then

$$V = \int\int\int_D dxdydz = \int\int\int_E 24dudvdw = 24\int\int\int_E dudvdw = 24\left(\frac{4\pi}{3}\right) = 32\pi,$$

where we have used the fact that the volume of a sphere of radius 1 is $\frac{4\pi}{3}$ to evaluate the final integral.

Nonlinear change of variables

Without going into the technical details, we will indicate how to proceed when the change of variables is not linear. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on a an open set U containing the closed bounded set D and $F : \mathbb{R}^n \to \mathbb{R}^n$ maps a closed bounded region E of $\mathbb{R}^n$ onto D so that every point of D corresponds to exactly one point of E. Writing $F(\mathbf{u}) = (F_1(\mathbf{u}), F_2(\mathbf{u}), \ldots, F_n(\mathbf{u}))$, we will assume that F_1, F_2, $\ldots$, and F_n are all differentiable on an open set W containing E. Although we will not study this type of function until Chapter 4, the natural candidate for the derivative of F is the matrix whose ith row is $\nabla F_i(\mathbf{u})$. Letting $x_i = F_i(u_1, u_2, \ldots, u_n)$, $i = 1, 2, \ldots, n$, we denote this matrix, called the *Jacobian matrix* of F,

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}. \tag{3.7.2}$$

Explicitly,

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} = \begin{bmatrix} \frac{\partial}{\partial u_1}F_1(\mathbf{u}) & \frac{\partial}{\partial u_2}F_1(\mathbf{u}) & \cdots & \frac{\partial}{\partial u_n}F_1(\mathbf{u}) \\ \frac{\partial}{\partial u_1}F_2(\mathbf{u}) & \frac{\partial}{\partial u_2}F_2(\mathbf{u}) & \cdots & \frac{\partial}{\partial u_n}F_2(\mathbf{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial u_1}F_n(\mathbf{u}) & \frac{\partial}{\partial u_2}F_n(\mathbf{u}) & \cdots & \frac{\partial}{\partial u_n}F_n(\mathbf{u}) \end{bmatrix}. \tag{3.7.3}$$

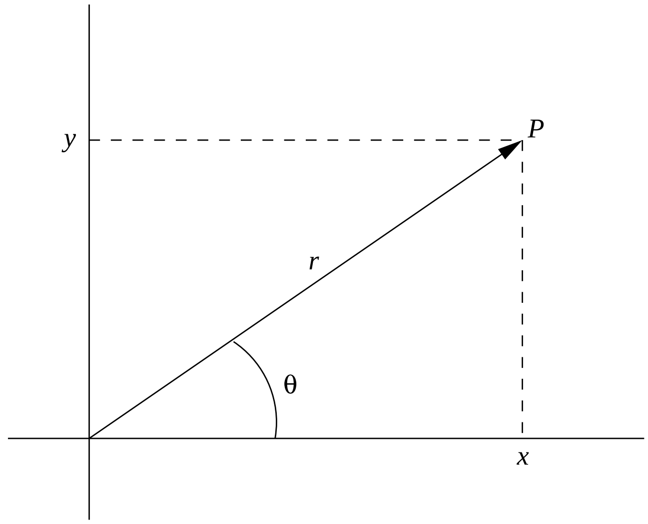


Figure 3.7.3 Polar and Cartesian coordinates for a point P

We shall see in Chapter 4 that

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}$$

is the matrix for the linear part of the best affine approximation to F at $(u_1, u_2, \ldots, u_n)$. Hence, for sufficiently small rectangles, the factor by which F changes the area of a rectangle when it maps it to a region will be approximately

$$\left|\det \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}\right|. \tag{3.7.4}$$

One may then show that, analogous to (3.7.1), we have

$$\begin{aligned} &\int \cdots \int \int_D f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n \\ &\quad = \int \cdots \int \int_E f(F(u_1, u_2, \ldots, u_n)) \left|\det \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}\right| dx_1 dx_2 \cdots dx_n. \end{aligned} \tag{3.7.5}$$

Note that (3.7.5) is just (3.7.1) with the matrix M replaced by the Jacobian of F.

We will now look at two very useful special cases of the preceding result. See Problems 22 and 23 for a third special case.

Polar coordinates

As an alternative to describing the location of a point P in the plane using its Cartesian coordinates (x, y), we may locate the point using r, the distance from P to the origin, and θ, the angle between the vector from $(0, 0)$ to P and the positive x-axis, measured in the counterclockwise direction from 0 to 2π (see Figure 3.7.3). That is, if P has Cartesian coordinates (x, y), with $x \neq 0$, we may define its *polar coordinates* (r, θ) by specifying that

$$r = \sqrt{x^2 + y^2} \tag{3.7.6}$$

and

$$\tan(\theta) = \frac{y}{x}, \tag{3.7.7}$$

where we take $0 \leq \theta \leq \pi$ if $y \geq 0$ and $\pi < \theta < 2\pi$ if $y < 0$. If $x = 0$, we let $\theta = \frac{\pi}{2}$ if $y > 0$ and $\theta = \frac{3\pi}{2}$ if $y < 0$. For $(x, y) = (0, 0)$, $r = 0$ and θ could have any value, and so is undefined. Conversely, if a point P has polar coordinates (r, θ), then

$$x = r\cos(\theta) \tag{3.7.8}$$

and

$$y = r\sin(\theta). \tag{3.7.9}$$

Note that the choice of the interval $[0, 2\pi)$ for the values of θ is not unique, with any interval of length 2π working as well. Although $[0, 2\pi)$ is the most common choice for values of θ, it is sometimes useful to use $(-\pi, \pi)$ instead.

Example If a point P has Cartesian coordinates $(-1, 1)$, then its polar coordinates are $\left(\sqrt{2}, \frac{3\pi}{4}\right)$.

Example A point with polar coordinates $\left(3, \frac{\pi}{6}\right)$ has Cartesian coordinates $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

In our current context, we want to think of the polar coordinate mapping

$$(x, y) = F(r, \theta) = (r\cos(\theta), r\sin(\theta)) \tag{3.7.10}$$

as a change of variables between the $r\theta$-plane and the xy-plane. This mapping is particularly useful for us because it maps rectangular regions in the $r\theta$-plane onto circular regions in the xy-plane. For example, for any $a > 0$, F maps the rectangular region

$$E = \{(r, \theta) : 0 \leq r \leq a, 0 \leq \theta < 2\pi\}$$

in the $r\theta$-plane onto the closed disk

$$D = \bar{B}^2((0, 0), a) = \{(x, y) : x^2 + y^2 \leq a\}$$

in the xy-plane (see Figure 3.7.5 below for an example). More generally, for any $0 \leq \alpha < \beta < 2\pi$, F maps the rectangular region

$$E = \{(r, \theta) : 0 \leq r \leq a, \alpha \leq \theta < \beta\}$$

in the $r\theta$-plane onto a region D in the xy-plane which is the sector of the closed disk $\bar{B}^2((0, 0), a)$ which lies between radii of angles α and β (see Figure 3.7.4). Another basic example is an annulus: for any $0 < a < b$, F maps the rectangular region

$$E = \{(r, \theta) : a \leq r \leq b, 0 \leq \theta < 2\pi\}$$

in the $r\theta$-plane onto the annulus

$$D = \{(x, y) : a \leq x^2 + y^2 \leq b\}$$

in the xy-plane. Figure 3.7.6 illustrates this mapping for the upper half of an annulus.

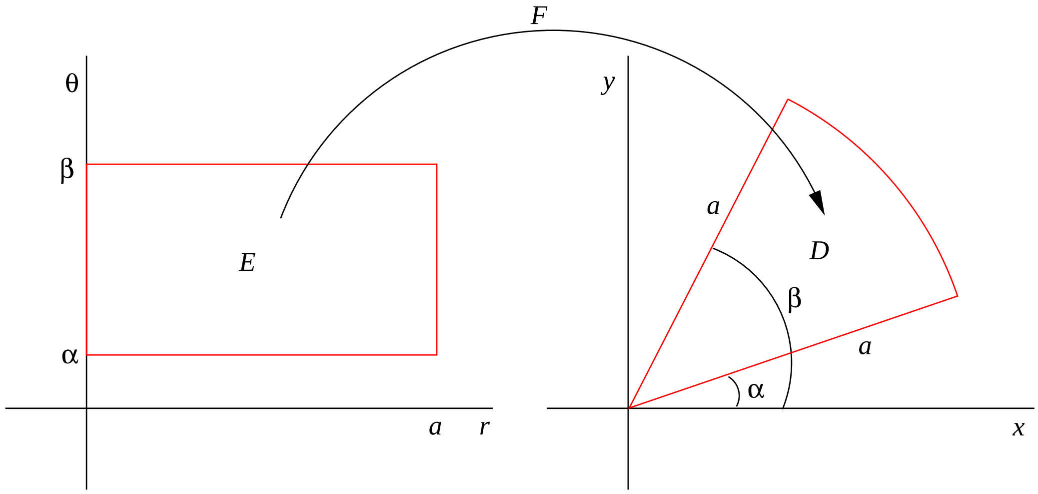


Figure 3.7.4 Polar coordinate change of variables

Example Let V be the volume of the region which lies beneath the paraboloid with equation $z = 4 - x^2 - y^2$ and above the xy-plane. In Section 3.6, we saw that

$$V = \int\int_D (4 - x^2 - y^2)dxdy = 8\pi,$$

where

$$D = \{(x, y) : x^2 + y^2 \leq 4\}.$$

The use of polar coordinates greatly simplifies the evaluation of this integral. With the polar coordinate change of variables

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta),$$

the closed disk D in the xy-plane corresponds to the closed rectangle

$$E = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

in the $r\theta$-plane (see Figure 3.7.5). Note that in describing E we have allowed $\theta = 2\pi$, but this has no affect on our outcome since a line has no area in $\mathbb{R}^2$. Moreover, if we let $f(x, y) = 4 - x^2 - y^2$, then

$$\begin{aligned} f(F(r,\theta)) &= f(r\cos(\theta), r\sin(\theta)) \\ &= 4 - r^2\cos^2(\theta) - r^2\sin(\theta) \\ &= 4 - r^2(\cos^2(\theta) + \sin^2(\theta)) \\ &= 4 - r^2, \end{aligned}$$

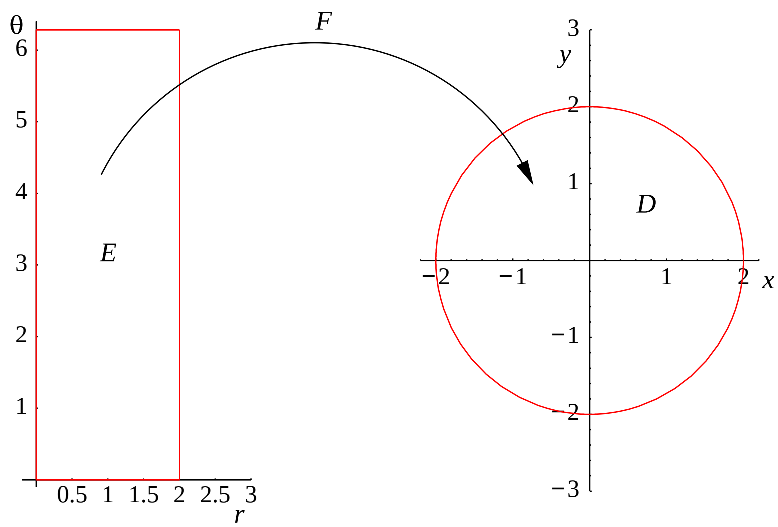


Figure 3.7.5 Polar coordinate change of variables maps $[0,2] \times [0,2\pi]$ to $\bar{B}^2((0,0),2)$

which also follows from the fact that $r^2 = x^2 + y^2$. Now

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \dfrac{\partial}{\partial r} r\cos(\theta) & \dfrac{\partial}{\partial \theta} r\cos(\theta) \\ \dfrac{\partial}{\partial r} r\sin(\theta) & \dfrac{\partial}{\partial \theta} r\sin(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}, \tag{3.7.11}$$

so

$$\det \frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2(\theta) + r\sin^2(\theta) = r(\cos^2(\theta) + \sin^2(\theta)) = r. \tag{3.7.12}$$

Hence, using (3.7.5), we have

$$\begin{aligned} \int\int_D (4 - x^2 - y^2)dxdy &= \int\int_E (4 - r^2) \left|\det \frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \int_0^2 \int_0^{2\pi} (4 - r^2) r d\theta dr \\ &= \int_0^2 2\pi(4r - r^3)dr \\ &= 2\pi \left(2r^2 - \frac{r^4}{4}\right)\Bigg|_0^2 \\ &= 2\pi(8 - 4) \\ &= 8\pi. \end{aligned}$$

Example Suppose D is the part of the region between the circles with equations $x^2+y^2 = 1$ and $x^2 + y^2 = 9$ which lies above the x-axis. That is,

$$D = \{(x,y) : 1 \le x^2 + y^2 \le 9, x \ge 0\}.$$

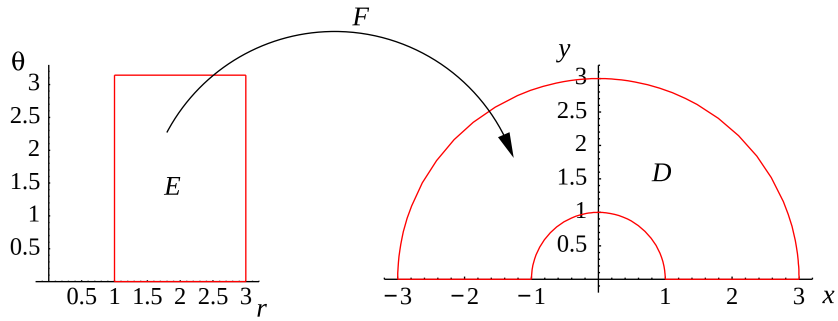


Figure 3.7.6 Polar coordinates map $[1,3]\times[0,\pi]$ to top half of an annulus

We wish to evaluate

$$\int\int_D e^{-(x^2+y^2)}dxdy.$$

Under the polar coordinate change of variables

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta),$$

the annular region D corresponds to the closed rectangle

$$E = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \pi\},$$

as illustrated in Figure 3.7.6. Moreover, $x^2 + y^2 = r^2$ and, as we saw in the previous example,

$$\left|\det \frac{\partial(x,y)}{\partial(r,\theta)}\right| = r.$$

Hence

$$\begin{aligned}\int\int_D e^{-(x^2+y^2)}dxdy &= \int\int_E re^{-r^2}drd\theta \\ &= \int_1^3\int_0^\pi re^{-r^2}d\theta dr \\ &= \int_1^3 \pi re^{-r^2}dr \\ &= -\frac{\pi}{2}e^{-r^2}\Big|_1^3 \\ &= \frac{\pi}{2}(e^{-1}-e^{-9}).\end{aligned}$$

Note that in this case the change of variables not only simplified the region of integration, but also put the function being integrated into a form to which we could apply the Fundamental Theorem of Calculus.

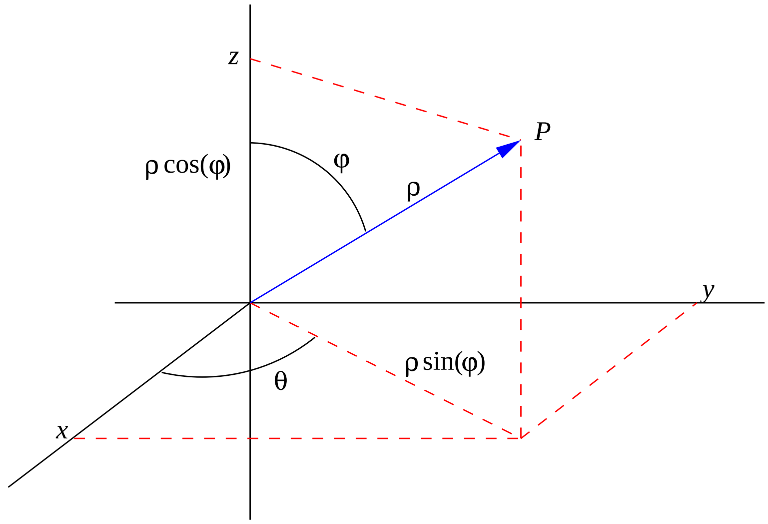


Figure 3.7.7 Spherical and Cartesian coordinates for a point P

Spherical coordinates

Next consider the following extension of polar coordinates to three space: given a point P with Cartesian coordinates (x, y, z), let ρ be the distance from P to the origin, θ be the angle coordinate for the polar coordinates of $(x, y, 0)$ (the projection of P onto the xy-plane), and let φ be the angle between the vector from the origin to P and the positive z-axis, measured from 0 to π. If $x \neq 0$, we have

$$\rho = \sqrt{x^2 + y^2 + z^2}, \tag{3.7.13}$$

$$\tan(\theta) = \frac{y}{x}, \tag{3.7.14}$$

and

$$\cos(\varphi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \tag{3.7.15}$$

where $0 \leq \theta < 2\pi$ and $0 \leq \varphi \leq \pi$. As with polar coordinates, if $x = 0$ we let $\theta = \frac{\pi}{2}$ if $y > 0$, $\theta = \frac{3\pi}{2}$ if $y < 0$, and θ is undefined if $y = 0$. See Figure 3.7.7. Conversely, given a point P with spherical coordinates (ρ, θ, φ), the projection of P onto the xy-plane will have polar coordinate $r = \rho \sin(\varphi)$. Hence the Cartesian coordinates of P are

$$x = \rho \cos(\theta) \sin(\varphi), \tag{3.7.16}$$

$$y = \rho \sin(\theta) \sin(\varphi), \tag{3.7.17}$$

and

$$z = \rho \cos(\varphi). \tag{3.7.18}$$

Example If a point P has Cartesian coordinates $(2, -2, 1)$, then its spherical coordinates satisfy

$$\rho = \sqrt{4 + 4 + 1} = 3,$$

$$\tan(\theta) = \frac{-2}{2} = -1,$$

and

$$\cos(\varphi) = \frac{1}{\sqrt{4+4+1}} = \frac{1}{3}.$$

Hence we have

$$\theta = \frac{7\pi}{4}$$

and

$$\varphi = \cos^{-1}\left(\frac{1}{3}\right) = 1.2310,$$

where we have rounded the value of φ to four decimal places. Hence P has spherical coordinates $\left(3, \frac{7\pi}{4}, 1.2310\right)$.

Example If a point P has spherical coordinates $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$, then its Cartesian coordinates are

$$x = 4\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{4}\right) = 4\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2},$$

$$y = 4\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{4}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{6},$$

and

$$z = 4\cos\left(\frac{3\pi}{4}\right) = 4\left(-\frac{1}{\sqrt{2}}\right) = -2\sqrt{2}.$$

Analogous to our work with polar coordinates, we think of the spherical coordinate mapping

$$(x,y,z) = F(\rho,\theta,\varphi) = (\rho\cos(\theta)\sin(\varphi), \rho\sin(\theta)\sin(\varphi), \rho\cos(\varphi)) \tag{3.7.19}$$

as a change of variables between $\rho\theta\varphi$-space and xyz-space. This mapping is particularly useful for evaluating triple integrals because it maps rectangular regions in $\rho\theta\varphi$-space onto spherical regions in xyz-space. For the most basic example, for any $a > 0$, F maps the rectangular region

$$E = \{(\rho,\theta,\varphi) : 0 \le \rho \le a, 0 \le \theta < 2\pi, 0 \le \varphi \le \pi\}$$

in $\rho\theta\varphi$-space onto the closed ball

$$D = \bar{B}^3((0,0,0),a) = \{(x,y,z) : x^2+y^2+z^2 \le a\}$$

in xyz-space. More generally, for any $0 < a < b$, $0 \le \alpha < \beta < 2\pi$, and $0 \le \gamma < \delta \le \pi$, F maps the rectangular region

$$E = \{(\rho,\theta,\varphi) : a \le \rho \le b, \alpha \le \theta < \beta, \gamma \le \varphi \le \delta\}$$

onto a region D in xyz-space which lies between the concentric spheres $S^2((0,0,0),a)$ and $S^2((0,0,0),b)$, and for which the angle θ lies between α and β and the angle φ between γ and δ. For example, if $\alpha = 0$, $\beta = \pi$, $\gamma = 0$, and $\delta = \frac{\pi}{2}$, then D is one-half of the region lying between two concentric hemispheres with radii a and b.

Before using the spherical coordinate change of variable in (3.7.19) to evaluate an integral using (3.7.5), we need to compute the determinate of the Jacobian of F. Now

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{bmatrix} \frac{\partial}{\partial\rho}\rho\cos(\theta)\sin(\varphi) & \frac{\partial}{\partial\theta}\rho\cos(\theta)\sin(\varphi) & \frac{\partial}{\partial\varphi}\rho\cos(\theta)\sin(\varphi) \\ \frac{\partial}{\partial\rho}\rho\sin(\theta)\sin(\varphi) & \frac{\partial}{\partial\theta}\rho\sin(\theta)\sin(\varphi) & \frac{\partial}{\partial\varphi}\rho\sin(\theta)\sin(\varphi) \\ \frac{\partial}{\partial\rho}\rho\cos(\varphi) & \frac{\partial}{\partial\theta}\rho\cos(\varphi) & \frac{\partial}{\partial\varphi}\rho\cos(\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\sin(\varphi) & -\rho\sin(\theta)\sin(\varphi) & \rho\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & \rho\cos(\theta)\sin(\varphi) & \rho\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -\rho\sin(\varphi) \end{bmatrix}, \tag{3.7.20}$$

so, expanding along the third row,

$$\begin{aligned} \det\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \cos(\varphi)(-\rho^2\sin^2(\theta)\sin(\varphi)\cos(\varphi) - \rho^2\cos^2(\theta)\sin(\varphi)\cos(\varphi)) \\ &\quad - \rho\sin(\varphi)(\rho\cos^2(\theta)\sin^2(\varphi) + \rho\sin^2(\theta)\sin^2(\varphi) \\ &= -\rho^2\sin(\varphi)\cos^2(\varphi)(\sin^2(\theta)+\cos^2(\theta)) - \rho^2\sin^3(\varphi)(\sin^2(\theta)+\cos^2(\theta)) \\ &= -\rho^2\sin(\varphi)\cos^2(\varphi) - \rho^2\sin^3(\varphi) \\ &= -\rho^2\sin(\varphi)(\cos^2(\varphi)+\sin^2(\varphi)) \\ &= -\rho^2\sin(\varphi). \end{aligned} \tag{3.7.21}$$

Now $\rho \geq 0$ and, since $0 \leq \varphi \leq \pi$, $\sin(\varphi) \geq 0$, so

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2\sin(\varphi). \tag{3.7.22}$$

Example In an earlier example we used the fact that the volume of a sphere of radius 1 is $\frac{4\pi}{3}$. In this example we will verify that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. Let V be the volume of

$$D = \bar{B}^3((0,0,0),a),$$

the closed ball of radius a centered at the origin in $\mathbb{R}^3$. Then

$$V = \int\int\int_D dxdydz.$$

Although we may evaluate this integral using Cartesian coordinates, we will find it significantly easier to use spherical coordinates. Using the spherical coordinate change of variables

$$x = \rho\cos(\theta)\sin(\varphi),$$

$$y = \rho\sin(\theta)\sin(\varphi),$$

and

$$z = \rho\cos(\varphi),$$

the region D in xyz-space corresponds to the region

$$E = \{(\rho, \theta, \varphi) : 0 \leq \rho \leq a, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$$

in $\rho\theta\varphi$-space. Using (3.7.22) in the change of variables formula (3.7.5), we have

$$\begin{aligned}
V &= \int\int\int_D dxdydz \\
&= \int\int\int_E \left|\det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| d\rho d\theta d\varphi \\
&= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\varphi) d\varphi d\theta d\rho \\
&= \int_0^a \int_0^{2\pi} (-\rho^2\cos(\varphi))\Big|_0^{\pi} d\theta d\rho \\
&= \int_0^a \int_0^{2\pi} (-\rho^2(-1-1)) d\theta d\rho \\
&= 2\int_0^a \int_0^{2\pi} \rho^2 d\theta d\rho \\
&= 4\pi \int_0^a \rho^2 d\rho \\
&= \frac{4\pi}{3}\rho^3\Big|_0^a \\
&= \frac{4}{3}\pi a^3.
\end{aligned}$$

Example Suppose we wish to evaluate

$$\int\int\int_D \log\sqrt{x^2+y^2+z^2}\ dxdydz,$$

where D is the region in $\mathbb{R}^3$ which lies between the two spheres with equations $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ and above the xy-plane. Under the spherical coordinate change of variables

$$x = \rho\cos(\theta)\sin(\varphi),$$

$$y = \rho \sin(\theta) \sin(\varphi),$$

and

$$z = \rho \cos(\varphi),$$

the region D in xyz-space corresponds to the region

$$E = \left\{ (\rho, \theta, \varphi) : 1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{2} \right\}$$

in $\rho\theta\varphi$-space. Using (3.7.22) in the change of variables formula (3.7.5), we have

$$\begin{aligned}
\int\int\int_D \log\sqrt{x^2+y^2+z^2}\; dxdydz &= \int\int\int_E \log(\rho) \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} \right| d\rho d\theta d\varphi \\
&= \int_1^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^2 \log(\rho) \sin(\varphi) d\varphi d\theta d\rho \\
&= \int_1^2 \int_0^{2\pi} (-\rho^2 \log(\rho) \cos(\varphi)) \Big|_0^{\frac{\pi}{2}} d\theta d\rho \\
&= \int_1^2 \int_0^{2\pi} (-\rho^2 \log(\rho))(0-1) d\theta d\rho \\
&= \int_1^2 \int_0^{2\pi} \rho^2 \log(\rho) d\theta d\rho \\
&= 2\pi \int_1^2 \rho^2 \log(\rho) d\rho.
\end{aligned}$$

We use integration by parts to evaluate this final integral: letting

$$\begin{array}{ll} u = \log(\rho) & dv = \rho^2 d\rho \\ du = \dfrac{1}{\rho}\, d\rho & v = \dfrac{\rho^3}{3}, \end{array}$$

we have

$$\begin{aligned}
\int\int\int_D \log\sqrt{x^2+y^2+z^2}\; dxdydz &= 2\pi \left(\frac{1}{3}\rho^3 \log(\rho) \Big|_1^2 - \frac{1}{3} \int_1^2 \rho^2 d\rho \right) \\
&= \frac{16}{3}\pi \log(2) - \frac{2\pi\rho^3}{9} \Big|_1^2 \\
&= \frac{16}{3}\pi \log(2) - \frac{14\pi}{9} \\
&= \frac{2\pi}{3} \left(8\log(2) - \frac{7}{3} \right).
\end{aligned}$$

Problems

1. Find the area of the region enclosed by the ellipse with equation $x^2 + 4y^2 = 4$.
2. Given $a > 0$ and $b > 0$, show that the area enclosed by the ellipse with equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
is πab.
3. Find the volume of the region enclosed by the ellipsoid with equation
$$\frac{x^2}{25} + y^2 + \frac{z^2}{4} = 1.$$
4. Given $a > 0$, $b > 0$, and $c > 0$, show that the volume of the region enclosed by the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
is $\frac{4}{3}\pi abc$.
5. Find the polar coordinates for each of the following points given in Cartesian coordinates.
(a) $(1, 1)$ (b) $(-2, 3)$
(c) $(-1, 3)$ (d) $(4, -4)$
6. Find the Cartesian coordinates for each of the following points given in polar coordinates.
(a) $(3, 0)$ (b) $\left(2, \frac{5\pi}{6}\right)$
(c) $(5, \pi)$ (d) $\left(4, \frac{4\pi}{3}\right)$
7. Evaluate
$$\int\int_D (x^2 + y^2)dxdy,$$
where D is the disk in $\mathbb{R}^2$ of radius 2 centered at the origin.
8. Evaluate
$$\int\int_D \sin(x^2 + y^2)dxdy,$$
where D is the disk in $\mathbb{R}^2$ of radius 1 centered at the origin.
9. Evaluate
$$\int\int_D \frac{1}{x^2 + y^2}\, dxdy,$$
where D is the region in the first quadrant of $\mathbb{R}^2$ which lies between the circle with equation $x^2 + y^2 = 1$ and the circle with equation $x^2 + y^2 = 16$.

10. Evaluate
$$\int\int_D \log(x^2+y^2)dxdy,$$
where D is the region in $\mathbb{R}^2$ which lies between the circle with equation $x^2+y^2=1$ and the circle with equation $x^2+y^2=4$.

11. Using polar coordinates, verify that the area of a circle of radius r is πr^2.

12. Let
$$I=\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx.$$
   (a) Show that
$$I^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}dxdy.$$
   (b) Show that
$$I^2=\int_0^{\infty}\int_0^{2\pi} re^{-\frac{r^2}{2}}d\theta dr.$$
   (c) Show that
$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}.$$

13. Find the spherical coordinates of the point with Cartesian coordinates $(-1,1,2)$.

14. Find the spherical coordinates of the point with Cartesian coordinates $(3,2,-1)$.

15. Find the Cartesian coordinates of the point with spherical coordinates $\left(2,\frac{3\pi}{4},\frac{2\pi}{3}\right)$.

16. Find the Cartesian coordinates of the point with spherical coordinates $\left(5,\frac{5\pi}{3},\frac{\pi}{6}\right)$.

17. Evaluate
$$\int\int\int(x^2+y^2+z^2)dxdydz,$$
where D is the closed ball in $\mathbb{R}^3$ of radius 2 centered at the origin.

18. Evaluate
$$\int\int\int_D \frac{1}{\sqrt{x^2+y^2+z^2}}\,dxdydz,$$
where D is the region in $\mathbb{R}^3$ between the two spheres with equations $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

19. Evaluate
$$\int\int\int_D \sin(\sqrt{x^2+y^2+z^2}\,)dxdydz,$$
where D is the region in $\mathbb{R}^3$ described by $x\geq 0$, $y\geq 0$, $z\geq 0$, and $x^2+y^2+z^2\leq 1$.

20. Evaluate
$$\int\int\int_D e^{-(x^2+y^2+z^2)}dxdydz,$$
where D is the closed ball in $\mathbb{R}^3$ of radius 3 centered at the origin.

21. Let D be the region in $\mathbb{R}^3$ described by $x^2+y^2+z^2 \le 1$ and $z \ge \sqrt{x^2+y^2}$.
    (a) Explain why the spherical coordinate change of variables maps the region
$$E = \left\{(\rho,\theta,\varphi) : 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{4}\right\}$$
    onto D.
    (b) Find the volume of D.

22. If a point P has Cartesian coordinates (x,y,z), then the *cylindrical coordinates* of P are (r,θ,z), where r and θ are the polar coordinates of (x,y). Show that
$$\left|\det \frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r.$$

23. Use cylindrical coordinates to evaluate
$$\int\int_D \sqrt{x^2+y^2}dxdydz,$$
where D is the region in $\mathbb{R}^3$ described by $1 \le x^2+y^2 \le 4$ and $0 \le z \le 5$.

24. A drill with a bit with a radius of 1 centimeter is used to drill a hole through the center of a solid ball of radius 3 centimeters. What is the volume of the remaining solid?

25. Let D be the set of all points in the intersection of the two solid cylinders in $\mathbb{R}^3$ described by $x^2+y^2 \le 1$ and $x^2+z^2 \le 1$. Find the volume of D.

The Calculus of Functions

of

Several Variables

Section 4.1

Geometry, Limits, and Continuity

In this chapter we will treat the general case of a function mapping $\mathbb{R}^m$ to $\mathbb{R}^n$. Since the cases $m = 1$ and $n = 1$ have been handled in previous chapters, our emphasis will be on the higher dimensional cases, most importantly when m and n are 2 or 3. We will begin in this section with some basic terminology and definitions.

Parametrized surfaces

If $f : \mathbb{R}^m \to \mathbb{R}^n$ has domain D, we call the set S of all points $\mathbf{y}$ in $\mathbb{R}^n$ for which $\mathbf{y} = f(\mathbf{x})$ for some $\mathbf{x}$ in D the *image* of f. That is,

$$S = \{f(\mathbf{x}) : \mathbf{x} \in D\}, \tag{4.1.1}$$

which is the same as what we have previously called the range of f. If $m = 1$, S is a curve as defined in Section 2.1. If $m > 1$ and $n > m$, then we call S an *m-dimensional* surface in $\mathbb{R}^n$. If we let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_n)$, then, for $k = 1, 2, \ldots, n$, we call the function $f_k : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_k(x_1, x_2, \ldots, x_m) = y_k$$

the *k-th coordinate function* of f. We call the system of equations

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_m), \\ y_2 &= f_2(x_1, x_2, \ldots, x_m), \\ \vdots &= \qquad\quad \vdots \\ y_n &= f_n(x_1, x_2, \ldots, x_m), \end{aligned} \tag{4.1.2}$$

a *parametrization* of the surface S. Note that f_k is the type of function we studied in Chapter 3. On the other hand, if we fix values of x_i for all $i \neq k$, then the function $\varphi_k : \mathbb{R} \to \mathbb{R}^n$ defined by

$$\varphi_k(t) = f(x_1, x_2, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_m) \tag{4.1.3}$$

is of the type we studied we Chapter 2. In particular, for each $k = 1, 2, \ldots, n$, φ_k parametrizes a curve which lies on the surface S. The following examples illustrate how the latter remark is useful when trying to picture a parametrized surface S.

Example Consider $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(s, t) = (t\cos(s), t\sin(s), t)$$

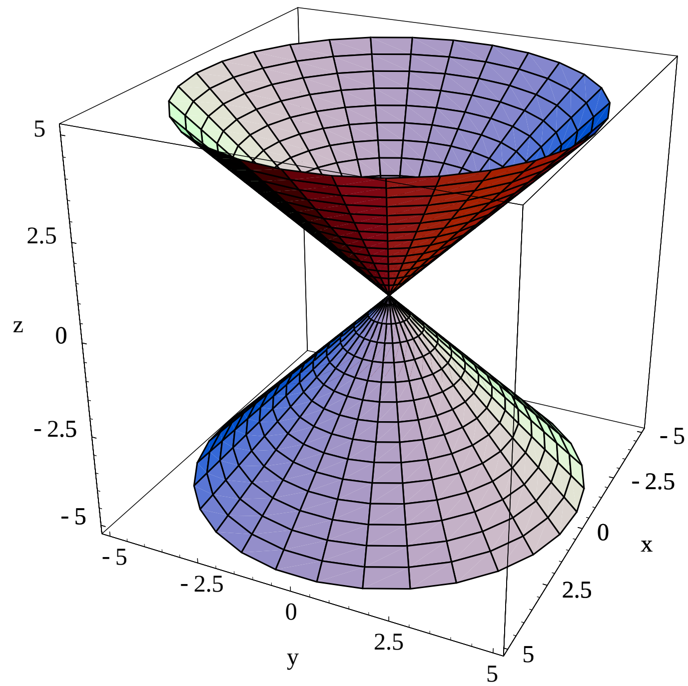


Figure 4.1.1 Cone parametrized by $f(s,t) = (t\cos(s), t\sin(s), t)$

for $0 \le s \le 2\pi$ and $-\infty < t < \infty$. The image of f is the surface S in $\mathbb{R}^3$ parametrized by the equations

$$\begin{aligned} x &= t\cos(s), \\ y &= t\sin(s), \\ z &= t. \end{aligned}$$

Note that for a fixed value of t, the function

$$\varphi_1(s) = (t\cos(s), t\sin(s), t)$$

parametrizes a circle of radius t on the plane $z = t$ with center at $(0,0,t)$. On the other hand, for a fixed value of s, the function

$$\varphi_2(t) = (t\cos(s), t\sin(s), t) = t(\cos(s), \sin(s), 1)$$

parametrizes a line through the origin in the direction of the vector $(\cos(s), \sin(s), 1$. Hence the surface S is a cone in $\mathbb{R}^3$, part of which is shown in Figure 4.1.1. Notice how the surface was drawn by plotting the curves corresponding to fixed values of s and t (that is, the curves parametrized by φ_1 and φ_2), and then filling in the resulting curvilinear "rectangles."

Example For a fixed $a > 0$, consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(s,t) = (a\cos(s)\sin(t), a\sin(s)\sin(t), a\cos(t))$$

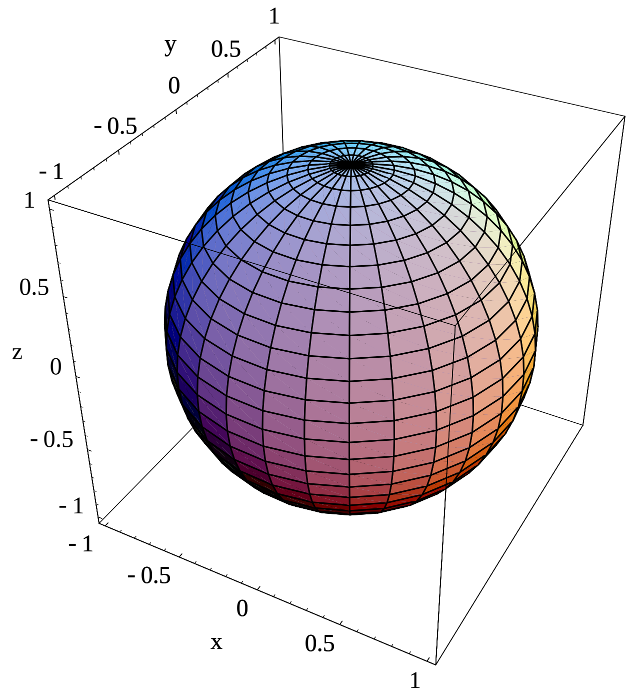


Figure 4.1.2 Unit sphere parametrized by $f(s,t) = (\cos(s)\sin(t), \sin(s)\sin(t), \cos(t))$

for $0 \le s \le 2\pi$ and $0 \le t \le \pi$. The image of f is the surface S in $\mathbb{R}^3$ parametrized by the equations

$$\begin{aligned} x &= a\cos(s)\sin(t), \\ y &= a\sin(s)\sin(t), \\ z &= a\cos(t). \end{aligned} \tag{4.1.4}$$

Note that these are the equations for the spherical coordinate change of variables discussed in Section 3.7, with $\rho = a$, $\theta = s$, and $\varphi = t$. Since a is fixed while s varies from 0 to 2π and t varies from 0 to π, it follows that S is a sphere of radius a with center $(0,0,0)$. Figure 4.1.2 displays S when $a = 1$. If we had not previously studied spherical coordinates, we could reach this conclusion about S as follows. First note that

$$\begin{aligned} x^2 + y^2 + z^2 &= a^2\cos^2(s)\sin^2(t) + a^2\sin^2(s)\sin^2(t) + a^2\cos^2(t) \\ &= a^2\sin^2(t)(\cos^2(s) + \sin^2(s)) + a^2\cos^2(t) \\ &= a^2(\sin^2(t) + \cos^2(t)) \\ &= a^2, \end{aligned}$$

from which it follows that every point of S lies on the sphere of radius a centered at the origin. Now for a fixed value of t,

$$\varphi_1(s) = (a\cos(s)\sin(t), a\sin(s)\sin(t), a\cos(t))$$

parametrizes a circle in the plane $z = a\cos(t)$ with center $(0, 0, a\cos(t))$ and radius $a\sin(t)$. As t varies from 0 to π, these circles vary from a circle in the $z = a$ plane with center

$(0, 0, a)$ and radius 0 (when $t = 0$) to a circle in the xy-plane with center $(0, 0, 0)$ and radius a (when $t = \frac{\pi}{2}$) to a circle in the $z = -a$ plane with center $(0, 0, -a)$ and radius 0 (when $t = \pi$). In other words, the circles fill in all the "lines of latitude" of the sphere from the "North Pole" to the "South Pole," and hence S is all of the sphere. One may also show that the functions

$$\varphi_2(t) = (a\cos(s)\sin(t), a\sin(s)\sin(t), a\cos(t))$$

parametrize the "lines of longitude" of S as s varies from 0 to 2π. Both the lines of "latitude" and "longitude" are visible in Figure 4.2.2.

Example Suppose $0 < b < a$ and define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(s, t) = ((a + b\cos(t))\cos(s), (a + b\cos(t))\sin(s), b\sin(t))$$

for $0 \le s \le 2\pi$ and $0 \le t \le 2\pi$. The image of f is the surface T parametrized by the equations

$$\begin{aligned} x &= (a + b\cos(t))\cos(s), \\ y &= (a + b\cos(t))\sin(s), \\ z &= b\sin(t). \end{aligned}$$

Note that for a fixed value of t,

$$\varphi_1(s) = ((a + b\cos(t))\cos(s), (a + b\cos(t))\sin(s), b\sin(t))$$

parametrizes a circle in the plane $z = b\sin(t)$ with center $(0, 0, b\sin(t)$ and radius $a+b\cos(t)$. In particular, when $t = 0$, we have a circle in the xy-plane with center $(0, 0, 0)$ and radius $a + b$; when $t = \frac{\pi}{2}$, we have a circle on the plane $z = b$ with center $(0, 0, b)$ and radius a; when $t = \pi$, we have a circle on the xy-plane with center $(0, 0, 0)$ and radius $a - b$; when $t = \frac{3\pi}{2}$, we have a circle on the $z = -b$ plane with center $(0, 0, -b)$ and radius a; and when $t = 2\pi$, we are back to a circle in the xy-plane with center $(0, 0, 0)$ and radius $a + b$. For fixed values of s, the curves parametrized by

$$\varphi_2(t) = ((a + b\cos(t))\cos(s), (a + b\cos(t))\sin(s), b\sin(t))$$

are not identified as easily. However, some particular cases are illuminating. When $s = 0$, we have a circle in the xz-plane with center $(a, 0, 0)$ and radius b; when $s = \frac{\pi}{2}$, we have a circle in the yz-plane with center $(0, a, 0)$ and radius b; when $s = \pi$, we have a circle in the xz-plane with center $(-a, 0, 0)$ and radius b; when $s = \frac{3\pi}{2}$, we have a circle in the yz-plane with center $(0, -a, 0)$ and radius b; and when $t = 2\pi$, we are back to a circle in the xz-plane with center $(a, 0, 0)$ and radius b. Putting all this together, we see that T is a *torus*, the surface of a doughnut shaped object. Figure 4.1.3 shows one such torus, the case $a = 3$ and $b = 1$.

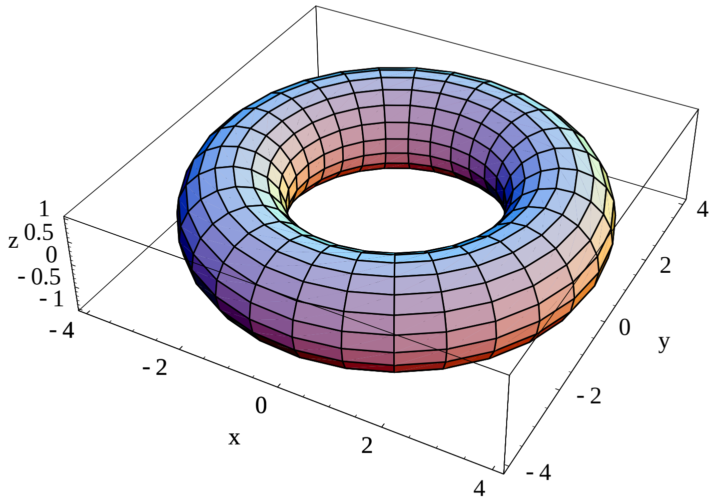


Figure 4.1.3 A torus: $f(s,t) = (3+\cos(t))\cos(s), (3+\cos(t))\sin(s), \sin(t))$

Vector fields

We call a function $f : \mathbb{R}^n \to \mathbb{R}^n$, that is, a function for which the domain and range space have the same dimension, a *vector field*. We have seen a few examples of such functions already. For example, the change of variable functions in Section 3.7 were of this type. Also, given a function $g : \mathbb{R}^n \to \mathbb{R}$, the gradient of g,

$$\nabla g(\mathbf{x}) = \left(\frac{\partial}{\partial x_1} g(\mathbf{x}), \frac{\partial}{\partial x_2} g(\mathbf{x}), \ldots, \frac{\partial}{\partial x_n} g(\mathbf{x})\right),$$

is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. As we saw in our discussion of gradient vector fields in Section 3.2, a plot showing the vectors $f(\mathbf{x})$ at each point in a rectangular grid provides a useful geometric view of a vector field f.

Example Consider the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(\mathbf{x}) = -\frac{\mathbf{x}}{\|\mathbf{x}\|^2}$$

for all $\mathbf{x} \neq \mathbf{0}$. Note that $f(\mathbf{x})$ is a vector of length

$$\left\|\frac{\mathbf{x}}{\|\mathbf{x}\|^2}\right\| = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|^2} = \frac{1}{\|\mathbf{x}\|}$$

pointing in the direction opposite that of $\mathbf{x}$. If $n = 2$, the coordinate functions of f are

$$f_1(x_1, x_2) = -\frac{x_1}{x_1^2 + x_2^2}$$

and

$$f_2(x_1, x_2) = -\frac{x_2}{x_1^2 + x_2^2}.$$

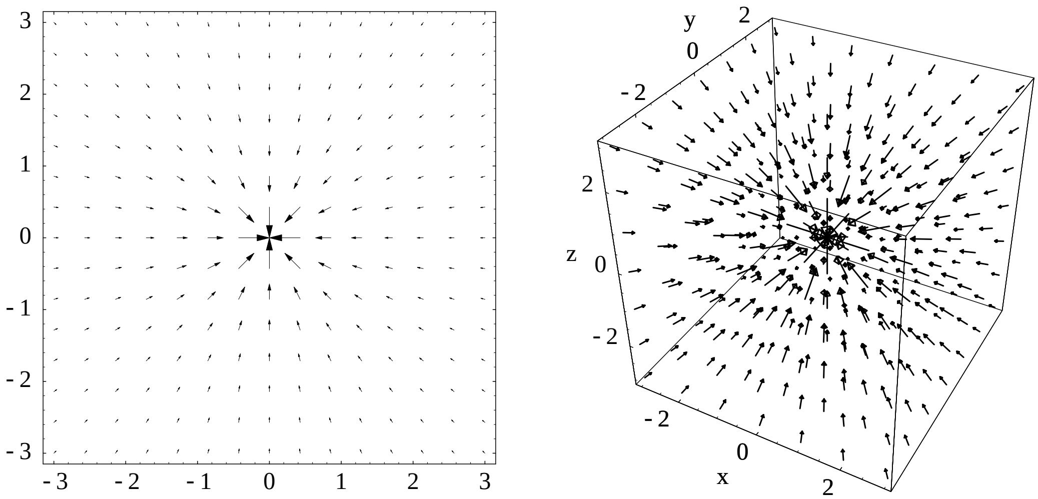


Figure 4.1.4 Vector field $f(\mathbf{x}) = \dfrac{\mathbf{x}}{\|\mathbf{x}\|^2}$ for $n = 2$ and $n = 3$

Figure 4.1.4 shows a plot of the vectors $f(\mathbf{x})$ for this case, drawn on a grid over the rectangle $[-3, 3] \times [-3, 3]$, and for the case $n = 3$, using the cube $[-3, 3] \times [-3, 3] \times [-3, 3]$. Note that these plots do not show the vectors $f(\mathbf{x})$ themselves, but vectors which have been scaled proportionately so they do not overlap one another.

Limits and continuity

The definitions of limits and continuity for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ follow the familiar pattern.

Definition Let $\mathbf{a}$ be a point in $\mathbb{R}^m$ and let O be the set of all points in the open ball of radius $r > 0$ centered at $\mathbf{a}$ except $\mathbf{a}$. That is,

$$O = \{\mathbf{x} : \mathbf{x} \in B^n(\mathbf{a}, r), \mathbf{x} \neq \mathbf{a}\}.$$

Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined for all $\mathbf{x}$ in O. We say the *limit* of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is $\mathbf{L}$, written $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{L}$, if for every sequence of points $\{\mathbf{x}_k\}$ in O,

$$\lim_{k \to \infty} f(\mathbf{x}_k) = \mathbf{L} \tag{4.1.5}$$

whenever $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$.

In Section 2.1 we saw that a sequence of points in $\mathbb{R}^n$ has a limit if and only if the individual coordinates of the points in the sequence each have a limit. The following proposition is an immediate consequence.

Proposition If $f_k : \mathbb{R}^m \to \mathbb{R}$, $k = 1, 2, \ldots, n$, is the kth coordinate function of $f : \mathbb{R}^m \to \mathbb{R}^n$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = (L_1, L_2, \ldots, L_n)$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{a}} f_k(\mathbf{x}) = L_k$$

for $k = 1, 2, \ldots, n$.

In other words, the computation of limits for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ reduces to the familiar problem of computing limits of real-valued functions, as we discussed in Section 3.1.

Example If

$$f(x, y, z) = (x^2 - 3yz, 4xz),$$

a function from $\mathbb{R}^3$ to $\mathbb{R}^2$, then, for example,

$$\lim_{(x,y,z)\to(1,-2,3)} f(x, y, z) = \left(\lim_{(x,y,z)\to(1,-2,3)} (x^2 - 3yz), \lim_{(x,y,z)\to(1,-2,3)} 4xz\right) = (19, 12).$$

Definition Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined for all $\mathbf{x}$ in some open ball $B^n(\mathbf{a}, r)$, $r > 0$. We say f is *continuous* at $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

The following result is an immediate consequence of the previous proposition.

Proposition If $f_k : \mathbb{R}^m \to \mathbb{R}$, $k = 1, 2, \ldots, n$, is the kth coordinate function of $f : \mathbb{R}^m \to \mathbb{R}^n$, then f is continuous at a point $\mathbf{a}$ if and only if f_k is continuous at $\mathbf{a}$ for $k = 1, 2, \ldots, n$.

In other words, checking for continuity for a function $f : \mathbb{R}^m \to \mathbb{R}^n$ reduces to checking the continuity of real-valued functions, a familiar problem from Section 3.1.

Example The function

$$f(x, y) = (3\sin(x + y), 4x^2y)$$

has coordinate functions

$$f_1(x, y) = 3\sin(x + y)$$

and

$$f_2(x, y) = 4x^2y.$$

Since, from our results in Section 3.1, both f_1 and f_2 are continuous at every point in $\mathbb{R}^2$, it follows that f is continuous at every point in $\mathbb{R}^2$.

Definition We say a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is *continuous* on an open set U if f is continuous at every point $\mathbf{u}$ in U.

Example We may restate the conclusion of the previous example by saying that

$$f(x, y) = (3\sin(x + y), 4x^2y)$$

is continuous on $\mathbb{R}^2$.

Problems

1. For each of the following, plot the surface parametrized by the given function.
   (a) $f(s,t) = (t^2\cos(s), t^2\sin(s), t^2)$, $0 \le s \le 2\pi$, $0 \le t \le 3$
   (b) $f(u,v) = (3\cos(u)\sin(v), \sin(u)\sin(v), 2\cos(v))$, $0 \le u \le 2\pi$, $0 \le v \le \pi$
   (c) $g(s,t) = ((4+2\cos(t))\cos(s), (4+2\cos(t))\sin(s), 2\sin(t))$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$
   (d) $f(s,t) = ((5+2\cos(t))\cos(s), 2(5+2\cos(t))\sin(s), \sin(t))$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$
   (e) $h(u,v) = (\sin(v), (3+\cos(v))\cos(u), (3+\cos(v))\sin(u))$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$
   (f) $g(s,t) = (s, s^2+t^2, t)$, $-2 \le s \le 2$, $-2 \le t \le 2$
   (g) $f(x,y) = (y\cos(x), y, y\sin(x))$, $0 \le x \le 2\pi$, $-5 \le y \le 5$
2. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and we define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(s,t) = (s,t,f(s,t))$. Describe the surface parametrized by F.
3. Find a parametrization for the surface that is the graph of the function $f(x,y) = x^2 + y^2$.
4. Make plots like those in Figure 4.1.4 for each of the following vector fields. Experiment with the rectangle used for the grid, as well as with the number of vectors drawn.
   (a) $f(x,y) = (y, -x)$
   (b) $g(x,y) = (y, -\sin(x))$
   (c) $f(u,v) = (v, u - u^3 - v)$
   (d) $f(x,y) = (x(1-y^2) - y, x)$
   (e) $f(x,y,z) = \left(10(y-x), 28x - y - xz, -\frac{8}{3}z + xy\right)$
   (f) $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$
   (g) $g(u,v,w) = -\frac{1}{(u-1)^2+(v-2)^2+(w-1)^2}(u-1, v-2, w-1)$
5. Find the set of points in $\mathbb{R}^2$ for which the vector field
$$f(x,y) = \left(4x\sin(x-y), \frac{4x+3y}{2x-y}\right)$$
is continuous.
6. For which points in $\mathbb{R}^n$ is the vector field
$$f(\mathbf{x}) = \frac{\mathbf{x}}{\log(\|\mathbf{x}\|)}$$
a continuous function?

The Calculus of Functions

of

Several Variables

Section 4.2

Best Affine Approximations

Best affine approximations

The following definitions should look very familiar.

Definition Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined on an open ball containing the point $\mathbf{c}$. We call an affine function $A : \mathbb{R}^m \to \mathbb{R}^n$ the *best affine approximation* to f at $\mathbf{c}$ if (1) $A(\mathbf{c}) = f(\mathbf{c})$ and (2) $\|R(\mathbf{h})\|$ is $o(\mathbf{h})$, where

$$R(\mathbf{h}) = f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h}). \tag{4.2.1}$$

Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is the best affine approximation to f at $\mathbf{c}$. Then, from our work in Section 1.5, there exists an $n \times m$ matrix M and a vector $\mathbf{b}$ in $\mathbb{R}^n$ such that

$$A(\mathbf{x}) = M\mathbf{x} + \mathbf{b} \tag{4.2.2}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. Moreover, the condition $A(\mathbf{c}) = f(\mathbf{c})$ implies $f(\mathbf{c}) = M\mathbf{c} + \mathbf{b}$, and so $\mathbf{b} = f(\mathbf{c}) - M\mathbf{c}$. Hence we have

$$A(\mathbf{x}) = M\mathbf{x} + f(\mathbf{c}) - M\mathbf{c} = M(\mathbf{x} - \mathbf{c}) + f(\mathbf{c}) \tag{4.2.3}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. Thus to find the best affine approximation we need only identify the matrix M in (4.2.3).

Definition Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined on an open ball containing the point $\mathbf{c}$. If f has a best affine approximation at $\mathbf{c}$, then we say f is *differentiable* at $\mathbf{c}$. Moreover, if the best affine approximation to f at $\mathbf{c}$ is given by

$$A(\mathbf{x}) = M(\mathbf{x} - \mathbf{c}) + f(\mathbf{c}), \tag{4.2.4}$$

then we call M the *derivative* of f at $\mathbf{c}$ and write $Df(\mathbf{c}) = M$.

Now suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ and A is an affine function with $A(\mathbf{c}) = f(\mathbf{c})$. Let f_k and A_k be the kth coordinate functions of f and A, respectively, for $k = 1, 2, \ldots, n$, and let R be the remainder function

$$\begin{aligned} R(\mathbf{h}) &= f(\mathbf{c} + \mathbf{h}) - A(\mathbf{c} + \mathbf{h}) \\ &= (f_1(\mathbf{c} + \mathbf{h}) - A_1(\mathbf{c} + \mathbf{h}), f_2(\mathbf{c} + \mathbf{h}) - A_2(\mathbf{c} + \mathbf{h}), \ldots, f_n(\mathbf{c} + \mathbf{h}) - A_n(\mathbf{c} + \mathbf{h})). \end{aligned}$$

Then

$$\frac{R(\mathbf{h})}{\|\mathbf{h}\|} = \left(\frac{f_1(\mathbf{c}+\mathbf{h}) - A_1(\mathbf{c}+\mathbf{h})}{\|\mathbf{h}\|}), \frac{f_2(\mathbf{c}+\mathbf{h}) - A_2(\mathbf{c}+\mathbf{h})}{\|\mathbf{h}\|}, \ldots, \frac{f_n(\mathbf{c}+\mathbf{h}) - A_n(\mathbf{c}+\mathbf{h})}{\|\mathbf{h}\|}\right),$$

and so

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|R(\mathbf{h}\|}{\|\mathbf{h}\|} = 0, \tag{4.2.5}$$

that is, A is the best affine approximation to f at $\mathbf{c}$, if and only if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f_k(\mathbf{c}+\mathbf{h}) - A_k(\mathbf{c}+\mathbf{h})}{\|\mathbf{h}\|} = 0 \tag{4.2.6}$$

for $k = 1, 2, \ldots, n$. But (4.2.6) is the statement that A_k is the best affine approximation to f_k at $\mathbf{c}$. In other words, A is the best affine approximation to f at $\mathbf{c}$ if and only if A_k is the best affine approximation to f_k at $\mathbf{c}$ for $k = 1, 2, \ldots, n$. This result has many interesting consequences.

Proposition If $f_k : \mathbb{R}^m \to \mathbb{R}$ is the kth coordinate function of $f : \mathbb{R}^m \to \mathbb{R}^n$, then f is differentiable at a point $\mathbf{c}$ if and only if f_k is differentiable at $\mathbf{c}$ for $k = 1, 2, \ldots, n$.

Definition If $f_k : \mathbb{R}^m \to \mathbb{R}$ is the kth coordinate function of $f : \mathbb{R}^m \to \mathbb{R}^n$, then we say f is C^1 on an open set U if f_k is C^1 on U for $k = 1, 2, \ldots, n$.

Putting our results in Section 3.3 together with the previous proposition and definition, we have the following basic result.

Theorem If $f : \mathbb{R}^m \to \mathbb{R}^n$ is C^1 on an open ball containing the point $\mathbf{c}$, then f is differentiable at $\mathbf{c}$.

Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ with best affine approximation A and $f_k : \mathbb{R}^m \to \mathbb{R}$ and $A_k : \mathbb{R}^m \to \mathbb{R}$ are the coordinate functions of f and A, respectively, for $k = 1, 2, \ldots, n$. Since A_k is the best affine approximation to f_k at $\mathbf{c}$, we know from Section 3.3 that

$$A_k(\mathbf{x}) = \nabla f_k(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}) + f_k(\mathbf{c}) \tag{4.2.7}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. Hence, writing the vectors as column vectors, we have

$$\begin{aligned} A(\mathbf{x}) &= \begin{bmatrix} A_1(\mathbf{x}) \\ A_2(\mathbf{x}) \\ \vdots \\ A_n(\mathbf{x}) \end{bmatrix} \\ &= \begin{bmatrix} \nabla f_1(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}) + f_1(\mathbf{c}) \\ \nabla f_2(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}) + f_2(\mathbf{c}) \\ \vdots \\ \nabla f_n(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}) + f_n(\mathbf{c}) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\mathbf{c}) & \frac{\partial}{\partial x_2} f_1(\mathbf{c}) & \cdots & \frac{\partial}{x_m} f_1(\mathbf{c}) \\ \frac{\partial}{\partial x_1} f_2(\mathbf{c}) & \frac{\partial}{\partial x_2} f_2(\mathbf{c}) & \cdots & \frac{\partial}{x_m} f_2(\mathbf{c}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_n(\mathbf{c}) & \frac{\partial}{\partial x_2} f_n(\mathbf{c}) & \cdots & \frac{\partial}{x_m} f_n(\mathbf{c}) \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ x_2 - c_2 \\ \vdots \\ x_m - c_m \end{bmatrix} + \begin{bmatrix} f_1(\mathbf{c}) \\ f_2(\mathbf{c}) \\ \vdots \\ f_m(\mathbf{c}) \end{bmatrix}. \quad (4.2.8)$$

It follows that the $n \times m$ matrix in (4.2.8) is the derivative of f.

Theorem If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at a point $\mathbf{c}$, then the derivative of f at $\mathbf{c}$ is given by

$$Df(\mathbf{c}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\mathbf{c}) & \frac{\partial}{\partial x_2} f_1(\mathbf{c}) & \cdots & \frac{\partial}{x_m} f_1(\mathbf{c}) \\ \frac{\partial}{\partial x_1} f_2(\mathbf{c}) & \frac{\partial}{\partial x_2} f_2(\mathbf{c}) & \cdots & \frac{\partial}{x_m} f_2(\mathbf{c}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_n(\mathbf{c}) & \frac{\partial}{\partial x_2} f_n(\mathbf{c}) & \cdots & \frac{\partial}{x_m} f_n(\mathbf{c}) \end{bmatrix}. \quad (4.2.9)$$

We call the matrix in (4.2.9) the *Jacobian matrix* of f, after the German mathematician Carl Gustav Jacob Jacobi (1804-1851). Note that we have seen this matrix before in our discussion of change of variables in integrals in Section 3.7.

Example Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (xyz, 3x - 2yz).$$

The coordinate functions of f are

$$f_1(x, y, z) = xyz$$

and

$$f_2(x, y, z) = 3x - 2yz.$$

Now

$$\nabla f_1(x, y, z) = (yz, xz, xy)$$

and

$$\nabla f_2(x, y, z) = (3, -2z, -2y),$$

so the Jacobian of f is

$$Df(x, y, z) = \begin{bmatrix} yz & xz & xy \\ 3 & -2z & -2y \end{bmatrix}.$$

Hence, for example,

$$Df(1, 2, -1) = \begin{bmatrix} -2 & -1 & 2 \\ 3 & 2 & -4 \end{bmatrix}.$$

Since $f(1,2,-1) = (-2,7)$, the best affine approximation to f at $(1,2,-1)$ is

$$\begin{aligned} A(x,y,z) &= \begin{bmatrix} -2 & -1 & 2 \\ 3 & 2 & -4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z+1 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} -2(x-1) - (y-2) + 2(z+1) - 2 \\ 3(x-1) + 2(y-2) - 4(z+1) + 7 \end{bmatrix} \\ &= \begin{bmatrix} -2x - y + 2z + 4 \\ 3x + 2y - 4z - 4 \end{bmatrix}. \end{aligned}$$

Tangent planes

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^3$ parametrizes a surface S in $\mathbb{R}^3$. If f_1, f_2, and f_3 are the coordinate functions of f, then the best affine approximation to f at a point (s_0, t_0) is given by

$$\begin{aligned} A(s,t) &= \begin{bmatrix} \frac{\partial}{\partial s} f_1(t_0, s_0) & \frac{\partial}{\partial t} f_1(t_0, s_0) \\ \frac{\partial}{\partial s} f_2(t_0, s_0) & \frac{\partial}{\partial t} f_2(t_0, s_0) \\ \frac{\partial}{\partial s} f_3(t_0, s_0) & \frac{\partial}{\partial t} f_3(t_0, s_0) \end{bmatrix} \begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix} + \begin{bmatrix} f_1(s_0, t_0) \\ f_2(s_0, t_0) \\ f_3(s_0, t_0) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial s} f_1(s_0, t_0) \\ \frac{\partial}{\partial s} f_2(s_0, t_0) \\ \frac{\partial}{\partial s} f_3(s_0, t_0) \end{bmatrix} (s - s_0) + \begin{bmatrix} \frac{\partial}{\partial t} f_1(s_0, t_0) \\ \frac{\partial}{\partial t} f_2(s_0, t_0) \\ \frac{\partial}{\partial t} f_3(s_0, t_0) \end{bmatrix} (t - t_0) + \begin{bmatrix} f_1(s_0, t_0) \\ f_2(s_0, t_0) \\ f_3(s_0, t_0) \end{bmatrix} \end{aligned} \tag{4.2.10}$$

If the vectors

$$\mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial s} f_1(s_0, t_0) \\ \frac{\partial}{\partial s} f_2(s_0, t_0) \\ \frac{\partial}{\partial s} f_3(s_0, t_0) \end{bmatrix} \tag{4.2.11}$$

and

$$\mathbf{w} = \begin{bmatrix} \frac{\partial}{\partial t} f_1(s_0, t_0) \\ \frac{\partial}{\partial t} f_2(s_0, t_0) \\ \frac{\partial}{\partial t} f_3(s_0, t_0) \end{bmatrix} \tag{4.2.12}$$

are linearly independent, then (4.2.10) implies that the image of A is a plane in $\mathbb{R}^3$ which passes through the point $f(s_0, t_0)$ on the surface S. Moreover, if we let C_1 be the curve on S through the point $f(s_0, t_0)$ parametrized by $\varphi_1(s) = f(s, t_0)$ and C_2 be the curve on S through the point $f(s_0, t_0)$ parametrized by $\varphi_2(t) = f(s_0, t)$, then $\mathbf{v}$ is tangent to C_1 at $f(s_0, t_0)$ and $\mathbf{w}$ is tangent to C_2 at $f(s_0, t_0)$. Hence we call the image of A the *tangent plane* to the surface S at the point $f(s_0, t_0)$.

Example Let T be the torus parametrized by

$$f(s,t) = ((3+\cos(t))\cos(s), (3+\cos(t))\sin(s), \sin(t))$$

for $0 \le s \le 2\pi$ and $0 \le t \le 2\pi$. Then

$$Df(s,t) = \begin{bmatrix} -(3+\cos(t))\sin(s) & -\sin(t)\cos(s) \\ (3+\cos(t))\cos(s) & -\sin(t)\sin(s) \\ 0 & \cos(t) \end{bmatrix}.$$

Thus, for example,

$$Df\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \begin{bmatrix} -\left(3+\frac{1}{\sqrt{2}}\right) & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Since

$$f\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(0, 3+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

the best affine approximation to f at $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ is

$$\begin{aligned} A(s,t) &= \begin{bmatrix} -\left(3+\frac{1}{\sqrt{2}}\right) & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} s-\frac{\pi}{2} \\ t-\frac{\pi}{4} \end{bmatrix} + \begin{bmatrix} 0 \\ 3+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} -\left(3+\frac{1}{\sqrt{2}}\right) \\ 0 \\ 0 \end{bmatrix} \left(s-\frac{\pi}{2}\right) + \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \left(t-\frac{\pi}{4}\right) + \begin{bmatrix} 0 \\ 3+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} x &= -\left(3+\frac{1}{\sqrt{2}}\right)\left(s-\frac{\pi}{2}\right), \\ y &= -\frac{1}{\sqrt{2}}\left(t-\frac{\pi}{4}\right) + 3 + \frac{1}{\sqrt{2}}, \\ z &= \frac{1}{\sqrt{2}}\left(t-\frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}, \end{aligned}$$

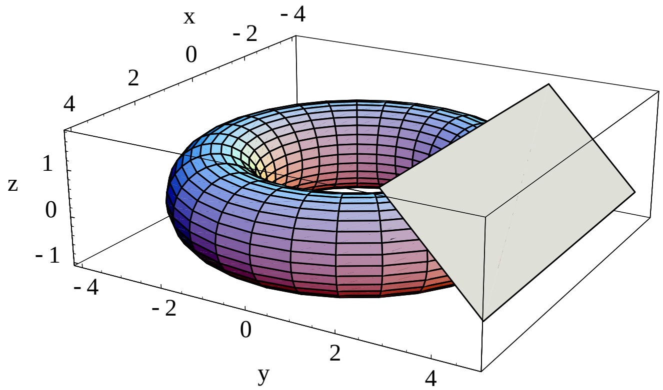


Figure 4.2.1 Torus with a tangent plane

are parametric equations for the plane P tangent to T at $\left(0, 3+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. See Figure 4.2.1.

Chain rule

We are now in a position to state the chain rule in its most general form. Consider functions $g : \mathbb{R}^m \to \mathbb{R}^q$ and $f : \mathbb{R}^q \to \mathbb{R}^n$ and suppose g is differentiable at $\mathbf{c}$ and f is differentiable at $g(\mathbf{c})$. Let $h : \mathbb{R}^m \to \mathbb{R}^n$ be the composition $h(\mathbf{x}) = f(g(\mathbf{x}))$ and denote the coordinate functions of f, g, and h by f_i, $i = 1, 2, \ldots, n$, g_j, $j = 1, 2 \ldots, q$, and h_k, $k = 1, 2, \ldots, n$, respectively. Then, for $k = 1, 2, \ldots, n$,

$$h_k(x_1, x_2, \ldots, x_m) = f_k(g_1(x_1, x_2, \ldots, x_m), g_2(x_1, x_2, \ldots, x_m), \ldots, g_q(x_1, x_2, \ldots, x_m)).$$

Now if we fix $m-1$ of the variables x_1, x_2, $\ldots$, x_m, say, all but x_j, then h_k is the composition of a function from $\mathbb{R}$ to $\mathbb{R}^q$ with a function from $\mathbb{R}^q$ to $\mathbb{R}$. Thus we may use the chain rule from Section 3.3 to compute $\frac{\partial}{\partial x_j} h_k(\mathbf{c})$, namely,

$$\begin{aligned} \frac{\partial}{\partial x_j} h_k(\mathbf{c}) &= \nabla f_k(g(\mathbf{c})) \cdot \left(\frac{\partial}{\partial x_j} g_1(\mathbf{c}), \frac{\partial}{\partial x_j} g_2(\mathbf{c}), \ldots, \frac{\partial}{x_j} g_q(\mathbf{c}) \right) \\ &= \frac{\partial}{\partial x_1} f_k(g(\mathbf{c})) \frac{\partial}{\partial x_j} g_1(\mathbf{c}) + \frac{\partial}{\partial x_2} f_k(g(\mathbf{c})) \frac{\partial}{\partial x_j} g_2(\mathbf{c}) + \\ &\qquad \cdots + \frac{\partial}{\partial x_q} f_k(g(\mathbf{c})) \frac{\partial}{\partial x_j} g_q(\mathbf{c}). \end{aligned} \tag{4.2.13}$$

Hence $\frac{\partial}{\partial x_j} h_k(\mathbf{c})$ is equal to the dot product of the kth row of $Df(g(\mathbf{c}))$ with the jth column of $Dg(\mathbf{c})$. Moreover, if g is C^1 on an open ball about $\mathbf{c}$ and f is C^1 on an open ball about $g(\mathbf{c})$, then (4.2.13) shows that $\frac{\partial}{\partial x_j} h_k$ is continuous on an open ball about $\mathbf{c}$. It follows from our results in Section 3.3 that h is differentiable at $\mathbf{c}$. Since $\frac{\partial}{\partial x_j} h_k$ is the entry in the kth row and jth column of $Dh(\mathbf{c})$, (4.2.13) implies $Dh(\mathbf{c}) = Df(g(\mathbf{c}))Dg(\mathbf{c})$. This result, the chain rule, may be proven without assuming that f and g are both C^1, and so we state the more general result in the following theorem.

Chain Rule If $g : \mathbb{R}^m \to \mathbb{R}^q$ is differentiable at $\mathbf{c}$ and $f : \mathbb{R}^q \to \mathbb{R}^n$ is differentiable at $g(\mathbf{c})$, then $f \circ g$ is differentiable at $\mathbf{c}$ and

$$D(f \circ g)(\mathbf{c}) = Df(g(\mathbf{c}))Dg(\mathbf{c}). \tag{4.2.14}$$

Equivalently, the chain rule says that if A is the best affine approximation to g at $\mathbf{c}$ and B is the best affine approximation to f at $g(\mathbf{c})$, then $B \circ A$ is the best affine approximation to $f \circ g$ at $\mathbf{c}$. That is, the best affine approximation to a composition of functions is the composition of the individual best affine approximations.

Example Suppose $g : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$g(s,t) = (\cos(s)\sin(t), \sin(s)\sin(t), \cos(t))$$

and $f : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$f(x,y,z) = (10xyz, x^2 - yz).$$

Then

$$Dg(s,t) = \begin{bmatrix} -\sin(s)\sin(t) & \cos(s)\cos(t) \\ \cos(s)\sin(t) & \sin(s)\cos(t) \\ 0 & -\sin(t) \end{bmatrix}$$

and

$$Df(x,y,z) = \begin{bmatrix} 10yz & 10xz & 10xy \\ 2x & -z & -y \end{bmatrix}.$$

Let $h(s,t) = f(g(s,t))$. To find $Dh\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, we first note that

$$g\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right),$$

$$Dg\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$Df\left(g\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\right) = Df\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) = \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & \frac{5}{2} \\ 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}.$$

Thus

$$\begin{aligned}
Dh\left(\frac{\pi}{4},\frac{\pi}{4}\right) &= Df\left(g\left(\frac{\pi}{4},\frac{\pi}{4}\right)\right) Dg\left(\frac{\pi}{4},\frac{\pi}{4}\right) \\
&= \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & \frac{5}{2} \\ 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{5}{2\sqrt{2}} \\ -\frac{1+\sqrt{2}}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

Problems

1. Find the best affine approximation for each of the following functions at the specified point $\mathbf{c}$.
   (a) $f(x,y) = (x^2+y^2, 3xy)$, $\mathbf{c} = (1,2)$
   (b) $g(x,y,z) = (\sin(x+y+z), xy\cos(z))$, $\mathbf{c} = \left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$
   (c) $h(s,t) = (3s^2+t, s-t, 4st^2, 4t-s)$, $\mathbf{c} = (-1,3)$
2. Each of the following functions parametrizes a surface S in $\mathbb{R}^3$. In each case, find parametric equations for the tangent plane P passing through the point $f(s_0,t_0)$. Plot S and P together.
   (a) $f(s,t) = (t\cos(s), t\sin(s), t)$, $(s_0,t_0) = \left(\frac{\pi}{2}, 2\right)$
   (b) $f(s,t) = (t^2\cos(s), t^2, t^2\sin(s))$, $(s_0,t_0) = (0,1)$
   (c) $f(s,t) = (\cos(s)\sin(t), \sin(s)\sin(t), \cos(t))$, $(s_0,t_0) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$
   (d) $f(s,t) = (3\cos(s)\sin(t), \sin(s)\sin(t), 2\cos(t))$, $(s_0,t_0) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$
   (e) $f(s,t) = ((4+2\cos(t))\cos(s), (4+2\cos(t))\sin(s), 2\sin(t))$, $(s_0,t_0) = \left(\frac{3\pi}{4}, \frac{\pi}{4}\right)$
3. Let S be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Define the function $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(s,t) = (s,t,f(s,t))$. We may find an equation for the plane tangent to S at $(s_0, t_0, f(s_0,t_0))$ using either the techniques of Section 3.3 (looking at S as the graph of f) or the techniques of this section (looking at S as a surface parametrized by F). Verify that these two approaches yield equations for the same plane, both in the special case when $f(s,t) = s^2+t^2$ and $(s_0,t_0) = (1,2)$, and in the general case.
4. Use the chain rule to find the derivative of $f \circ g$ at the point $\mathbf{c}$ for each of the following.
   (a) $f(x,y) = (x^2y, x-y)$, $g(s,t) = (3st, s^2-4t)$, $\mathbf{c} = (1,-2)$
   (b) $f(x,y,z) = (4xy, 3xz)$, $g(s,t) = \left(st^2-4t, s^2, \dfrac{4}{st}\right)$, $\mathbf{c} = (-2,3)$
   (c) $f(x,y) = (3x+4y, 2x^2y, x-y)$, $g(s,t,u) = (4s-3t+u, 5st^2)$, $\mathbf{c} = (1,-2,3)$

5. Suppose
$$x = f(u,v),$$
$$y = g(u,v),$$
and
$$u = h(s,t),$$
$$v = k(s,t).$$
   (a) Show that
$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s}$$
   and
$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial t}.$$
   (b) Find similar expressions for $\frac{\partial y}{\partial s}$ and $\frac{\partial y}{\partial t}$.
6. Use your results in Problem 5 to find $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ when
$$x = u^2 v,$$
$$y = 3u - v,$$
and
$$u = 4t^2 - s^2,$$
$$v = \frac{4t}{s}.$$
7. Suppose T is a function of x and y where
$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$
Show that
$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cos(\theta) + \frac{\partial T}{\partial y}\sin(\theta)$$
and
$$\frac{\partial T}{\partial \theta} = -\frac{\partial T}{\partial x} r\sin(\theta) + \frac{\partial T}{\partial y} r\cos(\theta).$$
8. Suppose the temperature at a point (x, y) in the plane is given by
$$T = 100 - \frac{20}{\sqrt{1 + x^2 + y^2}}.$$
   (a) If (r, θ) represents the polar coordinates of (x, y), use Problem 7 to find $\frac{\partial T}{\partial r}$ and $\frac{\partial T}{\partial \theta}$ when $r = 4$ and $\theta = \frac{\pi}{6}$.
   (b) Show that $\frac{\partial T}{\partial \theta} = 0$ for all values of r and θ. Can you explain this result geometrically?

9. Let T be the torus parametrized by

$$\begin{aligned} x &= (4 + 2\cos(t))\cos(s), \\ y &= (4 + 2\cos(t))\sin(s), \\ z &= 2\sin(t), \end{aligned}$$

for $0 \le s \le 2\pi$ and $0 \le t \le 2\pi$.

(a) If U is a function of x, y, and z, find general expressions for $\frac{\partial U}{\partial s}$ and $\frac{\partial U}{\partial t}$.

(b) Suppose

$$U = 80 - 40e^{-\frac{1}{20}(x^2+y^2+z^2)}$$

gives the temperature at a point (x, y, z) on T. Find expressions for $\frac{\partial U}{\partial s}$ and $\frac{\partial U}{\partial t}$ in this case. What is the geometrical interpretation of these quantities?

(c) Evaluate $\frac{\partial U}{\partial s}$ and $\frac{\partial U}{\partial t}$ in the particular case $s = \frac{\pi}{4}$ and $t = \frac{\pi}{4}$.

Section 4.3

Line Integrals

We will motivate the mathematical concept of a line integral through an initial discussion of the physical concept of work.

Work

If a force of constant magnitude F is acting in the direction of motion of an object along a line, and the object moves a distance d along this line, then we call the quantity Fd the *work* done by the force on the object. More generally, if the vector $\mathbf{F}$ represents a constant force acting on a object as it moves along a displacement vector $\mathbf{d}$, then

$$\mathbf{F} \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|} \tag{4.3.1}$$

is the magnitude of $\mathbf{F}$ in the direction of motion (see Figure 4.3.1) and we define

$$\left(\mathbf{F} \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|}\right) \|\mathbf{d}\| = \mathbf{F} \cdot \mathbf{d} \tag{4.3.2}$$

to be the work done by $\mathbf{F}$ on the object when it is displaced by $\mathbf{d}$.

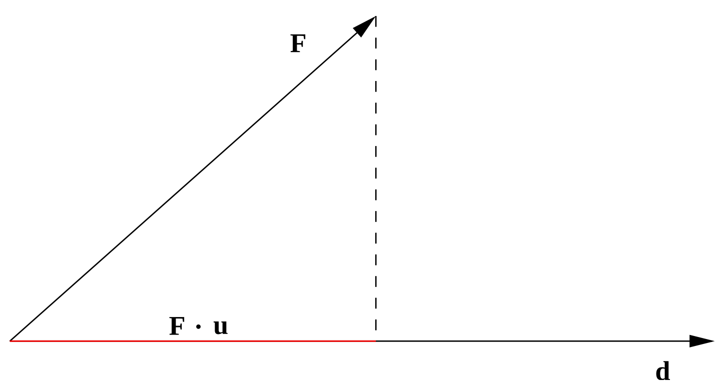


Figure 4.3.1 Magnitude of $\mathbf{F}$ in the direction of $\mathbf{d}$ is $\mathbf{F} \cdot \mathbf{u}$, where $\mathbf{u} = \dfrac{\mathbf{d}}{\|\mathbf{d}\|}$

We now generalize the formulation of work in (4.3.2) to the situation where an object P moves along some curve C subject to a force which depends continuously on position (but does not depend on time). Specifically, we represent the force by a continuous vector field, say, $F : \mathbb{R}^n \to \mathbb{R}^n$, and we suppose P moves along a curve C which has a smooth

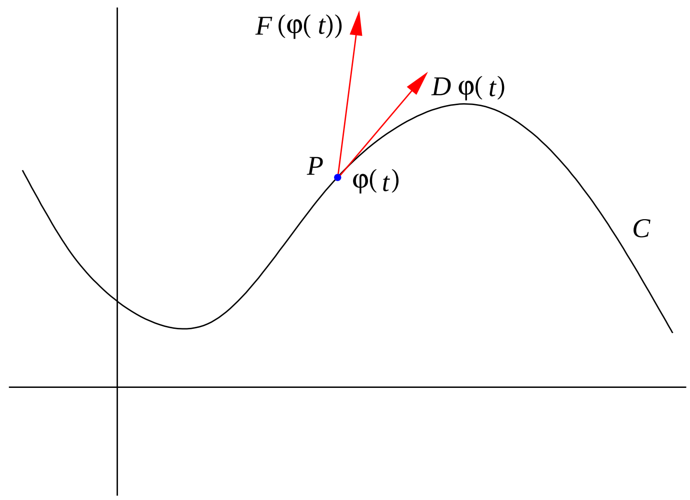


Figure 4.3.2 Object P moving along a curve C subject to a force F

parametrization $\varphi : I \to \mathbb{R}^n$, where $I = [a, b]$. See Figure 4.3.2. To approximate the work done by F as P moves from $\varphi(a)$ to $\varphi(b)$ along C, we first divide I into m equal subintervals of length

$$\Delta t = \frac{b-a}{m}$$

with endpoints $t_0 = a < t_1 < t_2 < \cdots < t_m = b$. Now at time t_k, $k = 0, 1, \ldots, m-1$, P is moving in the direction of $D\varphi(t_k)$ at a speed of $\|D\varphi(t_k)\|$, and so will move a distance of approximately $\|D\varphi(t_k)\|\Delta t$ over the time interval $[t_k, t_{k+1}]$. Thus we may approximate the work done by F as P moves from $\varphi(t_k)$ to $\varphi(t_{k+1})$ by the work done by the force $F(\varphi(t_k))$ in moving P along the displacement vector $D\varphi(t_k)\Delta t$, which is a vector of length $\|D\varphi(t_k)\|\Delta t$ in the direction of $D\varphi(t_k)$. That is, if we let W_k denote the work done by F as P moves from $\varphi(t_k)$ to $\varphi(t_{k-1})$, then

$$W_k \approx F(\varphi(t_k)) \cdot D\varphi(t_k)\Delta t. \tag{4.3.3}$$

If we let W denote the total work done by F as P moves along C, then we have

$$W = \sum_{k=0}^{m-1} W_k = \sum_{k=0}^{m-1} F(\varphi(t_k)) \cdot D\varphi(t_k)\Delta t. \tag{4.3.4}$$

As m increases, we should expect the approximation in (4.3.4) to approach W. Moreover, since $F(\varphi(t)) \cdot D\varphi(t)$ is a continuous function of t and the sum in (4.3.4) is a left-hand rule approximation for the definite integral of $F(\varphi(t)) \cdot D\varphi(t)$ over the interval $[a, b]$, we should have

$$W = \lim_{m\to\infty} \sum_{k=0}^{m-1} F(\varphi(t_k)) \cdot D\varphi(t_k)\Delta t = \int_a^b F(\varphi(t)) \cdot D\varphi(t)dt. \tag{4.3.5}$$

Example Suppose an object moves along the curve C parametrized by $\varphi(t) = (t, t^2)$, $-1 \leq t \leq 1$, subject to the force $F(x, y) = (y, x)$. Then the work done by F as the object moves from $\varphi(-1) = (-1, 1)$ to $\varphi(1) = (1, 1)$ is

$$\begin{aligned} W &= \int_{-1}^{1} F(\varphi(t)) \cdot D\varphi(t)dt \\ &= \int_{-1}^{1} F(t, t^2) \cdot (1, 2t)dt \\ &= \int_{-1}^{1} (t^2, t) \cdot (1, 2t)dt \\ &= \int_{-1}^{1} 3t^2 dt \\ &= t^3 \Big|_{-1}^{1} \\ &= 2. \end{aligned}$$

Example The function $\psi(t) = \left(\frac{t}{2}, \frac{t^2}{4}\right)$, $-2 \leq t \leq 2$, is also a smooth parametrization of the curve C in the previous example. Using the same force function F, we have

$$\begin{aligned} \int_{-2}^{2} F(\psi(t)) \cdot D\psi(t)dt &= \int_{-2}^{2} \left(\frac{t^2}{4}, \frac{t}{2}\right) \cdot \left(\frac{1}{2}, \frac{t}{2}\right) dt \\ &= \int_{-2}^{2} \frac{3t^2}{8}\, dt \\ &= \frac{t^3}{8}\Big|_{-2}^{2} \\ &= 2. \end{aligned}$$

This is the result we should expect: as long as the curve is traversed only once, the work done by a force when an object moves along the curve should depend only on the curve and not on any particular parametrization of the curve.

We need to verify the previous statement in general before we can state our definition of the line integral. Note that in these two examples, $\psi(t) = \varphi\left(\frac{t}{2}\right)$. In other words, $\psi(t) = \varphi(g(t))$, where $g(t) = \frac{t}{2}$ for $-2 \leq t \leq 2$. In general, if $\varphi(t)$, for t in an interval $[a, b]$, and $\psi(t)$, for t in an interval $[c, d]$, are both smooth parametrizations of a curve C such that every point on C corresponds to exactly one point in I and exactly one point in J, then there exists a differentiable function g which maps J onto I such that $\psi(t) = \varphi(g(t))$. Defining such a g is straightforward: given any t in $[c, d]$, find the unique value s in $[a, b]$ such that $\varphi(s) = \psi(t)$ (such a value s has to exist since C is the image of both ψ and φ). Then $g(t) = s$. Proving that g is differentiable is not as easy, and we will not provide a proof here. However, assuming that g is differentiable, it follows that for any continuous vector field F,

$$\begin{aligned}\int_c^d F(\psi(t)) \cdot D\psi(t)dt &= \int_c^d F(\varphi(g(t)) \cdot D(\varphi \circ g)(t))dt \\ &= \int_c^d F(\varphi(g(t)) \cdot D\varphi(g(t))g'(t)dt. \end{aligned} \tag{4.3.6}$$

Now if we let

$$\begin{aligned} u &= g(t), \\ du &= g'(t)dt, \end{aligned}$$

in (4.3.6), then

$$\int_c^d F(\psi(t)) \cdot D\psi(t)dt = \int_a^b F(\varphi(u)) \cdot D\varphi(u)dt \tag{4.3.7}$$

if $g(c) = a$ and $g(d) = b$ (that is, $\varphi(a) = \psi(c)$ and $\varphi(b) = \psi(d)$, and

$$\int_c^d F(\psi(t)) \cdot D\psi(t)dt = \int_b^a F(\varphi(u)) \cdot D\varphi(u)dt = -\int_b^a F(\varphi(u)) \cdot D\varphi(u)dt \tag{4.3.8}$$

if $g(c) = b$ and $g(d) = a$ (that is, $\varphi(a) = \psi(d)$ and $\varphi(b) = \psi(c)$. Note that the second case occurs only if ψ parametrizes C in the reverse direction of φ, in which case we say ψ is an *orientation reversing reparametrization* of φ. In the first case, that is, when $\varphi(a) = \psi(c)$ and $\varphi(b) = \psi(d)$, we say ψ is an *orientation preserving reparametrization* of φ. Our results in (4.3.7) and (4.3.8) then correspond to the physical notion that the work done by a force in moving an object along a curve is the negative of the work done by the force in moving the object along the curve in the opposite direction. From now on, when referring to a curve C, we will assume some orientation, or direction, has been specified. We will then use $-C$ to refer to the curve consisting of the same set of points as C, but with the reverse orientation.

Line integrals

Now that we know that, except for direction, the value of the integral involved in computing work does not depend on the particular parametrization of the curve, we may state a formal mathematical definition.

Definition Suppose C is a curve in $\mathbb{R}^n$ with smooth parametrization $\varphi : I \to \mathbb{R}^n$, where $I = [a, b]$ is an interval in $\mathbb{R}$. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field, then we define the *line integral* of F along C, denoted

$$\int_C F \cdot ds,$$

by

$$\int_C F \cdot ds = \int_a^b F(\varphi(t)) \cdot D\varphi(t)dt. \tag{4.3.9}$$

As a consequence of our previous remarks, we have the following result.

Proposition Using the notation of the definition,

$$\int_C F \cdot ds$$

depends only on the curve C and its orientation, not on the parametrization φ. Moreover,

$$\int_{-C} F \cdot ds = -\int_C F \cdot ds. \tag{4.3.10}$$

Example Let C be the unit circle centered at the origin in $\mathbb{R}^2$, oriented in the counter-clockwise direction, and let

$$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2}(-y,x).$$

To compute the line integral of F along C, we first need to find a smooth parametrization of C. One such parametrization is

$$\varphi(t) = (\cos(t), \sin(t))$$

for $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^{2\pi} F(\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t))dt \\
&= \int_0^{2\pi} \frac{1}{\cos^2(t) + \sin^2(t)}(-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))dt \\
&= \int_0^{2\pi} (\sin^2(t) + \cos^2(t))dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi.
\end{aligned}$$

Note that $\psi(t) = (\sin(t), \cos(t))$, $0 \leq t \leq 2\pi$, parametrizes $-C$, from which we can calculate

$$\begin{aligned}
\int_{-C} F \cdot ds &= \int_0^{2\pi} F(\sin(t), \cos(t)) \cdot (\cos(t), -\sin(t))dt \\
&= \int_0^{2\pi} \frac{1}{\sin^2(t) + \cos^2(t)}(-\cos(t), \sin(t)) \cdot (\cos(t), -\sin(t))dt \\
&= \int_0^{2\pi} (-\cos^2(t) - \sin^2(t))dt \\
&= -\int_0^{2\pi} dt \\
&= -2\pi,
\end{aligned}$$

in agreement with the previous proposition.

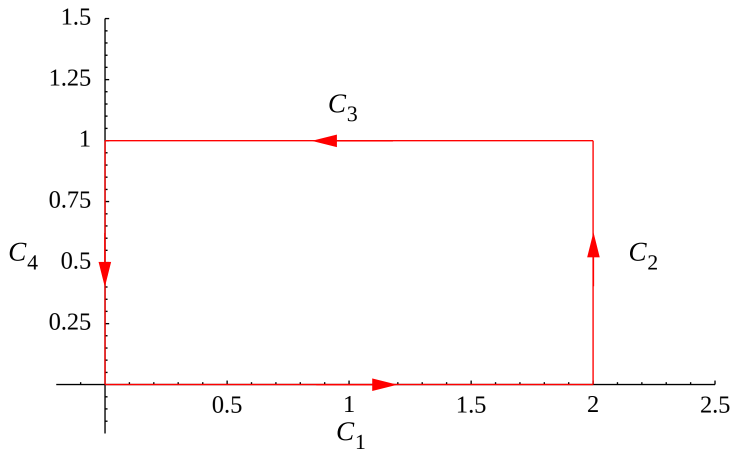


Figure 4.3.3 Rectangle with counterclockwise orientation

A *piecewise smooth* curve is one which may be decomposed into a finite number of curves, each of which has a smooth parametrization. If C is a piecewise smooth curve composed of the union of the curves $C_1, C_2, \ldots, C_m$, then we may extend the definition of the line integral to C by defining

$$\int_C F\cdot ds = \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds + \cdots + \int_{C_m} F\cdot ds. \tag{4.3.11}$$

The next example illustrates this procedure.

Example Let C be the rectangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, oriented in the counterclockwise direction, and let $F(x,y) = (y^2, 2xy)$. If we let C_1, C_2, C_3, and C_4 be the four sides of C, as labeled in Figure 4.3.3, then we may parametrize C_1 by

$$\alpha(t) = (t, 0),$$

$0 \le t \le 2$, C_2 by

$$\beta(t) = (2, t),$$

$0 \le t \le 1$, C_3 by

$$\gamma(t) = (2 - t, 1),$$

$0 \le t \le 2$, and C_4 by

$$\delta(t) = (0, 1 - t),$$

$0 \le t \le 1$. Then

$$\begin{aligned}\int_C F\cdot ds &= \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds + \int_{C_3} F\cdot ds + \int_{C_4} F\cdot ds \\ &= \int_0^2 F(t,0)\cdot(1,0)dt + \int_0^1 F(2,t)\cdot(0,1)dt + \int_0^2 F(2-t,1)\cdot(-1,0)dt\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 F(0,1-t)\cdot(0,-1)dt\\
&= \int_0^2 (0,0)\cdot(1,0)dt + \int_0^1 (t^2,4t)\cdot(0,1)dt + \int_0^2 (1,4-2t)\cdot(-1,0)dt\\
&\quad + \int_0^1 ((1-t)^2,0)\cdot(0,-1)dt\\
&= \int_0^2 0dt + \int_0^1 4tdt + \int_0^2 (-1)dt + \int_0^1 0dt\\
&= 2t^2\Big|_0^1 - 2\\
&= 2-2\\
&= 0.
\end{aligned}
$$

Note that it would be slightly simpler to parametrize $-C_3$ and $-C_4$, using

$$\varphi(t) = (1,t),$$

$0 \leq t \leq 2$, and

$$\psi(t) = (t,0),$$

$0 \leq t \leq 1$, respectively, than to parametrize C_3 and C_4 directly. We would then evaluate

$$\int_C F\cdot ds = \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds - \int_{-C_3} F\cdot ds - \int_{-C_4} F\cdot ds.$$

A note on notation

Suppose C is a smooth curve in $\mathbb{R}^n$, parametrized by $\varphi : I \to \mathbb{R}^n$, where $I = [a,b]$, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Our notation for the line integral of F along C comes from letting $s = \varphi(t)$, from which we have

$$\frac{ds}{dt} = D\varphi(t),$$

which we may write, symbolically, as

$$ds = D\varphi(t)dt.$$

Now suppose $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $F_1, F_2, \ldots, F_n$ are the component functions of φ and F, respectively. If we let

$$
\begin{aligned}
x_1 &= \varphi_1(t),\\
x_2 &= \varphi_2(t),\\
&\;\;\vdots\\
x_n &= \varphi_n(t),
\end{aligned}
$$

then we may write

$$\begin{aligned}\int_C F\cdot ds &= \int_a^b F(\varphi(t))\cdot D\varphi(t)dt\\ &= \int_a^b F(x_1(t),x_2(t),\ldots,x_n(t))\cdot(\varphi_1'(t),\varphi_2'(t),\ldots,\varphi_n'(t))dt\\ &= \int_a^b (F_1(x_1(t),x_2(t),\ldots,x_n(t))\varphi_1'(t)+F_2(x_1(t),x_2(t),\ldots,x_n(t))\varphi_2'(t))+\cdots\\ &\qquad + F_n(x_1(t),x_2(t),\ldots,x_n(t))\varphi_n'(t))dt\\ &= \int_a^b F_1(x_1(t),x_2(t),\ldots,x_n(t))\varphi_1'(t)dt+\int_a^b F_2(x_1(t),x_2(t),\ldots,x_n(t))\varphi_2'(t)dt\\ &\qquad +\cdots+\int_a^b F_n(x_1(t),x_2(t),\ldots,x_n(t))\varphi_n'(t)dt. \qquad (4.3.12)\end{aligned}$$

Suppressing the dependence on t, writing dx_k for $\varphi_k'(t)dt$, $k=1,2,\ldots,n$, and using only a single integral sign, we may rewrite (4.3.12) as

$$\int_C F_1(x_1,x_2,\ldots,x_n)dx_1+F_2(x_1,x_2,\ldots,x_n)dx_2+\cdots+F_n(x_1,x_2,\ldots,x_n)dx_n. \quad (4.3.13)$$

This is a common, and useful, notation for a line integral.

Example We will evaluate

$$\int_C ydx+xdy+z^2dz,$$

where C is the part of a helix in $\mathbb{R}^3$ with parametric equations

$$\begin{aligned}x&=\cos(t),\\ y&=\sin(t),\\ z&=t,\end{aligned}$$

$0\le t\le 2\pi$. Note that this is equivalent to evaluating

$$\int_C F\cdot ds,$$

where $F:\mathbb{R}^3\to\mathbb{R}^3$ is the vector field $F(x,y,z)=(y,x,z^2)$. We have

$$\begin{aligned}\int_C ydx+xdy+z^2dz &= \int_0^{2\pi}(\sin(t)(-\sin(t))+\cos(t)\cos(t)+t^2)dt\\ &= \int_0^{2\pi}(\cos^2(t)-\sin^2(t)+t^2)dt\\ &= \int_0^{2\pi}(\cos(2t)+t^2)dt\\ &= \frac{1}{2}\sin(2t)\Big|_0^{2\pi}+\frac{1}{3}t^3\Big|_0^{2\pi}\\ &= \frac{8\pi^3}{3}.\end{aligned}$$

Gradient fields

Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is C^1, then ∇f is a continuous vector field on $\mathbb{R}^n$. Suppose $\varphi : I \to \mathbb{R}^n$, $I = [a, b]$, is a smooth parametrization of a curve C. Then, using the chain rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \nabla f \cdot ds &= \int_a^b \nabla f(\varphi(t)) \cdot D\varphi(t)dt \\ &= \int_a^b \frac{d}{dt} f(\varphi(t))dt \\ &= f(\varphi(t))\Big|_a^b \\ &= f(\varphi(b)) - f(\varphi(a)).\end{aligned}$$

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 and $\varphi : I \to \mathbb{R}^n$, $I = [a, b]$, is a smooth parametrization of a curve C, then

$$\int_C \nabla f \cdot ds = f(\varphi(b)) - f(\varphi(a)). \tag{4.3.14}$$

Note that (4.3.14) shows that the value of a line integral of a gradient vector field depends only on the starting and ending points of the curve, not on which particular path is taken between these two points. Moreover, (4.3.14) provides a simple means for evaluating a line integral if the given vector field can be identified as the gradient of a scalar valued function. Another interesting consequence is that if the beginning and ending points of C are the same, that is, if $\mathbf{v} = \varphi(a) = \varphi(b)$, then

$$\int_C \nabla f \cdot ds = f(\varphi(b)) - f(\varphi(b)) = f(\mathbf{v}) - f(\mathbf{v}) = 0. \tag{4.3.15}$$

We call such curves *closed curves*. In words, the line integral of a gradient vector field is 0 along any closed curve.

Example If $F(x, y) = (y, x)$, then

$$F(x, y) = \nabla f(x, y),$$

where $f(x, y) = xy$. Hence, for example, for any smooth curve C starting at $(-1, 1)$ and ending at $(1, 1)$ we have

$$\int_C F \cdot ds = f(1, 1) - f(-1, 1) = 1 + 1 = 2.$$

Note that this agrees with the result in our first example above, where C was the part of the parabola $y = x^2$ extending from $(-1, 1)$ to $(1, 1)$.

Example If $f(x, y) = xy^2$, then

$$\nabla f(x, y) = (y^2, 2xy).$$

If C is the rectangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, then, since C is a closed curve,

$$\int_C y^2dx + 2xydy = 0,$$

in agreement with an earlier example. Similarly, if E is the unit circle in $\mathbb{R}^2$ centered at the origin, then we know that

$$\int_E y^2dx + 2xydy = 0,$$

with no need for further computations.

In physics, a force field F is said to be *conservative* if the work done by F in moving an object between any two points depends only on the points, and not on the path used between the two points. In particular, we have shown that if F is the gradient of some scalar function f, then F is a conservative force field. Under certain conditions on the domain of F, the converse is true as well. That is, under certain conditions, if F is a conservative force field, then there exists a scalar function f such that $F = \nabla f$. Problem 9 explores one such situation in which this is true. The function f is then known as a *potential function.*

Problems

1. For each of the following, compute the line integral $\displaystyle\int_C F \cdot ds$ for the given vector field F and curve C parametrized by φ.
   (a) $F(x,y) = (xy, 3x)$, $\varphi(t) = (t^2, t)$, $0 \leq t \leq 2$
   (b) $F(x,y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$, $\varphi(t) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$
   (c) $F(x,y) = (3x - 2y, 4x^2y)$, $\varphi(t) = (t^3, t^2)$, $-2 \leq t \leq 2$
   (d) $F(x,y,z) = (xyz, 3xy^2, 4z)$, $\varphi(t) = (3t, t^2, 4t^3)$, $0 \leq t \leq 4$
2. Let C be the circle of radius 2 centered at the origin in $\mathbb{R}^2$, with counterclockwise orientation. Evaluate the following line integrals.
   (a) $\displaystyle\int_C 3xdx + 4ydy$ (b) $\displaystyle\int_C 8xydx + 4x^2dy$
3. Let C be the part of a helix in $\mathbb{R}^3$ parametrized by $\varphi(t) = (\cos(2t), \sin(2t), t)$, $0 \leq t \leq 2\pi$. Evaluate the following line integrals.
   (a) $\displaystyle\int_C 3xdx + 4ydy + zdz$ (b) $\displaystyle\int_C yzdx + xzdy + xydz$
4. Let C be the rectangle in $\mathbb{R}^2$ with vertices at $(-1,1)$, $(2,1)$, $(2,3)$, and $(-1,3)$, with counterclockwise orientation. Evaluate the following line integrals.
   (a) $\displaystyle\int_C x^2ydx + (3y+x)dy$ (b) $\displaystyle\int_C 2xydx + x^2dy$

5. Let C be the ellipse in $\mathbb{R}^2$ with equation
$$\frac{x^2}{4} + \frac{y^2}{9} = 1,$$
with counterclockwise orientation. Evaluate $\int_C F \cdot ds$ for $F(x, y) = (4y, 3x)$.

6. Let C be the upper half of the circle of radius 3 centered at the origin in $\mathbb{R}^2$, with counterclockwise orientation. Evaluate the following line integrals.

   (a) $\int_C 3y dx$ (b) $\int_C 4x dy$

7. Evaluate
$$\int_C \frac{x}{x^2 + y^2}\, dx + \frac{y}{x^2 + y^2}\, dy,$$
where C is any curve which starts at $(1, 0)$ and ends at $(2, 3)$.

8. (a) Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field which is the gradient of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$. If F_k is the kth coordinate function of F, $k = 1, 2, \ldots, n$, show that
$$\frac{\partial}{\partial x_j} F_i(x_1, x_2, \ldots, x_n) = \frac{\partial}{\partial x_i} F_j(x_1, x_2, \ldots, x_n)$$
for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

   (b) Show that although
$$\int_C x dx + xy dy = 0$$
for every circle C in $\mathbb{R}^2$ with center at the origin, nevertheless $F(x, y) = (x, xy)$ is not the gradient of any scalar function $f : \mathbb{R}^n \to \mathbb{R}$.

   (c) Let
$$F(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$
for all (x, y) in the set $S = \{(x, y) : (x, y) \neq (0, 0)\}$. Let F_1 and F_2 be the coordinate functions of F. Show that
$$\frac{\partial}{\partial y} F_1(x, y) = \frac{\partial}{\partial x} F_2(x, y)$$
for all (x, y) in S, even though F is not the gradient of any scalar function . (Hint: For the last part, show that
$$\int_C F \cdot ds = 2\pi,$$
where C is the unit circle centered at the origin.)

9. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous vector field with the property that for any curve C,

$$\int_C F \cdot ds$$

depends only on the endpoints of C. That is, if C_1 and C_2 are any two curves with the same endpoints P and Q, then

$$\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds.$$

(a) Show that

$$\int_C F \cdot ds = 0$$

for any closed curve C.

(b) Let F_1 and F_2 be the coordinate functions of F. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \int_C F \cdot ds,$$

where C is any curve which starts at $(0, 0)$ and ends at (x, y). Show that

$$\frac{\partial}{\partial y} f(x, y) = F_2(x, y).$$

(Hint: In evaluating $f(x, y)$, consider the curve C from $(0, 0)$ to (x, y) which consists of the horizontal line from $(0, 0)$ to $(x, 0)$ followed by the vertical line from $(x, 0)$ to (x, y).)

(c) Show that $\nabla f = F$.

The Calculus of Functions

of

Several Variables

Section 4.4

Green's Theorem

Green's theorem is an example from a family of theorems which connect line integrals (and their higher-dimensional analogues) with the definite integrals we studied in Section 3.6. We will first look at Green's theorem for rectangles, and then generalize to more complex curves and regions in $\mathbb{R}^2$.

Green's theorem for rectangles

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 on an open set containing the closed rectangle

$$D = [a, b] \times [c, d],$$

and let F_1 and F_2 be the coordinate functions of F. If C denotes the boundary of D, oriented in the clockwise direction, then we may decompose C into the four curves C_1, C_2, C_3, and C_4 shown in Figure 4.4.1. Then

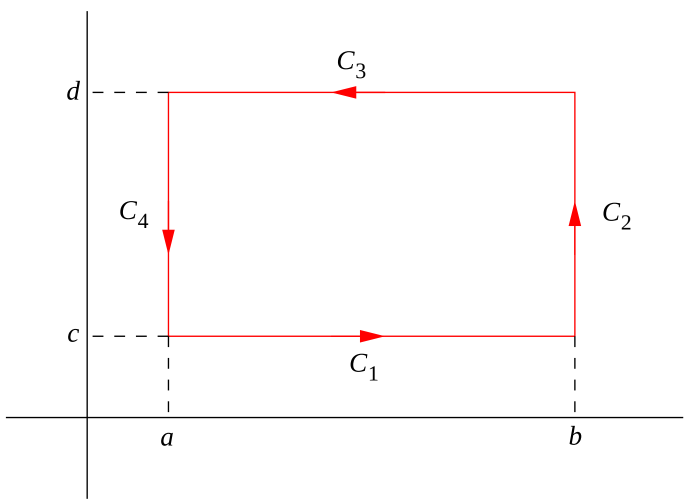


Figure 4.4.1 The boundary of a rectangle decomposed into four smooth curves

$$\alpha(t) = (t, c),$$

$a \leq t \leq b$, is a smooth parametrization of C_1,

$$\beta(t) = (b, t),$$

$c \le t \le d$, is a smooth parametrization of C_2,

$$\gamma(t) = (t, d),$$

$a \le t \le b$, is a smooth parametrization of $-C_3$, and

$$\delta(t) = (a, t),$$

$c \le t \le d$, is a smooth parametrization of $-C_4$. Now

$$\begin{aligned}\int_C F \cdot ds &= \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds + \int_{C_3} F \cdot ds + \int_{C_4} F \cdot ds \\ &= \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds - \int_{-C_3} F \cdot ds - \int_{-C_4} F \cdot ds, \end{aligned} \tag{4.4.1}$$

and

$$\int_{C_1} F \cdot ds = \int_a^b ((F_1(t,c), F_2(t,c)) \cdot (1,0) dt = \int_a^b F_1(t,c) dt, \tag{4.4.2}$$

$$\int_{C_2} F \cdot ds = \int_c^d ((F_1(b,t), F_2(b,t)) \cdot (0,1) dt = \int_c^c F_2(b,t) dt, \tag{4.4.3}$$

$$\int_{-C_3} F \cdot ds = \int_a^b ((F_1(t,d), F_2(t,d)) \cdot (1,0) dt = \int_a^b F_1(t,d) dt, \tag{4.4.4}$$

and

$$\int_{-C_4} F \cdot ds = \int_c^d ((F_1(a,t), F_2(a,t)) \cdot (0,1) dt = \int_c^c F_2(a,t) dt, \tag{4.4.5}$$

Hence, inserting (4.4.2) through (4.4.5) into (4.4.1),

$$\begin{aligned}\int_C F \cdot ds &= \int_a^b F_1(t,c) dt + \int_c^d F_2(b,t) dt - \int_a^b F_1(t,d) dt - \int_c^d F_2(a,t) dt \\ &= \int_c^d (F_2(b,t) - F_2(a,t)) dt - \int_a^b (F_1(t,d) - F_1(t,c)) dt. \end{aligned} \tag{4.4.6}$$

Now, by the Fundamental Theorem of Calculus, for a fixed value of t,

$$\int_a^b \frac{\partial}{\partial x} F_2(x,t) dx = F_2(b,t) - F_2(a,t) \tag{4.4.7}$$

and

$$\int_c^d \frac{\partial}{\partial y} F_1(t,y) dy = F_1(t,d) - F_1(t,c). \tag{4.4.8}$$

Thus, combining (4.4.7) and (4.4.8) with (4.4.6), we have

$$\begin{aligned}\int_C F\cdot ds &= \int_c^d\int_a^b \frac{\partial}{\partial x}F_2(x,t)dxdt - \int_a^b\int_c^d \frac{\partial}{\partial y}F_1(t,y)dydt\\ &= \int_c^d\int_a^b \frac{\partial}{\partial x}F_2(x,y)dxdy - \int_a^b\int_c^d \frac{\partial}{\partial y}F_1(x,y)dydx\\ &= \int_c^d\int_a^b \left(\frac{\partial}{\partial x}F_2(x,y) - \frac{\partial}{\partial y}F_1(x,y)\right)dxdy. \qquad (4.4.9)\end{aligned}$$

If we let $p = F_1(x,y)$, $q = F_2(x,y)$, and $\partial D = C$ (a common notation for the boundary of D), then we may rewrite (4.4.9) as

$$\int_{\partial D} pdx + qdy = \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)dxdy. \qquad (4.4.10)$$

This is Green's theorem for a rectangle.

Example If $D = [1,3]\times[2,5]$, then

$$\begin{aligned}\int_{\partial D} xydx + xdy &= \int\int_D \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}xy\right)dxdy\\ &= \int_1^3\int_2^5 (1-x)dydx\\ &= \int_1^3 3(1-x)dx\\ &= 3x\Big|_1^3 - \frac{3}{2}x^2\Big|_1^3\\ &= -6.\end{aligned}$$

Clearly, this is simpler than evaluating the line integral directly.

Green's theorem for regions of Type III

Green's theorem holds for more general regions than rectangles. We will confine ourselves here to discussing regions known as regions of Type III, but it is not hard to generalize to regions which may be subdivided into regions of this type (for an example, see Problem 12). Recall from Section 3.6 that we say a region D in $\mathbb{R}^2$ is of Type I if there exist real numbers $a < b$ and continuous functions $\alpha : \mathbb{R}\to\mathbb{R}$ and $\beta : \mathbb{R}\to\mathbb{R}$ such that

$$D = \{(x,y) : a \le x \le b, \alpha(x) \le y \le \beta(x)\}. \qquad (4.4.11)$$

We say a region D in $\mathbb{R}^2$ is of Type II if there exist real numbers c and d and continuous functions $\gamma : \mathbb{R}\to\mathbb{R}$ and $\delta : \mathbb{R}\to\mathbb{R}$ such that

$$D = \{(x,y) : c \le y \le d, \gamma(y) \le x \le \delta(y)\}. \qquad (4.4.12)$$

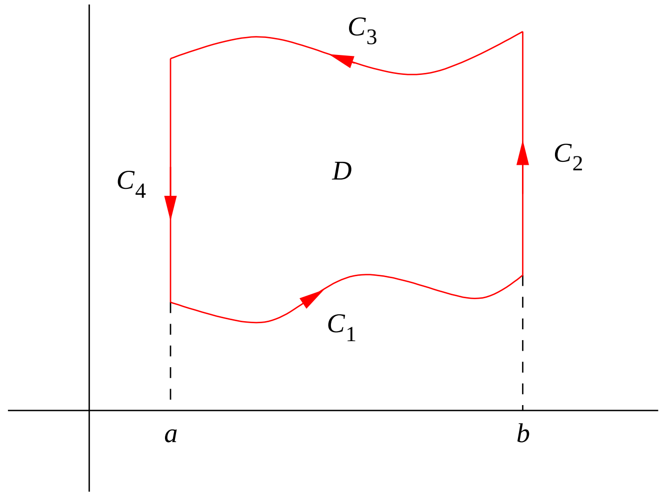


Figure 4.4.2 Decomposing the boundary of a region of Type I

Definition We call a region D in $\mathbb{R}^2$ which is both of Type I and of Type II a region of *Type III.*

Example In Section 3.6, we saw that the triangle T with vertices at $(0,0)$, $(1,0)$, and $(1,1)$ and the closed disk

$$D = \bar{B}^2((0,0),1) = \{(x,y) : x^2 + y^2 \leq 1\}$$

are of both Type I and Type II. Thus T and D are regions of Type III. We also saw that the region E beneath the graph of $y = x^2$ and above the interval $[-1,1]$ is of Type I, but not of Type II. Hence E is not of Type III.

Example Any closed rectangle in $\mathbb{R}^2$ is a region of Type III, as is any closed region bounded by an ellipse.

Now suppose D is a region of Type III and ∂D is the boundary of D, that is, the curve enclosing D, oriented counterclockwise. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field, with coordinate functions $p = F_1(x,y)$ and $q = F_2(x,y)$. We will first prove that

$$\int_{\partial D} p dx = -\int\int_D \frac{\partial p}{\partial y} dx dy. \tag{4.4.13}$$

Since D is, in particular, a region of Type I, there exist continuous functions α and β such that

$$D = \{(x,y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}. \tag{4.4.14}$$

In addition, we will assume that α and β are both differentiable (without this assumption the line integral of F along ∂D would not be defined). As with the rectangle in the previous proof, we may decompose ∂D into four curves, C_1, C_2, C_3, and C_4, as shown in Figure 4.4.2. Then

$$\varphi_1(t) = (t, \alpha(t)),$$

$a \leq t \leq b$, is a smooth parametrization of C_1,

$$\varphi_2(t) = (b, t),$$

$\alpha(b) \leq t \leq \beta(b)$, is a smooth parametrization of C_2,

$$\varphi_3(t) = (t, \beta(t)),$$

$a \leq t \leq b$, is a smooth parametrization of $-C_3$, and

$$\varphi_4(t) = (a, t),$$

$\alpha(a) \leq t \leq \beta(a)$, is a smooth parametrization of $-C_4$. Now

$$\int_{\partial D} p dx = \int_{C_1} p dx + \int_{C_2} p dx - \int_{-C_3} p dx - \int_{-C_4} p dx, \tag{4.4.15}$$

where

$$\int_{C_1} p dx = \int_a^b (F_1(t, \alpha(t)), 0) \cdot (1, \alpha'(t)) dt = \int_a^b F_1(t, \alpha(t)) dt, \tag{4.4.16}$$

$$\int_{C_2} p dx = \int_{\alpha(b)}^{\beta(b)} (F_1(b, t), 0) \cdot (0, 1) dt = \int_{\alpha(b)}^{\beta(b)} 0 dt = 0, \tag{4.4.17}$$

$$\int_{-C_3} p dx = \int_a^b (F_1(t, \beta(t)), 0) \cdot (1, \beta'(t)) dt = \int_a^b F_1(t, \beta(t)) dt, \tag{4.4.18}$$

and

$$\int_{-C_4} p dx = \int_{\alpha(a)}^{\beta(a)} (F_1(a, t), 0) \cdot (0, 1) dt = \int_{\alpha(a)}^{\beta(a)} 0 dt = 0. \tag{4.4.19}$$

Hence

$$\begin{aligned}\int_{\partial D} p dx &= \int_a^b F_1(t, \alpha(t)) dt - \int_a^b F_1(t, \beta(t)) dt \\ &= -\int_a^b (F_1(t, \beta(t)) - F_1(t, \alpha(t))) dt.\end{aligned} \tag{4.4.20}$$

Now, by the Fundamental Theorem of Calculus,

$$\int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial y} F_1(t, y) dy = F_1(t, \beta(t)) - F_1(t, \alpha(t)), \tag{4.4.21}$$

and so

$$\begin{aligned}\int_{\partial D} p dx &= -\int_a^b \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial y} F_1(t, y) dy dt \\ &= -\int_a^b \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial y} F_1(x, y) dy dx \\ &= -\int\int_D \frac{\partial p}{\partial y} \, dx dy.\end{aligned} \tag{4.4.22}$$

A similar calculation, treating D as a region of Type II, shows that

$$\int_{\partial D} q dy = \int\int_D \frac{\partial q}{\partial x}\, dxdy. \tag{4.4.23}$$

(You are asked to verify this in Problem 7.) Putting (4.4.22) and (4.4.23) together, we have

$$\begin{aligned}\int_{\partial D} F\cdot ds = \int_{\partial D} pdx + qdy &= -\int\int_D \frac{\partial p}{\partial y}\, dxdy + \int\int_D \frac{\partial q}{\partial x}\, dxdy \\ &= \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy. \end{aligned} \tag{4.4.24}$$

Green's Theorem Suppose D is a region of Type III, ∂D is the boundary of D with counterclockwise orientation, and the curves describing ∂D are differentiable. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field, with coordinate functions $p = F_1(x,y)$ and $q = F_2(x,y)$. Then

$$\int_{\partial D} pdx + qdy = \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy. \tag{4.4.25}$$

Example Let D be the region bounded by the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,3)$, as shown in Figure 4.4.3. If we orient ∂D in the counterclockwise direction, then

$$\begin{aligned}\int_{\partial D}(3x^2+y)dx + 5xdy &= \int\int_D \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(3x^2+y)\right) dxdy \\ &= \int\int_D (5-1)dxdy \\ &= 4\int\int_D dxdy \\ &= (4)(3) \\ &= 12, \end{aligned}$$

where we have used the fact that the area of D is 3 to evaluate the double integral.

The line integral in the previous example reduced to finding the area of the region D. This can be exploited in the reverse direction to compute the area of a region. For example, given a region D with area A and boundary ∂D, it follows from Green's theorem that

$$A = \int\int_D dxdy = \int_{\partial D} pdx + qdy \tag{4.4.26}$$

for any choice of p and q which have the property that

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 1. \tag{4.4.27}$$

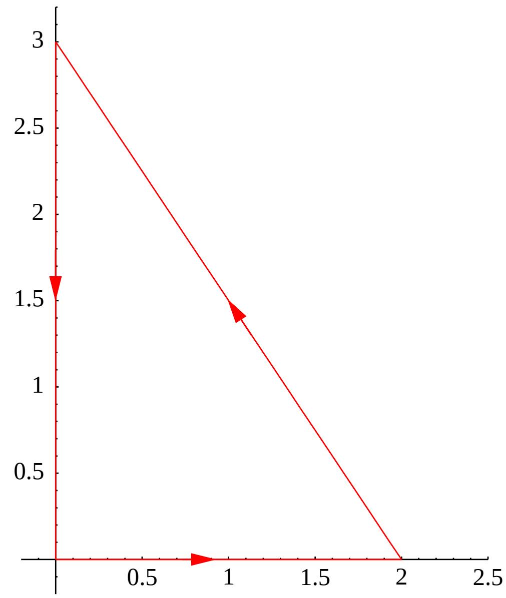


Figure 4.4.3 A triangle with counterclockwise orientation

For example, letting $p = 0$ and $q = x$, we have

$$A = \int_{\partial D} x dy \tag{4.4.28}$$

and, letting $p = -y$ and $q = 0$, we have

$$A = -\int_{\partial D} y dx. \tag{4.4.29}$$

The next example illustrates using the average of (4.4.28) and (4.4.29) to find A:

$$A = \frac{1}{2}\left(\int_{\partial D} x dy - \int_{\partial D} y dx\right) = \frac{1}{2}\int_{\partial D} x dy - y dx. \tag{4.4.30}$$

Example Let A be the area of the region D bounded by the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > 0$ and $b > 0$, as shown in Figure 4.4.4. Since we may parametrize ∂D, with counterclockwise orientation, by

$$\varphi(t) = (a\cos(t), b\sin(t)),$$

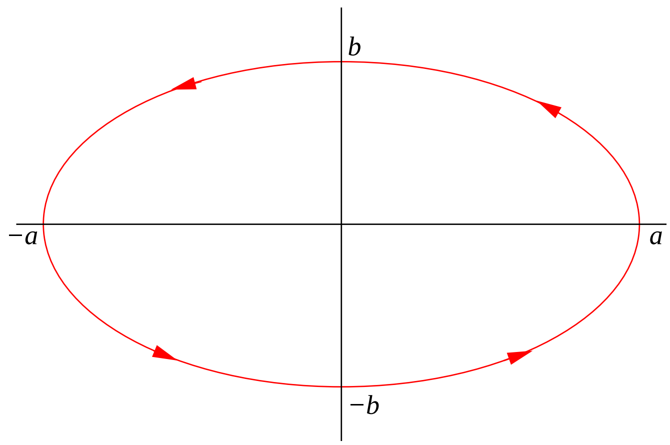


Figure 4.4.4 The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with counterclockwise orientation

$0 \leq t \leq 2\pi$, we have

$$\begin{aligned}
A &= \frac{1}{2} \int_{\partial D} x dy - y dx \\
&= \frac{1}{2} \int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t) dt \\
&= \frac{1}{2} \int_0^{2\pi} (ab\sin^2(t) + ab\cos^2(t)) dt \\
&= \frac{ab}{2} \int_0^{2\pi} dt \\
&= \left(\frac{ab}{2}\right)(2\pi) \\
&= \pi ab.
\end{aligned}$$

Problems

1. Let D be the closed rectangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, $(2,4)$, and $(0,4)$, with boundary ∂D oriented counterclockwise. Use Green's theorem to evaluate the following line integrals.

   (a) $\displaystyle\int_{\partial D} 2xy dx + 3x^2 dy$ (b) $\displaystyle\int_{\partial D} y dx + x dy$

2. Let D be the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, and $(0,4)$, with boundary ∂D oriented counterclockwise. Use Green's theorem to evaluate the following line integrals.

   (a) $\displaystyle\int_{\partial D} 2xy^2 dx + 4x dy$ (b) $\displaystyle\int_{\partial D} y dx + x dy$

   (c) $\displaystyle\int_{\partial D} y dx - x dy$

3. Use Green's theorem to find the area of a circle of radius r.
4. Use Green's theorem to find the area of the region D enclosed by the hypocycloid
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$
where $a > 0$. Note that we may parametrize this curve using
$$\varphi(t) = (a\cos^3(t), a\sin^3(t)),$$
$0 \le t \le 2\pi$.
5. Use Green's theorem to find the area of the region enclosed by one "petal" of the curve parametrized by
$$\varphi(t) = (\sin(2t)\cos(t), \sin(2t)\sin(t)).$$
6. Find the area of the region enclosed by the cardioid parametrized by
$$\varphi(t) = ((2 + \cos(t))\cos(t), (2 + \cos(t))\sin(t)),$$
$0 \le t \le 2\pi$.
7. Verify (4.4.23), thus completing the proof of Green's theorem.
8. Suppose the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ with coordinate functions $p = F_1(x, y)$ and $q = F_2(x, y)$ is C^1 on an open set containing the Type III region D. Moreover, suppose F is the gradient of a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$.
   (a) Show that
   $$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$$
   for all points (x, y) in D.
   (b) Use Green's theorem to show that
   $$\int_{\partial D} pdx + qdy = 0,$$
   where ∂D is the boundary of D with counterclockwise orientation.
9. How many ways do you know to calculate the area of a circle?
10. Who was George Green?
11. Explain how Green's theorem is a generalization of the Fundamental Theorem of Integral Calculus.
12. Let $b > a$, let C_1 be the circle of radius b centered at the origin, and let C_2 be the circle of radius a centered at the origin. If D is the annular region between C_1 and C_2 and F is a C^1 vector field with coordinate functions $p = F_1(x, y)$ and $q = F_2(x, y)$, show that
$$\int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = \int_{C_1} pdx + qdy + \int_{C_2} pdx + qdy,$$
where C_1 is oriented in the counterclockwise direction and C_2 is oriented in the clockwise direction. (Hint: Decompose D into Type III regions D_1, D_2, D_3, and D_4, each with boundary oriented counterclockwise, as shown in Figure 4.4.5.)

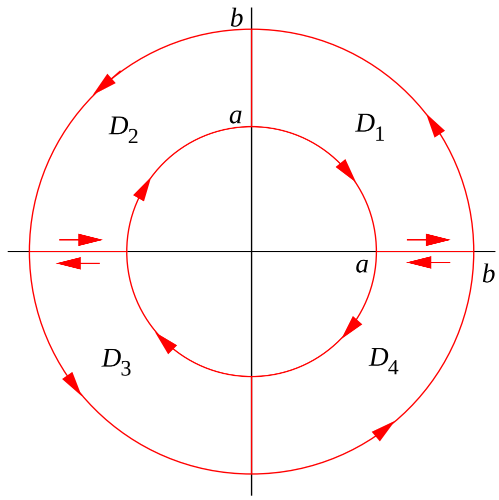


Figure 4.4.5 Decomposition of an annulus into regions of Type III