On Niemand's Factoring of Polynomials

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Professor Niemand

Late one night, I was considering the problem of reducing this to lowest terms:

$$\frac{x^4+2x^3-10x^2-11x-12}{x^4+3x^3-7x^2-21x-36}$$

I had not intended to think about this problem at all. I happened to see it in passing and the sight of it was upsetting to me. I felt as if I should be able to solve it easily. But no ideas came. I further made the mistake of copying it out of Todhunter's *Algebra* onto my scratch paper. And there it sat, filling me with darkness and doubt – I said it was late.

It's a division problem, obviously. But I was tired and saw it as a factoring problem. I felt I should be able to factor those two polynomials and had somehow missed or forgotten the method of factoring them.

It was at this moment that a cloud of tobacco smoke filled the doorway and not the smoke of a blend that should be smoked indoors. From out of this cloud came the bowl of an immense Meerschaum pipe, followed by its smoker -- a large, bearded man in an ill-fitting suit. I recognized him immediately, just as you would.

It was Herr Doktor Niemand. We've heard of him all our lives. He is smarter than Newton or Gauss. He can square the circle or trisect an angle with only a straight-edge and compass. And he regularly derives the roots of polynomials of arbitrary degrees greater than fourth. Just the man I wanted to see.

"Herr Niemand," I said. "Welcome. Come in. What brings you here?"

"You appear to be struggling," he replied. "And the sufferings of lesser mathematicians are pleasing to me."

Nobody ever said he was a nice man.

"I am struggling," I said and showed him my problem.

"Kinderspiel," he shrugged. "I see no difficulty here at all."

"Enlighten me," I replied.

"Very well," he said. "What is a polynomial?"

On Polynomials

"An algebraic expression?" I replied.

"Mein Gott," he rumbled, "and a man is a hairless biped which does not lay eggs."

"I have hair," I said.

"Less and less," he said, gesturing at my receding hairline. "Next you will tell me you lay eggs. What is a polynomial?"

"A series of values?" I ventured.

"Only from the vantage point of eternity. For mere mortals?"

"A value depending on x, then," I said.

"Very good, mein Kind," he said. "And of what nature is that value?"

"A number?"

"You are slow, but almost there," he said. "As you look at these polynomials, this $x^4+3x^3-7x^2-21x-36$, what does it remind you of?"

"Chaos and old night?" I said, yawning.

"Am I boring you?" he thundered.

"No, mein Herr," I answered promptly. "What should it remind me of?"

"Think back to arithmetic," he said. "You recently read Herr De Morgan's *Elements of Arithmetic*, did you not? Think at least that far back."

I did as instructed.

"I suppose it could remind me of De Morgan's explanation of our number system. Thirty-six ones, twenty-one tens, seven hundreds. And then you..."

"Excellent," said Niemand. "And therefore?"

"Alright," I said. "This could be a number. But the x's, the plus and minus signs, and the double digits confuse the issue. And how is this a particular number if we don't choose a value for x?"

"How indeed," he replied. "Let us take a simpler example."

When All X's are Tens

"Consider the following," said the learned professor: "

$$\frac{x^3+3x^2+3x+2}{3x^3+8x^2+5x+2}$$

Now there are no tricky minus signs or double digits to confuse your tiny mind. If we proceed with your remembrance of Herr De Morgan's arithmetic lesson, what is the value of x?"

"Ten," I said, for once sure of myself.

"You begin to show promise," smiled the doctor. "And the fraction becomes?"

"

$$\frac{1332}{3852}$$

," I said.

“Correct. Factor, please,” he commanded.

I factored, arriving at:

$$\frac{2^2 \cdot 3^2 \cdot 37}{2^2 \cdot 3^2 \cdot 41}$$

and showed Herr Niemand my result.

"Yes. Very good. And thus the Greatest Common Measure of these numbers?" he said.

"36," I replied.

Again I was pretty sure of myself.

"Now restore your x," he said, with a tone of our having arrived.

Apparently, I was still I transit.

"My what?" I asked.

"You are slow, mein Kind," he said, "and thick. But we overlook your failings. In our discussion so far, what is x?"

"Ten?"

I was beginning to doubt even the obvious.

"Yes. Of course. So restore it," he said. "Restore it to 36."

"3x+6?" I said, slowly.

"There may be hope for you yet," he said. But he shook his head doubtfully.

On the Nature of Factors

"Let us proceed," said Herr Niemand. "Please divide your polynomials by 3x+6."

I did so:

$$\frac{\frac{1}{3}x^2+\frac{1}{3}x+\frac{1}{3}}{x^2+\frac{2}{3}x+\frac{1}{3}}$$

"It divides evenly," I said.

"Of course," he replied. "Numbers are numbers. Factors are factors."

"But I'm not crazy about the fractional coefficients," I said.

"Consider your divisor," he said, pointing to my notes. "How many factors is that really?"

"Two," I said. "3 and x+2."

"True," he said, "and if (3)(x+2) is a factor, then (n)(x+2) is a factor, so to speak. The n merely varies the fractional coefficients."

"So the actual factor here is x+2," I said.

Light began to shine in the darkness and I redivided:

$$\frac{x^2+x+1}{3x^2+2x+1}$$

"Yes," he said, smiling and relighting his pipe.

"Then we can do it again," I said.

"Do what again?" he asked.

"Use the same process to derive the roots of these equations," I said.

"You can try," he said. "Proceed."

I proceeded.

$$\frac{111}{321} = \frac{3 \cdot 37}{3 \cdot 107}$$

"We have a problem," I said. "The numerator gives us $3 \cdot 3x+7$ and the denominator is $3 \cdot x^2+7$. That's $9x+21$ and $3x^2+21$."

"Which gives you 111 and 321," he said. "Where is your problem?"

"What about the roots?" I asked.

He shook his head as if pitying me.

"Liebes Kind," he said, "the numbers 37 and 107 are prime. And until you can divide them evenly..."

"And you can?" I asked.

And then I remembered.

"Of course, you can," I said.

He smiled his irritating smile.

"You could teach me," I said, hopefully.

"I tried that once," he replied. "A Herr Galois, as I recall. It ended badly."

"But he couldn't do that with primes," I said.

"He could," said Niemand. "I taught him. Unfortunately, as he was making his notes, his mind was pre-occupied with self-destruction. Such is not conducive to mathematical exposition."

I thought a moment.

"So if it hadn't been for the French Academy's obtuseness...?" I said.

Niemand shrugged.

"Let us return to your original problem," he said. "The witching hour approaches."

On Those Troublesome Minus Signs

I wrote the problem down afresh:

$$\frac{x^4+2x^3-10x^2-11x-12}{x^4+3x^3-7x^2-21x-36}$$

"Proceed," said Niemand again.

"What about the minus signs?" I asked.

"Have you forgotten your subtraction?" he asked. "Arithmetic is arithmetic."

"You speak as if everything were number," I mused.

"And everything we encounter here is indeed just that," he said.

"You are a Pythagorean then?" I said, trying not to laugh.

"Certainly, not," he said. "My sense was mathematical, not metaphysical. Consider hyperbolic spaces, which have been shown to be closely related to irrational numbers. Indeed, what are they but a restatement of the irrationals in Byzantine form? Upon a spherical space, all lines are fractions of unity and so we restate the rationals in space. Euclid then becomes a restatement of the natural numbers."

"But arcs of a meridian or lines in the plane can be incommensurables," I protested.

"Incummensurable to what?" he asked. "And therefore commensurable to what? But time presses. Proceed with our method."

I finally did the math and wrote the following:

$$\frac{10878}{10254} = \frac{2\cdot3\cdot7^2\cdot41}{2\cdot3\cdot7^2\cdot37}$$

"Consider your Greatest Common Measure," said Niemand.

"That would be 294," I said. "Or $2x^2+9x+4$. But that won't work. Because we're left with, say, 41 or 4x+1. And multiplying those two expressions gives us..."

"$8x^3+38x^2+25x+4$," he said. "Arithmetically correct. But not very helpful, as you can see. Consider your factors and look for one that will not give you fractional coefficients..."

"And whose last element is a factor of 12 and 36," I said.

"You do begin to show promise," he smiled. "In the earlier problem, 18 was a factor. But x+8 would not have worked because 8 is not a factor of 2."

"But neither was 6. Both 6 and 8 are multiples of 2," I said. "3x+6."

"Yes," he said. "But you can factor x+2 out of that GCM. That cannot be said of x+8."

"I see," I said. "So here given $2 \cdot 3 \cdot 7^2$ we have possible factors of 6, 14, 21, and 49. Fourteen should work. So factoring out x+4 gives us

$$\frac{x^3-2x^2-2x-3}{x^3-x^2-3x-9}$$

and that's as far as we can go. Because now we're down to 3·7·41 and 3·7·37. Or 2x+ 1 times either 4x+1 or 3x+7. Both give second degree results."

The Limits of Mortality

I sat back in my chair.

"So these are now in lowest terms?" I asked

"For you, perhaps," he said. "I would offer to finish it for you. But you wouldn't be able to write down the incommensurables. For a mere mortal, you certainly have..."

He was cut off by the clock striking midnight and disappeared, leaving only his unpleasant smoke behind. I wondered what it was he thought I had. Then I decided I'd probably be happier not knowing. At least I had grasped the basics of his heuristic. Niemand's Method can be used more broadly than factoring polynomials. For instance, you can find the square roots of polynomials. As simple examples:

$$x^2+2x+1 \text{ is } 121$$
$$\sqrt{121} = 11 \text{ or } x+1$$

$$x^2-2x+1 \text{ is } 81$$
$$\sqrt{81} = 9 = 10 - 1 \text{ or } x-1$$

In the case square roots, this just shows the basic reasoning. So if you were to want the square root of:

$$4x^4-12x^3+5x^2+6x+1 \text{ or } 28561$$
$$\sqrt{28561} = 169 \text{ or } x^2+6x+9$$

Not helpful. But you would realize that the first term must be $2x^2$ (because it has to square into $4x^4$) and the last term must be ±1 (because none of the coefficients smell like nines). So reasoning from the options 201 and 199, the middle term is, from the latter, clearly -3x and the result:

$$2x^2-3x-1 = 169$$

You can see that all of this is simply arithmetic combined with some extra thought to handle the minus signs. It would be nice if, somehow, Niemand's Method took us further and factored the "prime" polynomials. But as a mere mortal, I'll take what I can get.

Niemand's Standpunkt

It does strike me that something fundamental is going on here. Our civilization's mathematic, in the largest sense, is base ten. In these integral functions, or polynomials, the coefficients are base ten. So making x equal 10 seems somehow significant.

You can see in the last example that 28561 is attainable with infinitely many combinations of coefficients using positive or negative terms as needed. Then isn't $2x^4+8x^3+5x^2+6x+1$ somehow a more fundamental representation of $4x^4-12x^3+5x^2+6x+1$ in some sense?

I am aware that this rather turns our views of functions on its head. Since the advent of analytical geometry, calculus, and function theory, we look at a polynomial like this as a variable y which, as a function of x, takes on infinitely many values. Further, the function, so to speak, is its graph and vice versa. From this point of view, those infinite polynomials which at x=10 converge at 28561 are an infinity of spaghetti with a common intersection which they consider infinitely insignificant.

But in Niemand's view, this infinite set of functions would be the class that maps to 28561 of which $2x^4+8x^3+5x^2+6x+1$ is the fundamental representation. No other value of x, so far as I can see, is “fundamental” in this sense. If we made x=6, we'd be talking base six and would have to convert the coefficients -- if not go live in another civilization entirely. This seems to show a fundamental element in Niemand's approach given the form of our mathematic.

This gives rise to such questions as when x^2-2x+1, being 81, is somehow at bottom 8x+1, then what is the relation that must arise from a line and connect upwards into various curves? And what about the classes of constants?

Knowing what that element is would require perhaps a deeper basis of number theory and function theory than I currently have. And while more function theory is on my to-do list, I feel the same way about number theory that I do about many green vegetables. My aversion is foolish and irrational. But there you are.

So should someone with more talent and better tools actually take this somewhere, I expect them to give Herr Doktor Niemand due credit. I am indebted to Lewis Carroll for introducing me to him in Carroll's **Euclid and His Modern Rivals**. And nobody deserves more respect than Niemand does, which is technically a double negative.

Some Initial Noodling

Let us take a moment to look at the simplest examples of Neimand's Method of Factoring or NMF. Let us call the value at x = 10 the niemand value or nval so for x^2+4x+4 the nval is 114.

$x^2+15x+36$ nval(286) = 2·11·13 (x + 3)(x + 12) ∴ 13·22	$x^2+9x-36$ nval(154) = 2·7·11 (x - 3)(x + 12) ∴ 7·22
$x^2-15x+36$ nval(-14) = -1·2·7 (x - 3)(x - 12) ∴ 7·-2	$x^2-9x-36$ nval(-26) = -1·2·13 (x + 3)(x - 12) ∴ 13·-2
$2x^2+8x+6$ nval(286) = 2·11·13 (2x + 6)(x + 1) ∴ 26·11	$2x^2+8x+5$ nval(285) = 3·5·19 (2x + a)(x + b) integral ab must be 1·5 ∴ no integer soln for factors 3·5·19
$2x^2+8x+6 = 2(x^2+4x+3)$ nval(143) = 11·13 (x + 3)(x + 1) ∴ 13·11	x^2+x+3 nval(113)≡prime ∴ irreducible (complex factors)

Binomial Theorem

This section is more about using the NMF in general than in using it with the Binomial Theorem. It show how the heuristic is combined with reason to arrive at a known form. I was thinking about Niemand while working with Newton's Binomial Theorem. And in this case, Niemand shows itself an heuristic rather than a general solution. If we take

$$(x+2)^2$$

and use Niemand's x=10, 12^2 is 144 or

$$x^2+4x+4$$

which is the correct binomial expansion. But for

$$(x+2)^3$$

we have $12^3 = 1728$ and

$$x^3+7x^2+2x+8 \quad \text{nval}(1728)$$

is not the binomial expansion. But knowing the **form** of the binomial expansion, we know that the coefficient of unity for x^3 and 8 are correct. And we know that the second coefficient must be divisible by 2 and the third by 3. So taking one from the hundred's place and putting ten in the ten's place, we have

$$x^3+6x^2+12x+8 \quad \text{nval}(1728)$$

which is correct. As a final example, let's take

$$(x+3)^3$$

$13^3 = 2197$ or

$$2x^3+x^2+9x+7$$

If we fix the outer ones first, we have

$$x^3+11x^2+7x+27$$

which we can think of as

$$x^3+ax^2+bx+27$$

where a must be divisible by 3 and b by 9. So moving 2 from the 100s into the tens, we have

$$x^3+9x^2+27x+27$$

which is again the correct expansion.

Niemand's method is a heuristic because it does not express things generally. It's specific expression is, however, a useful tool. If in $(x+3)^3$ we let x=2, we get 125 or x^2+2x+5 and one cannot work backwards from this to $x^3+9x^2+27x+27$. I want to say that Niemand's approach is a slice through everything in base 10. So if x=2 we are somehow working with base 2 using base 10 (or vice versa, I'd have to think about it to be sure) and the result is a hairball.

Niemand's method is useful if and only if we know the **form** of things. Then, everything being number, what is true of Niemand is true of the binomial expansion or the factors of a polynomial. But we must know the true form of those things in order to use the NMF. And then his method greatly simplifies what we are doing in the same way that the elementary division algorithm simplifies division or using Euclid's algorithm simplifies creating a continued fraction.

Solving Equations

Consider some fifth degree equation factored as

$$(x-a)(x-b)(x-c)(x-d)(x-e)$$

If we make x = 10, we get

$$(10-a)(10-b)(10-c)(10-d)(10-e) = nval_{10}$$

We could continue in this way to get five equations of five unknowns:

$$(10-a)(10-b)(10-c)(10-d)(10-e) = nval_{10}$$
$$(1-a)(1-b)(1-c)(1-d)(1-e) = nval_{1}$$
$$(2-a)(2-b)(2-c)(2-d)(2-e) = nval_{2}$$
$$(3-a)(3-b)(3-c)(3-d)(3-e) = nval_{3}$$
$$(4-a)(4-b)(4-c)(4-d)(4-e) = nval_{4}$$

When you multiply (x+a)(x+b) you get x^2+(a+b)x + ab where a+b is the sum of the "product" of the roots chosen one at a time and ab is the sum of the product of the roots taken two at a time. You can multiply out (x+a)(x+b)(x+c) to see what I am talking about if you are not familiar with how combinatorics plays into things like polynomials and the binomial theorem. If we multiply out the equations in our above system, the meta-variables are sums or products of the roots.

Without doing any nasty multiplying, I am familiar enough with this business to know what kind of a hairball of algebra would be produced. Then you would have to solve for a, b, c, d, and e. All of which sounds unpleasant. But there are computer algebra programs out there in university computer labs and people with access to those computers. So if anyone would like to run such a thing on, first, a solveable quintic in Galoisean terms and, if that works, run it on an unsolveable quintic, and let's see what you get.

"Let's see what you get" means "email me the results no matter how badly it bombs out." And if it doesn't bomb out, don't forget to acknowledge those to whom credit is due, especially Her Doktor Niemand.

Of Prime Factors and the Factors of Primes

Sometimes the nval is a prime number like 113. I was hoping that this would tell us something about the roots. But the nval doesn't seem to relate numerically to the roots. It is only the 3 in x^2+x+3 that directly relates to surd factors. One line of inquiry is to determine whether some relation exists between the "110" or the two ones and the three that determines the form of the pair of surds or complex roots. Here the roots are $(-1\pm\sqrt{-11})/2$. But $2x^2$-9x+3 also has an nval of 113 and its roots are $(9\pm\sqrt{57})/4$. The 3 is effected by factors. If we make that last equation x^2-

9/2x+3/2 then the last term is the product of the roots as it should be. So with nvals, the final 3 is only a 3 if the leading coefficient is unity.

Still, if we take three digit number in the form 1n3 with n={0-9}, we get different factors of 3 and with x^2+6x+3 the nval is prime and the factors of 3 are $(-3\pm\sqrt{6})$. I'm probably not the first to notice this, but these varying factors suggest that for $\forall n,m\in\mathbf{N}$ any integer n can have any m factors, some of which may be mth roots of unity, and there are infinitely many different sets of these m factors for every n.

Along these lines of prime nvals, let's talk about a couple of common theorems used for factoring polynomial. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 : a_i \in \mathbf{Z}$$

The Rational Root Test says, if r≠0 and r/s (lowest terms)∈**Q** there are rational roots if r divides a_0 and s divides a_n. So for $4x^4-12x^3+x^2-4x+3$, r can be ±1,±3 and s can be ±1,±2,±4 and you have to plug them in. But here the nval is 28063 = 7·19·211. The 7 suggests (x-3) and f(3) is seen to be zero. The 19 can be x+9 or 2x-1 or nx - (10(n-1)+1)) and from 2x-1, f(1/2) = 0. Factor out these two and we are left with $2x^2+x+1$ for the 211. Because nval(211) can take many forms, you have to validate it somehow.

Eisenstein's Criterion says that if we have this same kind of f(x) which is non-constant and has coeff∈**Z**, if ∃prime p: p divides a_i [0-(n-1)] and does not divide a_n, then f(x) is irreducible in **Q[x]** (polynomials with rational coeff.) So $x^7+6x^5-15x^4+3x^2-9x+12$ is irreducible using p=5. The nval here is 10450222 = 2·53·311·317. The 2 as x-8 is quickly discarded. So we can see even more clearly than Sergei here (I assume this guy's the film director) that we are dealing with only surds and complex roots. If we can ever relate the factor of the nval to the roots **in any way**, we would have even more to work with, if not the roots themselves.

Regular Integral Functions

So what are the cases of $ax^2 + bx + c$ where the factors of the nval reveal the roots?
$2x^2 - 8x + 8$ nval $128 = 2^7 = 16\cdot 8$ or $(2x - 4)(x - 2)$ But this is just 2f(x) where f(x) is
$x^2 - 4x + 8$ nval $64 = 2^6 = 8\cdot 8$ or $(x - 2)(x - 2)$ or $(x - 2)^2$ We can see here that if the nval is a perfect power of n, it does not mean that we have an n-fold root. We may or we may not. But we do see that if $f(x) = ax^2 + bx + c$ and ${}^1/_a f(x)$ is a rifn (regular integral fn) then the factors of ${}^1/_a f(x) \cdot a$ are the factors of f(x) and the roots are identical. None of this is news. But it does indicate that Niemand's method is more useful if restricted to analysing rifn.

$2x^2 - 9x + 3$	nval 113	prime	roots are surds
$x^2 - {}^9/_2x + {}^3/_2$	nval 56.5	fractional	same roots
$2x^2 - 9x + 2$	nval $112 = 2^4 \cdot 7$	non-prime	roots are surds
$x^2 - {}^9/_2x + 1$	nval $56 = 2^3 \cdot 7$	non-prime	same roots
$x^2 + x + 3$	nval 113	prime	roots imaginary

$2x^2 + 2x + 6$ nval 226 non-prime same roots

In the purely non-prime example, all the nval tells us is that by inspection neither a power of 2 nor 7, giving $(x - 2^n)$ or $(x - 3)$ for factors, will give us a usable factor. So the roots will be surds. But in the third pair, where the base rifn has a prime nval, the roots are imaginary. It appears at this point that a prime nval of a rifn indicates imaginary roots. But I haven't validated this in general.

Perhaps I should say that while I have a serious interest in algebra and polynomials, Niemand's method isn't something I spend much time on. In fact, I'm a bit surprised it hasn't just gone away. But it continues to be useful in my work with integral functions and continues to tease me with possibilities. So in spite of its remaining a mere heuristic, things keep bringing me back to it and I share them as I go along.

nval and factors

The nval is consistent in polynomial division. Let's look at nvals of 144: $x^2 + 4x + 4$ factors into $(x + 2)^2$ or $12\cdot12 = 144$. $2x^2 - 6x + 4$ factors into $(x + 2)(2x - 10)$ r 24 and $12\cdot10 + 24 = 144$. But $2x^2 - 6x + 4$ also factors evenly into $(x - 2)(2x - 2)$ or $8\cdot18 = 144$. $3x^2 - 16x + 4$ factors into $(x - 2)(3x - 10)$ r -16 and $8\cdot20 - 16 = 144$.

We can use the nval backwards to investigate the possibility of factors. With $3x^2 - 16x + 4$ we would need $(3x - a)(x - b)$ just from the form of things and $-a \cdot -b = 4$. The nval is 144. So from the form of the factors, we can ask ourselves, using "?" as a digit's placeholder, can there be a $2? \cdot ? = 144 = 2^4 \cdot 3^2$. The only possibility for any 2? is $3\cdot8 = 24 = (3x - 6)$ and the -6 scotches our required 4.

$x^3 + 12x^2 - 32x - 256$ has an nval of $1624 = 2^3 \cdot 7 \cdot 29$. The 12, 32, and 256 encourage us to believe f(x) is divby [divisible by] $(x + 4)$ and synthetic division shows us that it is and with a quotient of $x^2 + 8x - 64$. $(x + 4)$ is a root of -4 and an nval of 14. $1624\div14 = 116$ and this is the nval of our quotient with factors of $2^2 \cdot 29$. The 64 and the 29 tell us there are no more integral factors and that the product of the surd roots is -64. The roots are $(-4 \pm 4\sqrt{5})/4$.

Here, the usefulness of Niemand's method for me is that this much analysis takes far less time than reading this last paragraph. One can very quickly use Niemand's method successfully or, just as quickly, see that it can't go any further. In my experience, using Niemand first in factoring polynomials or similar work saves time.