Elementary Mechanics

For Engineering Students

By

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FIRST EDITION
SECOND THOUSAND

NEW YORK:
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED
1914

Printed by Publishers Printing Co., New York

PREFACE

THIS little volume is the outcome of a series of lectures on elementary mechanics, delivered to the students in the engineering courses of the Cooper Union Day and Night Schools, with the intention of partially clearing the ground for some of the engineering subjects of the subsequent years.

In all cases, some knowledge of trigonometry and elementary physics has been assumed; and it is further assumed that the instructor using the text is thoroughly familiar with the students' pretraining in these branches. An attempt has been made to develop the formulæ from the most fundamental principles; it being the opinion of the author that in this way the student derives the greatest amount of good. No excuse is made, and none need be made, for employing the *poundal* as the unit force, and *g* as a proportionality factor.

A short chapter on the determination of maxima and minima values, by algebraic and trigonometric methods, is given with the hope that it will prove useful in solving problems, and also arouse an interest in mathematical analysis. It is, of course, understood that for any comprehensive treatment of either applied or theoretical mechanics, the application of the differential and integral calculus is absolutely necessary; but, in engineering schools, as a matter of economy in time, considerable physics must be taught before the student is familiar with the methods of the calculus. However, were it not a matter of economy it would still be desirable; for, it is undoubtedly true that those who have a fair knowledge of physical phenomena acquire the calculus more readily. The author is far from being in sympathy with those engineers who attempt to belittle the value of the calculus. When it is remembered that most engineering problems resolve themselves into the determination of maxima and minima values, and that so many of the factors entering into a problem are the ratios of variable quantities, it follows that those, other things being equal, whose minds are best equipped to deal with the mathematical relations of the quantities involved, will do the most efficient work.

Thanks are here expressed to Messrs. Riedel and Bateman for supplying the problems.

F. M. H.

COOPER UNION, March, 1910.

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CHAPTER I

MOTION

OUR concept of motion is the relative displacements that occur among bodies; but to describe motion it becomes necessary to do so with reference to some body *assumed* at rest—it being, of course, understood that, so far as we know, no body is actually at rest. The earth together with the other planets of the solar system, and their attendant satellites, are continually changing their relative positions; and furthermore, the sun is continually changing its position with respect to the so-called "fixed stars," and these stars again suffer displacement among themselves.

A body is said to be in motion when it occupies different positions at different intervals of time; or, what amounts to the same thing, when it is continually changing its position. In terrestrial mechanics the surface of the earth is assumed at rest, and bodies having a fixed position on the earth's surface are said to be at rest. Bodies that are continually changing their positions, relatively to the earth's surface, are said to be in motion.

Rectilinear Motion. In our preliminary definitions, we will consider the motion of a point on the body rather than the motion of the body, or consider the body of such dimensions that all parts of it may be considered as having precisely the same motion. Such a body is called *a particle*. To illustrate this, suppose some body to move so that a point in the body is always on the same straight line, this point then is said to have *rectilinear motion.* The body as a whole may, however, not have a rectilinear motion; for, it may at the same time be spinning about a line through this point as an axis. The body would have then both motion of *translation* and *rotation.* It is obvious that we are, at this part of the subject, not prepared to deal with combinations of this kind.

Uniform Motion. A body passing over equal distances in any arbitrary equal intervals of time is said to have uniform motion.

It must be remembered that the statement, "uniform motion is a motion such that equal distances are passed over in equal intervals of time," does not define uniform motion. For, this definition, though it includes uniform motion, does not exclude some other types of motion which are not uniform. This may perhaps be best illustrated by reference to a seconds pendulum. If we choose for our points of reference the extremities of its path and for the interval of time the second, then we have a case of equal distances passed over in equal intervals of time. This is equally true if one half second be taken for the interval of time. But the pendulum has by no means a uniform motion; for its motion is a maximum at the middle of its swing and at either extremity of its path its motion is momentarily zero.

Speed and Velocity. By speed is understood *the time rate of motion.* There is, however, another element to be considered, viz.: direction. When both speed and direction are specified it is called *Velocity.* Velocity is defined the same as speed; but it is specified with respect to the direction of the motion, *i.e.*, it is a *directed quantity.* Such a quantity is called a *Vector.* It is to be remembered that speed is merely a ratio; *i.e.*, the ratio of the distance passed over to the time consumed in passing over that distance.

Unit Velocity. Unit velocity is a velocity such that unit distance is passed over in unit time. In the c.g.s. (centimeter, gram, second) system, unit velocity is a velocity such that one centimeter is passed over per second.

In the F.P.S. (foot, pound, second) system, unit velocity is a velocity such that one foot is passed over per second.

Varied Motion. When the distances passed over in equal intervals of time are not equal, the motion is said to be varied.

Uniformly Varied Motion. When the velocity changes at a uniform rate, the motion is said to be uniformly varied; and the particle is said to have a constant *acceleration*. Acceleration, then, is rate of change of velocity. When the velocity of a particle is increasing, the acceleration is positive, when decreasing negative, and when it is neither increasing nor decreasing, *i.e.*, when the motion is uniform, the acceleration is zero.

In the c.g.s. system, unit acceleration is an acceleration such that the velocity changes at the rate of *one cm. per sec. per sec.* In the F.P.S. system, unit acceleration is an acceleration such that the velocity changes at the rate of *one foot per sec. per sec.*

Angular Measurements and Curvature of a Curve.

Radian. The unit of angular measure employed in mechanics is the Radian, and is the angle subtended by an arc equal to the radius. In any case, the measure of the angle is the ratio of the length of the arc to the radius. The length of the circumference of a circle being $2\pi r$, it follows from the definition that the sum of four right angles is equal to 2π radians.

Curvature. The ratio of the change in direction, measured in radians, to the change in length of a curve is a measure of the *curvature*.

To illustrate this: The direction of a curve at any point is the direction of the tangent to the curve at that point. If we take a second point indefinitely close to the first point and draw a second tangent, then the external angle between these two tangents measures the change in direction; this, divided by the length of the curve, between the two points, is a measure of the curvature of the curve.

If the curve is the circle, then this ratio is a constant; for the external angle between the tangents is equal to the angle at the centre, and the length of arc being proportional to the angle at the centre, it follows that the ratio of change in direction to change in length is a constant.

The change in direction in going once round a circle is four right angles or 2π radians. The length passed over is $2\pi r$. Hence, the curvature is $\frac{2\pi}{2\pi r} = \frac{1}{r}$. That is, the curvature of a curve is the reciprocal of the radius. The curvature of a straight line is zero, and the radius of curvature is infinity.

Angular Velocity.—Consider any plane figure rotating at a constant rate about an axis, normal to the plane of the figure, with a *period* T; *i.e.*, T being the time of one rotation. Any point on the plane will then be moving at a constant speed; but, the direction of motion will continually change at a uniform rate. In making one rotation, or what amounts to the same thing, during one period, the motion changes in direction by an amount equal to 2π radians. This change in direction, divided by the time consumed during the change, is a measure of the angular velocity of the point. That is, the angular velocity is numerically equal to $\frac{2\pi}{T}$, and is represented by the Greek letter ω. It will be observed that in the foregoing discussion, the distance of the point from the axis was not considered, and that the same result would have been obtained for any point. It therefore follows that all points on the body have the same angular velocity. Since, when a point is moving in the circumference of a circle, the direction of its motion at any point is in the direction of the tangent to the circle at that point, and further, since the radius drawn to the point changes in direction at the same rate that the tangent does, it follows that the angular velocity of a point is also measured by the rate of change of direction of the radius. We may then define angular velocity in general as the ratio of the angle swept out by a radius to the time consumed.

Since the circumference of a circle is equal to $2\pi r$, it follows that if a point is moving at a uniform rate in the circumference of a circle with a period

T, its linear velocity is $\frac{2\,\pi r}{T}$. That is, the linear velocity is equal to the product of angular velocity and radius.

Angular velocity may be uniform or variable. When the radius sweeps out equal angles in any arbitrary equal intervals of time, the rotating body has a *canstant angular velocity.*

When the angular velocity changes at a uniform rate, the rotating body has a *constant angular acceleration.* Angular acceleration has precisely the same relation to angular velocity that linear acceleration has to linear velocity. That is, angular acceleration is the rate of change of angular velocity.

We are now prepared to state some of these relations symbolically.

From the definition of uniform motion, we have

$$velocity = \frac{distance\ traveled}{time\ consumed};\ \text{or}\ v = \frac{s}{t}.$$

From which

$$s = vt.$$

In the same manner for constant acceleration;

$$acceleration = \frac{change\ in\ velocity}{time\ consumed};$$

or

$$a = \frac{v_2 - v_1}{t}; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v_1 is the initial and v_2 the final velocity, and a the acceleration.

The mean velocity of a body having a uniformly accelerated motion is obviously the half sum of the initial and final velocity; *i. e.*,

$$V = \frac{v_1 + v_2}{2}.$$

The distance passed over is numerically equal to the product of mean velocity and time. In symbols

$$s = Vt = \frac{v_1 + v_2}{2}t. \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying equation (1) by (2) and clearing of fractions, we have

$$2\,as = v^2{}_2 - v^2{}_1. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, if in equation (2) we substitute for v_2 its value, $v_1 + at$, we obtain

$$s = \frac{2\,v_1 + a\,t}{2}t = v_1 t + \frac{a\,t^2}{2}. \quad . \quad . \quad . \quad (4)$$

If the initial velocity is zero; *i.e.*, the body start from rest, we have the following equations:

$$v = a\,t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$s = \frac{v\,t}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$s = \frac{v^2}{2\,a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$s = \frac{a\,t^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The equations for angular motion are deduced in a precisely similar manner.

By definition for constant angular velocity;

$$\textit{angular velocity} = \frac{\textit{angle swept out}}{\textit{time consumed}},$$

or in symbols

$$\omega = \frac{\varphi}{t}; \text{ and } \varphi = \omega\, t.$$

In the same manner, for constant angular acceleration;

$$\textit{angular acceleration} = \frac{\textit{change in ang. vel.}}{\textit{time consumed}},$$

or

$$\alpha = \frac{\omega_2 - \omega_1}{t}; \quad . \quad . \quad . \quad . \quad (9)$$

where ω_1 is the initial, ω_2 the final angular velocity, and α the angular acceleration.

The mean angular velocity for constant angular acceleration is equal to the half sum of the initial and final angular velocities; *i.e.*,

$$\omega_a = \frac{\omega_1 + \omega_2}{2};$$

where ω_a is the mean angular velocity. The angle swept out is the mean angular velocity multiplied by the time, *i.e.*,

$$\varphi = \omega_a\, t = \frac{\omega_1 + \omega_2}{2}\, t. \quad . \quad . \quad . \quad (10)$$

Multiplying equation (9) by (10) and clearing of fractions, we obtain

$$2\, \alpha\, \varphi = \omega^2_2 - \omega^2_1. \quad . \quad . \quad . \quad . \quad (11)$$

Again, if in equation (10) we substitute for ω_2 its value, $\omega_1 + \alpha t$, we obtain

$$\varphi = \frac{2\omega_1 + \alpha t}{2} t = \omega_1 t + \frac{\alpha t^2}{2}. \quad . \quad . \quad (12)$$

If, as before, we assume the body starting from rest, we obtain the following relations:

$$\omega = \alpha t, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\varphi = \frac{\omega t}{2}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\varphi = \frac{\omega^2}{2\alpha}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\varphi = \frac{\alpha t^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Such are the algebraic equations that must necessarily follow from the definitions.

As was previously stated, linear velocity equals angular velocity multiplied by the radius, or $v = \omega r$; dividing by t we have

$$a = \alpha r; \text{ hence } v : a :: \omega : \alpha.$$

CHAPTER II

COMPOSITION AND RESOLUTION OF MOTIONS

THE direction in which a body is moving at any point of its path may always be represented by a straight line; and likewise, the distance over which a body has passed from a given fixed point may be represented to a given scale by the length of a line. In Fig. 1, let a particle move from A along the line $A\,b$ to the point B; then from B along the line BC to the point C. Now, the displacement of the particle from the point A is precisely the same as though the motion had taken place along the line AC by an amount equal to the length of the line AC; or along the path ADC. That is, it is immaterial, so far as final displacement is concerned, whether the particle first move in the direction AB by an amount AB and then in the direction BC by an amount BC; or, first move in the direction AD, parallel to BC, by an amount BC and then in the direction DC, parallel to AB, by an amount

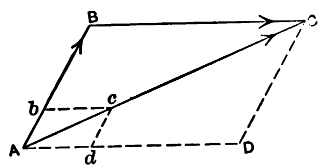

FIG. 1.

AB; or whether it move along the line AC by an amount AC. Suppose now, the particle to be at the point A, and moving at a uniform rate, relatively to the earth's surface, along the line AB, and simultaneously with this the plane surface upon which the particle is moving, to move at a uniform rate, relatively to the earth's surface, carrying the particle with it, in the direction AD. Assume that the relative rates are such that the particle moves on the surface from A to B, while the surface has moved by an amount equal to AD. The particle will then be situated at C, and the displacement will again be the same as though the particle had moved along the line AC from A to C. But in this case the particle has actually moved, relative to the earth's surface, along the line AC. For, take some fractional part of the line AB, such as Ab, and a like fractional part of the line AD, such as Ad. Draw the lines dc and bc respectively parallel to AB and AD. Then from the conditions, while the particle has moved from A to b, the surface has moved the amount Ad parallel to AD. Hence, the point c gives the position of the particle at that instant; but by construction, the triangles Adc and ADC are similar, hence the point c falls on the line AC; and this is true no matter what fractional part of AB is taken; hence the path of the particle is the line AC. It must be remembered that the line AC is fixed with respect to some surface assumed at rest, such as the earth's surface.

If now, we assume the diagram drawn to such a scale that the lengths $A B$ and $A D$ represent the distances passed over in a unit of time, then these lengths also represent the velocities in magnitude and direction. And, since $A C$ represents both in magnitude and direction the displacement of the particle in a unit of time, it follows that it also represents the resultant of the velocities $A B$ and $A D$.

If the velocities be variable, and at any instant the velocity in the direction $A B$ is equal to $A B$ and at the same instant the velocity in the direction $A D$ is equal to $A D$, then $A C$ represents in magnitude and direction the instantaneous resultant velocity. Since $A C$ represents the resultant velocity when the two component velocities are constant, it must also represent the resultant velocity at any instant; for, the velocities at the instant being such that the two displacements $A B$ and $A D$ would be produced in a unit of time, and the resultant of these two displacements being $A C$, which would also have been produced in unit time, it is obvious that $A C$ represents the instantaneous resultant velocity. We therefore have the following theorem:

The resultant of two concurrent coplanar velocities is represented in magnitude and direction by the diagonal of the parallelogram constructed upon the component velocities as sides; the component velocities being drawn from a common point to the same scale and in the proper directions.

If we assume constant accelerations in the two directions, and draw our diagram to such a scale that *A B* and *A D* represent the velocities acquired in the two directions in a unit of time, then the instantaneous resultant velocity at the end of a unit of time will be represented by *A C*. But, since this velocity has been acquired in a unit of time, the line *A C* also represents the acceleration. Accelerations may therefore be combined by the same methods as velocities.

It can readily be seen that the foregoing demonstrations can be extended to any number of coplanar components; for, it is only necessary to find the resultant for any two of them, combine this resultant with a third component, and so on, until the final resultant is obtained; the final resultant being independent of the order in which the components are taken.

As has been previously stated, quantities in which direction is specified, as well as magnitude, are *vector* quantities. Displacements, velocities, and accelerations are vector quantities. In Fig. 1, the lines *A B*, *B C*, and *A C* represent vectors. By referring to Fig. 1, it will be seen that the resultant of the two component vectors may also be obtained by laying off the vector *A B*, then from the terminal *B* laying off the vector BC, and by joining A and C we obtain the resultant. Or expressed vectorially

$$AB + BC = AC.$$

The student must consider carefully what this means. It, of course, does not mean that the length of the line AB plus the length of the line BC equals the length of the line AC. But it does mean that in *effect*, as regards displacements, velocities, or accelerations, the sum of AB and BC equals AC. This addition of vectors applies to any number of coplanar vectors; to find the resultant or vector sum, it is only necessary to lay off any vector in the proper direction and to proper scale, and from its terminal lay off a second vector, etc. The line then joining the beginning of the first vector and the terminal of the last vector, is the resultant, or vector sum. The vector sum of a number of vectors is independent of the order of addition.

Resolution. A displacement, a velocity, or an acceleration may be resolved into components just as well as components may be combined. If in Fig. 1, a particle start from A and move to C, its final displacement is independent of the path pursued in going from A to C. In the same manner, if AC represents the velocity of a particle at any instant, then since AC equals the components AB plus BC it can be resolved into these two components, or any other two components whose sum equals AC. But, since each of these components can again be resolved into two components, and these individual components can again be resolved, it follows that any vector can be resolved into any number of coplanar vectors whose sum is equal to the given vector.

Trigonometric Addition of Vectors. The resultant of two coplanar vectors is readily found by the formula:

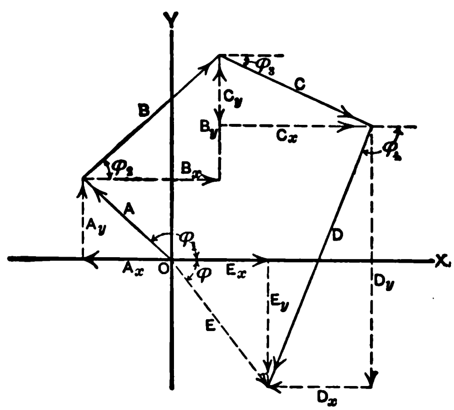

FIG. 2.

$R = (A^2 + B^2 + 2\,A\,B\cos\theta)^{\frac{1}{2}}$; where R is the resultant, A and B the two component vectors, inclined to each other by the angle θ. The formula gives, however, the magnitude only, and not the direction. To find the magnitude and direction of the resultant of two or any number of coplanar vectors, proceed as follows: Let in Fig. 2, A, B, C, and D be the given vectors; inclined respectively to the X axis by the angles φ_1, φ_2, φ_3, and φ_4. Multiplying each vector by the sine and cosine of its angle of inclination, the vertical components, A_y, B_y, C_y, and D_y are found; and likewise, the horizontal components A_x, B_x, C_x, and D_x. Taking now, the algebraic sum of the vertical components, E_y is found, and E_x for the algebraic sum of the horizontal components. The resultant, E, is now completely specified, it being the hypothenuse of the right triangle whose legs are E_y and E_x, and its inclination to the X axis is given by

$$\varphi = tan^{-1}\frac{E_y}{E_x}.$$

It is not necessary to draw the diagram; it is only necessary to multiply each vector by the sine and cosine of its angle of inclination, proper attention being given to the signs of the trigonometric functions, then taking the algebraic sum of the vertical components and of the horizontal components, the two quantities obtained being respectively the vertical and horizontal component of the resultant.

Centripetal Acceleration. Velocity has been defined as being constant when the rate of motion—speed—is constant, and further, when there is no change in direction. When there is a change in direction, even though the speed be constant, the velocity varies, and hence there is an acceleration.

A particle moving with a uniform speed in the circumference of a circle is such a case.

Let a particle be moving in the circumference of a circle whose centre is O, with a uniform speed. The speed being constant, the distance travelled is proportional to the time. Let us consider the change from the instant the body is passing through the point a, Fig. 3, with a velocity v in the direction af tangent to the curve at the point a. Had there been no acceleration, the body would have continued to travel in the direction af with constant speed; but due to an acceleration, it is continually deviated from the straight line and travels along the curve. By the time the particle has reached the point d, it has been displaced at right angles from its path at a, a distance $a\,e$. Assume the distance $a\,d$, along the arc, an immeasurably small fractional part of the circumference of the circle, then ae is very small compared with the radius. Under the assumed conditions, the distances along the arc and chord are sensibly equal. Also, the distance $a\,e$ along the radius will be so small that the acceleration, during the time required to bring about this displacement, is sensibly constant; hence, we may apply the formula for uniformly accelerated motion. By equation (8), Chapter I, we have $a\,e = \frac{a\,t^2}{2}$; where t is

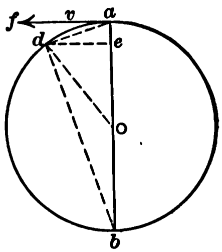

FIG. 3.

the time required to bring about the change. Further, we have $ad = vt$, and by the geometry of the figure, we have the following proportion:

$ba:ad::ad:ae$; from which, by substitution,

$$2r:vt::vt:\frac{at^2}{2}.$$

Solving, we have $a = \frac{v^2}{r}$; *i.e.*, the acceleration along the radius is equal to the square of the speed multiplied by the curvature of the curve.

It is to be remembered that distances in the figure are exaggerated, and that the assumptions made in the demonstration are true only when the distance ad becomes indefinitely small.

This formula is so important that it may be of value to deduce it by an entirely independent method.

In Fig. 4, let bc represent the direction and magnitude of the velocity of the particle at the instant it is passing through the point b. It is obvious that as the radius r rotates about the point O, bc is always at right angles to it; hence it may be represented by a second line od, passing through the center O, parallel to bc and equal to it in length. As v always represents in direction and magnitude the

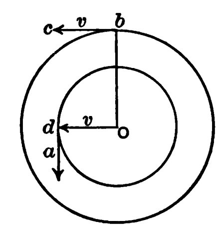

FIG. 4.

motion of the terminal of the radius r, so does a represent the motion of the extremity of v. Hence we may write, since the angular velocity of r and v is the same,

$$r : v :: v : a,$$

or

$$a = \frac{v^2}{r}. \qquad \qquad (1)$$

But since a is the rate of motion of the extremity of v, and v is the rate of motion of the extremity of r – or the rate of motion of the particle —it follows that a is the rate of a rate of motion; *i.e.*, an acceleration, and the theorem is proved.

Simple Harmonic Motion. S. H. M. is a motion along a fixed path, such that the acceleration towards a fixed point in the path is always proportional to the displacement, measured along the path, from the fixed point.

From this definition, it at once follows that no matter what the velocity of the particle, when passing through the fixed point, it must finally come to rest; since the acceleration is towards the fixed point. Furthermore, after coming to rest the particle will begin to move towards the fixed point with increasing velocity, and by the time it again reaches this point, will have a velocity precisely equal in magnitude, but opposite in direction to that when previously passing through the point. It will then move to an equal displacement on the opposite side of the fixed point, and so on. In other words, the motion is periodic; *i.e.*, repeats at regular intervals. The fixed point is called the *position of equilibrium*; there being no acceleration. The greatest distance from the fixed point, measured along the path, is the *amplitude*. The distance at any instant from the fixed point is the *displacement*. The time required to pass through a cycle; *ie.*, the time required to come to the original condition, which means being at the same point and going in the same direction, is a period and represented by T.

To make a complete general solution is far too complex mathematically for elementary mechanics; but a good idea may be obtained by studying some particular case in detail.

Consider a particle, Fig. 5, moving with a uniform speed in the circumference of a circle whose centre is C and radius r. Let the period be T, then the angular velocity is $\frac{2\pi}{T} = \omega$; and the angle at any instant is $\varphi = \omega t$; the angle and time being both measured from the origin, O. If the velocity, V, be resolved into two components, one at right angles and the other parallel to

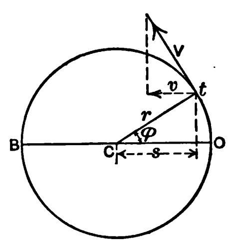

FIG. 5.

the diameter OB, the parallel component will be

$$v = V \sin \varphi = \omega r \sin \omega t; \quad . \quad . \quad . \quad (2)$$

ω and r being constant, it follows that the velocity parallel to the diameter is proportional to the sine of the angle. From equation (1) we have the radial acceleration equal to $\frac{V^2}{r}$. The component of this acceleration, parallel to the diameter, is

$$f = \frac{V^2}{r} \cos \varphi = \omega^2 r \cos \omega t. \quad . \quad . \quad (3)$$

But, $r \cos \omega t$ is equal to the displacement s; hence,

$$f = \omega^2 s = \frac{4\pi^2}{T^2} s. \quad . \quad . \quad . \quad . \quad (4)$$

That is, since ω^2 is a constant, the acceleration along the diameter varies directly as the displacement from the center of the circle. Therefore, when a point moves with a constant speed in the circumference of a circle, the projection of this motion onto a diameter of the circle is a simple harmonic motion. Students cannot become too familiar with these formulæ. There are no equations of more universal application in mechanics and physics than those of simple harmonic motion.

CHAPTER III

FORCE AND FRICTION
WORK AND ENERGY

Inertia and Force. So far, motion has been considered in the abstract, without considering the nature of the body moved.

As a result of experience we know that it requires an effort to change the rate of motion of a body; and further, that the intensity of the effort depends upon the body moved and the rate at which the change is brought about; *i.e.*, the acceleration.

Broadly stated, every body persists in maintaining whatever rate of motion it may happen to have; this is the chief characteristic of matter and is termed *inertia.* This idea must not be confused with inactivity.

As previously stated, an effort is required to accelerate matter, and this effort is termed *Force.*

We may then define force as that which changes or tends to change the velocity of a body.

Suppose we have two spheres of equal volume, but of different materials; say one of wood and the other of lead, lying on a perfectly smooth horizontal surface, and accelerate them equally, it will be found that a greater force is required for the leaden sphere than is required for the wooden one. Were they of the same material, we should find the forces equal. Were they of the same material but of different dimensions, it would be found that the greater force is required for the larger sphere. It is evident that the substances being the same, the larger sphere contains the greater quantity of matter. That is, by operating on bodies composed of the same material, we find that the greater the quantity of matter the greater the force. In the case of the two spheres of equal volume, one of wood and the other of lead, it is found that the leaden sphere requires a greater force than the wooden sphere when they are given equal accelerations; hence it is assumed that the leaden body contains a greater quantity of matter. The quantity of matter a body contains is called its *Mass.*

The mass per unit volume is the *Density; i.e.,*

$$Density = \frac{Mass}{Volume}, \text{ or } D = \frac{M}{V}.$$

We may now embody some of the foregoing statements regarding mass, force, and acceleration in the form of equations. That is, that the force is proportional to the mass and to the acceleration it produces. In symbols,

$$F \propto m\,a.$$

By introducing a constant we may use the equality sign, *i.e.*,

$$F = kma; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

k depending upon the units chosen, and may be made unity by choosing the proper units. Equation (1) shows that the mass of a body may be measured by the force required to produce a given acceleration; *i.e.*, by its inertia.

In the *c.g.s.* system, the unit mass is the gram, the unit acceleration is that when the velocity changes at the rate of 1 cm. per sec. per sec. Hence if we choose as our unit force that force which will accelerate the mass of 1 gram 1 cm. per sec. per sec., k in equation (1) becomes unity. This unit force is called the *dyne*.

In the F.P.S. system, the unit force is the *poundal*, and is that force which will accelerate the mass of 1 pound 1 foot per sec. per sec. There is another unit in common use; namely, the engineers' unit, called *pound*. This force is equal to the weight of a pound, and hence is a variable, depending upon latitude and elevation. In any case it is the force that will accelerate the mass of one pound g feet per sec. per sec.; where g is the acceleration due to the earth's gravitational field, for the particular locality. In this case, k in equation (1) becomes $\frac{1}{g}$; *i.e.*,

$$F = \frac{m\,a}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In most cases 32.2 is sufficiently accurate for the value of g, and in most engineering computations 32. may be used.

Composition and Resolution of Forces. Assume a body, whose mass is m, acted upon by two forces in such a manner that the two accelerations $A\,b$ and $A\,d$ (Fig. 6), are imparted. The resultant acceleration will be represented by $A\,c$. The force

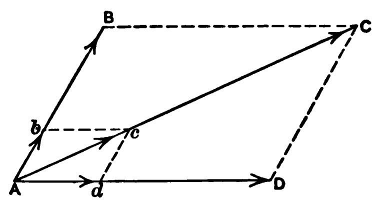

FIG. 6.

required to produce the acceleration $A\,b$ is equal to $m \times A\,b$. Let this be represented by $A\,B$. Likewise, let $A\,D$ represent to the same scale the product of m and $A\,d$; *i.e.*, the force required to produce the acceleration $A\,d$. The force required to produce the acceleration $A\,c$ is equal to the product of m and $A\,c$. Draw $B\,C$ and $D\,C$ respectively parallel to $A\,D$ and $A\,B$. Since $A\,b$ and $A\,d$ are, by construction, like fractional parts of $A\,B$ and $A\,D$, the point c must fall on the line AC. From the similarity of the figure we have

$$\frac{AC}{Ac} = \frac{AB}{Ab};$$

but $\frac{AB}{Ab} = m$ by construction. Therefore, $m \times Ac = AC$; *i.e.*, AC represents the force required to give the acceleration Ac to the mass m, and hence is the resultant of the forces AB and AD. The resultant of two or more concurrent coplanar forces is therefore found in the same manner that the resultant of displacements, velocities, or accelerations is found.

From what has been shown in the composition of forces, it is readily seen, that a force may be resolved into components the same as any other vector quantity.

Friction. When two surfaces are in contact and are caused to move relatively to each other, a resistance is experienced, which is a function of the normal pressure between the surfaces and of the rate of relative motion. This resistance is called the *force of friction*. The following statements are usually given as being approximately true.

(1) The force of friction is directly proportional to the normal pressure between the surfaces.

(2) The force of friction for any given normal pressure is independent of the area of the surfaces in contact.

(3) The force of friction is independent of the rate

The first and second statements are practically true for ordinary pressures. When the pressures become high there is a very wide departure; and at what pressure this takes place depends, of course, on the nature of the substances in contact. The third statement is not even practically true for most cases. In a great many cases when the rate of motion varies only slightly, then other things being equal, the force of friction is for practical purposes constant. But whenever there is considerable variation in the rate of motion there is also a measurable change in the force of friction.

Coefficient of Friction. The ratio of the force of friction to the normal pressure is called the coefficient of friction. In symbols

$$\mu = \frac{F}{N}, \text{ or } F = \mu N; \quad . \quad . \quad . \quad (3)$$

where μ is the coefficient, F the force, and N the normal pressure. From what has been stated, it is clear that this formula can be used only when the conditions under consideration approximate very closely, as regards intensity of pressure and rate of motion, to those conditions under which the coefficient of friction was determined.

Angle of Repose. If a plane, having a body resting upon it, be inclined so that the body is just on the point of sliding, the angle which the plane makes with the horizontal is called the angle of repose. It will now be shown that the coefficient of friction is equal to the tangent of the angle of repose.

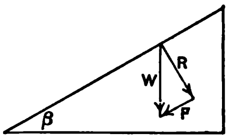

FIG. 7.

Let, in Fig. 7, the plane be inclined at an angle β such that the body, whose weight is W, is just on the point of sliding. Resolving the weight into two components, we obtain for the force parallel to the plane, which is equal to the force of friction,

$$F = W \sin \beta,$$

and for the normal pressure

$$R = W \cos \beta.$$

By definition, the coefficient of friction is the ratio of the force of friction to the normal pressure; hence

$$\mu = \frac{W \sin \beta}{W \cos \beta} = \tan \beta. \quad . \quad . \quad . \quad . \quad (4)$$

Assume, as in Fig. 8, a body of weight W lying upon a horizontal surface and having applied a force F making an angle φ with a normal to the plane. The normal pressure between the two surfaces then is

$$R = W + F \cos \varphi,$$

and the force tending to move the body is

$$f = F \sin \varphi.$$

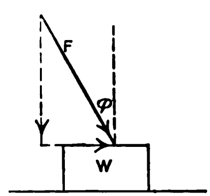

FIG. 8.

Now, for sliding to be impending, the force parallel to the surface must be equal to the product of the coefficient of friction and the normal pressure; hence, for the body to be on the point of sliding, we must have

$$F \sin \varphi = \mu \left(W + F \cos \varphi\right).$$

From which

$$\mu = \frac{F \sin \varphi}{W + F \cos \varphi}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the weight of the body be small in comparison with the applied force, then W in equation (5) may be neglected, and we have

$$\mu = \tan \varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Comparing equations (4) and (6) it is seen that for motion to be impending, the angle which the applied force makes with the normal must be equal to the angle of repose. For any angle smaller than this, there can be no motion, no matter what the intensity of the applied force.

Work and Energy. When a force overcomes a resistance through space it is said to do work; and the quantity of work is measured by the product of the displacement of the point of application of the force and the component of the force parallel to the displacement. In symbols,

$$W = F d;$$

where W is the work, d the displacement, and F the component of the applied force parallel to the displacement.

As an example, assume as in Fig. 9, the line of direction of the force F making an angle θ with the path in which the point of application of the force is constrained to move, and suppose further, that the

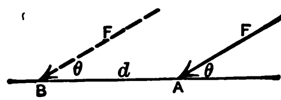

FIG. 9.

point of application A, is displaced through the distance d, to the point B.

The work then is

$$W = F \cos \theta \times d = F d \cos \theta.$$

$F \cos \theta$ being the component of the force parallel to the motion.

Suppose the force applied to a body which offers no resistance excepting its own inertia; then, from a previous formula, we have,

$$F = m a. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The distance passed over will be, if the force be constant so that the acceleration is constant, and the body start from rest,

$$d = \frac{a t^2}{2} = \frac{v^2}{2 a}. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Multiplying the equations (a) and (b), member by member, we have

$$F d = W = \frac{m v^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In this case, work is consumed in imparting motion to the mass m; but the mass in being brought to rest is able to do work in quantity precisely equal to that consumed in imparting the speed v. Work is therefore said to be stored, or the body possesses *Energy,—capability of doing work*—in virtue of its motion. The energy a body possesses due to its motion is called *Kinetic Energy.*

When a body is able to do work due to its position, its energy is said to be *potential.* Examples of potential energy are bodies above the earth's surface, compressed springs, etc.

Unit Work. In the c.g.s. system, the unit work is the *erg*, and is equal to the work done in bringing about a displacement of one cm. against a force of one dyne. The practical unit work is the *joule*, and is equal to 10^7 ergs.

In the F.P.S. system, the unit work is the *foot poundal*, and is equal to the work done in bringing about a displacement of one foot against a force of one poundal. The engineers' unit, the *ft. lb.*, is the work done in raising the weight of one pound one foot. One ft. lb., therefore, equals g foot poundals; or approximately, 1 ft. lb. = 32.2 foot poundals.

Conservation of Energy. The change in kinetic energy that a body undergoes in passing over a given path is equal to the work done in traversing that path. For, assume that at the beginning of the path the velocity of the body is v_1; then after traversing the distance s, if the acceleration is a, we have, by a previous demonstration, the following relation:

$$v^2 = v^2_1 + 2\,a\,s; \qquad . \quad . \quad . \quad . \quad (8)$$

where v is the velocity at the end of the path s. Multiplying both sides of equation (8) by $\frac{m}{2}$, where m is the mass of the body, we have

$$\frac{m\,v^2}{2} = \frac{m\,v^2_1}{2} + m\,a\,s. \qquad . \quad . \quad . \quad (9)$$

The second term of the right-hand member of equation (9) is the product of a force and distance, and therefore represents work. Writing equation (9) in another form, we have

$$\frac{m\,v^2_1}{2} = \frac{m\,v^2}{2} - m\,a\,s. \qquad . \quad . \quad . \quad (10)$$

If the acceleration is not constant, we may assume the path to be divided into n elements, each element of the path being indefinitely small, such that the acceleration for any element may be considered constant. Rewriting equation (10) for this condition, we have

$$\frac{m\,v^2_1}{2} = \frac{m\,v^2}{2}$$

$$-(m\,a_1\,s_1 + m\,a_2\,s_2 + m\,a_3\,s_3 + \ldots . + m\,a_n\,s_n); \quad . \quad (11)$$

where s_1, s_2, s_3, etc., represent the elements into which the path is divided, and a_1, a_2, a_3, etc., the corresponding accelerations while passing over those elements.

The left-hand member of equation (11), being the initial kinetic energy, is a constant quantity for any given case; hence, the right-hand member, which is the sum of the kinetic and potential energies, is a constant quantity for any body moving under the action of forces without collision with other bodies. And this is true for any body of a system of bodies; providing always that there are no collisions, and the forces acting are due to the mutual interaction of the bodies, and not to actions external to the system. It follows, then, that the total energy of such a system remains constant.

Potential and kinetic energy of masses are, however, not the only forms of energy. But by suitable processes, energy may be transformed from one form into another. One of the simplest cases of transformation of energy is the destruction of kinetic energy and the simultaneous evolution of heat through friction or impact.

If in any system, from which no energy escapes, and into which no energy enters, account be taken of all forms of energy, then no matter what transformations take place within the system, the sum total is a constant quantity. This is the principle of the *conservation of energy*. It is, so far as experience goes, consistent with all physical phenomena.

CHAPTER IV

MOMENTUM, PRINCIPLE OF MOMENTS, AND IMPACT

Momentum. By momentum is understood the quantity of motion a body possesses, and is expressed numerically by the product of mass and velocity. In symbols

$$M = m\,v. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If a body whose mass is m, having an initial velocity v_1, has applied a constant force F, its velocity will be augmented at a uniform rate. Let its velocity at the end of the time t be represented by v_2; then the change in momentum is

$$M = m\,(v_2 - v_1);$$

but

$$F = \frac{m\,(v_2 - v_1)}{t};$$

therefore

$$F\,t = M = m\,(v_2 - v_1). \quad . \quad . \quad . \quad (2)$$

Or the change in momentum is equal to the product of force and time; *i.e.*, the *Impulse.*

The Three Laws of Motion

(1) Every body continues at a uniform rate of motion in a right line, unless compelled to change its rate by some force external to it.

(2) Change of momentum is proportional to the impulse that produces it and in the same direction.

(3) *Action* and *reaction* are equal and opposite in direction.

Statement (2) includes statement (1); for, since statement (2) says that change in momentum is equal to the impulse that produces it, it follows that if the impulse is zero, *i.e.*, no external force, then there is no change in momentum, and hence no change in the rate of motion; and further, it implies that if the impulse is zero, there is no change in direction.

Statement (3) merely declares that the actions of two bodies on each other are always equal and opposite in direction. This action and reaction between two bodies considered jointly is termed *stress*.

Moment of a Force. In considering the action of a force on a rigid body, it is necessary to consider its *three elements;* viz.: *Intensity*, *Point of Application*, and *Direction*.

It is the result of experience that the tendency of a force, of given intensity, to produce rotation is independent of the point of application, so long as the direction in which the force acts remains unchanged. If, then, we understand by *the line of direction* of a force, the direction and position of the line along which the force acts, then the conditions in regard to the tendency of the force to produce rotation are completely specified by specifying its *intensity* and *its line of direction*, or as some prefer to call it, its line of action.

The moment of a force, with respect to an axis, is a measure of the tendency of the force to produce rotation about that axis. Numerically, it is equal to the product of the force and the perpendicular distance from the axis to the line of direction of the force. Calling this perpendicular distance the arm of the force, then the moment is equal to the product of the force and its arm. In symbols

$$C = F\,d;$$

where C is the moment, F the force, and d its arm. (As a matter of convenience, moments tending to produce anti-clockwise rotation are considered positive, and those tending to produce clockwise rotation, negative.)

Principle of Moments. Experiment shows that if a rigid body, capable of rotating about a fixed axis, is in equilibrium under the action of any number of forces lying in a plane perpendicular to the axis, then the sum of the moments, with respect to that axis tending to produce rotation in one direction, is equal to the sum of the moments tending to produce rotation in an opposite direction; or, *the algebraic sum of the moments is equal to zero.* This theorem is called the principle of moments.

The two conditions necessary so that a rigid body shall be in equilibrium under the action of coplanar forces are: ***the algebraic sum of the forces in any direction must be equal to zero, and the algebraic sum of the moments about any axis***, perpendicular to the plane of the forces, ***must be equal to zero.***

One particular case requires especial attention. Assume a body under the action of a number of coplanar forces acting in various directions, and that these forces be resolved into horizontal and vertical components. Assume further, that the algebraic sum of these components in any direction is equal to zero; but that the algebraic sum of the moments is not equal to zero. It is obvious that such a system of forces can not be balanced by one force; but that two equal and opposite forces must be applied, the sum of whose moments is equal and opposite to the sum of the moments of the system. Such a pair of forces is called a couple; that is, two equal and opposite forces not in the same straight line, constitute a couple, whose moment is equal to one of the forces multiplied by the perpendicular distance between their lines of direction. This may be shown as follows: Assume, as in Fig. 10, the two equal coplanar parallel forces oppositely directed, with the perpendicular distance d between their lines of direction. Choose any axis O at a perpendicular distance x from the line of direction of the force F_2.

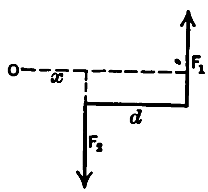

FIG. 10.

The moment of the force F_1, with respect to this axis, is

$$F_1 (d + x) = F_1 d + F_1 x.$$

The moment of the force F_2, with respect to the same axis, is

$$- F_2 x = - F_1 x;$$

since $F_1 = F_2$. Hence, the sum of the moments of the two forces, or the moment of the couple is

$$G = F_1 d + F_1 x - F_1 x = F_1 d.$$

It is clear that the same result will be obtained no matter where the axis is chosen, and hence the moment of the couple is equal to the product of one of the forces and the perpendicular distance between their lines of direction.

Centre of Mass. If two particles, m_1 and m_2, be rigidly connected and a force applied on the straight line joining them at such a point between the two particles that only linear acceleration is produced, the moments about the point of application, due to the reactions of the particles, must be equal and oppositely directed. Since the acceleration is the same for both particles, the reactions must be directly as their masses; and, consequently, for the moments to be equal, the arms must be to each other inversely as the masses of the particles. From which, we have (Fig. 11),

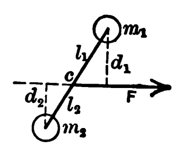

FIG. 11.

$$m_1 d_1 = m_2 d_2,$$

but for this to be true, we must have

$$m_1 l_1 = m_2 l_2,$$

and the point c is determined. The point c is the centre of mass of the two particles, and is so situated that if a force be applied at this point, in any direction whatsoever, the two particles will be equally accelerated parallel to the direction of the applied force; and, hence, no motion of rotation is produced in the system. If there be a third particle, m_3, then to find the centre of mass of the three particles, we assume a mass equal to $m_1 + m_2$ concentrated at c, and combine this with m_3, in precisely the same manner that m_1 and m_2 were combined. In a similar manner, the centre of mass for any number of particles is found. The centre of mass of a system of particles is then a point where the whole mass of the system may be considered concentrated, and is so situated in the system that if forces be applied whose resultant passes through the point, there will be no tendency to produce rotation. From the foregoing discussion it is evident that the centre of mass is a point so situated in the body, that if a plane be passed through it in any direction whatsoever, then the sum of the products, obtained by multiplying each particle by its perpendicular distance from the plane, on one side of the plane, must be equal to the sum obtained in a similar manner on the other side of the plane; or if we call the products on one side of the plane positive and on the other side of the plane negative, their algebraic sum must be zero.

Centre of Gravity. The force with which a body is attracted toward the centre of the earth is the resultant of the forces due to the particles of the body; and since the angle subtended at the centre of the earth, by any body of ordinary dimensions, is practically nil, the force with which the body is attracted is the resultant of a system of parallel forces.

The sum of these parallel forces, or the resultant, is, of course, the weight of the body. The point where the entire weight of the body may be considered to act is called the centre of gravity, and is determined by the principle of moments in precisely the same manner as the centre of mass. It is clear that in all ordinary cases the centre of mass and centre of gravity are defined by the same point. For bodies of regular geometric form and homogeneous in construction the centre of mass coincides with the geometrical centre. For bodies not so constituted the centre of mass or centre of gravity must be found experimentally.

Impact and Momentum. From the third law of motion it follows, that if two bodies impinge, the forces acting on the two bodies are equal; since action and reaction are equal. Furthermore, since the time of impact is the same for both bodies it follows that the impulse acting on the one is equal to the impulse acting on the other; therefore, ***the change in momentum in the one body is equal to the change in momentum in the other body.***

Suppose two bodies whose masses are m_1 and m_2 and having velocities u_1 and u_2, impinge in such a way that the velocities are changed to v_1 and v_2. Then since, from what has been previously shown, the changes in momentum are equal, it follows that

$$m_1(u_1 - v_1) = m_2(v_2 - u_2), \quad . \quad . \quad . \quad (3)$$

from which

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2. \quad . \quad . \quad (4)$$

That is, the sum of the momenta before impact is equal to the sum of the momenta after impact. This is the first law of impact.

If there is no energy absorbed during impact, the bodies are said to be perfectly elastic, and the kinetic energy of the system after impact is equal to the kinetic energy before impact; hence

$$m_1 u^2_1 + m_2 u^2_2 = m_1 v^2_1 + m_2 v^2_2, \quad . \quad . \quad (5)$$

and

$$m_1(u^2_1 - v^2_1) = m_2(v^2_2 - u^2_2),$$

from which

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2);$$

by equation (3)

$$m_1(u_1 - v_1) = m_2(v_2 - u_2),$$

hence

$$u_1 + v_1 = v_2 + u_2;$$

from which

$$u_1 - u_2 = v_2 - v_1,$$

and

$$\frac{v_2 - v_1}{u_1 - u_2} = 1. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is, the ratio of the difference of velocities after impact to the difference of velocities before impact is a constant, and equal to unity. In the case of two bodies impinging, which are not perfectly elastic, this ratio is less than unity, and must be determined by experiment. In any case, the ratio

$$\frac{v_2 - v_1}{u_1 - u_2} = e, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

is a constant for any two given bodies and is called the *coefficient of restitution.*

Conservation of Momentum. Let the two spherical bodies of masses m_1 and m_2, approach each other on the right line joining their centres with velocities, u_1 and u_2, as indicated in Fig. 12.

Let $a\,b$ be the plane of impact, and the full circles represent the position of the bodies with respect to the plane $a\,b$ one second before impact. Let the centre of mass of the two bodies at that instant be to the right of the plane by a distance x. The centre of mass of the body m_1, one second before impact, must be to the right of the plane of impact by the distance $u_1 + r_1$; the centre of mass of the body m_2, must at the same instant be to the left of the plane $a\,b$ by the distance $u_2 + r_2$. Writing now the equation for the centre of mass, we have

$$m_1 (u_1 + r_1 - x) = m_2 (u_2 + r_2 + x);$$

from which

$$\frac{m_1 u_1 - m_2 u_2}{m_1 + m_2} = x - \frac{m_1 r_1 - m_2 r_2}{m_1 + m_2}. \quad . \quad . \quad (8)$$

If now the dotted figures, displaced downward to avoid confusion, represent the positions of the two

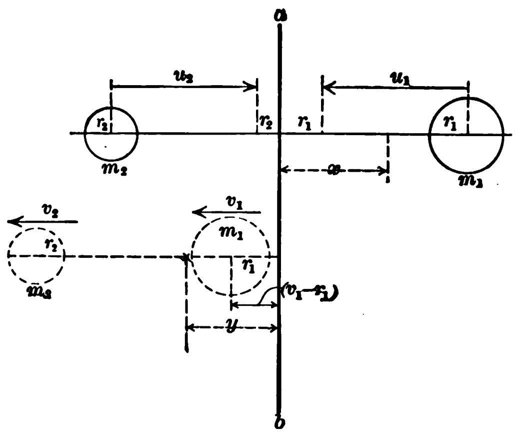

FIG. 12.

bodies one second after impact, and the centre of mass of the system is at that instant to the left of the plane $a\,b$ by the distance y, then by writing the equation for the centre of mass for this configuration, we obtain

$$m_1 (y - v_1 + r_1) = m_2 (v_2 + r_2 - y),$$

from which

$$\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = y + \frac{m_1 r_1 - m_2 r_2}{m_1 + m_2}. \quad . \quad . \quad (9)$$

The numerators in the left-hand members of equations (8) and (9) represent respectively the sum of the momenta before and after impact, and are therefore equal; hence the right-hand members of these two equations are equal. Now the fraction,

$$\frac{m_1 r_1 - m_2 r_2}{m_1 + m_2}$$

represents the distance that the centre of mass of the two bodies is to the right of the plane of impact at the instant of impact. Hence in the unit of time before impact the centre of mass passes over the distance

$$x - \frac{m_1 r_1 - m_2 r_2}{m_1 + m_2};$$

and in the unit of time after impact the centre of mass moved over the distance

$$y + \frac{m_1 r_1 - m_2 r_2}{m_1 + m_2}.$$

But, by equations (8) and (9), these two distances are equal, and hence the rate of motion of the centre of mass is unchanged by the impact.

If the two bodies do not approach each other on the same straight line, but on lines inclined to each other, then the impact will be oblique. In this case the velocities of the bodies may each be resolved into two components, one at right angles to the plane of impact and one parallel to it. The components parallel to the plane of impact will be unchanged, and those perpendicular to the plane of impact will obey the laws of direct impact. Hence, that part of the component of the velocity of the centre of mass perpendicular to the plane of impact is the same after impact as before, and the parallel component being unaffected, it follows that the velocity of the centre of mass is unchanged.

It will be readily seen how this demonstration can be extended to three or more bodies. This constitutes the fundamental law of momentum; viz.: The velocity of the centre of mass of a system cannot be altered by any internal forces; or in other words *the momentum of a system can be changed only by the action of a force, or forces, external to that system*

Loss of Energy during Impact. As previously stated, there is no loss of energy when perfectly elastic bodies impinge. If, however, the bodies are not perfectly elastic then there is a loss of energy depending upon the coefficient of restitution.

To obtain the expression for loss of energy during impact, it is convenient to first obtain equations for the final velocities in terms of the masses, the initial velocities, and the coefficient of restitution.

From the principle that the velocity of the centre of mass is unchanged by impact, we may write

$$(m_1 + m_2) V = m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2;$$

where V is the velocity of the centre of mass. From this we obtain

$$V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}, \quad . \quad . \quad . \quad . \quad (10)$$

and

$$V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \quad . \quad . \quad . \quad . \quad (11)$$

From equations (7) and (11), we obtain

$$\left. \begin{aligned} v_1 &= V - \frac{m_2}{m_1 + m_2} e (u_1 - u_2) \\ v_2 &= V + \frac{m_1}{m_1 + m_2} e (u_1 - u_2) \end{aligned} \right\} \quad . \quad . \quad (12)$$

Substituting in (12) for V the value as given in (10), we obtain

$$\left. \begin{aligned} v_1 &= \frac{m_1 u_1 + m_2 u_2 - m_2 e (u_1 - u_2)}{m_1 + m_2} \\ v_2 &= \frac{m_1 u_1 + m_2 u_2 + m_1 e (u_1 - u_2)}{m_1 + m_2} \end{aligned} \right\} \quad . \quad . \quad (13)$$

The kinetic energy, before impact, is equal to

$$\frac{1}{2} (m_1 u^2_1 + m_2 u^2_2);$$

and, after impact, it is equal to

$$\frac{1}{2} (m_1 v^2_1 + m_2 v^2_2).$$

The loss in kinetic energy then is

$$E = \frac{1}{2} [(m_1 u^2_1 + m_2 u^2_2) - (m_1 v^2_1 + m_2 v^2_2)]. \quad . \quad (14)$$

Substituting in equation (14), the values of v_1 and v_2 as given in (13), performing the indicated operations and simplifying, we obtain, finally

$$E = \frac{m_1 m_2}{2(m_1 + m_2)}(1 - e^2)(u_1 - u_2)^2. \quad . \quad (15)$$

CHAPTER V

MOMENT OF INERTIA

ASSUME a force F applied, as indicated in Fig. 13, and that the particles m_1, m_2, m_3, . . . m_n, are situated at distances r_1, r_2, r_3, . . . r_n, from the axis O, and are rigidly connected to this axis. Assume further, that all masses may be neglected but the masses of the particles, and that there is no friction; then the various masses will be accelerated, and every particle exerts a reaction due to its inertia.

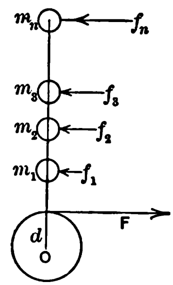

FIG. 13.

Let these reactions be represented by f_1, f_2, f_3. . . . f_n, and the linear accelerations by a_1, a_2, a_3, . . . a_n, we then have

$$\left.\begin{aligned} a_1 &= \frac{f_1}{m_1} \\ a_2 &= \frac{f_2}{m_2} \\ a_3 &= \frac{f_3}{m_3} \\ &\cdots\cdots \\ &\cdots\cdots \\ a_n &= \frac{f_n}{m_n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, from the Principle of Moments, we have

$$F d = f_1 r_1 + f_2 r_2 + f_3 r_3 + \ldots\ldots + f_n r_n$$

The bodies being rigidly connected, the angula celeration must be the same for all; hence, equation (1), we obtain

$$\alpha = \left\{ \begin{array}{l} \frac{a_1}{r_1} = \frac{f_1}{m_1 r_1} \\ \frac{a_2}{r_2} = \frac{f_2}{m_2 r_2} \\ \frac{a_3}{r_3} = \frac{f_3}{m_3 r_3} \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \frac{a_n}{r_n} = \frac{f_n}{m_n r_n} \end{array} \right\} ; \quad \ldots\ldots$$

from which

$$\left. \begin{array}{l} f_1 = \alpha m_1 r_1 \\ f_2 = \alpha m_2 r_2 \\ f_3 = \alpha m_3 r_3 \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ f_n = \alpha m_n r_n \end{array} \right\} \quad \ldots\ldots$$

Substituting these values of f_1, f_2, f_3, . . . f, equation (2) we obtain

$$F d = \alpha m_1 r^2_1 + \alpha m_2 r^2_2 + \alpha m_3 r^2_3 + \ldots\ldots + \alpha m$$

α being the same for all terms, we have

$$F d = \alpha (m_1 r^2_1 + m_2 r^2_2 + m_3 r^2_3 + \ldots\ldots + m_n r$$

$$= \alpha \Sigma m r^2 . \quad \ldots\ldots\ldots\ldots$$

It is obvious that the preceding demonstration holds for any number of masses; and that in any case where there is a rigid body capable of rotating about a fixed axis, we would obtain

$$F\,d = G = (\Sigma\, m\, r^2)\,\alpha; \quad . \quad . \quad . \quad (6)$$

in which G is the ***torque, or turning moment***, and $\Sigma\, m\, r^2$ is the summation of the products obtained by multiplying the mass of each particle by the square of its distance from the fixed axis.

It is obvious that for any given body with given axis, $\Sigma\, m\, r^2$ is a constant; hence, we may replace it by a symbol; *i.e.*,

$$G = I\,\alpha. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As previously defined, G is a measure of the tendency of a couple to produce rotation about a given axis; hence, it follows that I has precisely the same relation to motion of rotation that mass has to motion of translation. It has been stated that the inertia of a body may be measured by the force required to produce a given linear acceleration. In the same way, I may be measured by the torque, or turning moment, required to produce a given angular acceleration. Hence, I may be appropriately called *Moment of Inertia.*

We may then define the moment of inertia of a body with respect to an axis as a measure of the importance of the inertia of that body as regards its rotation about that axis.

It is of such importance to the engineering student to have a proper conception of moment of inertia, that it is advisable to derive formula (7) by an entirely independent method; *i.e.*, from the principle of energy.

Assume again, as previously, that the particles whose masses are m_1, m_2, m_3, . . . m_n, are rigidly connected to the axis O, as in Fig. 13, at distances r_1, r_2, r_3, . . . r_n. If at any instant the angular velocity of the system about the axis is ω, then the linear velocity of the various particles is ωr_1, ωr_2, ωr_3, . . . ωr_n; and the kinetic energy of the system is

$$E_k = \frac{1}{2} m_1 \omega^2 r^2_1 + \frac{1}{2} m_2 \omega^2 r^2_2 + \frac{1}{2} m_3 \omega^2 r^2_3 + \ldots\ldots + \frac{1}{2} m_n \omega^2 r^2_n \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

The angular velocity being the same in all terms, equation (8) may be simplified as follows:

$$E_k = \frac{\omega^2}{2} (m_1 r^2_1 + m_2 r^2_2 + m_3 r^2_3 + \ldots\ldots + m_n r^2_n)$$

$$= \frac{\omega^2}{2} \Sigma m r^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where $\Sigma m r^2$ has precisely the same significance it had in equation (6).

Hence, replacing it by the symbol I, we obtain

$$E_k = \frac{I \omega^2}{2}.^* \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

* If a constant force F is applied at the end of an arm l, such that the body starting from rest sweeps out an angle ψ, the work done is equal to $F l \psi = E_k$, the kinetic energy stored in the rotating body; *i.e.*,

$$F l \psi = \frac{I \omega^2}{2} = I a \psi \;\therefore\; F l = G = I a.$$

As has been previously shown, the kinetic energy of a body expressed in terms of its mass m, and velocity v, is

$$E_k = \frac{m v^2}{2}.$$

Comparing this expression with equation (10), it is at once seen that I, the moment of inertia, has the same relation to motion of rotation that mass has to motion of translation.

The moment of inertia of all homogeneous bodies that have regular geometric figures may be found by computation; but the student must take this for granted until he has learned to integrate. It will be shown later how to find, by experiment, the moment of inertia of any body that is not homogeneous or of regular geometric form.

Change of Axes. If the moment of inertia of a body with respect to a given axis is known it is always possible, by a simple computation, to find the moment of inertia about a new axis, parallel to the given axis, and at a fixed distance from it.

Let the moment of inertia of the body, as depicted in Fig. 14, about the axis $A B$, passing through the centre of mass C, be I_c. To find the moment of inertia about the axis $E D$, parallel to the axis $A B$, and at a distance a from it. Consider any particle of mass m, situated at a distance r from the centre of mass. From the figure, we have

$$x^2 = a^2 + r^2 + 2 a b,$$

and

$$m\,x^2 = m\,a^2 + m\,r^2 + m\,2\,a\,b.$$

Repeating this operation for each particle, we have

$$\Sigma\,m\,x^2 = \Sigma\,m\,a^2 + \Sigma\,m\,r^2 + \Sigma\,m\,2\,a\,b. \quad . \quad (11)$$

But, a being a constant, we may write equation (11) as follows:

$$\Sigma\,m\,x^2 = M\,a^2 + \Sigma\,m\,r^2 + 2\,a\,\Sigma\,m\,b; \quad . \quad (12)$$

where M is the mass of the body. Now $\Sigma\,m\,x^2$ is the moment of inertia about the axis $E\,D$, and may

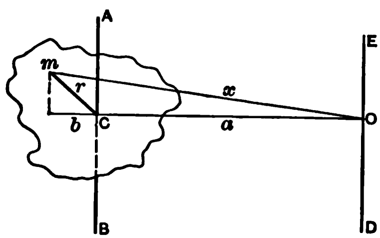

FIG. 14.

be represented by the symbol I_o. $\Sigma\,m\,r^2$ is the moment of inertia of the body about the axis $A\,B$; *i.e.*, about the axis passing through the centre of mass C. $\Sigma\,m\,b$ represents the moment of all the particles about the axis C and is by a previous proposition equal to zero.

Hence, equation (12) may be written,

$$I_o = I_c + M\,a^2. \quad . \quad . \quad . \quad . \quad (13)$$

Therefore, the moment of inertia with respect to any axis, parallel to an axis passing through the centre of mass, is equal to the moment of inertia with respect to the axis through the centre of mass, plus the mass multiplied by the square of the distance to the parallel axis.

This may also be proven by the principle of energy. Assume the body whose mass is M and centre of mass at C, Fig. 15, to be revolving with a uniform angular velocity ω, about the axis through O, in such a manner that any line on the body, such as $b\,d$, is always parallel to its first position. Its kinetic energy then is

$$W_1 = \frac{M\,\omega^2\,a^2}{2}. \quad . \quad (14)$$

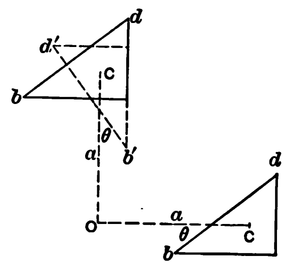

FIG. 15.

If, however, the body is rigidly connected and rotates about the axis through O, in such a manner that the line $b\,d$ makes a constant angle θ with the radius a, as depicted in the second position by $b'\,d'$, the body must rotate about the axis through C with an angular velocity equal to the angular velocity of the radius a about the axis O. The kinetic energy of the body, due to its rotation about the axis through C, is

$$W_2 = \frac{I_c\,\omega^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Adding equations (14) and (15) we obtain, for the total kinetic energy

$$W = \frac{\omega^2}{2}(I_c + M a^2). \quad . \quad . \quad . \quad (16)$$

As has been previously shown the kinetic energy of a rotating body is equal to the product of the moment of inertia, and one half the square of the angular velocity. Hence, the moment of inertia of the body about the axis through O, is equal to the moment of inertia about a parallel axis through the centre of mass, plus the mass multiplied by the square of the distance between the two axes, as has been shown by equation (13).

There are certain cases where it is possible, by a simple method, to find the moment of inertia of a figure about an axis which is not parallel to the axis about which the moment of inertia is known.

In Fig. 16, let $A B C$ be a triangle of mass M. Then the moment of inertia of the triangle, about the axis $X X$, is

$$I = \frac{M h^2}{6}; \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where h is the altitude of the triangle.

If the three sides of the triangle are given, or two sides and the included angle, or two angles and the included side, then the whole triangle is determinate, and hence it is possible to determine the area, and the mass m, per unit area.

Let it be required to find the moment of inertia of the triangle about a new axis $X_1 X_1$, lying in the same plane and passing through the point A, and making an angle θ with the axis $X X$.

The triangle $A B C$ being determinate, the altitude and area of the triangles $A B D$ and $A C D$ can be determined.

Assume that the mass per unit area of the triangle $A C D$ is the same as that of the triangle $A B C$.

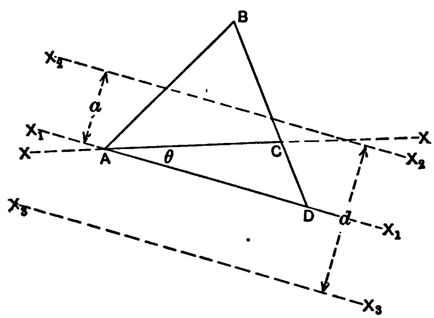

FIG. 16.

Then by a simple computation the mass of the triangle $A B D$ may be determined, and its altitude being known, its moment of inertia with reference to $X_1 X_1$ may be stated; *i.e.*,

$$I_1 = \frac{M_1 h^2_1}{6};$$

where M_1 is the mass and h_1 the altitude of the triangle $A B D$. In a like manner the moment of inertia of the triangle $A\,C\,D$ with respect to the axis $X_1 X_1$ is

$$I_2 = \frac{M_2 h^2_2}{6};$$

where M_2 is the mass and h_2 the altitude of the triangle $A\,C\,D$. Now the moment of inertia of the triangle $A\,B\,C$ with respect to the axis $X_1 X_1$ is equal to the moment of inertia of the triangle $A\,B\,D$, minus the moment of inertia of the triangle $A\,C\,D$ with respect to the same axis. In symbols:

$$I_3 = I_1 - I_2 = \frac{M_1 h^2_1}{6} - \frac{M_2 h^2_2}{6}. \quad . \quad (18)$$

Again, since the moment of inertia about any axis is equal to the moment of inertia about the axis passing through the centre of mass, plus the product of the mass and the square of the distance to the new axis parallel to the given axis, we have

$$I_c = I_3 - M\,a^2; \quad . \quad . \quad . \quad . \quad (19)$$

where a is the distance between $X_1 X_1$ and $X_2 X_2$, and I_c is the moment of inertia about the axis $X_2 X_2$ passing through the centre of mass. Let d be the distance between the axis $X_2 X_2$ and some parallel axis $X_3 X_3$. Then, by the same principle, the moment of inertia with respect to $X_3 X_3$ is

$$I_4 = I_c + M\,d^2 = I_3 - M\,a^2 + M\,d^2. \quad . \quad (20)$$

Since θ may be any angle, it follows that the moment of inertia of the triangle $A\,B\,C$ may be found with respect to any axis lying in the plane of the figure. Furthermore, it is obvious that the foregoing demonstration applies to any plane figure that is determinate, whose moment of inertia about an axis lying in the plane of that figure is determinate.

Radius of Gyration. Since, as has been previously stated, for any given body with given axis, the summation $\Sigma m r^2$ is a constant, we may write

$$I = \Sigma m r^2 = M K^2;$$

where M is the mass of the body, and K is the distance from the fixed axis where the mass M would have to be concentrated at a point so as to have a moment of inertia with respect to the axis equal to that of the body. K is called the radius of gyration of the body, with respect to the given axis, and necessarily changes in value for different axes.

CHAPTER VI

POWER AND ANGULAR MOMENTUM

THE rate at which work is being done is called *Power*. If the point of application of a force is displaced through a distance s, and the component of the force in the direction of the displacement is F, the work done is $F s$. If the motion is uniform, and t is the time consumed during the displacement, then the rate of doing work, that is, the power, is constant. In symbols

$$P = \frac{W}{t} = \frac{F s}{t} = F v. \quad . \quad . \quad . \quad (1)$$

From which we see that the power expended is numerically equal to the product of the component of the force in the direction of the motion and the speed.

In like manner, it can be shown that the power expended is numerically equal to the product of Torque and Angular Velocity.

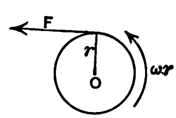

FIG. 17.

Let, as in Fig. 17, the drum whose centre is O, and radius r be acted upon by a constant force F, and let the resistance be such that the angular velocity maintained is a constant. The work done in a time t is

$$W = F \omega r t,$$

and the power is

$$P = \frac{W}{t} = F\,r\,\omega = G\,\omega. \quad . \quad . \quad . \quad (2)$$

Again, let a constant force F, act upon a mass m, which is perfectly free to move, then the acceleration is

$$a = \frac{F}{m} = a \textit{ constant.}$$

Since velocity is equal to the product of acceleration and time, it follows that it is a variable and changes at a uniform rate with the time; in other words, it is a function of the time. This being the case, that is, the force being constant, and the velocity variable, it follows that the power expended to produce the accelerated motion is variable. But power is equal to the product of velocity and force; hence, at any instant the value of the power expended is equal to the product of the force and the instantaneous velocity; *i.e.*,

$$p = F\,v;$$

where p represents instantaneous value of power and v represents instantaneous value of velocity.

In the French system, the unit power is doing work at the rate of one joule per second. This unit is called the *Watt*. In the English system, the unit power is the *Horse-power* and is equal to doing work at the rate of 33,000 ft. lbs. per minute.

Angular Momentum or Moment of Momentum. If a body of mass m, has a velocity in the circumference of a circle whose radius is r, its momentum is

$$m v = m \omega r;$$

where ω is the angular velocity.

The *angular momentum* then is

$$m v r = m \omega r^2. \quad . \quad . \quad . \quad . \quad (3)$$

If some constant force has been acting to produce the velocity v, there has been a constant acceleration, and the force is

$$F = m \frac{v}{t},$$

from which we have for torque

$$G = m \frac{v}{t} r = \frac{m \omega r^2}{t}. \quad . \quad . \quad . \quad . \quad (4)$$

Under the conditions stated, the velocity v, and the angular velocity ω, will be functions of the time; *i.e.*, will change at a uniform rate. The power, however, at any instant that is expended to produce the acceleration $\frac{v}{t} = \frac{\omega r}{t}$, is the product of the torque and angular velocity; *i.e.*,

$$p = G \omega = \frac{m \omega r^2}{t} \omega. \quad . \quad . \quad . \quad . \quad (5)$$

But, $\frac{m \omega r^2}{t}$ is the rate of change of angular momentum; therefore, the instantaneous value of power is equal to the product of rate of change of angular momentum and the instantaneous angular velocity.

CHAPTER VII

TENSION IN CORDS

If a cord support a weight, the tension in the cord at any section is, of course, equal to the weight supported plus the weight of the cord below the section. In practical problems, it is, however, seldom necessary to consider the weight of the cord since it is usually a very small fractional part of the total weight supported.

As an example, a Manilla rope, capable of supporting 7,000 lbs., weighs $\frac{1}{3}$ lb. per running foot. Assume a length of 100 ft., supporting a weight of 7,000 lbs.; the weight of the rope then is about 33.3 lbs., which is a trifle less than $\frac{1}{2}$ per cent of the total weight supported. Since the ultimate breaking strength of any specimen of material can never be predicted with certainty closer than 1 per cent or 2 per cent, it follows that in most cases the weight of the cord, cable, or rope may be neglected.

As previously stated, if a rope support a weight, the tension in the rope is equal to the weight. If, however, the body supported has an accelerated motion, the tension may have any value.

To fix the attention, assume a body supported as in Fig. 18. If the body be at rest the tension in the cord is

$$T = M g = W. \quad . \quad . \quad . \quad . \quad (1)$$

If the body be raised or lowered with a constant speed, the tension is still $M g$; for, since there is no acceleration there can be no additional force. Assume now that the drum has impressed upon it an acceleration, such that the body has an acceleration downward of g feet per sec. per sec. The tension, then, is zero; for the body is perfectly free to fall.

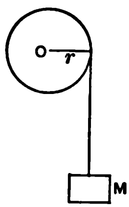

FIG. 18.

In general, if the acceleration downward is a, the tension is the unbalanced force; *i.e.*,

$$T = M (g - a). \quad . \quad . \quad . \quad . \quad (2)$$

If the acceleration is upward, it is considered negative, and therefore, for upward acceleration

$$T = M (g + a). \quad . \quad . \quad . \quad . \quad (3)$$

If the downward acceleration is greater than g, it follows from equation (2) that T is negative; that is, a pressure. This is possible only if the support is rigid, like a rod, and has impressed upon it a downward acceleration in excess of g.

Take a concrete case by assuming a spring balance suspended from the roof of an elevator cage, which is ascending with a constant acceleration of 16 ft. per sec. per sec. Assume further, that this spring balance is supporting a mass of 200 pounds, to find the weight registered by it. From formula (3) we have

$$T = 200\,(32 + 16)\ \textit{poundals};$$

g being assumed equal to 32 ft. per sec. per sec. From this we find

$$\begin{aligned} T &= 9{,}600\ \textit{poundals} \\ &= 300\ \textit{lbs.} \end{aligned}$$

If, on the other hand, the cage is descending with a constant acceleration of 16 ft. per sec. per sec., then the weight registered becomes

$$\begin{aligned} T &= 200\ (32 - 16)\ \textit{poundals} \\ &= 3{,}200\ \textit{poundals} \\ &= 100\ \textit{lbs.} \end{aligned}$$

Atwood's Machine. Assume, as in Fig. 19, two masses M and m, supported by a cord, whose mass is negligible, over a wheel, without mass, and perfectly free to rotate. To determine the tension in the cord and the acceleration of the system. The total mass moved is $(M + m)$; and the moving force is

$$F = M\,g - m\,g = (M - m)\,g. \quad . \quad (4)$$

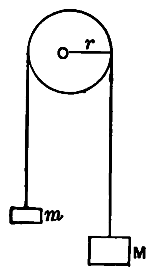

FIG. 19.

The acceleration is, the moving force divided by the mass moved, *i.e.*,

$$a = \frac{M - m}{M + m}\,g. \quad . \quad . \quad . \quad . \quad (5)$$

The tension in the cord, supporting the mass M, is

$$T = M(g - a) = M\left(g - \frac{M - m}{M + m} g\right)$$

$$= \frac{2 M m}{M + m} g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The tension in the cord, supporting the mass m, is

$$T_1 = m(g + a) = m\left(g + \frac{M - m}{M + m} g\right)$$

$$= \frac{2 M m}{M + m} g;$$

thus making T and T_1 equal. This must necessarily be so; since the wheel is assumed without mass and friction, the tension must be the same throughout the cord.

Assume now, the cord supporting the mass M wrapped over a cylinder of radius r, and moment of inertia I, to find the tension in the cord and the acceleration.

If T is the tension in the cord, then, as has been previously shown

$$T r = I \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$T r$ being the torque, I the moment of inertia, and α the angular acceleration. From which the linear acceleration is

$$a = \alpha r = \frac{T r^2}{I}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As previously shown

$$T = M(g - a) = M\left(g - \frac{T r^2}{I}\right).$$

From which we obtain

$$T = \frac{M I}{M r^2 + I} g. \quad . \quad . \quad . \quad . \quad (9)$$

Substituting in equation (8), the value for T as found in equation (9), we obtain

$$a = \frac{M r^2}{M r^2 + I} g. \quad . \quad . \quad . \quad . \quad (10)$$

Dividing by r, we obtain

$$\alpha = \frac{a}{r} = \frac{M r}{M r^2 + I} g. \quad . \quad . \quad . \quad (11)$$

If the drum is a solid homogeneous cylinder, of mass M_1, then

$$I = \frac{M_1 r^2}{2}.$$

Substituting this value in equations (9), (10), and (11), we obtain

$$T = \frac{M M_1 \frac{r^2}{2}}{M r^2 + M_1 \frac{r^2}{2}} g = \frac{M M_1}{2 M + M_1} g. \quad . \quad (12)$$

$$a = \frac{M r^2}{M r^2 + \frac{M_1 r^2}{2}} g = \frac{2 M}{2 M + M_1} g. \quad . \quad (13)$$

$$\alpha = \frac{M r}{M r^2 + I} g = \frac{2 M}{2 M r + M_1 r} g. \quad . \quad . \quad (14)$$

Assume, now, a drum of moment of inertia I and radius r, having wrapped around it a cord supporting a mass M on one side, and a mass m on the other side, as depicted in Fig. 20. Assuming no friction, we have

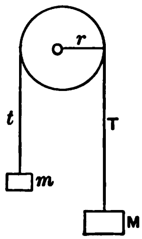

FIG. 20.

$$T = \frac{I \alpha}{r} + m (g + a) ; \quad . \quad (15)$$

and since $a = \alpha r$, we have

$$T = \frac{I \alpha}{r} + m (g + \alpha r). \quad . \quad (16)$$

Again

$$T = M (g - \alpha r); \quad . \quad . \quad (17)$$

hence, equating (16) and (17), we have

$$\frac{I \alpha}{r} + m (g + \alpha r) = M (g - \alpha r);$$

from which

$$I \alpha + m r g + m \alpha r^2 = M r g - M \alpha r^2.$$

Rearranging and factoring, we have

$$\alpha (I + m r^2 + M r^2) = (M - m) r g,$$

from which

$$\alpha = \frac{(M - m) r}{I + (M + m) r^2} g, \quad . \quad . \quad . \quad (18)$$

finally,

$$a = \alpha r = \frac{(M - m) r^2}{I + (M + m) r^2} g. \quad . \quad . \quad (19)$$

Substituting now, in equation (17), for a its value, we obtain

$$T = M(g - \alpha r) = M\left(g - \frac{(M - m)r^2}{I + (M + m)r^2}g\right)$$

$$= M g \frac{I + 2 m r^2}{I + (M + m) r^2}. \quad . \quad (20)$$

Also

$$t = m(g + a) = m\left(g + \frac{(M - m)r^2}{I + (M + m)r^2}g\right)$$

$$= m g \frac{I + 2 M r^2}{I + (M + m) r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

In this case there must be a difference in the tensions, since the drum has inertia; this difference is

$$t_1 = T - t = M g \frac{I + 2 m r^2}{I + (M + m) r^2}$$

$$- m g \frac{I + 2 M r^2}{I + (M + m) r^2} = \frac{I(M - m)}{I + (M + m) r^2} g. \quad . \quad (22)$$

This may be found directly, since the difference in the tensions must be equal to the tension required to produce the acceleration of the drum, *i.e.*,

$$t_1 = \frac{I\alpha}{r} = \frac{I}{r} \times \frac{(M - m) r}{I + (M + m) r^2} g$$

$$= \frac{I(M - m)}{I + (M + m) r^2} g;$$

which is the same as given in equation (22).

Reverting to the figure, we see that the moving force is $(M - m) g$, and the mass moved is, $M + m + K$; where K is the *equivalent mass* of the drum. The acceleration then is

$$a = \frac{M - m}{M + m + K} g. \quad . \quad . \quad . \quad (23)$$

As expressed in equation (19)

$$a = \frac{(M - m) r^2}{I + (M + m) r^2} g.$$

Hence, by equating (19) and (23), we have

$$\frac{1}{M + m + K} = \frac{r^2}{I + (M + m) r^2},$$

and

$$I + M r^2 + m r^2 = M r^2 + m r^2 + K r^2;$$

from which

$$K = \frac{I}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The value of K may also be found in a different manner. Let a constant force F be applied to the surface of a drum whose moment of inertia is I, and radius r, then

$$F r = I \alpha,$$

and

$$\alpha = \frac{F r}{I};$$

from which

$$a = \alpha r = \frac{F r^2}{I}.$$

But

$$a = \frac{F}{K};$$

where K is the equivalent mass of the drum, hence,

$$\frac{F}{K} = F \frac{r^2}{I};$$

from which

$$K = \frac{I}{r^2}, \text{ as before.}$$

CHAPTER VIII

MAXIMA AND MINIMA

It frequently becomes necessary, when dealing with equations involving two or more variables, to determine under what condition one of these variables attains either a maximum or a minimum value in terms of the other quantities involved. In general, this is done by the application of the differential calculus. It, however, is frequently the case that the conditions are such as to enable us to do this by inspection, or by elementary mathematics. A few examples will be given here to illustrate this.

Let it be required to determine the value of the angle which makes the product of its sine and cosine a maximum. Given the equation

$$y = \sin x \cos x; \quad . \quad . \quad . \quad . \quad (1)$$

to determine under what conditions y assumes its maximum value. Equation (1) may be written in the form of

$$y = \frac{1}{2} \sin 2x. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the sine has its maximum values when the angle is 90°, 450°, etc., and its minimum values when the angle is 270°, 630°, etc., we have y in equation

(2), and consequently, the product of the sine and cosine of x a maximum, when x equals 45°, 225°, etc.; and the minimum values occur for the product of sine and cosine when x equals 135°, 315°, etc.

Let it be required to show what must be the relation between two variable quantities, whose sum is a constant, such that their product is a maximum. Let

$$x + y = k; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k is a constant and x and y variables. Make k, as in Fig. 21, the diameter of a circle, inscribe the triangle $b\,c\,d$, and drop a perpendicular a, from the point c, dividing the diameter k into the segments x and y. Since $b\,c\,d$ is a right-angled triangle, we have

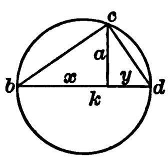

FIG. 21.

$$a^2 = x\,y. \quad . \quad . \quad . \quad (4)$$

If, now, the vertex c take various positions along the circumference of the circle, k remains constant, but x, y, and a will vary; and a attains its maximum value when equal to $\frac{k}{2}$. But when this occurs, x and y are equal and each equal to $\frac{k}{2}$. Hence, from equation (4), it follows that the product of two variables, whose sum is constant, is a maximum when the two variables are equal.

We may now draw the further conclusion that the triangle of maximum area which can be inscribed in semicircle is the one having equal legs. For the ea, which is measured by the product of the diamer k, and altitude a, is a maximum when a is a ıaximum, but this occurs when the legs $b\,c$ and $c\,d$ re equal. Stated more broadly, the area of any ight-angled triangle of given hypothenuse and variable ·gs is a maximum when the legs are equal.

It will now be shown that if a right-angled triangle ıave the perpendicular distance from the vertex of he right angle to the hypothenuse a constant, and ıe legs variable, the hypothenuse is a minimum when ıe legs are equal. Let, in Fig. 22, the triangle $b\,c\,d$ ıve the legs $b\,c$ and $c\,d$ ıual, and an altitude equal ı a, making the hypothenuse ıual to $2\,a$. Let the triıgle $b\,e\,f$ have the legs $b\,e$ ıd $e\,f$ not equal, then the ıgle θ is less than 45°. Calling the segments x ıd y, then the hypothenuse is equal to

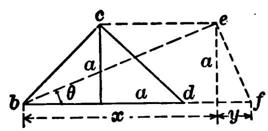

FIG. 22.

$$x + y = \frac{a}{tan\ \theta} + \frac{a}{tan\ (90 - \theta)};$$

om which

$$x + y = \frac{a}{sin\ \theta\ cos\ \theta};$$

ıt it has been shown that the product of the sine ıd cosine is a maximum when the angle is 45°.

Hence, since θ is less than $45°$, this product is less than $\frac{1}{2}$, and we have $x + y > 2a$. The hypothenuse is therefore a minimum when the legs or, what amounts to the same thing, the two segments are equal.

We may now draw the further conclusion, since $xy = a^2$, that when the product of two variables is a constant, their sum is a minimum when the variables are equal.

Given the particle a, Fig. 23, moving in the direction ah with the constant speed u and the particle b moving in the direction bh with the constant speed v, together with the distance ab and the angles θ

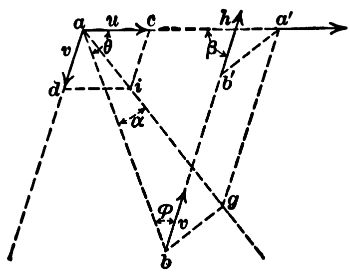

FIG. 23.

and φ, to determine what will be the position of the particles when the distance between them is a minimum. The relative motion of the two particles is precisely the same as though b were at rest and a had impressed upon it, together with its own velocity, a velocity ad, parallel and equal to v but in the opposite direction. The resultant velocity of a, under these conditions, is $a\,i$, and since b is at rest the particles are nearest together when the particle a is at the point g; the point g being found by dropping a perpendicular from b on to the line $a\,i$ produced. Actually at that instant the particle a is at a'; the point a' being on the intersection of the line $a\,h$ produced and a line through g parallel to $b\,h$. And the particle b will be at b', the point b' being on the intersection of the line $b\,h$ and a line through a' parallel to $g\,b$. The angles θ and φ being given, the angle β is determinate, and hence the magnitude and direction of $a\,i$ may be found. Knowing the direction of $a\,b$ and $a\,g$, the angle α is given; this, together with $a\,b$, determines $a\,g$ and $b\,g$, and from these $a\,a'$ and $b\,b'$ may be found.

Given the two particles a and b, Fig. 24, a having a constant speed v, in the direction $a\,c$ making an angle θ with the line $a\,b$; to determine what is the minimum speed which b may have and still meet the particle a. Let x be the speed with which the particle b is moving along some line $b\,c$; the path $b\,c$ making an angle φ with the line $a\,b$, and the point c being the intersection of their paths. If t is the time required for the particle a to travel from a to c, then $a\,c = v\,t$; and for the particles to meet

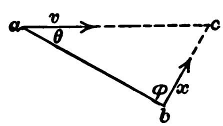

FIG. 24.

at c, we have $b\,c = x\,t$. From the law of sines we have

$$x\,t : v\,t :: sin\ \theta : sin\ \varphi,$$

from which

$$x = v\frac{sin\ \theta}{sin\ \varphi}.$$

For x to be a minimum, $sin\ \varphi$ must be a maximum, since v and $sin\ \theta$ are constant. But the maximum value for $sin\ \varphi$ is unity, and occurs when φ equals $90°$; hence, the minimum value for the speed of b is

$$x = v\ sin\ \theta.$$

Having given the coefficient of friction between a body and an inclined plane together with the angle of inclination of the plane, to determine the minimum force that will move the body, without acceleration, up the plane.

Let, in Fig. 25, W represent the weight of the body, then the normal pressure on the plane, due to the weight W, is

$$R_1 = W\ cos\ \theta;$$

where θ is the angle of inclination of the plane. If, now, F represents the required force making an angle φ with the plane, its normal component is

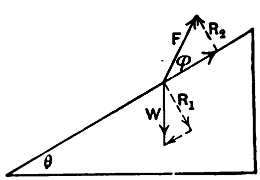

FIG. 25.

$$R_2 = F\ sin\ \varphi,$$

and the normal pressure on the plane is the algebraic sum of R_1 and R_2; *i.e.*,

$$R = W\ cos\ \theta - F\ sin\ \varphi. \quad . \quad . \quad . \quad . \quad (5)$$

The component of the force F, parallel to the plane must, to maintain a constant speed up the plane, be equal to the component of the weight parallel to the plane plus the force of friction. The force of friction being equal to the product of the normal pressure and coefficient of friction, we have

$$F \cos \varphi = W \sin \theta + \mu R; \quad . \quad . \quad . \quad (6)$$

where μ is the coefficient of friction. Substituting in equation (6) the value of R as given in equation (5), we obtain

$$F \cos \varphi = W \sin \theta + \mu W \cos \theta - \mu F \sin \varphi;$$

from which, solving for F, we find

$$F = W \frac{\sin \theta + \mu \cos \theta}{\cos \varphi + \mu \sin \varphi}. \quad . \quad . \quad . \quad (7)$$

The coefficient of friction is equal to the tangent of the angle of repose. If we then represent, by β, the angle of repose, we have

$$\mu = \frac{\sin \beta}{\cos \beta};$$

and substituting this value of μ, in equation (7), we obtain

$$F = W \frac{\sin \theta + \dfrac{\sin \beta}{\cos \beta} \cos \theta}{\cos \varphi + \dfrac{\sin \beta}{\cos \beta} \sin \varphi}.$$

This expression reduces to

$$F = W \frac{\sin (\theta + \beta)}{\cos (\varphi - \beta)}. \quad . \quad . \quad . \quad . \quad (8)$$

From equation (8) it is evident, since the numerator is a constant, that F is a minimum when φ and β are equal, making the denominator a maximum.

To determine the pitch of the thread of a jack-screw, having given the coefficient of friction, such that its efficiency shall be a maximum. The thread of the screw being an inclined plane and the applied force acting parallel to the base of the plane, we will first determine this force in terms of the weight lifted and the coefficient of friction.

Let, in Fig. 26, F be the force required to move the weight W at a uniform rate up the plane, and φ the angle of inclination of the plane. The total normal pressure on the plane, due to the weight W and the force F, is

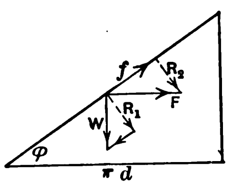

FIG. 26.

$$R = R_1 + R_2 = W \cos \varphi + F \sin \varphi. \quad (9)$$

The force component parallel to the plane, being the sum of the weight component parallel to the plane and the force of friction, we have

$$f = W \sin \varphi + \mu R. \quad . \quad . \quad . \quad (10)$$

Substituting in equation (10), the value of R as given in equation (9), we obtain

$$f = W \sin \varphi + \mu W \cos \varphi + \mu F \sin \varphi.$$

Dividing both sides of the equation by cos φ, we have

$$F = W \tan \varphi + \mu W + \mu F \tan \varphi;$$

and solving for F, we find

$$F = W \frac{\tan\varphi + \mu}{1 - \mu \tan\varphi}. \quad . \quad . \quad . \quad (11)$$

Substituting in equation (11), for μ, the value $\frac{\sin\beta}{\cos\beta}$; where β is the angle of repose, we obtain

$$F = W \frac{\frac{\sin\varphi}{\cos\varphi} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\beta \sin\varphi}{\cos\beta\cos\varphi}};$$

from which

$$F = W \frac{\sin(\beta + \varphi)}{\cos(\beta + \varphi)}. \quad . \quad . \quad . \quad (12)$$

If now d is the diameter of the screw, the height through which the weight is lifted, during one revolution, is

$$h = \pi d \tan\varphi;$$

and since the work done by the screw is measured by the product of the weight and height, we have

$$W_1 = W \pi d \tan\varphi. \quad . \quad . \quad . \quad . \quad (13)$$

The work done on the screw, during one revolution, is

$$W_2 = F \pi d = W \frac{\sin(\beta + \varphi)}{\cos(\beta + \varphi)} \pi d. \quad . \quad . \quad (14)$$

The efficiency of any machine being the ratio of the work done by the machine to the work done on the machine, we find, by dividing equation (13) by equation (14),

$$\eta = \frac{W_1}{W_2} = \frac{W \pi d \tan \varphi}{W \frac{\sin (\beta + \varphi)}{\cos (\beta + \varphi)} \pi d};$$

from which

$$\eta = \frac{\sin \varphi \cos (\beta + \varphi)}{\sin (\beta + \varphi) \cos \varphi}. \quad . \quad . \quad . \quad (15)$$

Since, $\sin x \cos y = \frac{1}{2} \sin (x + y) + \frac{1}{2} \sin (x - y)$,

equation (15) may be written in the form of

$$\eta = \frac{\sin (\beta + 2 \varphi) - \sin \beta}{\sin (\beta + 2 \varphi) + \sin \beta};$$

from which

$$\eta = 1 - \frac{2 \sin \beta}{\sin (\beta + 2 \varphi) + \sin \beta}. \quad . \quad . \quad (16)$$

η is a maximum, when the second term of the right-hand member of equation (16) is a minimum; and this occurs, since the numerator is a constant, when the denominator is a maximum; and since β is a constant, the denominator is a maximum when $\beta + 2 \varphi = 90°$. Hence for η to be a maximum

$$\varphi = 45° - \frac{\beta}{2}.$$

Having given, the direction in which a vessel is to sail, the direction of the wind, and furthermore, assuming no drift and the sail a plane surface, to determine the set of the sail such, that the component of the wind pressure producing motion, shall be a maximum.

Let, in Fig. 27, *c d* be the direction in which the vessel is to sail, and *P* represent in magnitude and direction the pressure of the wind on the sail *a b*. Resolving *P* into two components, the component parallel to the sail has no effect in producing motion; and the normal component is

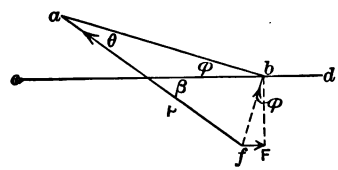

FIG. 27.

$$f b = P \sin \theta. \quad . \quad . \quad . \quad . \quad (17)$$

Now, resolving $f b$ into two components, one normal to the direction of the motion, which produces drift and is here neglected, and the other, parallel to the direction of the motion, and which produces the motion, is

$$F = f b \sin \varphi. \quad . \quad . \quad . \quad . \quad (18)$$

Combining equations (17) and (18), we obtain

$$F = P \sin \theta \sin \varphi. \quad . \quad . \quad . \quad . \quad (19)$$

Now, since the direction of the wind is fixed and also that of the motion of the vessel, we have the angle β a constant, and further, since

$$\varphi + \theta = \beta, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

we may eliminate one of the variable angles in equation (19). Substituting in equation (19), the value of φ as found from equation (20), we obtain

$$F = P \sin \theta \sin (\beta - \theta);$$

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from which

$$F = P\,(\sin\theta\cos\theta\sin\beta - \sin^2\theta\cos\beta). \quad . \quad (21)$$

Since

$$\sin\theta\cos\theta\sin\beta = \frac{1}{2}\sin 2\,\theta\sin\beta,$$

and also,

$$-\sin^2\theta\cos\beta = \frac{1}{2}(\cos 2\,\theta\cos\beta - \cos\beta);$$

equation (21) reduces to

$$F = \frac{P}{2}(\sin 2\,\theta\sin\beta + \cos 2\,\theta\cos\beta - \cos\beta);$$

from which

$$F = \frac{P}{2}[\cos(2\,\theta - \beta) - \cos\beta]. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Since $\cos\beta$ is a constant, the expression in the brace, and hence F, becomes a maximum when $2\,\theta = \beta$; and $\theta = \varphi = \frac{\beta}{2}$, which determines the set of the sail for the maximum force in the direction of motion.

CHAPTER IX

PENDULAR MOTION

Any body free to vibrate about a fixed axis under the action of gravity is called a *pendulum.*

The period of a pendulum, or time of *vibration,* represented by T, is the time required to pass through a cycle, *i.e.*, it is the time that elapses between any two successive identical positions when the body is moving in the same direction.

Half a period, or the time of an *oscillation,* represented by t, is the time required to pass through half a cycle.

The amplitude of the pendulum is the maximum displacement from the position of equilibrium.

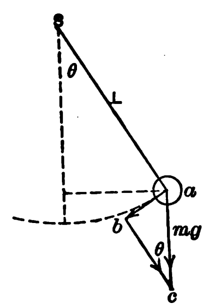


FIG. 28.

Simple Pendulum. Assume, as depicted in Fig. 28, a small particle of mass m, concentrated at a point, supported by a weightless cord whose length is L. The force acting vertically is constant and equal to $m\,g$, and may be represented by the line $a\,c$. Resolving this into two components, $a\,b$, parallel to the motion, and $b\,c$ at right angles to the motion, or parallel to the supporting cord, then the component producing motion is

$$a\,b = F = m\,g\,sin\,\theta. \quad . \quad . \quad . \quad . \quad (1)$$

Since acceleration equals force divided by mass, we have, for the acceleration along the arc

$$a = g\,sin\,\theta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the angular displacement of the body be small, the angle and sine are sensibly equal, and we have

$$a = g\,\theta. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But the displacement of the body from the position of equilibrium, measured along the path described by the body, is proportional to the angular displacement; hence, the acceleration is proportional to the displacement, and the body has a simple harmonic motion. Therefore

$$a = g\,\theta = \frac{4\,\pi^2}{T^2}\,s; \quad . \quad . \quad . \quad . \quad (4)$$

where s is the displacement. But,

$$s = L\,\theta;$$

and, substituting this value in equation (4), we have

$$g\,\theta = \frac{4\,\pi^2}{T^2}\,L\,\theta,$$

and

$$T^2 = 4\,\pi^2\,\frac{L}{g};$$

from which

$$T = 2\,\pi\,\sqrt{\frac{L}{g}}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Such an arrangement as we have just been considering is called a simple pendulum. It is, however, impossible to realize this condition practically; since any support that may be used has weight, and the supported mass is always a body of finite dimensions.

Physical Pendulum. Assume a rigid body, as in Fig. 29, whose centre of mass is at C, supported by an axis S, perpendicular to the plane of the paper. Let the moment of inertia of the body, whose mass is M, about the axis S be I, then

$$M\,g\,d = I\,\alpha; \quad . \quad . \quad (6)$$

where $M\,g\,d$ is the torque. Now,

$$d = r \sin\theta;$$

where θ is the angular displacement from the position of equilibrium, and r the distance from the axis of suspension to the centre of mass. Hence,

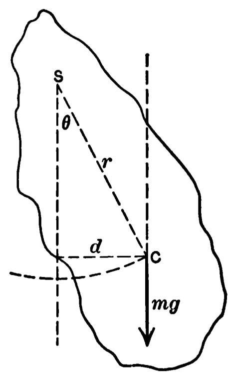


FIG. 29.

$$M\,g\,r \sin\theta = I\,\alpha. \quad . \quad . \quad . \quad . \quad (7)$$

If the angular displacement be small, the sine and angle are sensibly equal, and we may write:

$$M\,g\,r\,\theta = I\,\alpha; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which

$$\alpha = \frac{M\,g\,r\,\theta}{I}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The body being rigid, the angular acceleration for all points is the same at any instant, and varies directly as the angular displacement θ. But since angular acceleration and angular displacement are to each other directly as linear acceleration and linear displacement, it follows that every point in the body has a simple harmonic motion; therefore,

$$a = \alpha r = \frac{M g r^2 \theta}{I} = \frac{4 \pi^2}{T^2} s. \quad . \quad . \quad (10)$$

But,

$$s = r \theta;$$

hence,

$$\frac{M g r^2 \theta}{I} = \frac{4 \pi^2}{T^2} r \theta;$$

from which

$$T^2 = 4 \pi^2 \frac{I}{M g r},$$

and

$$T = 2 \pi \sqrt{\frac{I}{M g r}} = 2 \pi \sqrt{\frac{\frac{I}{M r}}{g}}. \quad . \quad . \quad (11)$$

Comparing equation (11) with equation (5), we find that $\frac{I}{M r}$ takes the place of L; hence $\frac{I}{M r}$ is the length of the equivalent simple pendulum.

Such an arrangement as just discussed is called a physical pendulum. The quantity $M r$ is called the statical moment; and the length of the physical pendulum then is numerically equal to the moment of inertia about the axis of suspension divided by the statical moment.

Kater's Pendulum. The most accurate method for determining the acceleration of gravity is by means of a pendulum. But the only quantity in equation (11) that is readily determined by experiment is the time.

It is, however, possible by employing a Kater's, or reversible pendulum, to determine the length, without knowing the moment of inertia or statical moment. The following discussion will make this clear.

In Fig. 30, let *a b* be a rigid rod supporting the two unequal masses m_1 and m_2, and let M be the mass of the whole system whose centre of mass is at C. Let this system be supported by an axis S, then the length of the equivalent simple pendulum is

$$L = \frac{I_s}{M r}; \quad . \quad . \quad . \quad (12)$$

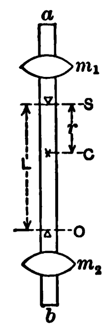


FIG. 30.

where I_s is the moment of inertia of the system about the axis S, and r the distance from the axis S to the centre of mass.

But, as has been previously shown,

$$I_s = I_c + M r^2;$$

where I_c is the moment of inertia about a parallel axis through the centre of mass. Hence, we have

$$L = \frac{I_c + M r^2}{M r}. \quad . \quad . \quad . \quad . \quad (13)$$

If the pendulum be now reversed and suspended by the axis O, parallel to the axis S, and at a distance L from it, we will have for the new length

$$L_1 = \frac{I_c + M\,(L - r)^2}{M\,(L - r)}. \quad . \quad . \quad . \quad (14)$$

From equation (13), we have

$$I_c = M\,r\,L - M\,r^2 = M\,r\,(L - r).$$

Substituting this value, for I_c in equation (14), we obtain

$$L_1 = \frac{M\,r\,(L - r) + M\,(L - r)^2}{M\,(L - r)};$$

from which

$$L_1 = r + L - r = L. \quad . \quad . \quad . \quad (15)$$

Showing that the length is the same, and therefore the time of vibration is the same when vibrating about the axis S as it is when vibrating about the axis O, at a distance L from S.

If, therefore, we take such an arrangement, as depicted in the figure, and adjust the axes S and O, until the time of vibration about the two axes is the same, it becomes necessary only to note the time, and measure the distance L; then by equation (5) g may be calculated.

Ballistic Pendulum. Assume a body, Fig. 31, such as $a\,b$ of mass M, and centre of mass at C. If a force F be applied to the body at a distance x from the centre of mass C, the effect of this force will be to produce a linear acceleration; which is

$$a = \frac{F}{M}.$$

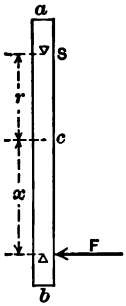


FIG. 31.

Further, there will be an angular acceleration about the centre of mass; which is

$$\alpha = \frac{F\,x}{I_c};$$

where I_c is the moment of inertia about the axis through C. The condition for the point S to be at rest is

$$a = \alpha\,r;$$

where r is the distance between the points S and C. Substituting for a and α, as given above, we have

$$\frac{F}{M} = \frac{F\,x\,r}{I_c};$$

from which

$$x = \frac{I_c}{M\,r}.$$

But L, the distance from S to the point of application of the force F, is $(x + r)$; therefore,

$$L = \frac{I_c}{M\,r} + r = \frac{I_c + M\,r^2}{M\,r}.$$

But $(I_c + M\,r^2)$ is the moment of inertia of the body about the axis through S, hence

$$L = \frac{I_s}{M\,r}.$$

Showing that the distance from the axis of suspension, where a blow must be struck, so that there shall be no jar on the axis, is the length of the equivalent simple pendulum; hence, the axis of oscillation is also the axis of percussion.

Graphical Representation of the Pendulum. Consider any irregular body, such as is depicted in Fig. 32, having its centre of mass at C, and suspended by an axis through S, perpendicular to the plane of the paper. If its moment of inertia about an axis, parallel to the axis through S, and passing through the centre of mass C is I_c, and the distance from the centre of mass to an axis of suspension is a, then by a previous equation the length of the pendulum is

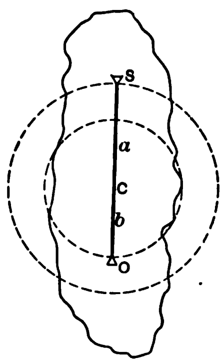


FIG. 32.

$$L = \frac{I_c + M a^2}{M a};$$

where M is the mass of the body.

If now, with C as centre and a as a radius we describe a circle, then the axis of suspension may be taken anywhere on the circumference of this circle for a constant time of vibration; for, the expression for the length of the pendulum is obviously constant.

Let, now, O be the axis of oscillation, then if we describe a circle with C as centre and b as radius, b being the distance from C to O, the time of vibration of the body when suspended at any point on the circumference of the circle whose radius is b will be constant and the same as when suspended on the circumference of the circle whose radius is a; and the length of the pendulum is

$$L = a + b. \qquad (16)$$

Also

$$L = \frac{I_c + M a^2}{M a} = \frac{I_c + M b^2}{M b}. \qquad (17)$$

Taking, now, the general equation for the length of the pendulum and writing it in a different form, we have

$$L = \frac{I_c}{M a} + a, \qquad (18)$$

since I_c and M are constant for the body under consideration, then if a be varied and approach infinity for its value, L approaches infinity for its value; and, if a approach zero for its value, L again approaches infinity for its value. But, for any other values of a, L will have a finite value. We will now show that L is a minimum when $a = b$. Let

$$I_c = M K^2;$$

where K is the radius of gyration with respect to the centre of mass, and is that distance from the centre of mass where the mass of the body would have to be concentrated at a point so as to have a moment of inertia I_c. In general

$$K = \sqrt{\frac{I}{M}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Let in the triangle $O\,S\,A$, Fig. 33, $S\,O$ be the length of the pendulum, a and b, respectively, the distances from the centre of mass to the axis of suspension and oscillation. Erect a perpendicular at C, and make it equal in length to K, the radius of gyration for the body about an axis through the centre of mass C. We then have

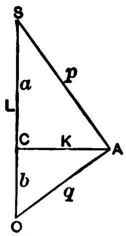


FIG. 33.

$$I_s = I_c + M\,a^2 = M\,K^2 + M\,a^2 = M\,(a^2 + K^2), \quad . \quad (20)$$

and

$$I_o = I_c + M\,b^2 = M\,K^2 + M\,b^2 = M\,(b^2 + K^2)\,; \quad . \quad (21)$$

where I_s and I_o, respectively, are the moments of inertia with respect to the axis through S and O.

By construction, we have

$$a^2 + K^2 = p^2,$$

and

$$b^2 + K^2 = q^2;$$

hence, by substituting in equations (20) and (21), we obtain

$$I_s = M\,p^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$I_o = M\,q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From equations (22) and (23), it follows that p and q, respectively, are the values for the radius of gyration for the body when suspended by the axis through S, and when suspended by the axis through O.

Since L, which is equal to $(a + b)$, is the same whether the body be suspended by an axis through S or O, we have

$$\frac{M\,p^2}{M\,a} = a + b,$$

and

$$\frac{M\,q^2}{M\,b} = a + b.$$

Therefore

$$p^2 = a^2 + a\,b,$$

and

$$q^2 = a\,b + b^2;$$

from which

$$p^2 + q^2 = (a + b)^2. \quad . \quad . \quad . \quad . \quad (24)$$

Equation (24) shows that the angle $S\,A\,O$ is a right angle. Now, a and b are variables and K is a constant, and as has been previously shown, the hypothenuse of a variable right triangle, when the perpendicular distance from the vertex of the right angle to the hypothenuse is fixed, is a minimum when the hypothenuse is divided equally and is double the perpendicular. Therefore, the minimum length of the pendulum is

$$L_m = 2\,K. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Equivalent Mass of the Pendulum. The equivalent mass of the pendulum must be a mass of such value that if concentrated at the point O, its moment of inertia with respect to the axis through S is the same as that of the body under consideration; and when reversed the mass concentrated at S must have a value such that its moment of inertia with respect to an axis through O, is the same as that of the body; and further, their relation must be such that their centre of mass falls at C.

Let m_o represent the mass to be concentrated at O, and m_s the mass to be concentrated at S; then

$$m_o (a + b)^2 = M (a^2 + K^2) = M (a^2 + a b),$$

and

$$m_o = \frac{M a}{a + b}. \quad . \quad . \quad . \quad . \quad (26)$$

Again

$$m_s (a + b)^2 = M (b^2 + K^2) = M (b^2 + a b),$$

and

$$m_s = \frac{M b}{a + b}. \quad . \quad . \quad . \quad . \quad (27)$$

From which, by adding equations (26) and (27),

$$m_o + m_s = M \left(\frac{a}{a + b} + \frac{b}{a + b}\right) = M.$$

To have the same centre of mass the moments about C must be equal; *i.e.*, $m_s a$ should be equal to $m_o b$. Multiplying equation (26) by b and equation (27) by a, we find the two expressions equal. Therefore the statical and dynamical conditions are completely expressed by assuming two masses; $M \frac{a}{a + b}$ situated at O, and $M \frac{b}{a + b}$ situated at S.

Conical Pendulum. Assume, as in Fig. 34, a mass m supported from an axis O, and caused to rotate such that the supporting cord L describes a cone. The mass then moves in the circumference of a circle of radius r, and the horizontal force acting upon the mass is $\frac{m v^2}{r}$; where v is the speed in the circumference of the circle described by the mass m.

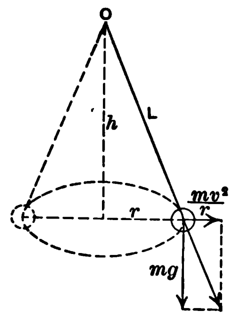


FIG. 34.

The vertical force is $m g$, and the condition of equilibrium is given by the fact that the direction of the supporting cord prolonged is the diagonal of the parallelogram constructed upon the forces $\frac{m v^2}{r}$ and $m g$ as sides; from the similarity of triangles

$$\frac{m v^2}{r} : m g : : r : h.$$

From which

$$h = \frac{m g r^2}{m v^2} = \frac{g r^2}{v^2}. \quad . \quad . \quad . \quad . \quad (28)$$

Let, now, n be the number of revolutions the mass makes per unit time in the circumference of the circle, then

$$v = 2 \pi r n,$$

and

$$v^2 = 4 \pi^2 r^2 n^2.$$

Substituting this value of v, in equation (28), we obtain

$$h = \frac{g r^2}{4 \pi^2 r^2 n^2} = \frac{g}{4 \pi^2 n^2}; \quad . \quad . \quad . \quad (29)$$

or, writing this in another form, we obtain

$$n = \frac{1}{2 \pi} \sqrt{\frac{g}{h}}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This is the equation for the conical or centrifugal governor.

Since the time of a revolution is $T = \frac{1}{n}$, it follows that

$$T = 2 \pi \sqrt{\frac{h}{g}}. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

If, now, r be small, so that h and L are sensibly equal, equation (31) becomes

$$T = 2 \pi \sqrt{\frac{L}{g}}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Showing that the period of a conical pendulum of small amplitude is equal to that of a simple pendulum of small amplitude.

CHAPTER X

FALLING BODIES AND PROJECTILES

SINCE the acceleration of gravity is sensibly a constant for ordinary heights above the earth's surface at any specified place, it follows that the formulæ deduced in Chapter I, for uniformly varied motion apply equally for bodies moving under the action of gravity; it only becomes necessary to replace a by g; where g is the acceleration due to gravity.

Making these substitutions, we obtain:

$$v_1 = v_o + g\,t, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$h = \frac{v^2_1 - v^2_o}{2\,g}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$h = v_o\,t + \frac{g\,t^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

v_o being the initial, and v_1 the final velocities, t the time, and h the height.

If the initial velocity be zero, these formulæ become,

$$v = g\,t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$h = \frac{v^2}{2\,g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$h = \frac{g\,t^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The relation between velocity and height is given by formula (5); *i.e.*, to produce a velocity v a body must fall from a height h such that $v = \sqrt{2gh}$; and similarly to rise to a height h, the body must be projected with a velocity v such that the same relation subsists. The time is fixed by either equation (4) or equation (6), depending upon whether v or h is given.

If a body be projected horizontally, its range depends upon its initial velocity, and height above the earth's surface; whereas, if it be projected vertically the height to which it will rise depends solely upon the initial velocity.

If the body be projected so that its initial velocity is inclined to the horizontal its height and range both depend upon the magnitude and direction of the initial velocity.

If V is the initial velocity and θ the angle of inclination, then the horizontal and vertical components are given as follows:

$$\left.\begin{array}{l} u = V \cos \theta \\ v = V \sin \theta \end{array}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where u is the horizontal, and v the vertical component.

The time required by the body to reach its highest point is $t = \frac{v}{g}$; and since, in falling, an equal interval is consumed, the total time, or time of flight is

$$T = \frac{2v}{g} = \frac{2V \sin \theta}{g}. \quad . \quad . \quad . \quad . \quad (8)$$

Since the horizontal component of the velocity is constant, the range is numerically equal to the product of the time and horizontal component. Designating the range by R, we have

$$R = \frac{2\,v}{g}\,u = \frac{2\,V^2 \sin\theta \cos\theta}{g}. \quad . \quad . \quad . \quad (9)$$

This is a maximum for a given speed, when the angle of inclination is 45°; since the product of the sine and the cosine of an angle is a maximum when the angle is 45°.

The actual velocity at any instant is numerically equal to the square root of the sum of the squares of the horizontal and vertical velocities at that instant.

Designating this by v_t, we have

$$v_t = \sqrt{u^2 + (v - g\,t)^2}, \quad . \quad . \quad . \quad (10)$$

which is a minimum, and equal to u when at the highest point; since at that instant $(v - g\,t) = 0$.

To obtain the equation for the path of the projectile, we let x equal the horizontal distance, and y the vertical distance; we then have

$$x = u\,t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$y = v\,t - \frac{g\,t^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substituting in equation (12), the value of t as found in equation (11), we obtain

$$u^2\,y = u\,v\,x - \frac{g}{2}\,x^2; \quad . \quad . \quad . \quad (13)$$

which is the equation of a parabola.

CHAPTER XI

ELASTICITY

FORCE has been defined as that which changes or tends to change the rate of motion of a body. But since to every action there is an equal and contrary reaction, it follows that there can never be a single force.

The mutual interaction of bodies changing or tending to change their rates of motion is called a *stress;* or, in other words, force is a stress considered in one of its aspects.

Heretofore, we have been considering bodies as being perfectly rigid. This is never the case. Whenever a body is under the action of a stress, there is produced a change in dimensions; which may be a change in volume, a change in shape, or as is usually the case, a change in volume and shape. This change is called a *strain.*

It is the result of experiment, known as Hooke's Law, that when a body serves to transmit a stress, then up to a certain limit, the strain produced is proportional to the applied stress; beyond this limit, the strain increases at a greater rate than the applied stress.

The force of restitution, or the resistance which a body offers to a stress producing deformation, is ascribed to its *elasticity*. Bodies which recover their original form upon the removal of the applied stress are said to be perfectly elastic. If, however, a body be deformed beyond the limit for which Hooke's Law holds, it will not return to its original size and shape. That point where a body ceases to obey Hooke's Law is called the *elastic limit*.

The ratio of the applied stress to the corresponding strain in a unit of a body is numerically equal to its *modulus of elasticity*.

There may be specified:

(1) *Elasticity of traction.*

(2) *Elasticity of torsion.*

(3) *Elasticity of flexure.*

(4) *Elasticity of volume.*

Modulus of Tractional Elasticity. If a body of cross section A, and length L is subjected to a stress S tending to compress or elongate it, then, up to the elastic limit, it is found that the elongation e, is directly proportional to the product of the applied stress and length, and inversely proportional to the cross section. In symbols

$$e \propto \frac{S\,L}{A};$$

and

$$\mu = \frac{S\,L}{A\,e} = a \textit{ constant}; \quad . \quad . \quad . \quad . \quad (1)$$

where μ is the modulus of tractional elasticity; and may be defined as the ratio of the stress per unit area to the corresponding strain per unit length.

Elasticity of Torsion. Theory indicates and experiment verifies that when a cylindrical body of radius r and length L be clamped at one end, and the other end be subjected to a couple G whose axis is the axis of the cylinder, then the amount of twist, or torsion θ, is proportional to the product of the couple and the length, and inversely proportional to the fourth power of the radius. In symbols

$$\theta \propto \frac{G L}{r^4}.$$

The exact relation, between the various magnitudes, is given by the formula

$$\theta = \frac{2 G L}{n \pi r^4}; \qquad (2)$$

where n is the modulus of rigidity. Writing this in another form, we have

$$n = \frac{2 G L}{\theta \pi r^4}. \qquad (3)$$

The *modulus of rigidity* n, may be determined in two ways, one is by direct measurement; *i.e.*, by subjecting a cylindrical body of known length and radius to a given torque and measuring the amount of torsion. These values substituted in equation (3) determine n. By taking a number of observations and plotting a curve, torques as abscissæ and amounts of torsions as ordinates, the limit of elasticity may be determined by noting the point where the curve departs from a straight line.

The modulus of rigidity may also be determined by clamping at one end a cylindrical body of known length and radius, and suspending from it a mass whose moment of inertia is determinate, and determining the period of the suspended body when vibrating about the axis of the cylinder.

Let τ equal the moment of torsion; *i.e.*, the moment of the couple which will twist the body through one radian. Then, since the amount of torsion is proportional to the torque, it follows that

$$\theta \tau = G = I \alpha; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I is the moment of inertia of the suspended body, and α its angular acceleration. This being the case it follows that the torque tending to restore the vibrating body to equilibrium is directly proportional to the angular displacement. And, since angular displacement and angular acceleration are directly proportional to linear displacement and linear acceleration, it follows that every point in the body has a simple harmonic motion. From this, it follows that

$$\alpha = \frac{4 \pi^2}{T^2} \theta. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But, from equation (4), we have

$$\tau = \frac{I \alpha}{\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting in equation (6), the value of α as given in equation (5), we obtain

$$\tau = \frac{4\pi^2}{T^2} I. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now

$$G = \theta \tau = \frac{4\pi^2}{T^2} I \theta.$$

Substituting this value of G, in equation (3), we obtain

$$n = \frac{8 \pi I L}{T^2 r^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If, in place of T, the time of vibration, we use t, the time of an oscillation, equation (8) becomes

$$n = \frac{2 \pi I L}{t^2 r^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From equation (7), it is seen that if the moment of torsion of a given wire be known, then the moment of inertia of the vibrating body is determined, no matter how irregular, providing the time of vibration is found. It is, however, not necessary to know the moment of torsion of the wire, provided we first determine the time of vibration of the body whose moment of inertia is sought, and then joining with this a body whose moment of inertia is known (or can be computed from its dimensions and mass), and again determining the time of vibration.

Let I_x be the moment of inertia of the first body, and T_1 its time of vibration; then, by equation (7), we have

$$\tau = \frac{4\pi^2}{T^2_1} I_x. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

If, now, we join with the body whose moment of inertia is I_x a body of moment of inertia I, and find the time of vibration T_2; then, since the moment of torsion is a constant, it follows that

$$\tau = \frac{4\pi^2}{T^2_2}(I_x + I). \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Combining equations (10) and (11), we obtain

$$\frac{I_x}{T^2_1} = \frac{I_x + I}{T^2_2};$$

from which

$$I_x = \frac{T^2_1}{T^2_2 - T^2_1} I. \quad . \quad . \quad . \qquad (12)$$

Elasticity of Flexure. Assume, as in Fig. 35, a rectangular beam, of width b and depth $2\,d$, bent

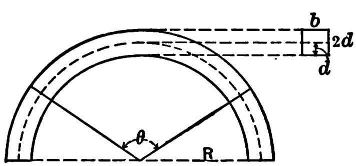


FIG. 35.

into the arc of a circle. Then the innermost fibres will be compressed, and the outermost fibres will be elongated; and, if the material offers the same resistance to compression that it does to elongation, the amount of compression of the innermost fibres will be equal to the amount of elongation of the outermost fibres; and furthermore, the amount of compression at any section at a distance x from the concave surface will be equal to the amount of elongation at a distance x from the convex surface; and at a distance d there will be neither compression nor elongation. A plane passed through the beam midway between, and parallel to the two surfaces, will not change in length when the beam is bent. This plane is called the ***neutral plane***. If the resistance offered to compression is not the same as that offered to elongation, then the neutral plane will not fall midway between the two surfaces.

Let L_0 be the original length of the beam, and R the radius of curvature of the neutral plane, then

$$L_0 = R\,\theta; \qquad (13)$$

where θ is the angle at the centre.

The length of the outermost fibre, after bending, becomes

$$L = (R + d)\,\theta$$
$$= R\,\theta + d\,\theta. \qquad (14)$$

Subtracting equation (13) from equation (14), we obtain the elongation; *i.e.*,

$$e = L - L_0 = d\,\theta. \qquad (15)$$

But, by definition

$$\mu = \frac{s\,L_0}{e}; \qquad (16)$$

where s is the stress per unit area. Substituting in equation (16), the values of L_o and e, as given by equations (13) and (15), we obtain

$$\mu = \frac{s\,R\,\theta}{d\,\theta} = \frac{s\,R}{d};$$

from which

$$s = \frac{\mu\,d}{R}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which is the expression for the stress per unit area on the concave and convex surfaces.

In like manner the stress per unit area, at any distance x from the neutral plane, is

$$s_x = \frac{\mu\,x}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

That is, the stresses at any section vary directly as the distances from the neutral plane; and, since they have opposite signs on opposite sides of the neutral plane, it follows that they constitute a torque, or turning moment, about the *neutral axis*. The neutral axis is defined by the intersection of the neutral plane with the section under consideration.

Let the beam whose width is b and depth $2\,d$ be divided, as in Fig. 36, into a number of sections of width b and indefinitely small depth $\frac{d}{n}$, such that the stress throughout the depth of the section may be considered constant. If s is the stress per unit area on the outermost fibre, or "skin stress," then the stress for unit area at a distance x from the neutral

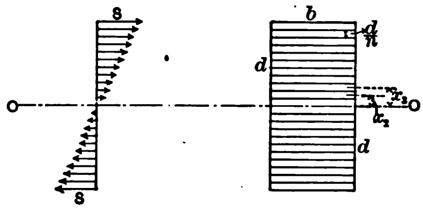


FIG. 36.

axis is, $s\frac{x}{d}$; and the stress for the element whose width is b, and depth $\frac{d}{n}$, at a distance x from the neutral axis, is

$$s_x = s\frac{x}{d}\left(b\frac{d}{n}\right) = s\frac{x}{d}a;$$

where a is the area of the section; and the turning moment of the section is

$$m = \frac{s\,x^2}{d}a.$$

If, now, we denote the distances from the neutral axis to the various elements by x_1, x_2, x_3, etc., the areas for the corresponding sections by a_1, a_2, a_3, etc., and by M the turning moment of the whole section, the turning moment, on one side of the neutral axis, then becomes

$$\frac{M}{2} = m_1 + m_2 + m_3 + \ldots\ldots + m_n$$

$$= \frac{s}{d}x^2_1 a_1 + \frac{s}{d}x^2_2 a_2 + \ldots\ldots + \frac{s}{d}x^2_n a_n;$$

that is,

$$\frac{M}{2} = \frac{s}{d}(x^2_1 a_1 + x^2_2 a_2 + x^2_3 a_3 + \ldots\ldots + x^2_n a_n);$$

and the turning moment for the whole section is,

$$M = \frac{2s}{d}(x^2_1 a_1 + x^2_2 a_2 + x^2_3 a_3 + \ldots\ldots + x^2_n a_n)$$

$$= \frac{s}{d} 2 \Sigma x^2 a. \qquad (19)$$

$2 \Sigma x^2 a$ denotes the result obtained by multiplying each elementary area by the square of its distance from the axis. It is the importance of the area with respect to the neutral axis, and may be appropriately called ***moment of area.*** In most text-books it is called moment of inertia and designated by I. This, however, is not a well-chosen expression, since it has nothing to do with inertia. To distinguish moment of area from moment of inertia we shall denote the former quantity by I_A. We then have

$$M = \frac{s}{d} I_A. \qquad (20)$$

Substituting in equation (20), the value of $\frac{s}{d}$ as obtained from equation (17), we have

$$M = \frac{\mu I_A}{R}. \qquad 21)$$

The moment of area may readily be found for all regular figures by integration.

It is possible to find this quantity, in certain cases, without the aid of the calculus; but all such methods are cumbersome. As a matter of illustration, the moment of area of a rectangular figure of width B and depth $2\,d$, will here be determined about an axis lying in the plane of the figure and half way between the top and bottom edges.

In Fig. 37, let $O\,O$ be the axis, and assume the rectangle to be divided into an indefinitely large number of strips of equal width, whose edges are all parallel to $O\,O$; then the width of each strip is indefinitely small and equal to $\frac{d}{n}$, where n is the number of strips for the half rectangle.

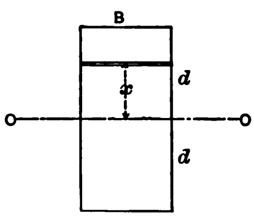


FIG. 37.

The moment of area of any strip at a distance x from the axis is, by what has been previously shown, equal to the product of area and square of distance from axis; in symbols

$$i_a = B\frac{d}{n}\,x^2; \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and the moment of area, of half the rectangle, becomes

$$\frac{I_A}{2} = \Sigma\, i_a = \frac{B\,d}{n}\,(x^2{}_1 + x^2{}_2 + x^2{}_3 + \ldots + x^2{}_n) \quad . \quad (23)$$

where x_1, x_2, x_3, etc., are the respective distances for the 1st, 2d, 3d, etc., strips from the axis.

Substituting for x_1, x_2, x_3, etc., their values; namely: $\frac{d}{n}$, $\frac{2d}{n}$, $\frac{3d}{n}$, etc., equation (23) becomes

$$\frac{I_A}{2} = \frac{B\,d}{n}\left(\frac{d^2}{n^2} + \frac{4\,d^2}{n^2} + \frac{9\,d^2}{n^2} + \ldots + \frac{n^2\,d^2}{n^2}\right)$$

$$= \frac{B\,d^3}{n^3}(1 + 4 + 9 + \ldots + n^2). \quad . \quad . \quad (24)$$

Now, the series, $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$, is equal to $\frac{n}{3}(n+1)\left(n + \frac{1}{2}\right)$. But, if n is indefinitely large, all quantities such as 1 and $\frac{1}{2}$ vanish with respect to it, and the sum of the series equals $\frac{n^3}{3}$. Substituting this value, in equation (24), we obtain

$$\frac{I_A}{2} = \frac{B\,d^3}{3};$$

and the moment of area of the whole rectangle, with respect to $O\,O$ as an axis, is

$$I_A = \frac{2\,B\,d^3}{3}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Deflection of a Rectangular Bar Clamped at one End. Assume a bar of length *L*, width *B*, and depth *D*, to be clamped rigidly at one end and have applied to it a force *F*, at the free end normal to the surface, as depicted in Fig. 38.

From equation (21), it is seen that the radius of curvature of a bent beam varies inversely as the turning moment. But at any section of a beam that is in equilibrium, the turning moment due to the internal forces must be equal to the *bending moment* due to the external forces. Now, the bending moment for a beam fastened at one end, and an applied force at

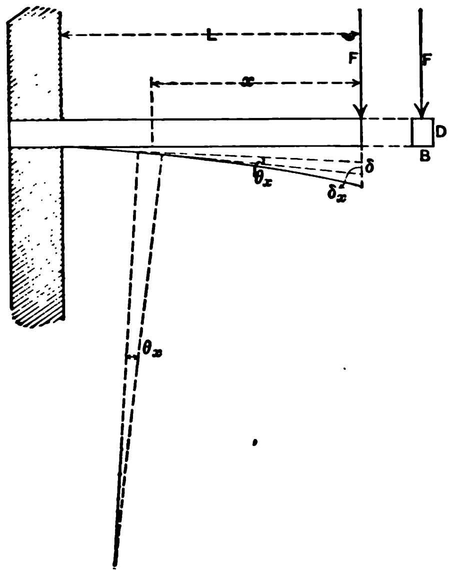


FIG. 38.

the other, varies directly as the distance from the free end. It therefore follows that the curvature varies; and is zero at the free end and a maximum at the clamped end. For a distance x from the applied force, we have for the radius of curvature

$$R_x = \frac{\mu I_A}{M_x} = \frac{\mu I_A}{F x}. \quad . \quad . \quad . \quad . \quad (26)$$

Assume the beam to be divided into a number of equal lengths, each equal to $\frac{L}{n}$; n being an indefinitely large number, so that the curvature may be assumed constant for each element.

The angle at the centre for an element at a distance x from the applied force is

$$\theta_x = \frac{L}{n\,R_x} = \frac{F\,x\,L}{n\,\mu\,I_A}.$$

If the total deflection is small, then the deflection due to an element is very small and equal to the product of the angle and distance from the free end; *i.e.*,

$$\delta_x = \theta_x\,x = \frac{F\,L\,x^2}{n\,\mu\,I_A}. \quad . \quad . \quad . \quad . \qquad (27)$$

But the total deflection is equal to the sum of the partial deflections; that is

$$\delta = \delta_1 + \delta_2 + \delta_3 + \ldots\ldots + \delta_n;$$

where δ_1, δ_2, δ_3, etc., are the deflections due to the elements at distances x_1, x_2, x_3, etc., from the free end. Hence, we have

$$\delta = \frac{F\,L}{n\,\mu\,I_A}\,({x^2}_1 + {x^2}_2 + {x^2}_3 + \ldots\ldots\ldots + {x^2}_n). \quad . \quad . \qquad (28)$$

Now x_1, x_2, x_3, etc., are respectively equal to $\frac{L}{n}$, $\frac{2\,L}{n}$, $\frac{3\,L}{n}$, etc.; making this substitution, in equation (28), we have

$$\delta = \frac{F\,L}{n\,\mu\,I_A}\left(\frac{L^2}{n^2} + \frac{4\,L^2}{n^2} + \frac{9\,L^2}{n^2} + \ldots\ldots + \frac{n^2\,L^2}{n^2}\right);$$

from which

$$\delta = \frac{F L^3}{n^3 \mu I_A} (1 + 4 + 9 + \ldots . + n^2). \quad . \quad . \quad . \quad (29)$$

But, as previously shown, when n is indefinitely large the sum of the series, in equation (29), becomes equal to $\frac{n^3}{3}$. Making this substitution, we finally obtain

$$\delta = \frac{F L^3}{3 \mu I_A}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Equation (25) gives $I_A = \frac{2 B d^3}{3}$. In deducing this formula the depth of the beam was designated by $2d$; if, in place of this, the depth be designated by D, equation (25) becomes

$$I_A = \frac{B D^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Substituting, in equation (30), for I_A its value, as given in equation (31), we have

$$\delta = \frac{4 F L^3}{\mu B D^3}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

This shows that when the deflections are small, they vary directly as the force and the cube of the length, and inversely as the width and the cube of the depth. These deductions are fully verified by experiment.

Consider, now, a rectangular bar of length L, width B, and depth D, supported at both ends and a force F applied at its middle section producing a deflection

δ. Since the bending moments at any two sections of the bar, on opposite sides of the middle section, and at equal distances from the point of support are equal, it follows that the curve assumed by the bar is symmetrical with respect to the middle section.

This is obvious from Fig. 39, L being the length of the bar, and the force F being applied at a distance $\frac{L}{2}$ from either point of support, it follows that the

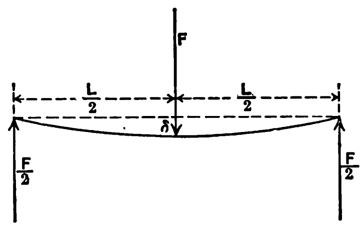


FIG. 39.

reaction on either support is $\frac{F}{2}$; and therefore, the bending moments, at equal distances from the points of support, are equal. Furthermore, the tangent to the curve at the middle section is parallel to the original position of the bar. The deflection, therefore, is the same as would be produced if the bar were clamped in the middle and subjected to a force $\frac{F}{2}$ at the end; the length of the bar being $\frac{L}{2}$. Making these substitutions in equation (32), *i.e.*, substituting for F, $\frac{F}{2}$, and for L, $\frac{L}{2}$, we obtain

$$\delta = \frac{F L^3}{4 \mu B D^3}. \quad . \quad . \quad . \quad . \quad (33)$$

By equation (31)

$$12 I_A = B D^3.$$

Substituting, in equation (33), for $B D^3$ its value, we have

$$\delta = \frac{F L^3}{48 \mu I_A}. \quad . \quad . \quad . \quad . \quad (34)$$

Equations (30) and (34) show that, other things being equal, the deflection varies inversely as the moment of area of the section. But, since the moment of area of a section is found by taking the sum of the products obtained by multiplying each elementary area by the square of its distance from the neutral axis, it follows that for any given sectional area, the moment of area may be increased, and the deflection decreased, by distributing the material in such a manner that the greater part of it is at a maximum distance from the neutral axis. It is for this reason that a plate will support a greater load turned edgewise than when lying flat; and for the same reason an I beam of given sectional area will support a greater load than a rectangular beam of equal sectional area.

Elasticity of Volume. Liquids differ from solids, *since solids* offer resistance to changes in form; and liquids, such as water, gasoline, alcohol, etc., offer practically no resistance to changes in form; but they do, like solids, offer resistance to changes in volume. As an example, water always takes the form of the containing vessel; but, to bring about a diminution in volume without change of temperature, a pressure must be applied. Experiment shows that the change per unit volume is directly proportional to the change in the applied pressure; *i.e.*,

$$\frac{v}{V} \propto p;$$

where V is the original volume, v the diminution in volume, and p the applied pressure per unit area. Rewriting, we have

$$\frac{p}{\frac{v}{V}} = \frac{pV}{v} = \mu \text{ a constant;}$$

where μ is defined as the *modulus of voluminal elasticity*. For water, the modulus of voluminal elasticity is found to be about 300,000 lbs. per sq. in.; whereas, for steel, the modulus of tractional elasticity is about 28,000,000 lbs. per sq. in.

Gases. Like liquids, gases offer no resistance to changes in form; but differ from liquids, inasmuch as a liquid merely takes the form of the containing vessel and a gas tends to fill the whole space in which it is enclosed. That is, as the pressure on a gas is decreased, the volume continually increases, and finally, if the pressure be made indefinitely small, the volume becomes indefinitely large. At constant temperature, for the so-called permanent gases, such as air, hydrogen, oxygen, etc., the volume varies inversely with the pressure; or, in other words, the product of pressure and volume equals a constant. This is known as Boyle's law. In symbols

$$pv = k; \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where p is the pressure per unit area, v the corresponding volume, and k a constant, whose numerical value depends upon the units chosen.

Assume now, the temperature remaining constant, that the pressure receives an indefinitely small increment $\triangle p$, in consequence of which the volume suffers a change equal to $-\triangle v$; hence, since the product of pressure and volume is constant, we have

$$(p + \triangle p)(v - \triangle v) = k.$$

Expanding

$$pv - p\triangle v + v\triangle p - \triangle p \triangle v = k. \quad . \quad (36)$$

Subtracting equation (35) from equation (36), and rearranging, we obtain

$$v\frac{\triangle p}{\triangle v} = p + \triangle p. \quad . \quad . \quad . \quad . \quad (37)$$

If we had assumed a decrement in pressure and a consequent increment in volume, we would have found the following:

$$v\frac{\triangle p}{\triangle v} = p - \triangle p. \quad . \quad . \quad . \quad . \quad (38)$$

Now, the nearer $\triangle p$ and consequently $\triangle v$ approach zero for their values the nearer the left-hand members of equations (37) and (38) approach the ratio of the change in pressure to the corresponding change per unit volume; and in the limit, just as the pressure is beginning to suffer a change, the right-hand members are equal to each other, and necessarily equal to p, and the left-hand members are rigidly equal to the ratio of change in pressure to the corresponding change per unit volume. Therefore, the modulus of voluminal elasticity of a gas obeying Boyle's law is numerically equal to the pressure.

CHAPTER XII

STATICS

It was stated, in Chap. IV, when dealing with the principle of moments, that experiment shows, that the tendency of a given force to produce rotation about an axis is independent of the point of application, but depends solely upon the intensity of the force and its arm. Assume, as depicted in Fig. 40, the two non-parallel coplanar forces *a b*, and *c d*, applied to a rigid body perfectly free to move; the points of application being *a* and *c*. Since the tendency of a force to produce rotation is not altered by shifting the point of application along the *line of direction* of the force, the two forces, *a b* and *c d*, may be replaced, respectively, by the forces *O g* and *O h*; where *O g* = *a b*, and *O h* = *c d*, and *O* is the point of intersection of *b a* and *d c* produced. Now, since neither force has an arm with respect to an axis through *O*, there can be no rotation about this axis. If then a third force, lying in the same plane equal in magnitude to *O f*,

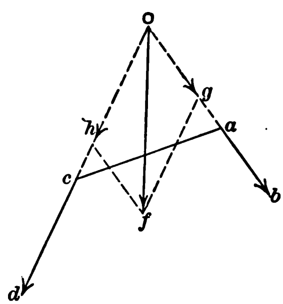


Fig. 40.

which is the vector sum of *a b* and *c d*, be applied along the line *O f*, but opposite in direction, the tendency of the forces *a b* and *c d* to produce linear acceleration will be balanced. Since the three forces have no tendency to produce either an angular acceleration about an axis through *O*, or a linear acceleration of the point *O*, the three forces are in equilibrium. If, on the other hand, a force equal to *O f*, and opposite in direction, whose line of direction does not pass through the point *O*, be applied to the body, there will be no tendency to produce linear acceleration; but there will be a tendency to produce angular acceleration, since the two forces constitute a couple. Hence, for three non-parallel coplanar forces to be in equilibrium, *their lines of direction must intersect at a common point, and the intensity of any one of the three forces must be equal and opposite to the vector sum of the two remaining forces.*

Force Polygon. It was shown, in Chapter III, that the resultant of two or more concurrent coplanar forces may be found by vector addition. That is, if we begin at any point *O*, and draw, assuming a certain scale, a line in the direction of one of the forces, and from the terminal of this line draw a second line, representing to the same scale and in a proper direction a second force, and from the terminal of this second line, a third line representing in a similar manner a third force, and continue in this manner until all the forces have been represented, the line then joining the point O and the terminal of the last line drawn, represents in direction and magnitude the resultant of all the forces. And for the system of forces to be in equilibrium, a force equal in magnitude to the resultant and opposite in direction must be applied on the line of direction of the resultant. From this, it follows that if a number of coplanar forces are in equilibrium, and a vector diagram be drawn, as just described, the resulting figure is a *closed polygon.* This may be further illustrated as follows: Assume the forces, F_1, F_2, F_3, ...F_7, when plotted as just described, to form the closed polygon O, p, q, ... u, as depicted in Fig. 41. The line R_1, joining the points O and q, represents the resultant of the two forces F_1 and F_2, and therefore may replace these two forces; similarly, the line R_2 represents the resultant of R_1 and F_3, and therefore may replace them. Continuing in this manner there finally remain the three forces, R_4, F_6, and F_7; but the resultant of R_4 and F_6 is equal and opposite to F_7 and passes through the point O; hence the system is in equilibrium.

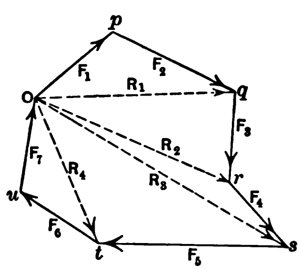


FIG. 41.

Funicular Polygon. The figure assumed by a closed flexible cord when in equilibrium under the application of a number of coplanar forces is termed a *funicular polygon*. Let, as in Fig. 42, the forces F_1, F_2, F_3, ... F_7, which are in equilibrium, be applied to a closed flexible cord in such a manner that the cord assumes the form of a polygon, O, p, q, ... u. The two following statements may

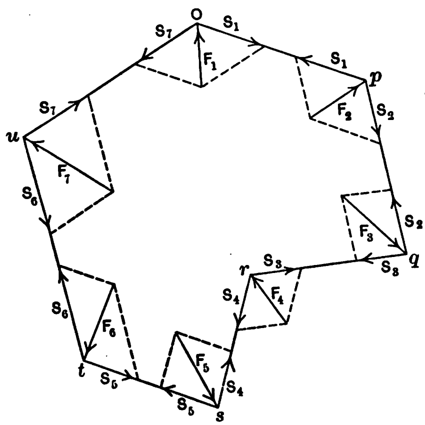


FIG. 42.

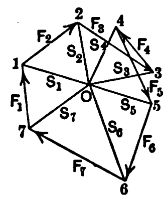


FIG. 43.

then be made. (1) The system being in equilibrium, the applied forces, F_1, F_2, F_3, ... F_7, must give a closed polygon. (2) Since all points of application are in equilibrium, the vector sum of the three forces at these points must be zero; *i.e.*, the applied force F_1 must be equal and opposite to the resultant of the two stresses in the cord, viz., S_1 and S_7; likewise, the applied force F_2 must be equal and opposite to the resultant of the stresses, S_1 and S_2; likewise for F_3, etc.

Ray Polygon. If now, in Fig. 43, we lay off the force F_1, then S_1 and S_7 must form a triangle 1 O 7 with F_1; laying off from the terminal of F_1 the force F_2, then S_1 and S_2 must combine with it to form the triangle 2 O 1. Hence the two triangles 1 O 7 and 2 O 1 have the side S_1 in common. Likewise, if we lay off the force F_3 from the terminal of F_2 and combine with it the two stresses S_2 and S_3 we obtain the triangle 3 O 2, having the side S_2 in common with the triangle 2 O 1. Proceeding in this manner it is found that each triangle has one side in common with the triangle preceding; hence, since the force polygon closes, the lines drawn from the points 1, 2, 3, etc., parallel to the stresses S_1, S_2, S_3, etc., must meet in a point O, termed the pole; and the lengths of the lines radiating from the pole determine the stresses in the sides of the original polygon.

A little consideration will show that if the applied forces acting upon a closed cord are in equilibrium and their directions and magnitudes are known, then the assumption of the directions of two consecutive sides of the polygon determines the directions and stresses for the whole polygon; also, if the shape of the polygon and the directions of the applied forces are given, then the assumption of the magnitude of one of the applied forces determines all the others.

The foregoing demonstration holds equally well for an articulated frame, when in equilibrium under the action of forces applied to the joints. The only difference being that some or all of the members may be under compression instead of, as in the case of the cord, under tension.

From what has been said about the force polygon, it is clear, that if the forces acting on a frame are

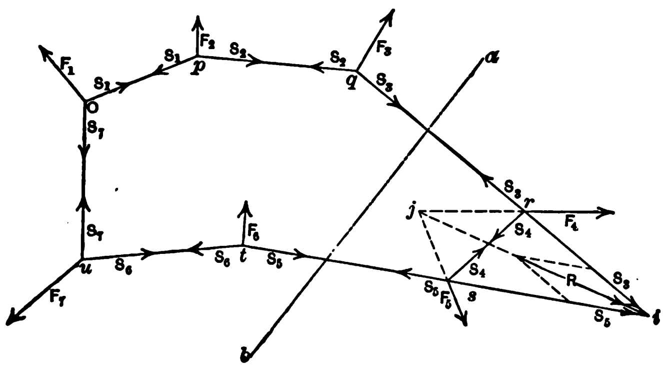


FIG. 44.

not in equilibrium, then the closing side of the polygon determines the direction and magnitude of the equilibrant.

Assume, as in Fig. 44, the frame *O*, *p*, *q*, *r*, *s*, *t*, *u*, in equilibrium under the action of the forces F_1, F_2, F_3, ... F_7; and further, that two of the sides, such as *q r* and *t s*, are cut across. Since the forces F_1, F_2, F_3, F_6, and F_7 are in equilibrium with the two stresses acting along *q r* and *s t*, it follows that the resultant of these two stresses is the equilibrant of all the forces applied to the left of *a b*; viz., F_1, F_2, F_3, F_6, and F_7, and must lie on a line passing through the point *i*, the intersection of *q r* and *t s* produced. If we now draw the triangle of forces for the two stresses S_3, S_4, and the force F_4, and the triangle of forces for the two stresses S_5, S_4, and the force F_5, the side S_4 will be in common, and therefore the two stresses S_3 and S_5 are balanced by the two forces F_4 and F_5. Hence, the resultant of S_3 and S_5 has the same magnitude and line of direction as the resultant of F_4 and F_5 and its line of direction must pass through the point *j*, the intersection of the lines of direction of F_4 and F_5.

If we have a given system of forces, such as F_1, F_2, F_3, F_6, and F_7, which are not in equilibrium, the magnitude and line of direction of the equilibrant can readily be determined as follows: Consider the given forces applied to the joints of an articulated frame and assume the directions of two consecutive sides of the frame, such as *O p* and *O u*, this determines, as previously stated, the directions and stresses for all the members of the frame; hence, S_3 and S_5 are known. But, as has just been shown, the resultant of S_3 and S_5 is the equilibrant of the given forces. If it develops that S_3 and S_5 are nearly parallel, so that it is impracticable to get the point of intersection, then if we assume *q i* and *t i* to be cut by a third member, such as *r s*, under an assumed stress S_4, the two forces to be applied at the joints *r* and *s*, namely F_4 and F_5, may be found. For, since S_3 and S_4 are given in both magnitude and direction, F_4 is determined; and similarly F_5. But, as has been shown in the previous demonstration, the resultant of F_4 and F_5 is the equilibrant of the given forces. The resultant of the given forces may, of course, be found, both in magnitude and direction, by the polygon of forces; but this does not give *the line of direction.*

Parallel Forces. If the forces applied to the joints of an articulated frame are parallel, then the force polygon reduces to a straight line, and necessarily, to be in equilibrium, the algebraic sum of the forces must be zero.

Assume, as in Fig. 45, the three parallel forces F_1, F_2, and F_3 applied to the jointed frame at the points *b*, *c*, and *d*; and further, that the frame is supported at the points *a* and *e* by the reactions R_1 and R_2 parallel to the applied forces. The conditions here represented are similar to a chain or cable supporting weights. *O p q* is the *ray polygon* obtained by constructing the triangle of forces for the points *d*, *c*, and *b*, as previously described. Since the point *e* is in equilibrium the three forces, S_5, S_4, and R_2 must combine to form a triangle; hence by drawing *O s*, in Fig. 46, parallel to *a e*, in Fig. 45, the stress S_5 is determined by the length of the line

$O\,s$, and the reaction R_2 by the length of the line $s\,p$. Similarly the reaction R_1 is determined by the length of the line $q\,s$. Again, since one of the three forces, acting at any point, is vertical, the horizontal components of the stresses for the two adjacent members must be equal and opposite. But, since all the applied forces are vertical, it follows that the horizontal

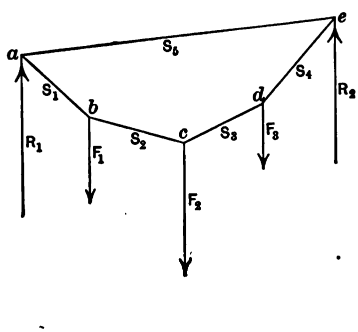


FIG. 45.

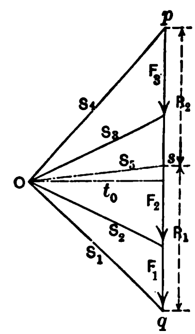


FIG. 46.

component for the stresses throughout the frame is constant, and is determined by the length of the line t_o, in Fig. 46; *i.e.*, the normal from O to the line $p\,q$, called the "polar distance."

Uniform Horizontal Loading. If a perfectly flexible cord, supported at two points, has applied equal weights uniformly distributed between the points of support, and the weight of the cord is negligible in comparison with the weights, the curve assumed by the cord is a parabola. The curve which a perfectly flexible non-stretchable cord, supported at two points, assumes under its own weight is a *catenary*. The equation of this curve, however, cannot be deduced without the aid of the calculus. If, however, the deflection is small in comparison with the length between supports, as is the case in a belt or cable drive, such that we may, without appreciable error, assume uniform horizontal loading, the equation is readily determined.

Assume, as in Fig. 47, the half span x, to be divided into n equal parts, and the deflection y, of the cord a, b, c, to be so small, in comparison with

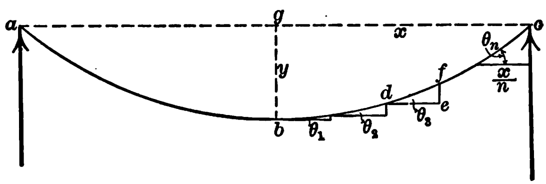


FIG. 47.

the span, that the weight of the cord for equal horizontal distances is practically constant throughout. If x be divided into an indefinitely large number of parts, such that $\frac{x}{n}$ is very small, then in any triangle, such as $d\,e\,f$, the chord and tangent practically coincide, and the deflection is given by

$$y = \frac{x}{n} \tan \theta_1 + \frac{x}{n} \tan \theta_2 + \ldots\ldots + \frac{x}{n} \tan \theta_n$$

$$= \frac{x}{n} (\tan \theta_1 + \tan \theta_2 + \ldots\ldots + \tan \theta_n). \quad (1)$$

Since all the applied forces, namely the weights of the various elements of the cord, act vertically, the horizontal tension must be constant throughout. The vertical tension at any point is equal to the weight of the cord included between the point b and the point under consideration. If the tension in the cord, at any point, be resolved into its two components, the horizontal component will be a constant, which we will denote by t_o, and the vertical will be as just stated, the weight of the cord from the point b to the point under consideration. But by assumption, we have constant loads for equal horizontal distances; hence, for the point f, the vertical component is

$$w \frac{3x}{n};$$

where w is the weight per unit length. The horizontal component being constant and equal to t_o, we have for the slope at f,

$$\tan \theta_3 = w \frac{3x}{n t_o}.$$

In a similar manner, we find

$$\tan \theta_1 = w \frac{x}{n t_o}, \tan \theta_2 = w \frac{2x}{n t_o}, \tan \theta_n = w \frac{nx}{n t_o}.$$

Substituting the values of the tangents in equation (1), and factoring the common part $\frac{wx}{n t_o}$, we obtain

$$y = \frac{w x^2}{n^2 t_o} (1 + 2 + 3 + \ldots + n)$$

$$= \frac{w x^2}{2 n^2 t_o} (n + 1) n. \quad \quad (2)$$

It is, of course, obvious that the smaller the length $\frac{x}{n}$ becomes, and consequently the larger n becomes, the nearer equation (2) represents the exact conditions. Assume n so large that 1 vanishes in comparison with it; equation (2) then becomes

$$y = \frac{w\,x^2}{2\,t_o}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the equation of a* parabola.

We will now determine t_o in terms of the total span S, the corresponding deflection D, and the

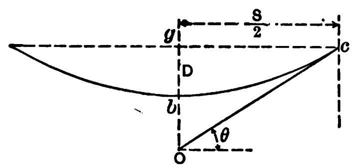


FIG. 48.

weight per unit length. Since, in a parabola, the subtangent is bisected at the vertex, we have in the

* Equation (3) is easily found by integration. For any point whose co-ordinates are x and y, the vertical component is $w\,x$; and the horizontal component being t_o, we have

$$\tan \theta_x = \frac{d\,y}{d\,x} = \frac{w\,x}{t_o};$$

from which

$$y = \frac{w\,x^2}{2\,t_o} + c.$$

The constant of integration being found to be zero from the condition that $x = 0$, when $y = 0$.

triangle $O\,g\,c$, in Fig. 48, $g\,b$ equal to $b\,O$; *i.e.*, $O\,g$ equal to $2\,D$. Hence, we have

$$tan\ \theta = \frac{4\,D}{S}.$$

But, since the slope is also equal to the vertical component divided by the horizontal component, we have

$$tan\ \theta = \frac{4\,D}{S} = \frac{w\,S}{2\,t_o};$$

from which

$$t_o = \frac{w\,S^2}{8\,D}. \qquad \qquad (4)$$

Three Forces Meeting in a Point. Problems whose solutions involve the principle that three coplanar forces to be in equilibrium must have their lines of direction meet in a point, being of such frequent occurrence it will be well to consider a few concrete cases. The simplest case is that of a weight supported as shown in Fig. 49. The three forces meeting in the point c being: the tension in the member supporting the weight W, and the stresses in the members $a\,c$ and $b\,c$, $a\,c$ being under compression and $b\,c$ under tension. The point c being in equilibrium, the vector sum of the two stresses along $a\,c$ and $b\,c$ must be equal and opposite to W, as shown by the triangle of forces $c\,d\,e$; where $c\,d$ and $d\,e$ are respectively the reactions of the members $b\,c$ and $a\,c$. Since the triangles $a\,b\,c$ and $e\,c\,d$ are similar, it follows that the stresses in the members, $a\,c$, $a\,b$, and $b\,c$, are to each other directly as the sides of the triangle formed by the members. But the stress in $a\,b$ is necessarily

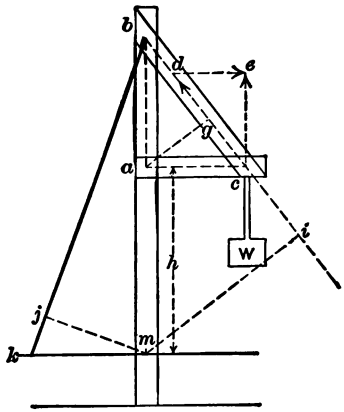


FIG. 49.

equal to W. Hence, denoting the stress in $a\,c$ by S_1, and in $b\,c$ by S_2, we have

$$W : a\,b :: S_1 : a\,c,$$

from which

$$S_1 = W\frac{a\,c}{a\,b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly

$$S_2 = W\frac{b\,c}{a\,b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The same results will, of course, be obtained if the problem be solved by the principle of moments. Taking moments, about the point a, we find

$$S_2 \times a\,g = W \times a\,c. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But, the triangles $a\,b\,c$ and $g\,a\,c$ are similar; hence

$$a\,g = \frac{a\,c \times a\,b}{b\,c}.$$

Substituting this value of $a\,g$ in equation (7), we find

$$S_2 = W\frac{b\,c}{a\,b},$$

which is the same as equation (6). Similarly, taking moments about the point b, we find

$$S_1 \times a\,b = W \times a\,c,$$

from which

$$S_1 = W\frac{a\,c}{a\,b},$$

which is the same as equation (5).

The moment tending to turn the support $b\,m$, in a clockwise direction, about the point m, is

$$W \times a\,c = S_2 \times m\,i - S_1\,h.$$

In order to balance this moment about the point m, a tie rod $b\,k$ may be used which must be under a tension S_3, such that

$$S_3 \times m\,j = W \times a\,c = S_2 \times m\,i - S_1\,h.$$

Bar Supported by a Horizontal and Vertical Surface. Assume a bar resting with one end on a horizontal surface, and the other against a vertical surface, in such a manner that it lies in a plane normal to the two surfaces. We have here three forces; *i.e.*, the reactions of the two surfaces, and a force, equal to the weight of the bar, applied at its centre of gravity acting vertically downward. Remembering, that when there is no friction, the reactions must be normal to the surfaces, it follows that equilibrium cannot obtain for perfectly smooth surfaces; for, in such a case, the lines of direction of two of the forces are parallel and the third acts at right angles to them. If, however, the horizontal surface is rough, equilibrium will obtain, providing the normal reaction of the vertical surface is not greater than the force of friction on the horizontal surface. Assume, as depicted in Fig. 50, the bar *a b*, whose weight is *W*, and whose centre of gravity is at *G*, having the end *a* resting against a perfectly smooth vertical surface, and the end *b*, resting upon the rough horizontal surface *O b*. The vertical surface, *O a*, being perfectly smooth, the re-

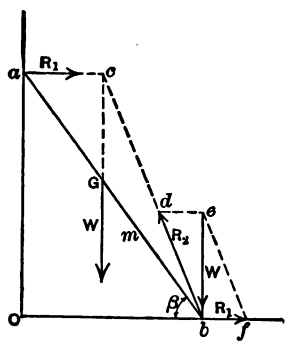


FIG. 50.

action R_1 must be normal to it. Hence, for equilibrium to obtain, the line of direction of R_2 must pass through the point *c*, the intersection of R_1, and *W*.

The magnitude of W being known, the two reactions are found by constructing the parallelogram $b\ d\ e\ f$. It is evident from the figure that as the angle β decreases, R_1 increases; but, for the equilibrium to obtain, the balancing force due to friction must be equal and opposite to R_1. Since the greatest value the force of friction can have is the product of weight and coefficient of friction, it follows that when the angle β has been decreased to a value such that

$$\mu = \frac{R_1}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

sliding will be impending. For all values of β less than this, equilibrium is impossible. To determine the critical value for the angle β, *i.e.*, that value when sliding is impending, take moments about the point of support b. For equilibrium to obtain, we must have

$$W\ m \cos\beta = R_1\ l \sin\beta; \quad . \quad . \quad . \quad (9)$$

where m is the distance from the centre of gravity to the point of support b, and l the length of the bar. From equation (9), we find

$$\tan\beta = \frac{W\ m}{R_1\ l},$$

and substituting for R_1, its value, as obtained from equation (8), we obtain

$$\tan\beta = \frac{m}{\mu\ l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For all values of β greater than that given by equation (10), equilibrium will obtain.

PROBLEMS

CHAPTER I

1. A body moving uniformly passes over a distance of 10 feet in 2 seconds. What is its speed? How long will it take to travel 25 feet?

Ans. 5 ft. per sec.; 5 sec.

2. A particle has a uniform speed of 30 kilometers per day. How long will it take to travel 3,500 millimeters?

Ans. 10.08 sec.

3. A body starts from rest with a constant acceleration of 10 ft. per sec. per sec. Determine the distance passed over in the 3d, 5th, and 7th seconds, and the total distance passed over in 10 seconds.

Ans. 25 ft.; 45 ft.; 65 ft.; and 500 ft.

4. The velocity of a body changes uniformly from 10 ft. per sec. to 25 ft. per sec. in 3 seconds. What is its constant acceleration? When is its velocity 75 ft. per sec.? How long will it have been in motion, assuming it to have started from rest? What space will it have passed over?

Ans. 5 ft. per sec. per sec.; in 10 sec.; 15 sec.; 562.5 ft.

5. A body changes speed from 100 meters per second to 60

meters per second in going 40 meters. What is the acceleration, assuming it to be constant? With the same acceleration, how far will the body have moved before coming to rest? In what time will it come to rest?

Ans. – 80 meters per sec. per sec.; 22.5 meters; 0.75 sec.

6. At a given instant a body is found to have a velocity of 200 ft. per sec. Ten seconds later it is found to have a velocity of 500 ft. per sec. What is its acceleration, assuming it constant? What space did it cover in the 10 seconds?

Ans. 30 ft. per sec. per sec.; 3,500 ft.

7. A body starting from rest with a uniformly accelerated motion passes over a distance of 36 kilometers in 2 hours. What is its acceleration in cm. per sec. per sec.? What was its velocity, and how far had it travelled, 15 minutes after starting?

Ans. 5/36 cm. per sec. per sec.; 125 cm. per sec.; 562.5 meters.

8. The velocity of a particle changes uniformly from 30 ft. per sec. to 20 ft. per sec. in passing over 25 ft. What is its acceleration? How long will it be before coming to rest, and what distance will it have traversed in that time, if its retardation is constant?

Ans. – 10 ft. per sec. per sec.; 2 sec.; 20 ft.

9. With what acceleration, and how far must a body move to have a speed of 30 m. p. h. in 30 seconds after starting from rest? What retardation would destroy this speed in 10 seconds? How far would the body have travelled?

Ans. 1.467 ft. per sec. per sec.; 1/8 mile; 4.4 ft. per sec. per sec.; 1/24 mile.

10. A body moving with a speed of 40 m. p. h. is retarded uniformly and brought to rest in 500 ft. What was the retardation in miles per hour per sec., and in feet per sec. per sec.?

Ans. 2.346; 3.44.

11. What is the curvature of a circle whose diameter is eight feet?

Ans. 1/4 radian per foot.

12. The direction of motion of a particle is changed uniformly by 0.25 radians in passing over 5 feet. What curve has it described and what are its dimensions?

Ans. Circle; 40 ft. diameter.

13. What will be the change in direction of a particle moving 10 ft. in the circumference of a circle 100 ft. in diameter?

Ans. 11.46°.

14. What distance has a body moved in the circumference of a circle of 25 ft. radius, if its change in direction of motion was 0.6 radians?

Ans. 15 ft.

15. A rotating disc makes 3,000 r.p.m. Find its angular velocity. Find the linear speed of a point 2 ft. from the axis of rotation.

Ans. 100 π radians per sec.; 200 π ft. per sec.

16. The linear speed of a point on a rotating body is 90 ft. per minute and its distance from the axis of rotation is 5 ft. How long will it take the body to sweep out 54 radians?

Ans. 3 minutes.

17. A bucket is raised by a rope passsing over a sheave near the end of a derrick boom, at the uniform speed of ten feet per second. If the rope winds up on a drum four feet in diameter, then, neglecting the thickness of the rope, what is the angular velocity of the drum? If the given velocity were acquired in three seconds, what would be the angular acceleration of the drum?

Ans. 5 rad. per sec.; 1.67 rad. per sec. per sec.

18. A flywheel starting from rest is found in twenty seconds to be revolving 150 times per minute. What is its angular acceleration, assuming it a constant? What would be its angular velocity at the end of one minute from rest?

Ans. $\pi/4$ rad. per sec. per sec.; 15π rad. per sec.

19. A rotating body, having an angular acceleration of 10 rad. per sec. per sec., has been in motion 10 sec. What is its angular velocity and how many rotations has it made? What time will elapse and how many rotations will be made, before its angular velocity is 900 rad. per sec.?

Ans. 100 rad. per sec.; $250/\pi$; 80 sec.; $20{,}000/\pi$.

20. A rotating body starting from rest has been in motion n seconds. If its angular acceleration is α how many rotations will be made during the next t seconds?

Ans. $\dfrac{\alpha t}{4\pi}(2n + t)$.

CHAPTER II

1. Add together the four vectors, ten north, fifteen east, seven south, and twelve west.

Ans. $3\sqrt{2}$ *N.E.*

2. Resolve the vector twenty into two vectors making angles of 30° and 60° on each side of it.

Ans. $10\sqrt{3}$; 10.

3. Resolve the vector *A* into the vectors *B*, *C*, *D*, *E*, and *F*; having assumed the directions of the vectors *B*, *C*, *D*, and *E*.

4. Two cars are moving along level lines inclined at an angle of 45°, with speeds of 20 and 30 m.p.h. If the cars are respectively 500 and 600 feet from the crossing point, show by diagram the motion of each car as it appears to the driver of the other.

5. A captain wished to sail a ship, whose speed is 12 m.p.h. in a *Southeast* direction. There is a current running 5 m.p.h. which sets the ship due *West*, off her course. In what direction must she be headed in order to sail in the *S.E.* direction? Show by diagram.

6. In problem (5) what will be the actual speed of the ship in the *S.E.* direction? If the ship had been headed *S.E.*, where would she have been at the end of one hour?

Ans. 7.93 m.p.h.; 9.17 miles from starting point, and 3.54 miles out of her course.

7. A boat steams across a river at right angles to the course of the river with a speed of 10 m.p.h. If the boat reaches the opposite shore 2 miles below the starting point, and if the river is 4 miles wide, what was the speed of the current?

Ans. 5 m.p.h.

8. A point moves in the circumference of a circle, whose diameter is 20 ft., with a uniform angular velocity of 5 radians per sec. What is the centripetal acceleration?

Ans. 250 ft. per sec. per sec.

9. If in problem (8), the point started from rest and moved with a uniform angular acceleration of 2 radians per sec. per sec., what would be the centripetal acceleration at the end of five seconds?

Ans. 1,000 ft. per sec. per sec.

10. If a point moves with a uniform speed of 10 ft. per second in the circumference of a circle of diameter 40 feet, what will be the velocity and the acceleration of the projection of the point upon a diameter, when the point has moved through $\pi/4$ radians from the extremity of the diameter?

Ans. $5\sqrt{2}$ ft. per sec.; $5/\sqrt{2}$ ft. per sec. per sec.

CHAPTER III

1. A picture whose weight is 21 lbs. is suspended by a cord hung over a peg. Each branch of the cord makes an angle of 30° with the vertical. What is the tension in the cord?

Ans. $7\sqrt{3}$ lbs.

2. An inclined plane has a rise of 1 in 10. Assuming no friction, what force, acting parallel to the plane, will just support a weight of 100 lbs.? What force parallel to the base?

Ans. 9.95 lbs.; 10 lbs.

3. A body weighing 50 lbs. rests on a plane inclined at an angle of 30°. Assuming the coefficient of friction to be 0.3, what force, acting parallel to the plane, will draw the body up the plane with uniform motion?

Ans. 37.99 lbs.

4. A body weighing 100 lbs. rests upon a plane inclined at an angle of 45°. Assuming the coefficient of friction to be 0.1, what force, parallel to the base, will draw the body up the plane with uniform motion?

Ans. 122.2 lbs.

5. A mass of 1 gram, perfectly free to move, starts from rest under the action of a constant force of 1 dyne. In what time is 1 erg of work performed?

Ans. $\sqrt{2}$ sec.

6. A mass suspended from a railway car by a cord 3 ft. long, rises a vertical height of 0.1 inches upon starting. If the mass

is in equilibrium in this position, what is the acceleration of the car?

Ans. 28.68 inches per sec. per sec.

7. A mass of 10 pounds, resting on a horizontal plane, is moved 8 feet in 8 seconds, starting from rest. What force, parallel to the plane, was necessary if the coefficient of friction is 0.3?

Ans. 3.08 lbs.

8. A weight of 100 lbs. rests on a plane, inclined $\sin^{-1} 0.6$ with the horizontal. What happens, respectively, when forces of 20, 40, 80, and 100 lbs. are applied to the body up and parallel to the plane? Coefficient of friction 0.25.

9. If a carriage be slipped from a train moving at 30 m.p.h. up a plane inclined $\sin^{-1} 0.02$ with the horizontal, how far, friction being neglected, will it move before beginning to run back?

Ans. 1512.5 ft.

10. A mass of 10 pounds is whirled in a horizontal plane by a cord 10 feet long capable of carrying but 50 lbs. How many revolutions per minute are necessary to break it?

Ans. 38.2.

11. 400 masses of 6 pounds each are distributed around the circumference of a rotating body at a mean distance of 3 ft. from the axis. What will be the tension in a cord wrapped round them when the system is making 200 r.p.m.?

Ans. 15,708 lbs.

12. A 400-ton train travels round a curve 1 mile in radius at

40 m.p.h. What is the horizontal component of the pressure on the rails?

Ans. 16,300 lbs.

13. If the centre of gravity of the train in the preceding problem be midway between the rails (5 ft. gauge), and 5 ft. above them, what must be the speed so that the train is on the point of turning over?

Ans. 198.17 m.p.h.

14. Assuming, in problem (12), the centre of gravity midway between rails, then how much would it be necessary to incline the track in order that there be equal pressure on them?

Ans. $\tan^{-1}0.02037$.

15. Find the number of vibrations that would be executed per minute by a mass of 5 pounds attached to a spring, obeying Hooke's Law, if a weight of 4 lbs. causes an elongation of 10 inches.

Ans. 53 (very nearly).

16. A mass of 5 pounds when attached to a spring obeying Hooke's Law executes 240 vibrations in 3 minutes. What is the force required to elongate the spring 1 ft.?

Ans. 11 lbs. (very nearly).

17. A mass of 5 pounds attached to a spring obeying Hooke's Law, vibrates with an amplitude of 2.5 ft., executing 100 vibrations in 3 minutes. When the displacement is 18 inches, find the value of the acceleration, velocity, kinetic energy, potential energy, and total energy.

Ans. 18.3 ft. per sec. per sec.; 6.98 ft. per sec.; 121.9 ft. poundals; 68.5 ft. poundals; 190.4 ft. poundals.

18. Find the value of the quantities named in the preceding problem when the mass is at a position such that 0.15 seconds elapse before reaching the equilibrium position.

Ans. 15.23 ft. per sec. per sec.; 7.56 ft. per sec.; 142.8 ft. poundals; 47.6 ft. poundals.

19. The velocity of a body moving with a S.H.M. is 20 ft. per sec. when 3 ft. from the equilibrium position, and 15 ft. per sec. when 4 ft. from it. What are the maximum values of the displacement, velocity, and acceleration?

Ans. 5 ft.; 25 ft. per sec.; 125 ft. per sec. per sec.

20. A weight of 100 lbs. rests on a platform which moves with an S.H.M., having an amplitude of 4 ft., and a period of 4 seconds. When the platform is 2 ft. above the equilibrium position and moving upward, what is the pressure exerted by the weight? What is the pressure, at the same position, when moving downward?

21. How long will it take a force of 1,000 lbs. to stop a 200-ton mass moving at 60 m.p.h.? What work will be done?

Ans. 18 min. 20 sec.; 24.44 H.P. hours.

22. What is the constant force required to stop a mass of 200 tons, moving at 60 m.p.h., in 100 feet? How much work has been done?

Ans. 242 tons; 484×10^5 ft. lbs.

23. A weight of 500 lbs. rests upon a plane having an inclination of 30°. If the coefficient of friction is 0.1, how much work will be done in drawing the weight, with uniform speed, 10 ft. up the plane?

Ans. 2,933 ft. lbs.

24. A mass of 10 grams, at rest but perfectly free to move, is acted upon for 5 seconds by a constant force of 50 dynes. What kinetic energy will the mass have at the end of 10 seconds?

Ans. 3,125 ergs.

25. A weight of 200 lbs. falls from a height of 15 feet upon the head of a pile, which, under the action of the blow, sinks 3 inches into the ground. What was the resistance?

Ans. 12,000 lbs.

CHAPTER IV

1. A force of 5 poundals acts upon a mass of 10 pounds for 2 minutes. How much will the momentum of the body be changed?

Ans. 600 F.P.S. units.

2. What force in 5 seconds will change the speed of a 100 gram mass from 40 cm. per sec. to 100 cm. per sec.?

Ans. 1,200 dynes.

3. Masses of 5 and 10 pounds, having velocities of 8 and 5 ft. per second respectively, collide. What are their velocities after impact if the coefficient of restitution is unity? What if 0.5? What if zero? Illustrate conditions by diagrams.

4. Solve problem (3) when the 10-pound mass has a velocity of − 5 ft. per sec.

5. Solve problem (4) when the 5-pound mass has a velocity of 10 ft. per sec.

6. Solve problem (5) when the 10-pound mass has a velocity of − 5 ft. per sec.

7. Solve problem (3) when the 5-pound mass has a velocity of 12 ft. per sec.

8. Solve problem (7) when the 10-pound mass has a velocity of − 5 ft. per sec.

9. An inelastic mass impinges directly upon another which is at rest and twenty times as great. What was its initial velocity if, after impact, both move a distance of 3 feet in 2 seconds?

Ans. 31.5 ft. per sec.

10. A one-ounce bullet is fired with a velocity of 1,600 ft. per sec. from a 20-pound rifle, which is held against a mass of 180 pounds. With what velocity did the rifle "kick" back?

Ans. 6 inches per sec.

11. A body is dropped from a height of 16 ft. and bounces a height of 9 ft. What is the coefficient of restitution? To what height will the body bounce the next time?

Ans. 0.75; $5\frac{1}{16}$ ft.

12. A jet of water from an orifice 1 sq. inch in section impinges against a wall. What is the force exerted if 120 gallons are delivered per minute?

Ans. 20.07 lbs.

13. What force is exerted upon a gun delivering 200 one-ounce bullets per minute with a speed of 1,600 ft. per sec.?

Ans. 10.42 lbs.

14. A projectile whose mass is 100 pounds is fired into a target, whose mass is 20,000 pounds, with a velocity of 1,000 ft. per sec. If the target be free to move, find the loss in energy during impact.

Ans. 1,555,000 ft. lbs.

15. A square board 2 feet on a side and weighing 3 lbs., has placed at the corners A, B, C, and D, weights of 1 lb., 2 lbs.

3 lbs., and 4 lbs., respectively. .Where must the boa
supported in order to remain in a horizontal position?

Ans. 1.307 ft. from the side A B, 1.0 ft. from the side

16. A uniform bar 10 ft. long and weighing 10 lbs.
weights of 5, 6, 7, and 8 lbs. suspended from it at distan
1, 2, 3, and 4 feet respectively, from one end. Find the e
brant.

Ans. 36 lbs. acting upward 3 ft. 4 inches from the end
which measurements were taken.

CHAPTER V

Moments of Inertia

I. Hollow cylinder of mass M, length l, internal radius r_1, and external radius r_2. Moment of inertia with respect to axis of cylinder $= \frac{M}{2}({r^2}_1 + {r^2}_2)$.

II. Solid sphere, mass M, and radius r. Moment of inertia with respect to a diameter $= \frac{2}{5} M r^2$.

III. Cone, mass M, radius r, and height h in the direction of axis. Moment of inertia with respect to its axis $= \frac{3 M r^2}{10}$.

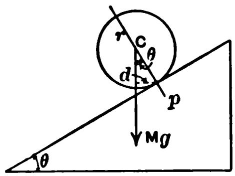
IV. Rectangular plate, mass M, length l, width b, and depth d, in the direction of axis. Moment of inertia with respect to a perpendicular axis passing through centre of mass $= \frac{M}{12}(l^2 + b^2)$.

V. Thin rectangular plate, mass M, length l, and breadth b, in direction of axis. Moment of inertia with respect to one end as an axis $= \frac{M l^2}{3}$.

VI. Thin triangular plate, mass M, altitude h, and base b. Moment of inertia with base as axis $= \frac{M h^2}{6}$.

VII. Moment of inertia of a thin plate with respect to an axis normal to the figure is equal to the sum of the moments of inertia with respect to two axes, coplanar with the figure, at right angles to each other, and whose intersection is coincident with the normal axis.

VIII. To show that the acceleration of a body rolling, on a circular section, down an inclined plane, is less than $g \sin \theta$; where θ is the angle of inclination of the plane. Let, as depicted in the accompanying figure, the homogeneous body roll on the circular section whose centre is at C; and let its moment of inertia, about an axis normal to this section, and passing through the centre of mass C, be MK^2; where M is the mass of the body and K the radius of gyration for this particular axis.



The moment of inertia then, about the parallel instantaneous axis, through the point of contact p, is

$$I = M\,(K^2 + r^2).$$

The torque, about the axis through p, is

$$G = Mgr \sin \theta;$$

and since angular acceleration is given by the ratio of torque to moment of inertia, we have

$$\alpha = \frac{M g r \sin \theta}{M(K^2 + r^2)} = \frac{r}{K^2 + r^2} g \sin \theta.$$

Multiplying by r, we find for the linear acceleration

$$a = \alpha r = \frac{r^2}{K^2 + r^2} g \sin \theta;$$

which shows, since $\frac{r^2}{K^2 + r^2}$ is less than unity, that a is less than $g \sin \theta$. It furthermore shows, since M is eliminated, that the acceleration is independent of the mass; and since, in any case $K = kr$; where k is some constant, we may write

$$a = \frac{r^2}{k^2 r^2 + r^2} g \sin \theta = \frac{1}{k^2 + 1} g \sin \theta;$$

which shows that the acceleration is independent of the radius of the section on which the body rolls.

It will prove instructive to the student to demonstrate these results from the principle of energy.

1. Find the moment of inertia of a thin square plate with respect to a diagonal as axis.

2. Find the moment of inertia of a thin trapezoidal plate about its base as axis.

3. Find the moment of inertia of a thin circular plate with respect to a diameter as axis.

4. Find the moment of inertia of a thin iron plate (density = 480 pounds per cu. ft.), 3 feet long and 1 square inch in cross-section, about one end.

Ans. 30 pound ft.^{2}

5. Find the moment of inertia of a plate whose mass is 10 pounds, the dimensions being five feet by two feet by one-eighth inch, about an axis through the centre of mass parallel to the five-foot side.

Ans. 3 1/3 pound ft.^{2}

6. What is the moment of inertia of a thin plate of iron, 10 feet long and 1 square inch in cross-section, about an axis in the plane of the plate, parallel to a short edge, and 2 feet from it, assuming the density of iron to be 480 pounds per cubic foot?

Ans. 577.8 pound ft.^{2}

7. What is the moment of inertia of a thin rectangular plate, 2 feet by 6 feet, whose mass is 4 pounds, about a long edge? About a short edge? About an axis through the centre, perpendicular to its plane?

Ans. 5 1/3 pound ft.^{2}; 48 pound ft.^{2}; 13 1/3 pound ft.^{2}

8. A wheel, consisting of a solid disk of stone, 4 feet in

diameter and 6 inches thick, makes 120 revolutions per minute. If the density of the stone is 150 pounds per cubic foot, what kinetic energy does the wheel possess?

Ans. $150\,\pi^2$ ft. lbs.

9. A cast-iron flywheel has a rim 1 inch thick, 12 inches wide, and 4 ft. mean diameter; 6 spokes, 19.5 inches long and 4 inches by 3 inches in section. The hub is 10 inches external diameter, 4 inches internal diameter, and 10 inches thick. What is the moment of inertia of the flywheel, the density of cast iron being 480 pounds per cubic foot?

	Mass.	*I.*
Rim	502 pounds	2,008 pound ft.^{2}
6 spokes	390 pounds	679 pound ft.^{2}
Hub	183 pounds	18.4 pound ft.^{2}
Total	1,075 pounds	2705.4 pound ft.^{2}

10. A hollow cylinder 6 inches long, is free to vibrate about a knife-edge support, passing through it. If its external diameter is 36 inches and its internal diameter is 18 inches, what is the moment of inertia about its support? Density of material 400 pounds per cubic ft.

Ans. Mass, 1,060 pounds; I, 2,087 pound ft.^{2}

11. A body free to rotate about an axis has its speed changed from 900 r.p.m. to 600 r.p.m. in 90 rotations. If its moment of inertia is 4,000 pound ft.^{2}, what constant torque brought about the change?

Ans. 545.4 lb. ft.

12. What is the moment of inertia of a body free to rotate, if a constant torque of 4,000 lb. ft. is necessary

to produce a speed of 1,200 r.p.m., in 90 seconds, starting from rest?

Ans. 91,670 pound ft.^{2}

13. What is the constant torque required to stop a body, whose moment of inertia is 500 pound ft.^{2}, making 1,200 r.p.m., in 60 rotations?

Ans. 327 lb. ft.

14. A rotating body mounted on a shaft, 4 inches in diameter, is making 240 r.p.m. and is being retarded by the friction of the bearings which support it. The coefficient of friction is 0.01, mass of rotating system 2,500 pounds, and its moment of inertia 7,500 pound ft.^{2} What time elapses before coming to rest? How much work has been done?

Ans. 23 min. 33.7 sec.; 74,000 ft. lbs.

15. A flywheel, whose mass is 2,000 pounds and radius of gyration 3 ft., takes 2 minutes to come to rest from a speed of 240 r.p.m. What is the retardation, and coefficient of friction at the bearings? Diameter of shaft 4 inches.

Ans. 0.21 rad. per sec. per sec.; 0.354.

16. What energy is possessed by the fly wheel in problem (9), if it makes 300 r.p.m.?

Ans. 41,460 ft. lbs.

17. A car weighing 42 tons including its 8 wheels of 500 pounds each, is moving at 30 m.p.h. If the diameter of each wheel is 3 ft., and its radius of gyration is 1 ft., what is the kinetic energy possessed by the car?

Ans. 2,595,000 ft. lbs.

CHAPTER VI

1. Assuming an efficiency of 75 per cent, what quantity of water, per minute, will a 40 H.P. engine raise from a mine 300 ft. deep?

Ans. 52.9 cu. ft.

2. A mass of 25 kilograms, perfectly free to move, is under the action of a constant force. Its velocity changes from 2 meters per second to 4 meters per second in passing over 3 meters. Find the power in watts, H.P., and kilogram-meters per sec., that is being expended when the velocity is 4 meters per sec.

Ans. 200; 0.268; 20.4.

3. A mass of 200 tons, having a velocity of 50 m.p.h. is retarded uniformly at 2 miles per hour per sec. What is the mean rate in kilowatts at which its kinetic energy is destroyed? What work in kilowatt-hours will be performed?

Ans. 1823.5; 12.66.

4. A mass of 500 kilograms, starting from rest, is made to move with a uniformly accelerated motion up a plane, inclined 30° with the horizontal, and passes over a distance of 8 meters in 8 seconds. If the coefficient of friction is 0.25, find the total work done during the eight seconds. What power is being expended at the end of the eighth second?

Ans. 29,080 joules; 9.748 H.P.

5. What power must be expended to propel, at 15 m.p.h., a 200-ton mass, up a plane inclined with the horizontal $\sin^{-1} 0.05$? What if the friction be 15 lbs. per ton?

Ans. 800 H.P.; 920 H.P.

6. To propel a mass of 400 tons along a horizontal surface at 60 miles an hour requires 950 H.P. What is the coefficient of friction?

Ans. 0.0074.

7. The angular velocity of a rotating mass changes in 5 seconds from 100 radians per second to 40 radians per second. If the mass is 1,000 pounds and its radius of gyration 5 ft., find the time rate at which its angular momentum is changing.

Ans. 300,000 F.P.S. units.

8. What is the time rate at which work is being done at the end of the 5 seconds in the previous problem?

Ans. 508.6 K.W.

9. An engine is doing work, at the rate of 40 H.P., in maintaining a constant speed of 300 r.p.m. against the force of friction applied to the circumference of its flywheel by means of a Prony brake. If the centre of the flywheel and the platform of the balance, upon which the lever arm of the brake rests, are in the same horizontal plane, then what will be the reading of the balance, if the distance between the centre of the flywheel and the point of contact on the platform is 4.5 ft.?

Ans. 155.6 lbs.

10. A force of 50 lbs. friction exists at the circumference of a pulley 8 inches in diameter. If a constant speed of 1,525 r.p.m. is maintained, what is the H.P. expended?

Ans. 4.84.

11. If the pulley in the previous problem is hollow and capable of containing 4 pounds of water, how long will it take to raise the water 160° F., assuming no heat losses?

Ans. 3 min. 7 sec.

CHAPTER VII

1. Posts are placed at the corners of a square. A rope is passed completely around them. In what direction would the posts fall if unable to withstand the pressure?

2. An elevator car weighing 2,000 lbs. is made to ascend with a constant acceleration of 16 ft. per sec. per sec. What is the tension in the rope hauling the cage? If the elevator car were falling with a constant acceleration of 32 ft. per sec. per sec., what would be the tension in the rope?

Ans. 3,000 lbs.; zero lbs.

3. A 100 lb. weight rests upon the floor of an elevator car, which is descending with a constant acceleration of 2 ft. per sec. per sec. What pressure does the weight exert upon the floor? If the elevator car were ascending with a uniform speed of 16 ft. per sec. what pressure would the weight exert upon the floor?

Ans. 93.75 lbs.; 100 lbs.

4. In an Atwood's Machine a rope is led over a pulley and has attached to the ends masses of 8 pounds and 7 pounds respectively. Assuming the equivalent mass of the pulley to be 1 pound and neglecting the mass of the rope, what is the acceleration of the system? What is the tension in each branch of the rope?

Ans. 2.0 ft. per sec. per sec.; 7.5 lbs. and 7.437 lbs.

5. Masses of 40 and 50 grams are connected together by a thin cord and hung over a frictionless pulley, whose mass may

be neglected. What distance will be traversed in 2 seconds when starting from rest? What is the tension in the cord?

Ans. 218 cm.; 44.4 gms.

6. If the masses, in the previous problem, had but passed over a distance of 196 cm. in the 2 seconds, and the diameter of the pulley is 10 cm., what would be the moment of inertia of the pulley? Its equivalent mass? The tension in the cords?

Ans. 250 gram $cm.^2$; 10 grams; 45 gms.; 44 gms.

7. A rotating body, together with the shaft upon which it is mounted, has a moment of inertia of 5,000 pound $ft.^2$ What weight must be suspended from a rope wrapped round the shaft to produce a speed of 90 r.p.m. in one minute? Diameter of shaft 12 inches.

Ans. 49.2 lbs.

8. Two masses of 0.4 and 0.6 pounds, respectively, are supported by a cord, passing over a frictionless pulley, whose radius is 3 inches. It is found that the masses in starting from rest pass over a distance of 16 ft. in 4 seconds. What is the moment of inertia of the pulley? What is its equivalent mass?

Ans. 0.1375 pound $ft.^2$; 2.2 pounds.

9. The coefficient of friction between a mass of 10 pounds and a horizontal plane is 0.2. The mass starting from rest moves over a distance of 24.6 ft. in 2 seconds, and is propelled by a cord parallel to the plane, passing over a pulley 6 inches in diameter, supporting a weight of 12 lbs. What is the moment of inertia of the pulley?

Ans. 0.251 pound $ft.^2$

10. A mass is drawn up a plane, inclined θ° with the horizontal, by a cord parallel to the plane, passing over a frictionless pulley (radius r, moment of inertia I), suspending a weight W. If the coefficient of friction between the mass and plane is μ, what is the acceleration? What are the tensions in the cord?

11. A solid drum, whose mass is 500 pounds, has a diameter of 4 feet. There is wound about the drum a rope supporting a load of 1,000 lbs. Assuming no friction, what H.P. is expended at the instant the weight is 48 ft. above its initial position, if the time consumed was 4 seconds and the acceleration constant? What is the tension in the rope?

Ans. 53.9; 1187.5 lbs.

CHAPTER IX

1. What is the length of a simple pendulum that will make one oscillation per second where $g = 32$ *ft. per sec. per sec.?*

Ans. 38.9 inches.

2. What will be the time of vibration of a simple pendulum, whose length is one meter, where $g = 980$ *cm. per sec. per sec.?*

Ans. 2.007 seconds.

3. Find the time of vibration of a thin rod 4 ft. long when vibrating about an axis 6 inches from one end. What is the equivalent length of the simple pendulum?

Ans. 1.717 sec.; 2.389 ft.

4. What is the radius of gyration of the rod in problem (3), as suspended?

Ans. 1.893 ft.

5. What will be the minimum time of vibration of the bar of problem (3)?

Ans. 1.688 seconds.

6. If the mass of the pendulum in problem (3) is 2 pounds, what are the masses which, when concentrated at the axis of suspension and oscillation respectively, will constitute a pendulum having the same characteristics?

Ans. 0.745 pounds; 1.255 pounds.

7. What is the time of vibration of the hollow cylinder, of problem (10), Chapter V? What is the length of the equivalent simple pendulum?

Ans. 1.8 seconds; 2.625 ft.

8. A thin rectangular bar 6 ft. long is suspended by a cord 9 ft. long. At what point must a blow be struck to make the system vibrate smoothly about the point of suspension of the cord? About the point of suspension of the bar?

Ans. 2.75 ft. from bottom; 2.00 ft. from bottom.

9. A mass, attached to a cord 5 ft. long, rotates in a horizontal plane making 27 r.p.m. How high will the mass be from its position of rest? What will be the velocity of the mass in its path?

Ans. 1 ft.; 8.48 ft. per sec.

10. What is the time of vibration of a rectangular plate, six feet by eight feet, about one corner, the axis being perpendicular to the plane of the plate?

Ans. 2.868 seconds.

11. Determine where else the plate of problem (10) must be suspended in order that it will vibrate in the same time.

Ans. $1\frac{2}{3}$ ft. from the centre of the plate.

12. A one-pound projectile is fired into a suspended block of wood, whose mass is 319 pounds, and causes it to rise, without rotation, a vertical height of 6 inches. What was the velocity of the projectile at the instant of the impact?

Ans. 1,810.2 feet per sec.

CHAPTER X

1. A body falling from rest passes over 496 ft. during a certain second. How long had it been in motion?

Ans. 15 sec.

2. A freely falling body starting from rest has been in motion n seconds. What will be the space traversed by it during the next t seconds?

Ans. $\frac{g t}{2}(2 n + t)$.

3. A body is dropped from an elevator ascending with a speed of 20 ft. per sec. How long will it take to reach its highest point, and then fall 100 ft.? What will its velocity be at the end of that time?

Ans. $3\frac{1}{8}$ sec.; 80 ft. per sec.

4. If an elevator is descending with a speed of 20 ft. per sec., how long will it take a body dropped from it to fall 100 ft.? What will be its velocity at the end of that time?

Ans. 1.952 sec.; 82.46 ft. per sec.

5. A body, projected vertically, has an upward velocity of 100 ft. per sec. after being in motion for 5 seconds. How high is it? How much further will it continue to rise? What time will elapse before reaching the ground?

Ans. 900 ft.; $156\frac{1}{4}$ ft.; $11\frac{1}{4}$ sec.

6. A body is dropped from a height of 100 ft. and at the same time another body is projected vertically upward with a velocity sufficient to carry it to that point. When and where will the bodies pass each other?

Ans. 75 ft. from bottom, in 1.25 sec.

7. Two masses are let fall from the same place one second apart. How long a time will elapse before the masses are 32 ft. apart?

Ans. $1\frac{1}{2}$ seconds after the first mass is let fall.

8. A body is projected vertically upward with a velocity of 40 ft. per sec. To what height will it rise? How long will it be before reaching the level from which it was projected? At the instant the body is 20 ft. from that level a second body is dropped from there. At what distance below the level will the bodies meet?

Ans. 25 ft.; 2.5 sec.; 20 ft.

9. An elevator car is ascending at the uniform rate of 32 ft. per second, and when 240 ft. above the floor of the building a ball is kicked off. In what time will the ball reach the floor?

Ans. 5.0 sec.

10. In problem (9), how high will the elevator car be when the ball strikes the floor?

Ans. 400 ft.

11. A balloon is sinking at the uniform rate of 10 ft. per second. A ball is thrown upward with a velocity, relative to the balloon, of 74 ft. per second. When the ball is at its highest point, how far down from it will the balloon be?

Ans. 84 ft.

12. What is the actual velocity of a projectile in its path at an elevation y?

Ans. $(u^2 + v^2 - 2gy)^{\frac{1}{2}}$

13. A projectile is shot over the ocean from the top of a hill 1,600 ft. high, with a horizontal velocity of 1,200 ft. per sec. Neglecting the curvature of the earth, where will the projectile strike the water? How soon will it strike the water?

Ans. 12,000 ft. from the projection of the cannon's mouth on the plane of the ocean; 10 sec.

14. A projectile is shot out with a velocity of 300 ft. per sec., and after travelling 1,000 ft. arrives at the same level with a velocity of 250 ft. per sec. What was the average resistance of the air?

Ans. 0.43 (nearly) of the weight of the projectile.

15. A gun is elevated at an angle of 60° to the horizon. A projectile shot out reaches the ground in 54 1/8 seconds. Find the initial velocity and range.

Ans. 1,000 ft. per sec.; 27,062.5 ft.

16. A projectile is shot out of a gun, elevated at an angle of 30°, with a velocity of 800 ft. per sec. Find the time of flight, maximum height, and range.

Ans. 25 sec.; 2,500 ft.; 17,300 ft.

17. With the same gun as in problem (16), but elevated at an angle so as to give the maximum range, find the time of flight, height to which the projectile will rise, and the range.

Ans. 35.355 sec.; 5,000 ft.; 20,000 ft.

18. A projectile is shot out at an angle of 45° to the horizon, with a velocity of 1,414.2 ft. per sec. To what height will it rise? What will be its range?

Ans. 15,625 ft.; 62,500 ft.

19. A baseball is struck at an angle which will give the maximum range with a velocity of 100 feet per second. A fence 12 ft. high is at a distance such that the ball will just clear it. What is the distance?

Ans. 300 ft. or 12.5 ft.

20. A projectile after being in motion for 31 seconds strikes the ground at an angle of 30°. Assuming it to have been fired from the same horizontal plane, what was its velocity?

Ans. 992 ft. per sec.

21. A projectile having a range of 30,000 ft. was in motion 50 seconds. If its initial velocity was 1,000 ft. per sec., how high did it rise?

Ans. 10,000 ft.

22. What was the velocity, in ft. per sec., of the projectile in the previous problem at an elevation of 5,000 ft.? At 10,000 ft.? After being in motion 12.5 seconds? After 25 seconds?

Ans. 824.6; 600; 721.1; 600.

23. A mass of 25 pounds is whirled in a vertical plane by a string 10 ft. long until it breaks. If the string is capable of supporting but 45 lbs., and the centre of the circle, which the mass describes, is 74 feet above a horizontal surface, what will be the range of the mass?

Ans. 32 ft.

CHAPTER XI

1. A brass rod 80 cm. long is elongated 1.8 mm. by a load of 400 lbs. If its diameter is 3 mm., what is the modulus of elasticity?

Ans. 1.12×10^{12} dynes per sq. cm.

2. The modulus of tractional elasticity of wrought iron is 15,000 tons per sq. inch, and the safe working load for tensile stresses is 10,000 lbs. per sq. inch. What will be the elongation per foot of a bar so loaded?

Ans. 0.004 inches.

3. Allowing the elongation per foot as found from the previous problem to be a safe working practice, what must be the diameter of a wrought-iron bar to support a load of 5 tons?

Ans. 1.13 inches.

4. What is the largest force that can be safely sustained by a phosphor bronze wire 0.1 inches in diameter, if an elongation of 0.0084 inches per foot is allowable? $\mu = 7{,}000$ tons per sq. inch.

Ans. 77 lbs.

5. Find the modulus of rigidity of a steel wire 75 cm. long, 4 mm. in diameter, if a force of 5 lbs., having a moment arm of 10 cm., produces a twist of 47°.

Ans. 8.09×10^{11} c.g.s. units.

6. A torque of 1 dyne-cm. applied to a quartz fibre, 10 cm. long, produces a twist of 360°. If the modulus of rigidity of

quartz is 2.9×10^{11} c.g.s. units, what is the diameter of the fibre?

Ans. 0.0273 mm.

7. Find and compare the forces in microdynes, which when having a moment arm of 1 cm., and applied to quartz fibres 10 cm. long, of diameters 0.002, 0.003, 0.004, 0.005, 0.006 mm., will produce twists of 1 radian. Modulus of rigidity for quartz 2.9×10^{11} dynes per sq. cm.

Ans. 4.56; 23.1; 72.9; 178; 369.

8. Find the period of a torsion pendulum, consisting of a cylindrical mass of 3 kilograms 10 cm. in diameter, suspended by a phosphor bronze wire 2 meters long, 0.5 mm. in diameter, whose modulus of rigidity is 3.6×10^{11} c.g.s. units.

Ans. 36.61 sec.

9. What is the moment of torsion of the wire in the preceding problem?

Ans. 1,104 c.g.s. units.

10. Find the moment of inertia of the body which when added to the cylindrical mass of the torsion pendulum, in the preceding problem, makes the period 50 seconds.

Ans. 32,400 gram $cm.^2$

11. If the modulus of tractional elasticity of the wire, in the preceding problem, is 9.3×10^{11} dynes per sq. cm., find the elongation produced by the added mass of 3 kilograms.

Ans. 3.22 mm.

12. Find the modulus of rigidity of the wire of a torsion pendulum 185 cm. long, 0.5 mm. in diameter, which has a

period of 50 seconds; the rotating mass having a moment of inertia of 86,000 gram cm.^{2}

Ans. 4.09×10^{11} c.g.s. units.

13. A steel shaft 50 ft. long, and 3 inches in diameter, transmits 10 H.P. at 250 r.p.m. How much is it twisted if its modulus of rigidity is 6,000 tons per sq. inch?

Ans. 0.908°.

14. What diameter steel shaft is necessary to transmit 50 H.P. at 200 r.p.m.? The permissible twist is 1° for every 20 diameters contained in the length, and the modulus of rigidity of steel is 6,000 tons per sq. inch.

Ans. 2.48 inches.

15. What horse-power is being transmitted by a steel shaft 320 mm. in diameter, making 75 r.p.m., if the amount of torsion for a length of 25 meters is 17.5 mm., measured along the circumference of the shaft? Modulus of rigidity of steel 8×10^{11} c.g.s. units.

Ans. 3,793.

16. The formula for the angle of torsion may be written, $\theta = K \dfrac{G L}{n d^4}$. Find the numerical value of K, so that θ will be expressed in degrees when G is measured in lb.ft., L in feet, n in tons per sq. inch, and d, the diameter, in inches.

Ans. 42.02.

17. The angle of torsion in degrees in a shaft, d inches in diameter, whose modulus of rigidity is n tons per sq. inch, when

transmitting $H.P.$ horse-power, a distance of L feet, at N r.p.m., is $\theta^\circ = C\frac{L\ HP}{n\,N\,d^4}$. What is the numerical value of C?

Ans. 220,700.

18. With the aid of the constant found in problem (17), recalculate problem (13).

19. A rectangular bar 0.75 inches wide, and 0.25 inches thick, is lying flat; when it is supported at points 3 ft. apart, it is deflected 0.177 inches by a load of 5 lbs. midway between them. What is the modulus of elasticity?

Ans. 14,060 tons per sq. inch.

20. Compare the deflection of a rectangular beam supported at each end lying flat, with that obtained when turned on edge, other conditions remaining the same.

21. A rectangular beam, whose section is 8 inches by 4 inches, is supported at the ends, and is loaded at the middle by a force, applied parallel to the 8-inch side. If the resulting deflection be one inch, what would the deflection be if the same force were applied at the middle, parallel to the 4-inch side?

Ans. 4 inches.

22. A beam of rectangular cross-section, 20 feet long, 18 inches deep, and 6 inches wide, is supported at the ends and loaded at the middle by a force of 1,000 lbs. If the modulus of elasticity is 1,500,000 lbs. per sq. inch, what is the deflection?

Ans. 0.0658 inches.